VOLUME I

Solutions Manual

TO ACCOMPANY

CALCULUS

SECOND EDITION

BY MUNEM & FOULIS

WORTH PUBLISHERS, INC.

Solutions Manual

VOLUME I

Solutions Manual

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TO ACCOMPANY

CALCULUS

with Analytic Geometry

SECOND EDITION AND
BRIEF EDITION

WORTH PUBLISHERS, INC.

SOLUTIONS MANUAL.

Volume I

to accompany

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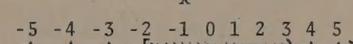
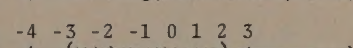
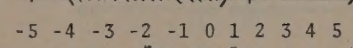
CONTENTS

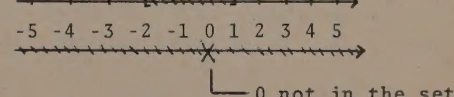
1 FUNCTIONS AND LIMITS

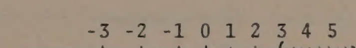
Problem Set 1.1, page 8

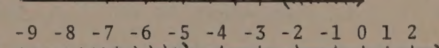
1. (a) True. The product of two positive numbers is positive.
   (b) True. $x < 3$ and $3 < y$, so $x < y$ by the transitive law.
   (c) False. Put $x = -1$ and $y = 0$. Then $x < y$, but $-5x = 5$, $-5y = 0$ and so $-5x > -5y$.
   (d) True. For suppose that $x > 3$. Then since $x > 0$, $x^2 > 3x$. Also, since $3 > 0$, $3x > 3 \cdot 3 = 9$. From $x^2 > 3x$ and $3x > 9$, we conclude that $x^2 > 9$, contradicting $x^2 \le 9$. Hence, $x > 3$ must be false, so that $x \le 3$ as claimed.
   (e) True. If $x \ge 2$, then $x > 0$ and, since $y > x$, we have $y > 0$ by the transitive law.
2. Suppose first that $x > 0$. Multiplying both sides of the latter inequality by x (which is permitted since $x > 0$), we obtain $x^2 > 0$. Now suppose that $x < 0$. Multiplying both sides of the latter inequality by x and reversing the inequality sign since $x < 0$, we again obtain $x^2 > 0$.
3. Since $(3)(233) = 699 < 700 = (28)(25)$, it follows that $\frac{(3)(233)}{(28)(233)} < \frac{(28)(25)}{(28)(233)}$; that is, $\frac{3}{28} < \frac{25}{233}$.
4. Suppose that $0 < x < y$. Then $x > 0$ and $y > 0$, so that $xy > 0$. Dividing both sides of $x < y$ by the positive number xy, we obtain $\frac{1}{y} < \frac{1}{x}$; that is, $\frac{1}{x} > \frac{1}{y}$.
5. (a) The condition $-x > 0$ holds precisely when x is negative. This can be seen by multiplication of both sides of the inequality by -1, causing the inequality to reverse.
   (b) The condition $-x < 0$ holds precisely when x is positive. Again, multiply both sides by -1, reversing the sense of the inequality sign.
   (c) The condition $-x = 0$ holds precisely when $x = 0$. This is seen by multiplying both sides of the equation by -1.
6. (a) From $x < y$ and $x > 0$, we have $x^2 < xy$. From $x < y$ and $y > 0$, we have $xy < y^2$. Therefore, $x^2 < y^2$ by the transitive law.
   (b) Suppose that $0 < x < y$. If $0 < \sqrt{y} < \sqrt{x}$, we would have $y < x$ by part (a); hence we must have $\sqrt{x} \le \sqrt{y}$. If $\sqrt{x} = \sqrt{y}$, then $x = y$; hence we must have $\sqrt{x} < \sqrt{y}$.
7. No. For example, take $x = -3$. Then $x^2 \ge 9$.
8. Yes. Assume that $x > 0$. If $\frac{1}{x} \le 0$, then multiplying by x, $1 \le 0$, which is a contradiction. Hence, $\frac{1}{x} > 0$.
9. (a) 

   (b) 

   (c) 

   (d) 

   (e) 

   (f) 

10. (a) (2,5) together with [6,8].

(b) [2,6)

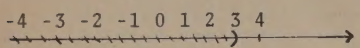
(c) (-1,2) together with (4,7).

(d) [2,3) together with (3,7].

(e) $(-\infty,-1]$ together with $(2,\infty)$.

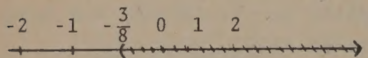
11. $10x - 4x < 18$, $6x < 18$, $x < \frac{18}{6}$, $x < 3$.

In interval notation: $(-\infty,3)$.



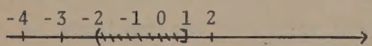
12. $\frac{9}{4} < \frac{5}{2} + \frac{2}{3}x$, $\frac{9}{4} - \frac{5}{2} < \frac{2}{3}x$, $-\frac{1}{4} < \frac{2}{3}x$, $-\frac{3}{8} < x$.

In interval notation: $(-\frac{3}{8},\infty)$.

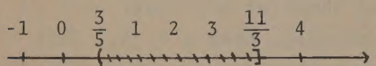


13. $2 \leq 5 - 3x < 11$, $2 - 5 \leq -3x < 11 - 5$, $-3 \leq -3x < 6$, $3 \geq 3x > -6$, $-\frac{6}{3} < \frac{3x}{3} \leq \frac{3}{3}$, $-2 < x \leq 1$.

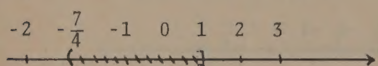
In interval notation: $(-2,1]$.



14. $3 < 5x \leq 2x + 11$, so that $3 < 5x$ and $5x \leq 2x + 11$. From $3 < 5x$, conclude that $\frac{3}{5} < x$. From $5x \leq 2x + 11$, conclude that $5x - 2x \leq 11$, $3x \leq 11$, $x \leq \frac{11}{3}$. Thus, $\frac{3}{5} < x \leq \frac{11}{3}$. In interval notation: $(\frac{3}{5}, \frac{11}{3}]$.

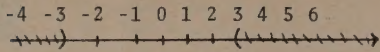


15. $3 > -4 - 4x \geq -8$, $-3 < 4 + 4x \leq 8$, $-3 - 4 < 4x \leq 8 - 4$, $-7 < 4x \leq 4$, $-\frac{7}{4} < x \leq 1$. In interval notation: $(-\frac{7}{4},1]$.

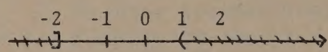


16. $8 - 4 < \frac{3}{x} - \frac{2}{x}$, $4 < \frac{1}{x}$; therefore, $x > 0$ and $x < \frac{1}{4}$. In interval notation: $(0,\frac{1}{4})$.

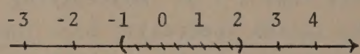
17. $x^2 > 9$ is equivalent to $\sqrt{x^2} > 3$, that is, to $|x| > 3$. Since $|x| > 3$ is equivalent to $x > 3$ or $x < -3$, the solution in interval form is $(-\infty,-3)$ together with $(3,\infty)$



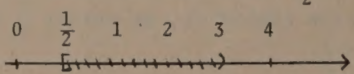
18. $\frac{3}{1 - x} \leq 1$, $0 \leq 1 - \frac{3}{1 - x}$, $0 \leq \frac{1 - x - 3}{1 - x}$, $0 \leq \frac{-x - 2}{1 - x}$, $0 \leq \frac{x + 2}{x - 1}$, so either $x + 2 = 0$, or else $x + 2$ and $x - 1$ have the same algebraic sign. If $x + 2 > 0$ and $x - 1 > 0$, then $x > -2$ and $x > 1$, that is, $x > 1$. If $x + 2 < 0$ and $x - 1 < 0$, then $x < -2$ and $x < 1$, that is, $x < -2$. Since $x + 2 = 0$ when $x = -2$, then the solution set is $(-\infty,-2]$ together with $(1,\infty)$.



19. Since $x^2 - x - 2 = (x - 2)(x + 1)$, the inequality is equivalent to $(x - 2)(x + 1) < 0$; that is, to the condition that $x - 2$ and $x + 1$ have opposite algebraic signs. Thus, either $x - 2 > 0$ and $x + 1 < 0$ or else $x - 2 < 0$ and $x + 1 > 0$. In the first case, $x > 2$ and $x < -1$, which is impossible. Thus, the second case must hold, so $x < 2$ and $x > -1$; that is, x belongs to the interval $(-1,2)$.



20. $\frac{5}{3 - x} \geq 2$. In order for the inequality to hold, $\frac{5}{3 - x}$ must be positive; so $3 - x > 0$ and $3 > x$. Assume, then, that $3 - x > 0$. Thus, $\frac{5}{3 - x} \geq 2$ is equivalent to $5 \geq 2(3 - x)$; that is, $5 \geq 6 - 2x$, or, $2x \geq 1$. Hence, the solution set consists of all values of x with $x < 3$ and $x \geq \frac{1}{2}$. In interval notation: $[\frac{1}{2},3)$.



21. $x^2 \leq 4$ is equivalent to $\sqrt{x^2} \leq 2$, that is, to $|x| \leq 2$. Since $|x| \leq 2$ is equivalent

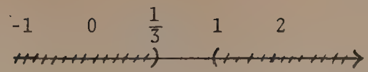
to $-2 \le x \le 2$, the solution in interval form is: $[-2,2]$.

-2 -1 0 1 2

22. $\frac{3+x}{3-x} - 1 \le 0$, $\frac{3+x-(3-x)}{3-x} \le 0$, $\frac{2x}{3-x} \le 0$. Equality is obtained when $x = 0$. Otherwise, $\frac{2x}{3-x} < 0$ and so $2x$ and $3 - x$ have opposite algebraic signs; that is, either $2x < 0$ and $3 - x > 0$, or else $2x > 0$ and $3 - x < 0$. Hence, since the former condition is equivalent to $x < 0$ and the latter to $x > 3$, the solution set consists of $(-\infty,0]$ together with $(3,\infty)$.

23. $2x^2 + 5x - 12 < 0$; $(2x - 3)(x + 4) < 0$. $2x - 3$ and $x + 4$ have opposite algebraic signs; that is, either $2x - 3 < 0$ and $x + 4 > 0$ or else $2x - 3 > 0$ and $x + 4 < 0$. Thus, either $x < \frac{3}{2}$ and $x > -4$ or $x > \frac{3}{2}$ and $x < -4$. Since the latter case is impossible, the solution set consists of $(-4,\frac{3}{2})$.

24. $0 < \frac{x-1}{2x-1}$ is equivalent to the condition that $x - 1 > 0$ and $2x - 1 > 0$ or else $x - 1 < 0$ and $2x - 1 < 0$; that is, $x > 1$ or else $x < \frac{1}{2}$. Similarly $\frac{x-1}{2x-1} < 2$; that is, $0 < 2 - \frac{x-1}{2x-1}$, or, $0 < \frac{3x-1}{2x-1}$ is equivalent to $x > \frac{1}{2}$ or else $x < \frac{1}{3}$. Thus, the solution set for $0 < \frac{x-1}{2x-1}$ is the pair of intervals $(-\infty,\frac{1}{2})$ and $(1,\infty)$, while the solution set for $\frac{x-1}{2x-1} < 2$ is the pair of intervals $(-\infty,\frac{1}{3})$ and $(\frac{1}{2},\infty)$. The solution set for $0 < \frac{x-1}{2x-1} < 2$ consists of all real numbers belonging to <u>both</u> of these solution sets; hence, it consists of the pair of intervals $(-\infty,\frac{1}{3})$ and $(1,\infty)$.



25. $x - 3 = \pm 2$, $x = 3 \pm 2$; $x = 5$ or $x = 1$.

26. $x - 5 = \pm(3x - 1)$, $x \mp 3x = 5 \mp 1$; $-2x = 4$ or $4x = 6$; $x = -2$ or $x = \frac{3}{2}$.

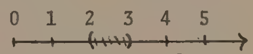
27. $3y + 2 = \pm 5$; $y = \frac{-2 \pm 5}{3}$; $y = -\frac{7}{3}$ or $y = 1$.

28. $t - 2 = \pm(3 - 5t)$; $t \pm 5t = 2 \pm 3$; $6t = 5$ or $-4t = -1$; $t = \frac{5}{6}$ or $t = \frac{1}{4}$.

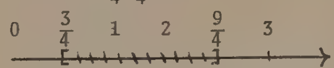
29. $5x = \pm(3 - x)$, $5x \pm x = \pm 3$; $6x = 3$ or $4x = -3$; $x = \frac{1}{2}$ or $x = -\frac{3}{4}$.

30. $|y^2 + y - 6| = 0$ so that $y^2 + y - 6 = 0$ or $(y + 3)(y - 2) = 0$. Thus, $y + 3 = 0$ or $y - 2 = 0$. Hence, $y = -3$ or $y = 2$.

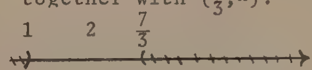
31. $-1 < 2x - 5 < 1$, $5 - 1 < 2x < 5 + 1$, $\frac{4}{2} < x < \frac{6}{2}$, $2 < x < 3$. The solution set is $(2,3)$.



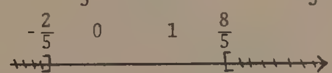
32. $-3 \le 4x - 6 \le 3$, $\frac{3}{4} \le x \le \frac{9}{4}$. The solution set is $[\frac{3}{4},\frac{9}{4}]$.



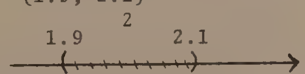
33. $|3t - 5| > 2$. Hence, $3t - 5 > 2$ or $3t - 5 < -2$. Thus, $3t > 7$ or $3t < 3$, and so $t > \frac{7}{3}$ or $t < 1$. Interval notation: $(-\infty,1)$ together with $(\frac{7}{3},\infty)$.



34. $|3 - 5s| \ge 5$, so that $3 - 5s \ge 5$ or $3 - 5s \le -5$. Thus, $-5s \ge 2$ or $-5s \le -8$, and so $s \le -\frac{2}{5}$ or $s \ge \frac{8}{5}$. Interval notation: $(-\infty,-\frac{2}{5}]$ together with $[\frac{8}{5},\infty)$.



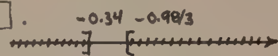
35. $x - 2 < 0.1$, so that $-0.1 < x - 2 < 0.1$. Thus, $1.9 < x < 2.1$. Interval notation: $(1.9, 2.1)$



36. $3u + 1 \ge 0.02$, so that $3u + 1 \ge 0.02$ or $3u + 1 \le -0.02$. Now, $3u \ge -0.98$ or $3u \le -1.02$;

$u \geq -\frac{0.98}{3}$ or $u \leq -0.34$

Interval notation $\left[-\frac{0.98}{3}, \infty\right)$ together with $(-\infty, -0.34]$.

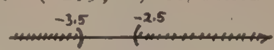


37. $|x + 3| > 0.5$,

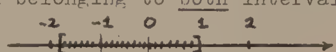
so that $x + 3 > 0.5$ or $x + 3 < -0.5$;

thus, $x > -2.5$ or $x < -3.5$.

Interval notation: $(-2.5, \infty)$ together with $(-\infty, -3.5)$



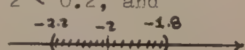
38. $-|9 - 2x| \leq 7x \leq |9 - 2x|$. The condition $-|9 - 2x| \leq 7x$ is equivalent to $-7x \leq |9 - 2x|$; that is, to $-7x \leq 9 - 2x$ or $9 - 2x \leq 7x$. Hence, the solution set of $-|9 - 2x| \leq 7x$ is $\left[-\frac{9}{5}, \infty\right)$. The condition $7x \leq |9 - 2x|$ is equivalent to $7x \leq 9 - 2x$ or $7x \leq -(9 - 2x)$. Hence, the solution set of $7x \leq |9 - 2x|$ is $(-\infty, 1]$. The solution set of $-|9 - 2x| \leq 7x \leq |9 - 2x|$ consists of all values of x belonging to both intervals: $\left[-\frac{9}{5}, 1\right]$.



39. $|x + 2| < 0.2$

so that $-0.2 < x + 2 < 0.2$, and

$-2.2 < x < -1.8$.



Interval notation: $(-2.2, -1.8)$

40. (a) By the triangle inequality,

$|x| = |x - y + y| \leq |x - y| + |y|$,

so $|x| - |y| \leq |x - y|$.

(b) Since (a) holds for any two numbers x and y, it must hold when x and y are interchanged. Thus,

$|y| - |x| \leq |y - x|$. Since

$|y - x| = |-(x - y)| = |x - y|$

and since $|y| - |x| = -(|x| - |y|)$, then $-(|x| - |y|) \leq |x - y|$. Hence, recalling (a), we can conclude that

$\pm(|x| - |y|) \leq |x - y|$; that is,

$||x| - |y|| \leq |x - y|$.

41. 

Area = 1.5 ℓ square meters

We want $2000 \leq (1.5\ \ell)400 \leq 3500$;

that is $2000 \leq 600\ \ell \leq 3500$. Thus,

$\frac{2000}{600} \leq \ell \leq \frac{3500}{600}$, and so $\frac{10}{3} \leq \ell \leq \frac{35}{6}$

42. Since $a < b$, the $b - a$ is positive. Since $a \leq x \leq b$, then $-b \leq -x \leq -a$ and $0 \leq b - x \leq b - a$. Divide the latter inequality by $b - a$ and conclude that $0 \leq \frac{b - x}{b - a}$, $n \leq 1$. Put $t = \frac{b - x}{b - a}$, noting that $0 \leq t \leq 1$ and $t(b - a) = b - x$, so $x = ta + (1 - t)b$.

43. Let m be the number of minutes for a phone call. Then, $6.05 \leq 2.25 + (m - 3)(0.38) \leq 8.71$; $6.05 \leq 2.25 + 0.38m - 1.14 \leq 8.71$; $6.05 \leq 1.11 + 0.38m \leq 8.71$; $4.94 \leq 0.38m \leq 7.6$; $\frac{4.94}{0.38} \leq m \leq \frac{7.6}{0.38}$. Thus, $13 \leq m \leq 20$.

44. Let I be the income after deductions. Then $3484 \leq 3260 + 0.28(I - 19{,}200) \leq 4044$;

$224 \leq 0.28I - 5376 \leq 784$;

$5600 \leq 0.28I \leq 6160$;

$\frac{5600}{0.28} \leq I \leq \frac{6160}{0.28}$;

thus, $20{,}000 \leq I \leq 22{,}00$.

45. $\frac{1314}{220} = 5 + \frac{214}{220}$ tanks of gas. Therefore, it would require 6 full tanks of gas.

46. (a) $|x - y| = |(x - 2) + (2 - y)| \leq |x - 2| + |2 - y| = |x - 2| + |y - 2| < \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

(b) $|x - y| = |(x + 2) + (-1)(y + 2)| \leq |x + 2| + |(-1)(y + 2)| = |x + 2| + |y + 2| < \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

(c) $|y + 2| = |(x + 2) + (y - x)| \leq$
$|x + 2| + |y - x| = |x + 2| +$
$|x - y| < \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

47. In order to satisfy the specifications, we must have $400 \leq \frac{1200R}{1200+R} \leq 900$. Since R must be positive, $1200 + R > 0$, so we have $400(1200 + R) \leq 1200R \leq 900(1200 + R)$ or $480{,}000 + 400R \leq 1200R \leq 1{,}080{,}000 + 900R$

Now we break the compound inequality into 2 simple inequalities:

(1) $480{,}000 + 400R \leq 1200R$
$480{,}000 \leq 800R$
$600 \leq R$

(2) $1200R \leq 1{,}080{,}000 + 900R$
$300R \leq 1{,}080{,}000$
$R \leq 3600$

Therefore we must have $600 \leq R$ and at the same time $R \leq 3600$. Hence, $600 \leq R \leq 3600$.

Problem Set 1.2, page 14

1. (a) Q_I (b) Q_{II}
(c) Q_I (d) Q_{IV}
(e) Q_{III} (f) y axis
(g) x axis (h) y axis

2. A in Q_I, B in Q_{II}, C in Q_{IV}, D in Q_{IV}.

3. (a) $Q = (3,-2)$ (b) $R = (-3,2)$
(c) $S = (-3,-2)$

4. (a) $Q = (-4,3)$ (b) $R = (4,-3)$
(c) $S = (4,3)$

5. (a) $Q = (-1,-3)$ (b) $R = (1,3)$
(c) $S = (1,-3)$

6. (a) $Q = (\frac{\sqrt{3}}{2},\frac{1}{2})$ (b) $R = (-\frac{\sqrt{3}}{2},\frac{1}{2})$
(c) $S = (-\frac{\sqrt{3}}{2},\frac{1}{2})$

7. $\sqrt{(7-1)^2 + (10-2)^2} = \sqrt{6^2+8^2} = \sqrt{10^2} = 10$

8. $d = \sqrt{(11-7)^2 + (2-(-1))^2} = \sqrt{16+9} = 5$

9. $d = \sqrt{(3-(-1))^2 + (7-7)^2} = \sqrt{16+0} = 4$

10. $\sqrt{(-4-0)^2 + (7-(-8))^2} = \sqrt{16+225} = \sqrt{241}$

11. $\sqrt{(-6-3)^2 + (3-(-5))^2} = \sqrt{81+64} = \sqrt{145}$

12. $d = \sqrt{(0-(-4))^2 + (4-0)^2} = \sqrt{16+16} = 4\sqrt{2}$

13. $d = \sqrt{(0-(-8))^2 + (0-(-6))^2} = \sqrt{64+36} = 10$

14. $\sqrt{(t-t)^2+(4-8)^2} = \sqrt{0+16} = \sqrt{16} = 4$

15. $\sqrt{(-3-(-7))^2+(-5-(-8))^2} = \sqrt{16+9} = \sqrt{25} = 5$

16. $\sqrt{(-\frac{1}{2} + 3)^2 + (-\frac{3}{2} - (-\frac{5}{2}))^2} =$
$\sqrt{(\frac{5}{2})^2 + (1)^2} = \sqrt{\frac{25}{4} + 1} = \sqrt{\frac{29}{4}} = \frac{\sqrt{29}}{2}$

17. $\sqrt{(2-5)^2 + (-t-t)^2} = \sqrt{9-4t^2}$

18. $\sqrt{(a-(a+1))^2 + (b+1-b)^2} = \sqrt{(-1)^2+(1)^2} =$
$\sqrt{1+1} = \sqrt{2}$

19. $\sqrt{(-2.714-3.135)^2 + (7.111-4.982)^2} =$
$\sqrt{34.210801 + 4.532641} =$
$\sqrt{38.743442} \approx 6.224$

20. $\sqrt{(\pi+\sqrt{17})^2 + (\frac{53}{4} - \frac{211}{5})^2} =$
$\sqrt{52.77584109 + 838.1025} =$
$\sqrt{890.8783411} \approx 29.848$

21. (a) $|\overline{AB}| = \sqrt{(5-1)^2 + (1-1)^2} = 4$
$|\overline{AC}| = \sqrt{(5-1)^2 + (7-1)^2} = \sqrt{16+36}$
$= \sqrt{52}$
$|\overline{BC}| = \sqrt{(5-5)^2 + (7-1)^2} = 6$
But $(\sqrt{52})^2 = 4^2 + 6^2$. By the converse of the Pythagorean theorem, the given points are vertices of a right triangle.

(b) Area $= \frac{1}{2} \cdot 4 \cdot 6 = 12$

22. (a) $|\overline{AB}| = \sqrt{(-1 - 3)^2 + (-2 - (-2))^2}$
$= \sqrt{16} = 4.$
$|\overline{AC}| = \sqrt{(-1 - (-1))^2 + (-2 - (-7))^2}$
$= \sqrt{25} = 5.$
$|\overline{BC}| = \sqrt{(3 - (-1))^2 + (-2 - (-7))^2}$
$= \sqrt{16 + 25} = \sqrt{41}.$
But $(\sqrt{41})^2 = 4^2 + 5^2$. Hence, the given points are vertices of a right triangle.

(b) Area $= \frac{1}{2} \cdot 4 \cdot 5 = 10.$

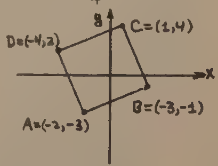
23. (a) $|\overline{AB}| = \sqrt{(0 - (-3))^2 + (0 - 3)^2} = \sqrt{18}.$
$|\overline{AC}| = \sqrt{(2 - 0)^2 + (2 - 0)^2} = \sqrt{8}.$
$|\overline{BC}| = \sqrt{(-3 - 2)^2 + (3 - 2)^2} = \sqrt{26}.$
But $(\sqrt{26})^2 = (\sqrt{18})^2 + (\sqrt{8})^2$.
Hence, the given points are vertices of a right triangle.

(b) Area $= \frac{1}{2} \cdot \sqrt{18}\,\sqrt{8} = \frac{1}{2} \cdot 3\sqrt{2} \cdot 2\sqrt{2}$
$= 3 \cdot 2 = 6.$

24. (a) $|\overline{AB}| = \sqrt{(-2 - 9)^2 + (-5 - \frac{1}{2})^2}$
$= \frac{11}{2}\sqrt{5}.$
$|\overline{AC}| = \sqrt{(-2 - 4)^2 + (-5 - \frac{21}{2})^2} = \frac{\sqrt{1105}}{2}$
$|\overline{BC}| = \sqrt{(9 - 4)^2 + (\frac{1}{2} - \frac{21}{2})^2} = \sqrt{125}$
Is $(\frac{\sqrt{1105}}{2})^2 = (\frac{11}{2}\sqrt{5})^2 + (\sqrt{125})^2$?
Yes, since the right side equals $\frac{605}{4} + 125 = \frac{605+500}{4} = \frac{1105}{4}$, and the left side is $\frac{1105}{4}$ also.
Therefore, the given points are the vertices of a right triangle.

(b) Area $= \frac{1}{2} \cdot \frac{11}{2}\sqrt{5}\,\sqrt{125} = \frac{11}{4}\sqrt{5} \cdot 5\sqrt{5}$
$= \frac{11}{4} \cdot 25 = \frac{275}{4}.$

25.

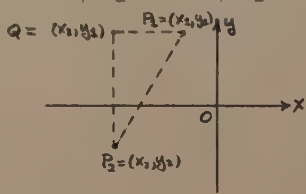


$|\overline{AB}| = \sqrt{(3 + 2)^2 + (-1 + 3)^2} = \sqrt{29}$
$|\overline{BC}| = \sqrt{(3 - 1)^2 + (-1 - 4)^2} = \sqrt{29}$
$|\overline{CD}| = \sqrt{(-4 - 1)^2 + (2 - 4)^2} = \sqrt{29}$
$|\overline{DA}| = \sqrt{(-4 + 2)^2 + (2 + 3)^2} = \sqrt{29}$
So ABCD is a rhombus. Now $|\overline{AC}| = \sqrt{(1 + 2)^2 + (4 + 3)^2} = \sqrt{58}.$
Since $(\sqrt{58})^2 = (\sqrt{29})^2 + (\sqrt{29})^2$, $\triangle$ ACD is a right triangle with right angle at D. Thus, ABCD is a square.

26. Call $(x_1, y_1) = P_1$ and $(x_2, y_2) = P_2$ and $(x_1 - x_2, y_1 - y_2) = P_3$.
Now $|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. $|\overline{P_3O}|$
$= \sqrt{((x_1 - x_2) - 0)^2 + ((y_1 - y_2) - 0)^2}$
$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Hence, $|\overline{P_1P_2}| = |\overline{P_3O}|$.

27. $|\overline{AB}| = \sqrt{(-6 - (-5))^2 + (5 - 1)^2}$
$= \sqrt{1 + 16} = \sqrt{17}.$
$|\overline{BC}| = \sqrt{(-6 - (-2))^2 + (5 - 4)^2}$
$= \sqrt{16 + 1} = \sqrt{17}.$
Triangle ABC is isosceles.

28. Suppose $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are in the second and third quadrants as shown. Let the hypotenuse $\overline{P_1P_2} = d$.
Notice that $|\overline{P_1Q}| = |x_2 - x_1| = |x_1 - x_2|$ and that $|\overline{QP_2}| = |y_1 - y_2|$. Hence,
$d^2 = |x_2 - x_1|^2 + |y_1 - y_2|^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$ by the Pythagorean theorem. But $|x_1 - x_2|^2 = (x_1 - x_2)^2$ and $|y_1 - y_2|^2 = (y_1 - y_2)^2$.
So $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.
Therefore, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ as before.



29. $\sqrt{(-2 - t)^2 + (3 - t)^2} = 5;$

$\sqrt{4 + 4t + t^2 + 9 - 6t + t^2} = 5;$

$13 - 2t + 2t^2 = 25;$

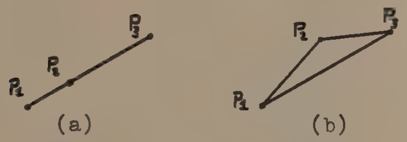
$2t^2 - 2t - 12 = 0;$

$t^2 - t - 6 = 0;$

$(t - 3)(t + 2) = 0;$

$t = 3$ or $t = -2$

30.



Notice that in Figure (a) $|\overline{P_1P_2}| + |\overline{P_2P_3}| = |\overline{P_1P_3}|$; however, in Figure (b), where P_2 does not lie on the segment between P_1 and P_3, we have a triangle, and so the sum of two sides is greater than the third side. Here, $|\overline{P_1P_2}| + |\overline{P_2P_3}| \neq |\overline{P_1P_3}|$

31. $|\overline{P_1P_3}| = \sqrt{(3 - 2)^2 + (-1 - 1)^2} = \sqrt{1 + 4}$
$= \sqrt{5}.$

$|\overline{P_1P_2}| = \sqrt{(\frac{5}{2} - 2)^2 + (0 - 1)^2} = \sqrt{\frac{1}{4} + 1}$
$= \sqrt{\frac{5}{4}} = \sqrt{\frac{5}{2}}.$

$|\overline{P_2P_3}| = \sqrt{(3 - \frac{5}{2})^2 + (-1 - 0)^2}$
$= \sqrt{\frac{1}{4} + 1} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}.$

Now $\sqrt{5} = \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}$, so that $|\overline{P_1P_3}| = |\overline{P_1P_2}| + |\overline{P_2P_3}|$. Therefore, P_2 does lie on the line segment joining P_1 to P_3.

32. $|\overline{P_1P_3}| = \sqrt{(-\frac{7}{2} - 2)^2 + (0 - 11)^2}$
$= \sqrt{\frac{121}{4} + 121} = \sqrt{\frac{605}{4}}.$

$|\overline{P_1P_2}| = \sqrt{(-\frac{7}{2} - (-1))^2 + (0 - 5)^2}$
$= \sqrt{\frac{25}{4} + 25} = \sqrt{\frac{125}{4}}.$

$|\overline{P_2P_3}| = \sqrt{(2 - (-1))^2 + (11 - 5)^2}$
$= \sqrt{9 + 36} = \sqrt{45}.$

$|\overline{P_1P_2}| + |\overline{P_2P_3}| = \sqrt{\frac{125}{4}} + \sqrt{\frac{180}{4}}$
$= \frac{5\sqrt{5}}{2} + \frac{6\sqrt{5}}{2} = \frac{11\sqrt{5}}{2}.$

But $\sqrt{\frac{605}{4}} = \frac{11\sqrt{5}}{2} = |\overline{P_1P_3}|$. Hence, $|\overline{P_1P_2}| + |\overline{P_2P_3}| = |\overline{P_1P_3}|$. So P_2 lies on the line segment between P_1 and P_3.

33. $|\overline{P_1P_3}| = \sqrt{(2 - (-1))^2 + (3 - (-1))^2}$
$= \sqrt{9 + 16} = 5.$

$|\overline{P_1P_2}| = \sqrt{(2 - 3)^2 + (3 - (-3))^2}$
$= \sqrt{1 + 36} = \sqrt{37}.$

$|\overline{P_2P_3}| = \sqrt{(3 - (-1))^2 + (-3 - (-1))^2}$
$= \sqrt{16 + 4} = \sqrt{20}.$

$|\overline{P_1P_3}| \neq |\overline{P_1P_2}| + |\overline{P_2P_3}|$ since $\sqrt{37}$ is bigger than 5. So P_2 does not lie on the line segment between P_1 and P_3.

34. $|\overline{AS}| = \sqrt{(52 - 47)^2 + (71 - 83)^2}$
$= \sqrt{5^2 + 12^2} = \sqrt{13^2} = 13.$

35. (a) $x^2 + (y - 2)^2 = 9$

(b) $(x + 1)^2 + (y - 4)^2 = 4$

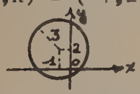
(c) $(x - 3)^2 + (y - 4)^2 = 25$

36. (a) $r^2 = (1 - (-2))^2 + (6 - 2)^2 = 25$, so the equation is $(x - 1)^2 + (y - 6)^2 = 25.$

(b) Let (h,k) be the center. Then the equation is $(x - h)^2 + (y - k)^2 = 16$. Since $(-3,0)$ is on the circle, $(-3 - h)^2 + (0 - k)^2 = 16$. Since $(5,0)$ is also on the circle, $(5 - h)^2 + (0 - k)^2 = 16$. Subtracting the latter equation from the former gives $(-3 - h)^2 = (5 - h)^2$, $5 - h = \pm(-3 - h)$. Use of the + sign gives no solution, so $5 - h = -(-3 - h)$, $h = 1$. Since $h = 1$ and $(-3 - h)^2 + (0 - k)^2 = 16$, $16 + k^2 = 16$, $k = 0$. Thus, the equation is $(x - 1)^2 + (y - 0)^2 = 16.$

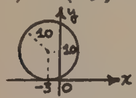
36. (c) The center is the midpoint of the line segment from (3,7) to (-3,-1). Thus, the center is $(h,k)=(\frac{3-3}{2},\frac{7-1}{2})$ $= (0,3)$. The radius is $r = \sqrt{(0-3)^2 + (3-7)^2} = \sqrt{9+16}$ $= \sqrt{25} = 5$. The equation is $(x-0)^2 + (y-3)^2 = 25$.

37. $(h,k) = (-1,2)$, $r = 3$.



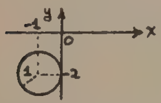
$(x+1)^2+(y-2)^2=9$

38. $(h,k) = (-3,10)$, $r = 10$.



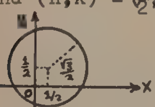
$(x+3)^2+(y-10)^2=100$

39. Complete the squares to get $(x+1)^2 + (y+2)^2 = 1$. Thus $r = 1$ and $(h,k) = (-1,-2)$.



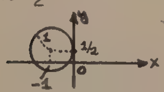
$(x+1)^2+(y+2)^2=1$

40. Complete the squares to get $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{3}{2}$. Thus $r = \sqrt{\frac{3}{2}}$ and $(h,k) = (\frac{1}{2}, \frac{1}{2})$.



$(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{3}{2}$

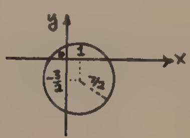
41. Dividing by 4 gives $x^2+y^2+2x-y+\frac{1}{4} = 0$. Now complete the squares to get $(x+1)^2+(y-\frac{1}{2})^2 = 1$. Thus, $r = 1$, $(h,k) = (-1,\frac{1}{2})$.



$(x+1)^2+(y-\frac{1}{2})^2=1$

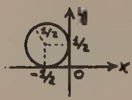
42. Dividing by 3 gives $x^2+y^2-2x+3y = 9$. Complete the squares to get $(x-1)^2+(y+\frac{3}{2})^2 = \frac{49}{4}$, $r = \frac{7}{2}$, $(h,k) = (1,-\frac{3}{2})$.

$(x-1)^2+(y+\frac{3}{2})^2= \frac{49}{4}$

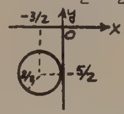


43. Dividing by 4 gives $x^2+y^2+x-y+\frac{1}{4} = 0$. Complete the squares to get $(x+\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{4}$. Thus, $r = \frac{1}{2}$ and $(h,k) = (-\frac{1}{2},\frac{1}{2})$.

$(x+\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{4}$



44. Dividing by 4 gives $x^2+y^2+3x+5y+\frac{25}{4} = 0$. Complete the squares to get $(x+\frac{3}{2})^2 + (y+\frac{5}{2})^2 = \frac{9}{4}$, $r = \frac{3}{2}$, $(h,k) = (\frac{3}{2}, \frac{5}{2})$.



$(x+\frac{3}{2})^2+(y+\frac{5}{2})^2 = \frac{9}{4}$

45. Let the equation be $x^2+y^2+Ax+By+C = 0$. Substituting the coordinates of the three given points gives $-3A + B + C = -10$, $7A + B + C = -50$ and $-7A + 5B + C = -74$. Solving these three simultanious equations gives $A = -4$, $B = -20$, $C = -2$, so the equation is $x^2+y^2-4x-20y-2 = 0$. Completing the squares gives $(x-2)^2 + (y-10)^2 = 106$.

46. $A + 7B + C = -50$, $8A + 6B + C = -100$ and $7A - B + C = -50$. The solution is $A = -8$, $B = -6$, $C = 0$. The equation is $x^2+y^2-8x-6y = 0$. Completing the squares gives $(x-4)^2 + (y-3)^2 = 25$.

47. Let the center be $(h,0)$. The equation of the circle is $(x-h)^2+y^2 = 17$. Since $(0,1)$ is on this circle, $(0-h)^2+1^2 = 17$, $h^2 = 16$, $h = \pm 4$. One circle is $(x+4)^2 + y^2 = 17$ and the other is $(x-4)^2 + y^2 = 17$.

48. $\sqrt{(x-6)^2 + (y-0)^2} = 2\sqrt{(x-0)^2 + (y-3)^2}$,

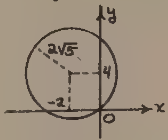
$(x-6)^2 + y^2 = 4\left[x^2 + (y-3)^2\right]$,

$x^2-12x+36+y^2 = 4\left[x^2+y^2-6y+9\right]$,

$3x^2+3y^2+12x-24y = 0$, $x^2+y^2+4x-8y = 0$, so

(a) $(x+2)^2 + (y-4)^2 = 20$.

(b) A circle of radius $r=2\sqrt{5}$ centered at $(-2,4)$.



49. The distance between the centers of the circles is $\sqrt{(0-20)^2 + (0-0)^2} = 20$ while the sum of the two radii is $10 + 12 = 22 > 20$. Therefore, the two circles do overlap.

50. As in Problem 49, the condition for overlap is $\sqrt{(a-c)^2 + (b-d)^2} < r+R$.

51. $x^2-2hx+h^2+y^2-2ky+k^2 = r^2$, or, $x^2+y^2+Ax+By+C = 0$ where $A = -2h$, $B = -2k$, and $C = h^2 + k^2 - r^2$.

52. $x^2+Ax+\frac{A^2}{4}+y^2+By+\frac{B^2}{4} = \frac{A^2}{4}+\frac{B^2}{4}-C$, or,

$(x+\frac{A}{2})^2+(y+\frac{B}{2})^2 = \frac{A^2+B^2-4C}{4}$. If $A^2+B^2-4C>0$, then the equation becomes

$(x-h)^2+(y-k)^2 = r^2$ with $h = -\frac{A}{2}$, $k = -\frac{B}{2}$, $r = \frac{\sqrt{A^2+B^2-4C}}{2}$.

Problem Set 1.3, page 21

1. The slope is $\frac{7-2}{3-6} = -\frac{5}{3}$.

2. The slope is $\frac{-6-(-2)}{5-3} = -\frac{4}{2} = -2$.

3. The slope is $\frac{7-1}{14-2} = \frac{6}{12} = \frac{1}{2}$.

4. The slope is $\frac{2-(-1)}{2-(-4)} = \frac{3}{6} = \frac{1}{2}$.

5. The slope is $\frac{8-3}{6-(-5)} = \frac{5}{11}$.

6. The slope is $\frac{3-(-1)}{1-(-1)} = \frac{4}{2}$.

7. $y - 4 = 2(x - 5)$.

8. $y - 1 = -4(x - 6)$.

9. $y - 2 = \frac{1}{4}(x - 3)$.

10. $y = -1$.

11. $y - (-2) = -3(x - 7)$ or $y + 2 = -3(x - 7)$.

12. $y - 2 = -\frac{2}{3}(x - 0)$ or $y - 2 = -\frac{2}{3}x$.

13. $y - \frac{2}{3} = 0(x - \frac{1}{2})$ or $y = \frac{2}{3}$.

14. $m = \frac{11-1}{7+1} = \frac{10}{8} = \frac{5}{4}$, $y - 1 = \frac{5}{4}(x + 1)$.

15. $m = \frac{8-2}{4-3} = 6$, $y - 2 = 6(x - 3)$.

16. Slope of $\overline{AB}$ is $\dfrac{\frac{3}{5} - 1}{-\frac{2}{3} - \frac{1}{3}} = \dfrac{-\frac{2}{5}}{-1} = \frac{2}{5}$.

so desired equation is $y-2 = \frac{2}{5}(x-7)$.

17. Slope of line containing the given points is $\frac{4-4}{-4+3} = 0$. The desired equation is $y - 4 = 0(x + 4)$.

18. Slope of $\overline{AB} = \dfrac{\frac{1}{3} - \frac{2}{3}}{-\frac{2}{5} - \frac{3}{5}} = \dfrac{-\frac{1}{3}}{-1} = \frac{1}{3}$; slope of line perpendicular to $\overline{AB}$ is -3; desired equation is $y - 2 = -3(x + 1)$.

19. (a) $y = 3$

(b) $x = -2$

20. $m = \frac{y_2 - y_1}{x_2 - x_1}$ so that the equation with slope m and passing through (x_1, y_1) has equation $y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$ or

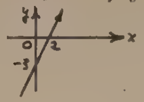
$$y = \frac{(y_2-y_1)x}{x_2-x_1} - \frac{x_1(y_2-y_1)}{x_2-x_1} + y_1 \text{ or}$$

$$y = \frac{y_2-y_1}{x_2-x_1} \cdot x + \frac{-x_1y_2+x_1y_1+x_2y_1-x_1y_1}{x_2-x_1}.$$

Therefore,

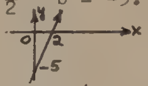
$$y = \frac{y_2-y_1}{x_2-x_1} \cdot x + \frac{x_2y_1-x_1y_2}{x_2-x_1}.$$

21. $3x - 2y = 6$, $-2y = -3x + 6$, $y = \frac{3}{2}x - 3$;
$m = \frac{3}{2}$ $b = -3$.



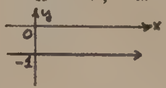
$y = \frac{3}{2}x - 3$

22. $5x - 2y - 10 = 0$, $-2y = -5x + 10$,
$y = \frac{5}{2}x - 5$; $m = \frac{5}{2}$ $b = -5$.



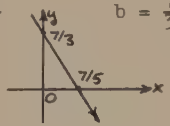
$y = \frac{5}{2}x - 5$

23. $y + 1 = 0$, $y = 0 \cdot x - 1$; $m = 0$ $b = -1$.



$y = -1$

24. $x = -\frac{3}{5}y + \frac{7}{5}$, $5x = -3y + 7$,
$3y = -5x + 7$, $y = -\frac{5}{3}x + \frac{7}{3}$;
$m = -\frac{5}{3}$ $b = \frac{7}{3}$



$y = -\frac{5}{3}x + \frac{7}{3}$

25. $m = -3$ $b = 5$ $(0,5)$ is on line
(a) $y - 5 = -3(x - 0)$
(b) $y = -3x + 5$
(c) $3x + y = 0$

26. $m = \frac{4}{5}$ $(-3,0)$ on line
(a) $y - 0 = \frac{4}{5}(x + 3)$
(b) $y = \frac{4}{5}x + \frac{12}{5}$
(c) $5y = 4x + 12$ or $-4x + 5y - 12 = 0$

27. Slope of line equals $\frac{5-0}{0-3} = -\frac{5}{3}$
(a) $y - 5 = -\frac{5}{3}(x - 0)$
(b) $y - 5 = -\frac{5}{3}x$ or $y = -\frac{5}{3}x + 5$
(c) $5x + 3y - 45 = 0$

28. Slope of line is $\frac{-6-\frac{5}{3}}{\frac{2}{5}-\frac{7}{2}} = \frac{-180-50}{12-105}$

$= \frac{-230}{-93} = \frac{230}{93}$

(a) $y + 6 = \frac{230}{93}(x - \frac{2}{5})$
(b) $y = \frac{230}{93}x - \frac{92}{93} - 6 = \frac{230}{93}x - \frac{650}{93}$
(c) $93y = 230x - 650$ or $230x-93y-650=0$

29. $2x-5y+3 = 0$, $-5y = -2x-3$, $y = \frac{2}{5}x + \frac{3}{5}$;
Slope of line is $\frac{2}{5}$.
(a) $y + 4 = \frac{2}{5}(x - 4)$
(b) $y + 4 = \frac{2}{5}x - \frac{8}{5}$ or,
$y = \frac{2}{5}x - \frac{28}{5}$
(c) $5y = 2x - 28$ or $2x - 5y - 28 = 0$

30. $y = \frac{2}{3}$
(a) $y - \frac{2}{3} = 0(x + 3)$
(b) $y = \frac{2}{3}$
(c) $3y - 2 = 0$

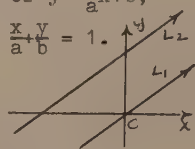
31. Since $5x + 3y - 1 = 0$, $3y = -5x + 1$,
$y = -\frac{5}{3}x + \frac{1}{3}$, and its slope is $-\frac{5}{3}$;
slope of desired perpendicular line is $\frac{3}{5}$.
(a) $y - \frac{2}{3} = \frac{3}{5}(x + 3)$
(b) $y = \frac{3}{5}x + \frac{9}{5} + \frac{2}{3}$ or $y = \frac{3}{5}x + \frac{37}{15}$
(c) $15y = 9x + 37$ or $9x - 15y + 37 = 0$

32. Midpoint of $\overline{AB}$ is $(\frac{3+7}{2}, \frac{-2+6}{2}) = (5,2)$
Slope of $\overline{AB}$ is $\frac{6+2}{7-3} = \frac{8}{4} = 2$
Slope of perpendicular bisector is $-\frac{1}{2}$.
(a) $y - 2 = -\frac{1}{2}(x - 5)$
(b) $y = -\frac{1}{2}x + \frac{5}{2} + 2$ or $y = -\frac{1}{2}x + \frac{9}{2}$
(c) $2y = -x + 9$ or $x + 2y - 9 = 0$

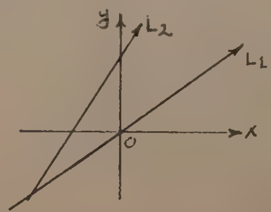
33. $3x + By - 5 = 0$, so that $By = -3x + 5$,
and $y = -\frac{3}{B}x + \frac{5}{B}$.
We want $\frac{5}{B} = -4$, so $B = -\frac{5}{4}$.

34. $(a,0)$ and $(0,b)$ are on the line, so the slope of the line is $\frac{b-0}{0-a} = -\frac{b}{a}$. The equation is $y-0 = -\frac{b}{a}(x-a)$ or $y=-\frac{b}{a}x+b$; and dividing by b, we get $\frac{x}{a}+\frac{y}{b} = 1$.

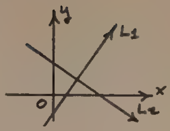
35. L_1 and L_2 are parallel.



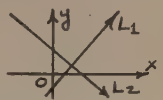
36. L_1 and L_2 are neither parallel nor perpendicular.



37. L_1 and L_2 are perpendicular.



38. L_1 and L_2 are perpendicular.



39. The slope of $\overline{AB}$ is $\frac{-1+2}{1+5} = \frac{1}{6}$, and the slope of $\overline{CD}$ is $\frac{4-3}{4+3} = \frac{1}{6}$. Hence, $\overline{AB}$ is parallel to $\overline{CD}$. Now the slope of $\overline{BC}$ is $\frac{4+1}{4-1} = \frac{5}{3}$ and the slope of $\overline{AD}$ is $\frac{-2-3}{-5+2} = \frac{-5}{-3} = \frac{5}{3}$; so $\overline{BC}$ is parallel to $\overline{AD}$. Thus, ABCD is a parallelogram.

40. Let (h,k) be the center of a circle tangent to $3x + y = 6$ at $(3,-3)$. Slope of line is -3 since $y = -3x + 6$. So slope of line containing radius at point of tangency is $\frac{1}{3}$. We have $(h-3)^2 + (k+3)^2 = 10$ since distance between center (h,k) and point on the circle is $\sqrt{10}$. Also $\frac{k+3}{h-3} = \frac{1}{3}$. Solving for $h - 3$, we get $h-3 = 3(k+3)$. Substituting above, we have
$9(k + 3)^2 + (k + 3)^2 = 10$ or
$10(k + 3)^2 = 10$ or $(k + 3)^2 = 1$;
thus, $k + 3 = \pm 1$, so that $k = -4$ or $k = -2$. When $k = -4$, $h - 3 = 3(-1)$, so $h = 0$; when $k = -2$, $h - 3 = 3(1)$, so $h = 6$.
Thus, the desired circles are
$(x - 6)^2 + (y + 2)^2 = 10$ and
$x^2 + (y + 4)^2 = 10$.

41. (a) The line containing $(5,-2)$ and $(1,4)$ has slope $\frac{4+2}{1-5} = \frac{6}{-4} = -\frac{3}{2}$. So we want $\frac{3-1}{d+2} = \frac{2}{3}$; that is, $6 = 2d + 4$, $2 = 2d$, so $d = 1$.

(b) We want $\frac{3-1}{k+2} = -\frac{3}{2}$; $4 = -3k - 6$, $3k = -10$, $k = -\frac{10}{3}$.

42. Let $P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$. We show that P is on the line segment $\overline{P_1P_2}$ by showing that $|\overline{P_1P}| + |\overline{P_2P}| = |\overline{P_1P_2}|$. We will also show that $|\overline{P_1P}| = |\overline{P_2P}|$. Thus P is the midpoint of $\overline{P_1P_2}$. Now,

$$|\overline{P_1P_2}| = \sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2},$$

$$|\overline{P_1P}| = \sqrt{\left(x_1 - \frac{x_1}{2} - \frac{x_2}{2}\right)^2 + \left(y_1 - \frac{y_1}{2} - \frac{y_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \text{ and}$$

$$|\overline{P_2P}| = \sqrt{\left(x_2 - \frac{x_1}{2} - \frac{x_2}{2}\right)^2 + \left(y_2 - \frac{y_1}{2} - \frac{y_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Since $(x_1-x_2)^2 = (x_2-x_1)^2$ and $(y_1-y_2)^2 = (y_2-y_1)$, it is clear that $|\overline{P_1P}| + |\overline{P_2P}| = |\overline{P_1P_2}|$ and that $|\overline{P_1P}| = |\overline{P_2P}|$, and we are done.

43. (a) $\left(\frac{8+7}{2}, \frac{1+3}{2}\right) = \left(\frac{15}{2}, 2\right)$.

(b) $\left(\frac{9 + (-5)}{2}, \frac{3 + 7}{2}\right) = (2,5)$.

(c) $\left(\frac{-1 + 5}{2}, \frac{1 + 3}{2}\right) = (2,2)$.

(d) $\left(\frac{1 + 5}{2}, \frac{-3 + 8}{2}\right) = (3, \frac{5}{2})$.

44. Set $m_1x + b_1 = m_2x + b_2$

$$(m_1 - m_2)x = b_2 - b_1$$

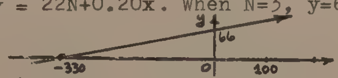
$$x = \frac{b_2 - b_1}{m_1 - m_2}.$$ Substituting into the first equation, we have

$$y = m_1\left(\frac{b_2 - b_1}{m_1 - m_2}\right) + b_1;$$

$$y = \frac{m_1b_2 - m_1b_1 + m_1b_1 - m_2b_1}{m_1 - m_2}; \quad y = \frac{m_1b_2 - m_2b_1}{m_1 - m_2};$$

So $\left(\frac{b_2-b_1}{m_1-m_2}, \frac{m_1b_2-m_2b_1}{m_1 - m_2}\right)$ is the point of intersection of the lines $y = m_1x + b_1$ and $y = m_2x + b_2$.

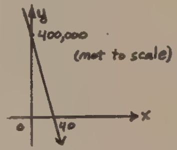
45. $y = 22N + 0.20x$. When $N=3$, $y=66 + 0.20x$.



46. (a) The slope of the line $y=3x-5$ is 3, so the line perpendicular to it has slope $-\frac{1}{3}$. The desired equation is $y-3=-\frac{1}{3}(x+4)$.

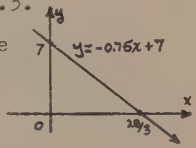
46. (b) Solve $y=3x-5$ and $y-3=-\frac{1}{3}(x+4)$ simultaneously. Substituting the first equation into the second, we have $3x-5-3=-\frac{1}{3}(x+4)$
$9x-24=-x-4$, $10x=20$, $x=2$. So
$y=3(2)-5=1$. $(x_1,y_1)=(2,1)$.
(c) $d=\sqrt{(-4-2)^2+(3-1)^2}=\sqrt{36+4}=\sqrt{40}=2\sqrt{10}$.

47. $y=400{,}000(1-\frac{x}{40})$. In 1995 when $x=20$,
$y=400{,}000(1-\frac{20}{40})$
$=400{,}000(\frac{1}{2})=200{,}000$.



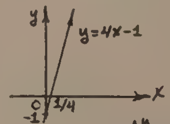
48. By $=-Ax-C$, $y=-\frac{A}{B}x-\frac{C}{D}$;
$Ay=Bx-D$, $y=\frac{B}{A}x-\frac{D}{A}$. The first line has slope $-\frac{A}{B}$ and the second has slope $\frac{B}{A}$. Since the slopes are negative reciprocals, the lines are perpendicular.

49. $y=mx+b$. In 1980, $x=0$ and $y=7$. Thus, $y=mx+7$. Since $m=-0.75$, we have
$y=-0.75x+7$.
When $y=0$, $x=\frac{7}{0.75}=\frac{28}{3}\approx 9.3$.
Therefore, the year the lake has no pollution will be about $1980+9=1989$.

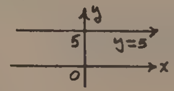


Problem Set 1.4, page 29

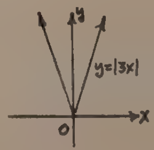
1. $f(-3)=2(-3)+1=-6+1=-5$.
2. $F(\frac{7}{3})=\dfrac{\frac{7}{3}-2}{\frac{7}{3}+7}=\dfrac{7-6}{3.14}=\dfrac{1}{42}$.
3. $h(-\frac{1}{3})=\sqrt{3(-\frac{1}{3})+5}=\sqrt{-1+5}=\sqrt{4}=2$.
4. $H(-4)=|2-5(-4)|=|2+20|=|22|=22$.
5. $G(\sqrt[3]{31})=\sqrt[3]{(\sqrt[3]{31})^3-4}=\sqrt[3]{31-4}=\sqrt[3]{27}=3$.
6. $[h(-1)]^2=[\sqrt{3(-1)+5}]^2=-3+5=2$.
7. $g(0)=0^2-3.0-4=-4$.
8. $b(\frac{1}{a})=2(\frac{1}{a})+1=\frac{2}{a}+1$.
9. $H(C+2)=|2-5(C+2)|=|2-5C-10|$
$=|-5C-8|=|-(5C+8)|$
$=|5C+8|$.
10. $f(\frac{x-1}{2})=2(\frac{x-1}{2})+1=x-1+1=x$.
11. $F(\frac{a}{3})=\dfrac{\frac{a}{3}-2}{a+7}=\dfrac{a-6}{3(a+7)}$.
12. $G(\sqrt{b})=\sqrt[3]{(\sqrt{b})^3-4}=\sqrt[3]{b^{3/2}-4}$.
13. $g(4.718)=(4.718)^2-3(4.718)-4$
$=22.259524-14.154-4=4.105524$.
14. $h(2.003)=\sqrt{3(2.003)+5}$
$=\sqrt{11.009}\approx 3.317981314$.
15. (a) All reals.
(b) $g(x)=\frac{1}{x+2}$. All reals except $x=-2$.
(c) Non-negative reals.
(d) $F(x)=\sqrt{5-3x}$. We want $5-3x\geq 0$,
$5\geq 3x$, so $x\leq\frac{5}{3}$.
(e) $G(x)=\frac{7}{5-6x}$, $5-6x\neq 0$ or $6x\neq 5$ or $x\neq\frac{5}{6}$;
domain is all reals except $x=\frac{5}{6}$.
(f) $K(x)=\frac{1}{(4-5x)^{\frac{1}{2}}}$. The domain consists of all real x, such that $4-5x>0$, $-5x>-4$, so that $x<\frac{4}{5}$.
16. (a) All reals
(b) $x+|x|\neq 0$. All positive reals.
(c) $x^2-4=0$ when $x=\pm 2$. All reals except $+2$ and -2.
(d) $\frac{x-2}{x-4}>0$. Domain consists of all reals >4 or ≤ 2.
17. (a) Function (b) Not a function
(c) Function (d) Not a function
18. Not graph of a function, does not pass the vertical line test.



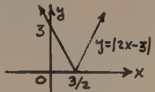
19. Domain: All reals
Range: All reals
20. Domain: All reals
Range: The set consisting of the number 5.



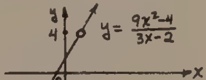
21. The domain is the set of all real numbers. The range is the interval $[0,\infty)$. (The graph is symmetric about the y axis.)



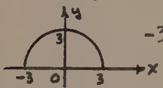
22. The domain is the set of all real numbers. The range is the interval $[0,\infty)$.



23. The domain is the two intervals $(-\infty,\frac{2}{3})$ and $(\frac{2}{3},\infty)$. The range is the two intervals $(-\infty,4)$ and $(4,\infty)$ since $y = \frac{(3x+2)(3x-2)}{3x-2} = 3x+2$ for all x except $\frac{2}{3}$.

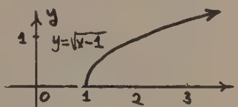


24. The domain consists of all real x such that $9 - x^2 \geq 0$, $x^2 \leq 9$; that is, $-3 \leq x \leq 3$.

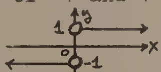


range: $[0,3]$

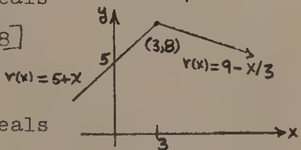
25. The domain consists of all real x such that $x-1\geq 0$, $x\geq 1$.
Range: $[0,\infty)$



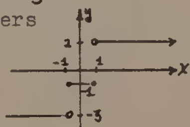
26. Domain: All reals except 0
Range: Set consisting of -1 and 1



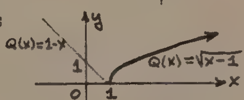
27. Domain: All reals
Range: $(-\infty,8]$



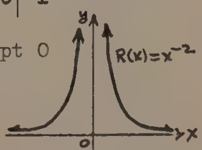
28. Domain: All reals
Range: The set of numbers -3, -1, and 2.



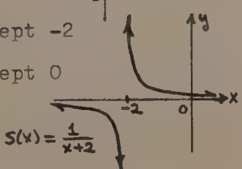
29. Domain: All reals
Range: $[0,\infty)$



30. Domain: All reals except 0
Range: $(0,\infty)$

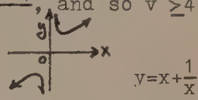


31. Domain: All reals except -2
Range: All reals except 0



32. (a) The domain is the two intervals $(-\infty,0)$ and $(0,\infty)$.

(b) The range of f is the two intervals $[2,\infty)$ and $(-\infty,-2]$. (Solving $v=x+\frac{1}{x}$ for x, we have $x = \frac{v\pm\sqrt{v^2-4}}{2}$, and so $v^2\geq 4$, and $v\geq 2$ or $v\leq -2$.)



33. (a) $f(x)=\frac{4(x+h)-1-(4x-1)}{h} = \frac{4x+4h-4x}{h} = 4$

(b) $\frac{f(x+h)-f(x)}{h} = \frac{5-5}{h} = \frac{0}{h} = 0, \quad h \neq 0$

(c) $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+3-(x^2+3)}{h}$
$= \frac{x^2+2hx+h^2+3-x^2-3}{h}$
$= \frac{2hx+h^2}{h} = 2x+h \quad (h\neq 0)$

34. (a) $\frac{(x+h)^2+(x+h)-(x^2+x)}{h}$
$= \frac{x^2+2xh+h^2+x+h-x^2-x}{h} = \frac{2xh+h^2+h}{h}$
$= 2x + h + 1$

(b) $\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{1}{h} \cdot \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}(\sqrt{x+h})}$
$= \frac{1}{h} \cdot \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}(\sqrt{x+h})} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$
$= \frac{1}{h}\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$
$= \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$

(c) $\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{x-(x+h)}{h(x+h)x} = \frac{-h}{h(x+h)x} = \frac{-1}{x(x+h)}$

35. $f(0) = \frac{9}{5}\cdot 0+32 = 32$ $\quad f: 0\rightarrow 32$
$f(15) = \frac{9}{5}\cdot 15+32 = 27+32 = 59$ $\quad f: 15\rightarrow 59$
$f(-10) = \frac{9}{5}(-10)+32 = -18+32 = 14$ $\quad f: -10\rightarrow 14$
$f(55) = \frac{9}{5}(55)+32 = 99+32 = 131$ $\quad f: 55\rightarrow 131$

36. $f(x) = x$; that is, $\frac{9}{5}x + 32 = x$,
so $\frac{9}{5}x - x = -32$
or $\frac{4}{5}x = -32$
so $x = -40$.

-40°C is the same temperature as -40°F.

37. $p(x)' = 32 - \frac{3x}{50}$ $\qquad 50 \le x \le 500$

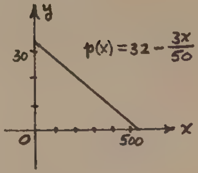
$p(50) = 32 - 3 = 29$

$p(100) = 32 - 6 = 26$

$p(200) = 32 - 12 = 20$

$p(400) = 32 - 24 = 8$

$p(500) = 32 - 30 = 2$



38. $A = 2x^2 + 4xh$. But $V = x^2h = 100$, so

$h = \frac{100}{x^2}$. Hence, $A = 2x^2 + 4x(\frac{100}{x^2})$, and

$A = 2x^2 + \frac{400}{x}$.

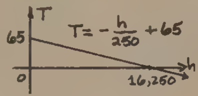
39. Let $T = ah + b$. Then we know that

$5 = a(15{,}000) + b$ and, also,

$-15 = a(20{,}000) + b$.

Subtracting the second equation from the first, we get $20 = -5000a$

so that $a = \frac{20}{-5000} = \frac{1}{250}$.



Then, since $b = 5 - a(15{,}000)$, we have

$b = 5 - 15{,}000(-\frac{1}{250}) = 5 + 60 = 65.$

Hence, $T = -\frac{1}{250}h + 65$, $h \ge 0$.

When $h = 30{,}000$, $T = -\frac{1}{250}(30{,}000) + 65$

$= -120 + 65 = -55'$

40. $T = 60t$ $\quad V = 1000v$ thus

$$\frac{V}{1000} = \begin{cases} T & \text{for } 0 \le \frac{T}{60} < 5 \\ 300 & \text{for } \frac{T}{60} \ge 5 \end{cases} \quad \text{or,}$$

$$V = \begin{cases} 1000T & \text{for } 0 \le T < 300 \\ 300{,}000 & \text{for } T \ge 300 \end{cases}$$

$V = \frac{1000v}{60} = \frac{50v}{3}$ and $60T - t$. Therefore,

$$\frac{50V}{3} = \begin{cases} 60(60T) & \text{for } 0 \le 60T \le 5 \\ 300 & \text{for } 60T \ge 5 \end{cases} \quad \text{or,}$$

$$V = \begin{cases} \frac{3(3600T)}{50} & \text{for } 0 \le T \le \frac{5}{60} \\ \frac{3(300)}{50} & \text{for } T \ge \frac{5}{60}. \end{cases}$$ That is,

$$V = \begin{cases} 216T & \text{for } 0 \le T \le \frac{1}{12} \\ 18 & \text{for } T \ge \frac{1}{12} \end{cases}$$

41. $P = 9.9 \times 10^4 h$

42. $T(0.1) = 2\pi\sqrt{\frac{0.1}{9.807}} = 0.6344710062$

≈ 0.634

$T(1) = 2\pi\sqrt{\frac{1}{9.807}} = 2.006373489$

≈ 2.006

$T(1.5) = 2\pi\sqrt{\frac{1.5}{9.807}} = 2.457295641$

≈ 2.46

$T(0.2484) = 2\pi\sqrt{\frac{0.2484}{9.807}} = 0.9999713941$

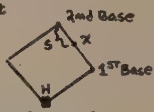
≈ 1.000

43. As indicated in the diagram, at time t,

$s = \sqrt{90^2 + (90-30t)^2}$

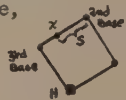
$= 30\sqrt{18 - 6t + t^2}$

44. For $0 \le t \le 3$, see problem 43



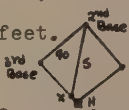
For $3 < t \le 6$

From first to x takes $t - 3$ seconds; thus, the distance to x is $30(t - 3)$ feet. Hence, $s = 90 - 30(t - 3) = 180 - 30t$.



For $6 < t \le 9$

From second to x takes $t-6$ seconds; in this case, we have $s = 30t - 180$ feet.



For $9 < t \le 12$

To get from third to x takes $t-9$ seconds, so $(t-9)30$ feet is the distance from third base to x. Hence, from the diagram,

$s = \sqrt{90^2 + 30^2(t-9)^2} = 30\sqrt{9 + (t - 9)^2}$

45. The graph falls below the x axis between 0 and 0.31.

46. Consider x intercepts: Let $y = 0$, so that $x^3 + 3x^2 = 0$. Then $x^2(x + 3) = 0$ and $x = 0$ or $x = -3$. Thus, $(-3,0)$ is a point on the graph. Hence, the sketch shown is incorrect.

Problem Set 1.5, page 39

1. (a) Domain: All reals,

Range: $(-\infty, 2]$; even

(b) Domain: $[-5,5]$

Range: $[-3,3]$; neither

(c) Domain: $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

Range: $[-1,1]$; odd

(d) Domain: All reals

Range: $[-2,1]$; neither

(e) Domain: All reals

Range: All reals; neither

(f) Domain: All reals

Range: $(-\infty,2]$; even

(g) Domain: All reals

Range: All reals; neither

(h) Domain: All reals except 0

Range: All reals except 0; odd

2. Yes; $f(x) = 0$, since clearly $f(x)=-f(x)$ and $f(x) = f(-x)$.

3. $f(-x) = (-x)^4 + 3 = x^4 + 3 = f(x)$. The function is even.

4. $g(-x) = -(-x)^4 + 2(-x)^2 + 1$
$= -x^4 + 2x^2 + 1 = g(h)$. The function is even.

5. $f(-x) = (-x)^4 + (-x) = x^4 - x \neq x^4 + x$ and $x^4 - x \neq -(x^4 + x)$, unless $x = 0$. So the function is neither even nor odd.

6. $g(-t) = (-t)^2 + |-t| = t^2 + |t| = g(t)$. The function is even.

7. $F(-x) = 5(-x)^3 + 7(-x) = -5x^3 - 7x$
$= -(5x^3 + 7x) = -F(x)$. The function is odd.

8. $f(-t) = -(-t)^3 + 7(-t) = +t^3 - 7t$
$= -(-t^3 + 7t) = -f(t)$. The function is odd.

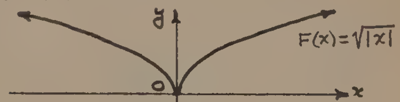
9. The domain of h is $[0,\infty)$, and h does not have the property that if x is in the domain, then $-x$ is in the domain. Hence, h is neither even nor odd.

10. $f(-y) = \frac{\sqrt{(-y)^2 + 1}}{|-y|} = \frac{\sqrt{y^2 + 1}}{|y|} = f(y)$. The function is even.

11. $f(-x) = \frac{-x + 1}{x^2 + 1}$, so that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. The function is neither even nor odd.

12. (a) $F(-x) = \sqrt{|-x|} = \sqrt{|x|} = F(x)$. Thus, $F(x)$ is even.

(b)



13. It is a polynomial function of degree 2; $a_2 = 6$, $a_1 = -3$, $a_0 = -8$.

14. It is not a polynomial function.

15. It is a polynomial function of degree 3; $a_3 = -1$, $a_2 = 1$, $a_1 = -5$, $a_0 = 6$.

16. It is a polynomial function of degree 0; $a_0 = \frac{1}{2}$.

17. It is a polynomial function of degree 4; $a_4 = \sqrt{2}$, $a_3 = -\frac{1}{5}$, $a_2 = 0$, $a_1=0$, $a_0=20$.

18. It is a polynomial function of degree 117; $a_{117} = 210$, $a_{116} = a_{115} = \dots = a_2 = 0$, $a_1 = -11$, $a_0 = -40$.

19. It is a polynomial function of undefined degree. $a_0 = 0$.

20. It is a polynomial function, $h(x) = \sqrt[3]{(x-2)^3} = x - 2$, of degree 1. $a_1 = 1$, $a_0 = -2$.

21. The constant function $f(x) = 2$ is a rule which assigns the real number 2 to each number x in the domains of f.

22. No, since $f(x) = \frac{1}{x} + \frac{x - 1}{x} = \frac{x}{x} = 1$ for all x except 0, but the domain is not $\mathbb{R}$.

23. Subtracting $7 = -3m + b$ from $5 = 2m + b$, we get $-2 = 5m$. $m = -\frac{2}{5}$, and so $b = \frac{29}{5}$. $f(x) = -\frac{2}{5}x + \frac{29}{5}$.

24. We want $m(2x + 3) + b = 2(mx + b) + 3$ or $2mx + 3m + b = 2mx + 2b + 3$ and so $b = 3m - 3$ or $m = \frac{b+3}{3}$. Define $f(x) = mx + (3m - 3) = m(x + 3) - 3$ for any real $m \neq 0$. For example, for $m = 4$, $f(x) = 4x + 9$ (or define $f(x) = \frac{b+3}{3}x + b$ for any real $b \neq -3$).

25. We want $m(5x) + b = f(mx + b)$ or $5mx + b = 5mx + 5b$ or $4b = 0$, and so $b = 0$. Define $f(x) = mx$ for any real $m \neq 0$. For example, $f(x) = 2x$ or $f(x) = -3x$.

26. We want $m(x + 7) + b = (mx + b) + (7m+b)$,

$$mx + 7m + b = mx + 2b + 7m,$$

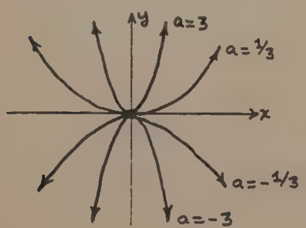
$0 = b$. Define $f(x) = mx$ for any real $m \neq 0$.

27. Let $f(x) = mx + b$, $m \neq 0$. If $r = -\frac{b}{m}$, then $f(r) = m(-\frac{b}{m}) + b = 0$; so every nonconstant linear function has a root, namely, $-\frac{b}{m}$.

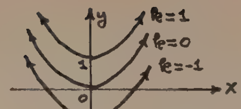
28. For $f(x) = mx + b$, $f(tc + (1 - t)d)$

$$\begin{aligned}
&= m\left[tc + (1 - t)d\right] + b \\
&= mtc + md - mtd + b \\
&= mtc + tb - tb + md - mtd + b \\
&= t(mc + b) + b(1 - t) + md(1-t) \\
&= tf(c) + (1 - t)(b + md) \\
&= tf(c) + (1 - t)(md + b) \\
&= tf(c) + (1 - t)f(d).
\end{aligned}$$

Conversely, let $c=1$, $d=0$ to get $f(t)=mt+b$ where $m = f(1)-f(0)$ and $b = -f(0)$.

29. 

$f(x) = ax^2$

30. 

$f(x) = 2x^2 + k$

31. $f(x) = x^2 + 2x - 4 = x^2 + 2x + 1 - 5$
$= (x + 1)^2 - 5 = 1\cdot(x + 1)^2 - 5.$

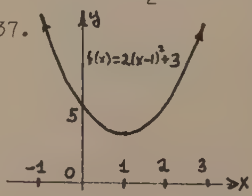
32. $f(x) = x^2 - 12x + 5 = x^2 - 12x + 36-36+5$
$= x^2 - 12x + 36 - 31 = (x-6)^2 - 31$
$= 1 \cdot (x - 6)^2 - 31.$

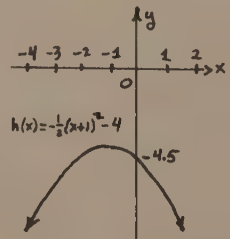
33. $f(x) = 3x^2 - 10x - 2 = 3(x^2 - \frac{10}{3}x) - 2$
$= 3(x^2 - \frac{10}{3}x + \frac{25}{9} - \frac{25}{9}) - 2$
$= 3(x^2 - \frac{10}{3}x + \frac{25}{9}) - 3 \cdot \frac{25}{9} - 2$
$= 3(x - \frac{5}{3})^2 - \frac{31}{3}.$

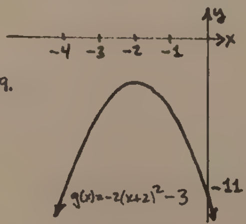
34. $f(x) = 2x^2 + 3x - 1 = 2(x^2 + \frac{3}{2}x) - 1$
$= 2(x^2 + \frac{3}{2}x + \frac{9}{4} - \frac{9}{4}) - 1$
$= 2(x^2 + \frac{3}{2}x + \frac{9}{4}) - 2 \cdot \frac{9}{4} - 1$
$= 2(x + \frac{3}{2})^2 - \frac{11}{2}.$

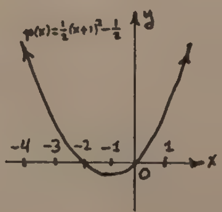
35. $f(x) = -2x^2 + 6x + 3 = -2(x^2 - 3x) + 3$
$= -2(x^2 - 3x + \frac{9}{4} - \frac{9}{4}) + 3$
$= -2(x^2 - 3x + \frac{9}{4}) - 2(-\frac{9}{4}) + 3$
$= -2(x - \frac{3}{2})^2 + \frac{15}{2}.$

36. $f(x) = \frac{3}{2}x^2 - 6x - 7 = \frac{3}{2}(x^2 - 4x) - 7$
$= \frac{3}{2}(x^2 - 4x + 4 - 4) - 7$
$= \frac{3}{2}(x^2 - 4x + 4) - \frac{3}{2} \cdot 4 - 7$
$= \frac{3}{2}(x - 2)^2 - 13$

37. 

38. 

39. 

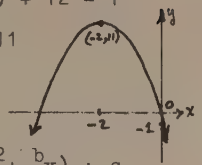
40. 

41. (a) $f(x) = -3(x^2 + 4x) - 1$

$= -3(x^2 + 4x + 4) + 12 - 1$

$= -3(x + 2)^2 + 11$

(b) $(-2,11)$



42. $f(x) = ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c$

$= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c$

$= a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$

Let $h = -\frac{b}{2a}$ and let $k = -\frac{b^2}{4a} + c$. Then

$f(h) = f(-\frac{b}{2a}) = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c$

$= \frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c = k$. Thus,

$f(x) = a(x - h)^2 + k$.

43. (a) $(f+g)(x) = f(x)+g(x) = (2x-5)+(x^2+1)$

$= x^2+2x - 4$

$(f-g)(x) = f(x)-g(x) = 2x-5-(x^2+1)$

$= -x^2 + 2x - 6$

$(f\cdot g)(x) = f(x)\cdot g(x) = (2x-5)(x^2+1)$

$= 2x^3 - 5x^2 + 2x - 5$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{2x-5}{x^2+1}$

(b) $(f+g)(x) = f(x)+g(x) = \sqrt{x}+x^2+4$

$(f-g)(x) = f(x)-g(x) = \sqrt{x}-(x^2+4) = \sqrt{x}-x^2-4$

$(f\cdot g)(x) = f(x)g(x) = \sqrt{x}(x^2+4)$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x^2+4}$

(c) $(f+g)(x) = f(x)+g(x) = 3x+5+7-4x = -x+12$

$(f-g)(x) = f(x)-g(x) = 3x+5-(7-4x) = 7x-2$

$(f\cdot g)(x) = f(x)\cdot g(x) = (3x+5)(7-4x)$

$= 12x^2 + x + 35$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{3x+5}{7-4x}$

(d) $(f+g)(x) = f(x)+g(x) = \sqrt{x+3}+\frac{1}{x}$

$(f-g)(x) = f(x)-g(x) = \sqrt{x+3}-\frac{1}{x}$

$(f\cdot g)(x) = f(x)\cdot g(x) = \frac{\sqrt{x+3}}{x}$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = x\sqrt{x+3}$

(e) $(f+g)(x) = f(x)+g(x) = |x|+|x-2|$

$(f-g)(x) = f(x)-g(x) = |x|-|x-2|$

$(f\cdot g)(x) = f(x)\cdot g(x) = |x|\,|x-2| = |x(x-2)|$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{|x|}{|x-2|} = \left|\frac{x}{x-2}\right|$

(f) $(f+g)(x) = f(x)+g(x) = ax+b+cx+d$

$= (a+c)x+b+d$

$(f-g)(x) = f(x)-g(x) = ax+b-(cx+d)$

$= (a-c)x+b-d$

$(f\cdot g)(x) = f(x)\cdot g(x) = (ax+b)(cx+d)$

$= acx^2+(bc+ad)x+bd$

$(\frac{b}{g})(x) = \frac{f(x)}{g(x)} = \frac{ax+b}{cx+d}$

44. Given that $f(-x) = f(x)$ and $g(-x) = g(x)$.

$(f+g)(-x) = f(-x)+g(-x) = f(x)+g(x)$

$= (f+g)(x)$, so $f+g$ is even.

$(f-g)(-x) = f(-x)-g(-x) = f(x)-g(x)$

$= (f-g)(x)$, so $f-g$ is even.

$(f\cdot g)(-x) = f(-x)g(-x) = f(x)g(x)$

$= (f-g)\,x$, so $f-g$ is even.

$(\frac{f}{g})(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{g(x)} = (\frac{b}{g})(x)$,

so $\frac{f}{g}$ is even.

45. (a) Rational (b) Not Rational

(c) Rational (d) Rational

(e) Not Rational

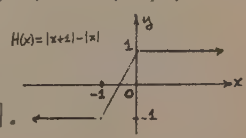
46. $f(x) = \frac{x}{1-x} - \frac{1}{1+x} = \frac{x(1+x) - (1-x)}{(1-x)(1+x)}$

$= \frac{x^2 + 2x - 1}{1 - x^2}$.

So f is a rational function, with domain the intervals $(-\infty,-1)$ and $(-1,1)$ and $(1,\infty)$.

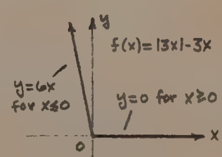
47. The domain is $\mathbb{R}$.

The range is $[-1,1]$.

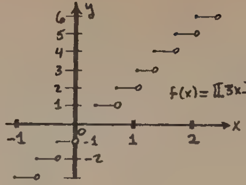


48. The domain is $\mathbb{R}$.

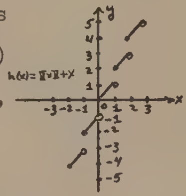
The range is $[0,\infty)$.



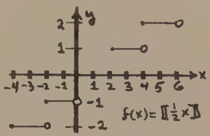
49. $f(x) = [\![3x]\!]$
The domain is $\mathbb{R}$.
The range is the set of all integers.



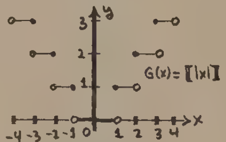
50. The domain is $\mathbb{R}$.
The range is the intervals $[0,1)$, $[2,3)$, $[4,5)$, $[6,7)$ and so forth; and $[-2,-1)$, $[-4,-3)$, $[-6,-5)$, $[-8,-7)$ and so forth.



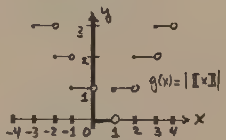
51. The domain is $\mathbb{R}$.
The range is all intergers.



52. The domain is $\mathbb{R}$.
The range is all non-negative integers.



53. The domain is $\mathbb{R}$.
The range is all non-negative integers.



54. (a) $g(-x) = \dfrac{f(-x)+f(-(-x))}{2} = \dfrac{f(-x)+f(x)}{2}$
$= g(x)$. So g is even.

(b) $h(-x) = \dfrac{f(-x)-f(-(-x))}{2} = \dfrac{f(-x)-f(x)}{2}$
$= -\left[\dfrac{f(x)-f(-x)}{2}\right] = -h(x)$.
So h is odd.

(c) $g(x) + h(x) = \dfrac{f(x)+f(-x)}{2} + \dfrac{f(x)-f(-x)}{2}$
$= \dfrac{2f(x)}{2} = f(x)$.

(d) Since $f(x) = G(x)+H(x)$ and $f(x) = g(x) + h(x)$, it follows that $G(x)+H(x)=g(x)+h(x)$. Since G, H, g, and h are even or odd, then $-x$ is in the domain of each function. Hence,
$G(-x) + H(-x) = g(-x) + h(-x)$ and
$G(x) - H(x) = g(x) - h(x)$ since G and g are even and H and h are odd. Adding
$G(x) + H(x) = g(x) + h(x)$ and
$G(x) - H(x) = g(x) - h(x)$,
we get $2G(x) = 2g(x)$, and so $G(x) = g(x)$. Substitution into either equation results in $H(x) = h(x)$.

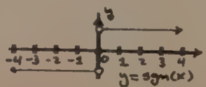
(e) If $f(x) = g(x)$ holds for all x, then f is even since g is. Now assume f is even; that is, $f(x) = f(-x)$. Hence,
$f(-x) = g(-x) + h(-x)$ becomes
$f(x) = g(x) - h(x)$, since g is even and h is odd. Adding the two equations
$f(x) = g(x) + h(x)$ and $f(x) = g(x) - h(x)$,
we get $2f(x) = 2g(x)$, so $f(x) = g(x)$ for all x.

(f) If $f(x) = h(x)$ holds for all x, then f is odd since h is. Now suppose f is odd; that is $f(-x) = -f(x)$. So
$f(-x) = g(-x) + h(-x)$ becomes
$-f(x) = g(x) - h(x)$. Now subtracting this latter equation from
$f(x) = g(x) + h(x)$, we get $2f(x)=2h(x)$, and thus $f(x) = h(x)$ for all x.

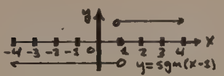
55. (a) $\operatorname{sgn}(-2) = \dfrac{|-2|}{-2} = -1$; $\operatorname{sgn}(-3) = -1$;
$\operatorname{sgn}(0) = 0$; $\operatorname{sgn}(2) = 1$;
$\operatorname{sgn}(3) = 1$; $\operatorname{sgn}(151) = 1$.

(b) $x \operatorname{sgn} x = x\dfrac{|x|}{x} = |x|$ for $x \neq 0$.
If $x = 0$, $x \operatorname{sgn} x = 0\cdot 0 = |0| = |x|$.

(c) If $ab \neq 0$, then $\operatorname{sgn}(ab)$
$= \dfrac{|ab|}{ab} = \dfrac{|a|}{a}\dfrac{|b|}{b} = \dfrac{|a|}{a} \cdot \dfrac{|b|}{b}$
$= \operatorname{sgn} a \operatorname{sgn} b$. If $ab = 0$, then $a = 0$ or $b = 0$. Say $a = 0$. Then
$\operatorname{sgn}(ab) = \dfrac{|ab|}{ab} = 0\cdot\dfrac{|b|}{b} = \operatorname{sgn} a \cdot \dfrac{|b|}{b}$
$= \operatorname{sgn} a \operatorname{sgn} b$. Similarly, if $b=0$, then $\operatorname{sgn}(ab) = \operatorname{sgn} a \operatorname{sgn} b$.

(d) 

(e) The domain is all $\mathbb{R}$. The range is the set of numbers -1, 0, and 1.

(f) 

(g) The sgn function is discontinuous because its graph is not one connected piece.

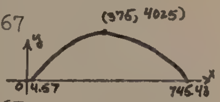
56. $P(x) = R(x) - C(x) = 25x + \frac{x^2}{250} - (100 + 3x + \frac{x^2}{30})$

$P(x) = -\frac{11}{375}x^2 + 22x - 100$

$P(350) = 4006.666\ldots \approx 4006.67$

$P(375) = 4025$

$P(400) = 4006.666\ldots \approx 4006.67$



Solving $-\frac{11}{375}x^2 + 22x - 100 = 0$, that is,

$11x^2 - 8250x + 37500 = 0$, we get

$x \approx 4.57$ or $x \approx 745.43$.

Smallest value of production level is about 4.57.

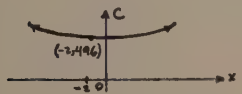
Largest value of production level is about 745.43.

Profit is maximum when $x = 375$.

57. (a) $C = F + V$

(b) $C(x) = 500 + x^2 + 4x = x^2 + 4x + 500$

$= (x+2)^2 + 496$



Problem Set 1.6, page 50

1. $s = r\theta = 2(1.65) = 3.30$ m.
2. $s = r\theta = 1.8(8) = 14.4$cm.
3. $\theta = \frac{s}{r} = \frac{12}{9} = \frac{4}{3}$ radians.
4. $r = \frac{s}{\theta} = \frac{4\pi}{\frac{\pi}{2}} = 8$ km.
5. $s = r\theta = 12(\frac{5\pi}{18}) = \frac{10\pi}{3}$ in.
6. $\theta = \frac{s}{r} = \frac{13\pi}{5}$ radians.
7. (a) $30^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{6}$

(b) $45^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{4}$

(c) $90^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{2}$

(d) $120^\circ \times \frac{\pi}{180^\circ} = \frac{2\pi}{3}$

(e) $-150^\circ \times \frac{\pi}{180^\circ} = -\frac{5\pi}{6}$

(f) $520^\circ \times \frac{\pi}{180^\circ} = \frac{26\pi}{9}$

(g) $72^\circ \times \frac{\pi}{180^\circ} = \frac{2}{5}\pi$

(h) $67.5^\circ \times \frac{\pi}{180^\circ} = \frac{3}{8}\pi$

(i) $-330^\circ \times \frac{\pi}{180^\circ} = -\frac{11}{6}\pi$

(j) $450^\circ \times \frac{\pi}{180^\circ} = \frac{5}{2}\pi$

(k) $21^\circ \times \frac{\pi}{180^\circ} = \frac{7}{60}\pi$

(l) $-360^\circ \times \frac{\pi}{180^\circ} = -2\pi$

8. (a) $7^\circ \times \frac{\pi}{180^\circ} = 0.1222$

(b) $33.333^\circ \times \frac{\pi}{180^\circ} = 0.5818$

(c) $-11.227^\circ \times \frac{\pi}{180^\circ} = -0.1959$

(d) $571^\circ \times \frac{\pi}{180^\circ} = 9.9658$

(e) $1229^\circ \times \frac{\pi}{180^\circ} = 21.4501$

(f) $0.0425^\circ \times \frac{\pi}{180^\circ} = 0.0007$

9. (a) $\frac{\pi}{2} \cdot \frac{180^\circ}{\pi} = 90^\circ$ (b) $\frac{\pi}{3} \cdot \frac{180^\circ}{\pi} = 60^\circ$

(c) $\frac{\pi}{4} \cdot \frac{180^\circ}{\pi} = 45^\circ$ (d) $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ$

(e) $\frac{2\pi}{3} \cdot \frac{180^\circ}{\pi} = 120^\circ$ (f) $-\pi \cdot \frac{180^\circ}{\pi} = -180^\circ$

(g) $\frac{3\pi}{5} \cdot \frac{180^\circ}{\pi} = 108^\circ$ (h) $\frac{-5\pi}{2} \cdot \frac{180^\circ}{\pi} = -450^\circ$

(i) $\frac{9\pi}{4} \cdot \frac{180^\circ}{\pi} = 405^\circ$ (j) $-\frac{3\pi}{8} \cdot \frac{180^\circ}{\pi} = -67.5^\circ$

(k) $7\pi \cdot \frac{180^\circ}{\pi} = 1260^\circ$ (l) $-\frac{\pi}{14} \cdot \frac{180^\circ}{\pi} = -\frac{90^\circ}{7}$

10. (a) $\frac{2}{3} \cdot \frac{180^\circ}{\pi} = 38.1972^\circ$

(b) $-2 \cdot \frac{180^\circ}{\pi} = -114.5916^\circ$

(c) $200 \cdot \frac{180^\circ}{\pi} = 11459.1559^\circ$

(d) $\frac{7\pi}{12} \cdot \frac{180^\circ}{\pi} = 105^\circ$

(e) $(2.7333) \cdot \frac{180^\circ}{\pi} = 156.6066^\circ$

(f) $(1.5708) \cdot \frac{180^\circ}{\pi} = 90.0002^\circ$

11. (a) $-135°, -\frac{3\pi}{4}$ since $\frac{3}{8} \cdot 360° = 135°$ and $135° \cdot \frac{\pi}{180°} = \frac{3\pi}{4}$

(b) $1500°, \frac{25\pi}{3}$ since $4(360°) + \frac{1}{6}(360°) = 1440° + 60° = 1500°$

(c) $120°, \frac{2\pi}{3}$ since $\frac{20}{60} \times 360° = 120°$

12. $(\frac{1}{60})° \times \frac{\pi}{180°} = \frac{\pi}{10,800}$ radians. Thus $s = r\theta = 2.09 \times 10^7 \times (\frac{\pi}{10.800}) = \frac{209 \times 10^3\pi}{108} \approx 6079.56$ feet

13. (a) $A = \frac{\frac{3\pi}{14} \cdot 49}{2} = \frac{21\pi}{4}$ sq. cm.

(b) $A = \frac{\frac{13\pi}{9} \cdot 81}{2} = \frac{117\pi}{2}$ sq. in.

14. $135° \times \frac{\pi}{180°} = \frac{3\pi}{4}$ radians

$A = \frac{\frac{3\pi}{4} \times 70^2}{2} = \frac{3675\pi}{2}$ sq. km.

15. $\sin \frac{2\pi}{7} = 0.781831483$ $\quad \csc \frac{2\pi}{7} = 1.279048008$

$\cos \frac{2\pi}{7} = 0.623489802$ $\quad \sec \frac{2\pi}{7} = 1.603875472$

$\tan \frac{2\pi}{7} = 1.253960338$ $\quad \cot \frac{2\pi}{7} = 0.797473389$

16. $\sin \frac{5\pi}{21} = 0.680172738$ $\quad \csc \frac{5\pi}{21} = 1.470214762$

$\cos \frac{5\pi}{21} = 0.733051872$ $\quad \sec \frac{5\pi}{21} = 1.364159944$

$\tan \frac{5\pi}{21} = 0.927864404$ $\quad \cot \frac{5\pi}{21} = 1.077743683$

17. $\sin (-\frac{17\pi}{3}) = 0.866025404$ $\quad \csc (-\frac{17\pi}{3}) = 1.154700538$

$\cos (-\frac{17\pi}{3}) = 0.5050809$ $\quad \sec (-\frac{17\pi}{3}) = 2$

$\tan (-\frac{17\pi}{3}) = 1.732050809$ $\quad \cot (-\frac{17\pi}{3}) = 0.577350269$

18. $\sin 7° = 0.121869343$ $\quad \csc 7° = 8.205509048$

$\cos 7° = 0.992546152$ $\quad \sec 7° = 1.007509826$

$\tan 7° = 0.122784561$ $\quad \cot 7° = 8.144346428$

19. $\sin (-1.7764) = 0.978937918$ $\quad \csc 1.7764 = 1.021515238$

$\cos (-1.7764) = -0.204158156$ $\quad \sec 1.7764 = -4.898163370$

$\tan (-1.7764) = -4.794997853$ $\quad \cot 1.7764 = -0.208550667$

20. $\sin (-231.4°)=0.781520472$ $\quad \csc(-231.4°)=1.279557011$

$\cos (-231.4°)=0.623879597$ $\quad \sec(-231.4°)=-1.602873383$

$\tan (-231.4°)=-1.252678364$ $\quad \cot(-231.4°)=-0.798289512$

21. $\sin 48° = 0.743144826$ $\quad \csc 48° = 1.345632730$

$\cos 48° = 0.669130606$ $\quad \sec 48° = 1.494476550$

$\tan 48° = 1.110612515$ $\quad \cot 48° = 0.900404044$

22. $\sin 16.19° = .278823499$ $\quad \csc 16.19° = 3.586498283$

$\cos 16.18° = .960342364$ $\quad \sec 16.19° = 1.041295310$

$\tan 16.19° = .290337602$ $\quad \cot 16.19° = 3.444266240$

23. Evaluate sin 30 on the calculator; if the result is 0.50, then 30 must be 30 degrees and if the result is not 0.50, then the calculator must be in radian mode.

25. (a) $(1 - \cos t)(1 + \cos t) = 1 - \cos^2 t = \sin^2 t.$

(b) $2 \sin t \cos t \csc t = 2 \sin t \cos t(\frac{1}{\sin$

$= 2 \cos t.$

(c) $\sec^2 t(\csc^2 t - 1)(\sin t + 1) - \csc t$

$= \frac{1}{\cos^2 t} \cot^2 t(\sin t + 1) - \csc t$

$= \frac{1}{\cos^2 t} (\frac{\cos^2 t}{\sin^2 t})(\sin t + 1) - \csc t$

$= \frac{1}{\cos^2 t} \cot^2 t(\sin t + 1) - \csc t$

$= \frac{1}{\sin t} + \frac{1}{\sin^2 t} - \csc t$

$= \csc t + \csc^2 t - \csc t = \csc^2 t.$

(d) $\frac{1 + \cot^2 t}{\sec^2 t} = \frac{\csc^2 t}{\sec^2 t} = \frac{(\frac{1}{\sin^2 t})}{(\frac{1}{\cos^2 t})} = \frac{\cos^2 t}{\sin^2 t}$

$= \cot^2 t.$

(e) $\frac{\cos t - 1}{\sec t - 1} = \frac{\cos t - 1}{\frac{1}{\cos t} - 1} = \frac{(\cos t - 1)\cos}{(\frac{1}{\cos t} - 1)\cos}$

$= \frac{(\cos t - 1)\cos t}{1 - \cos t} = \frac{(\cos t - 1)\cos t}{-(\cos t - 1)}$

$= -\cos t.$

26. (a) $\cos(\frac{\pi}{2} - t) = \cos \frac{\pi}{2} \cos t + \sin \frac{\pi}{2} \sin t$

$= 0 \cos t + 1 \sin t = \sin t.$

(b) $\sin(\frac{\pi}{2} - t) = \sin \frac{\pi}{2} \cos t - \cos \frac{\pi}{2} \sin t$

$= 1 \cos t - 0 \sin t = \cos t.$

27. (a) $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$

$\cos 75° = \cos(45° + 30°)$

$= \cos 45° \cos 30° - \sin 45° \sin 30°$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$

(b) $\tan 75° = \frac{\sin 75°}{\cos 75°} = \frac{\frac{\sqrt{2}}{4}(\sqrt{3}+1)}{\frac{\sqrt{2}}{4}(\sqrt{3}-1)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{(\sqrt{3}+1)^2}{2}$

$\cot 75° = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{2}$

$\sec 75° = \frac{1}{\cos 75°} = \frac{4}{\sqrt{2}(\sqrt{3}-1)} = \frac{2\sqrt{2}}{\sqrt{3}-1}$

$= \frac{2\sqrt{2}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{2\sqrt{2}(\sqrt{3}+1)}{\sqrt{3}-1}$

$= \sqrt{2}(\sqrt{3} + 1)$

$\csc 75° = \frac{1}{\sin 75°} = \frac{4}{\sqrt{2}(\sqrt{3}+1)} = \frac{2\sqrt{2}}{\sqrt{3}+1}$

$= \sqrt{2}(\sqrt{3} - 1)$

28. $\cos 3t = \cos(t+2t)$

$= \cos t \cos 2t - \sin t \sin 2t$

$= \cos t(\cos^2 t - \sin^2 t) - \sin t(2 \sin t \cos t)$

$= \cos^3 t - \cos t \sin^2 t - 2 \sin^2 t \cos t = \cos^3 t -$

$3 \cos t \sin^2 t = \cos^3 t - 3 \cos t(1 - \cos^2 t)$

$= \cos^3 t - 3 \cos t + 3 \cos^3 t$

$= 4 \cos^3 t - 3 \cos t.$

Thus, let $x = \cos \frac{\pi}{9}$ and put $t = \frac{\pi}{9}$ in the above identity. Then $4x^3 - 3x = \cos(\frac{3\pi}{9}) = \cos \frac{\pi}{3}$

$= \frac{1}{2}$. Multiplying the latter equation by 2, we obtain $8x^3 - 6x = 1$ or $8x^3 - 6x - 1 = 0$.

29. (a) $\frac{\sin^2 2t}{(1 + \cos 2t)^2} + 1 = \frac{\sin^2 2t + (1+\cos 2t)^2}{(1+\cos 2t)^2}$

$= \frac{\sin^2 2t + 1 + 2\cos 2t + \cos^2 2t}{(1 + \cos 2t)^2}$

$= \frac{\sin^2 2t + \cos^2 2t + 1 + 2\cos 2t}{(1 + \cos 2t)^2}$

$= \frac{1+1+2\cos 2t}{(1+\cos 2t)^2} = \frac{2+2\cos 2t}{(1+\cos 2t)^2} = \frac{2(1+\cos 2t)}{(1+\cos 2t)^2}$

$= \frac{2}{1+\cos 2t} = \frac{2}{2\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$

(b) $\frac{\cos^4 t - \sin^4 t}{\sin 2t}$

$= \frac{(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)}{\sin 2t} = \frac{1 \cdot \cos 2t}{\sin 2t}$

$= \cot 2t.$

(c) $\cos^2 2t - \sin^2 t$

$= (1-2\sin^2 t)^2 - \sin^2 t = 1-4\sin^2 t + 4\sin^4 t - \sin^2 t$

$= 4\sin^4 t - 5\sin^2 t + 1 = (4\sin^2 t - 1)(\sin^2 t - 1)$

$= (1-4\sin^2 t)(1-\sin^2 t)$

$= (1+2\sin t)(1-2\sin t)\cos^2 t$

$= (1 - 4 \sin^2 t)\cos^2 t$

$= [1 - 4(1 - \cos^2 t)]\cos^2 t = (-3 + 4 \cos^2 t)\cos^2 t.$

(d) $\tan t - \csc t(1 - 2 \cos^2 t)\sec t$

$= \tan t - \csc t(-\cos 2t)\sec t$

$= \tan t + \frac{\cos 2t}{\sin t \cos t}$

$= \tan t + \frac{\cos^2 t - \sin^2 t}{\sin t \cos t}$

$= \tan t + \frac{\cos^2 t}{\sin t \cos t} - \frac{\sin^2 t}{\sin t \cos t}$

$= \tan t + \frac{\cos t}{\sin t} - \frac{\sin t}{\cos t} = \cot t$

(e) $\cos(s - t)\cos t - \sin(s - t)\sin t$

$= \cos[(s - t) + t] = \cos s$

30. $\sin s = \pm\sqrt{1-\cos^2 s} = \pm\sqrt{1-\frac{16}{25}} = \pm\sqrt{\frac{9}{25}} = \pm \frac{3}{5}.$

$\cos t = \pm\sqrt{1-\sin^2 t} = \pm\sqrt{1-\frac{144}{169}} = \pm\sqrt{\frac{25}{169}} = \pm\frac{5}{13}$

Since s and t are second quadrant angles, then $\sin s > 0$ and $\cos t < 0$; hence,

$\sin s = \frac{3}{5}$ and $\cos t = -\frac{5}{13}.$

(a) $\sin(s-t) = \sin s \cos t - \cos s \sin t$

$= (\frac{3}{5})(-\frac{5}{13}) - (-\frac{4}{5})(\frac{12}{13}) = \frac{33}{65}.$

(b) $\cos(s+t) = \cos s \cos t - \sin s \sin t$

$= (-\frac{4}{5})(-\frac{5}{13}) - (\frac{3}{5})(\frac{12}{13}) = -\frac{16}{65}.$

(c) $\cos(s-t) = \cos s \cos t + \sin s \sin t$

$= (-\frac{4}{5})(-\frac{5}{13}) + (\frac{3}{5})(\frac{12}{13}) = \frac{56}{65}$; hence,

$\cot(s-t) = \frac{\cos(s-t)}{\sin(s-t)} = \frac{(\frac{56}{65})}{(\frac{33}{65})} = \frac{56}{33}.$

31. $\sin\theta = \frac{3}{5}$ $\csc\theta = \frac{5}{3}$

$\cos\theta = \frac{4}{5}$ $\sec\theta = \frac{5}{4}$

$\tan\theta = \frac{3}{4}$ $\cot\theta = \frac{4}{3}$

32. $3^2+5^2 = h^2$; so $h = \sqrt{9+25} = \sqrt{34}$ = hypotenuse.

$\sin\theta = \frac{3}{\sqrt{34}}$ $\csc\theta = \frac{\sqrt{34}}{3}$

$\cos\theta = \frac{5}{\sqrt{34}}$ $\sec\theta = \frac{\sqrt{34}}{5}$

$\tan\theta = \frac{3}{5}$ $\cot\theta = \frac{5}{3}$

33. $s^2+3^2 = 4^2$; so $s^2 = 16-9 = 7$.

Thus, $s^2 = \sqrt{7}$ = adj. side.

$\sin\theta = \frac{3}{4}$ $\csc\theta = \frac{4}{3}$

$\cos\theta = \frac{\sqrt{7}}{4}$ $\sec\theta = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$

$\tan\theta = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$ $\cot\theta = \frac{\sqrt{7}}{3}$

34. $2^2+9^2 = h^2$; so $h = \sqrt{4+81} = \sqrt{85}$ = hypotenuse.

$\sin\theta = \frac{9}{\sqrt{85}}$ $\csc\theta = \frac{\sqrt{85}}{9}$

$\cos\theta = \frac{2}{\sqrt{85}}$ $\sec\theta = \frac{\sqrt{85}}{2}$

$\tan\theta = \frac{9}{2}$ $\cot\theta = \frac{2}{9}$

35. $2^2+5^2 = 4^2$ so $s^2 = 16-4 = 12$.

Thus, $s = \sqrt{12} = 2\sqrt{3}$ = adj. side.

$\sin\theta = \frac{2}{4} = \frac{1}{2}$ $\csc\theta = 2$

$\cos\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ $\sec\theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\tan\theta = \frac{2}{(\sqrt{3})} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\cot\theta = \sqrt{3}$

36. $(s\sqrt{5})^2 + (2\sqrt{3})^2 = 20 + 12 = 32 = h^2$;

so $h = \sqrt{32} = 4\sqrt{2}$ = hypotenuse.

$\sin\theta = \frac{2\sqrt{3}}{4\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}}$ $\csc\theta = \frac{2\sqrt{2}}{\sqrt{3}}$

$\cos\theta = \frac{2\sqrt{5}}{4\sqrt{2}} = \frac{\sqrt{5}}{2\sqrt{2}}$ $\sec\theta = \frac{2\sqrt{2}}{\sqrt{5}}$

$\tan\theta = \frac{2\sqrt{3}}{2\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}$ $\cot\theta = \frac{\sqrt{5}}{\sqrt{3}}$

37. $3^2+s^2=(3\sqrt{2})^2 = 18$; so $s = \sqrt{9} = 3$ = adj. side.

$\sin\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\csc\theta = \sqrt{2}$

$\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\sec\theta = \sqrt{2}$

$\tan\theta = \frac{3}{3} = 1$ $\cot\theta = 1$

38. $(\frac{7}{2})^2+8^2 = \frac{49}{4} + 64 = \frac{305}{4} = h^2$

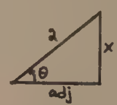
so $h = \frac{\sqrt{305}}{2}$ = hypotenuse

$\sin\theta = \frac{8}{\frac{\sqrt{305}}{2}} = \frac{16}{\sqrt{305}}$ $\csc\theta = \frac{\sqrt{305}}{16}$

$\cos\theta = \frac{\frac{7}{2}}{\frac{\sqrt{305}}{2}} = \frac{7}{\sqrt{305}}$ $\csc\theta = \frac{\sqrt{305}}{7}$

$\tan\theta = \frac{8}{\frac{7}{2}} = \frac{16}{7}$ $\cot\theta = \frac{7}{16}$

39. Let θ be the angle shown in the figure so that $\sin\theta = \frac{x}{2}$ and $2\sin\theta = x$.

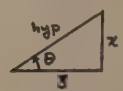


By the Pythagorean theorem,

adj $= \sqrt{2^2 - x^2} = \sqrt{4 - x^2}$.

Thus, $\cos\theta = \frac{\sqrt{4-x^2}}{2}$ so that $\sec\theta = \frac{2}{\sqrt{4-x^2}}$

40. Let θ be the angle shown in the figure so that $\tan\theta = \frac{x}{3}$ and $3\tan\theta = x$.



By the Pythagorean theorem,

hyp $= \sqrt{3^2 + x^2} = \sqrt{9 + x^2}$.

Thus, $\sin\theta = \frac{x}{\sqrt{9+x^2}}$; hence, $\csc\theta = \frac{\sqrt{9+x^2}}{x}$.

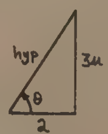
41. $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$

42. Let θ be the angle shown in the figure so that $\tan\theta = \frac{3u}{2}$ and $2\tan\theta = 3u$.

By the Pythagorean theorem,

hyp $= \sqrt{2^2 + (3u)^2} = \sqrt{4 + 9u^2}$.

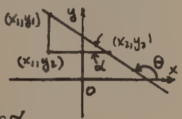
Therefore, $\sin\theta = \frac{3u}{\sqrt{4 + 9u^2}}$.



43. $\theta = 0$, line is parallel to x axis - say equation is $y = c$; then

$m = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{c - c}{x_2-x_1} = 0$; but

$m = \tan\theta = \tan 0 = 0$. Thus formula holds for $\frac{\pi}{2} < \theta < \pi$.

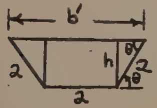


$\tan\alpha = \dfrac{y_1-y_2}{x_2-x_1}$ $\qquad \theta = \pi - \alpha$

$\tan\theta = \tan(\pi-\alpha) = \dfrac{\tan\pi - \tan\alpha}{1+\tan\pi\tan\alpha}$

$= \dfrac{0-\tan\alpha}{1+0\cdot\tan\alpha} = -\tan\alpha$

So $\tan\theta = -\tan\alpha = \dfrac{-(y_1-y_2)}{x_2-x_1} = \dfrac{y_2-y_1}{x_2-x_1} = m$

44. 

Area $= \frac{h}{2}(b+b')$

$= \frac{2\sin\theta}{2}(2+b')$

$= \sin\theta(2+b')$

$= \sin\theta(2+2\cos\theta+2+2\cos\theta)$

$= \sin\theta(4+4\cos\theta)$

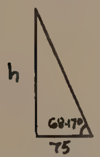
$= 4\sin\theta(1+\cos\theta)$

$\sin\theta = \frac{h}{2}$

or $h = 2\sin\theta$

$\cos\theta = \frac{u}{2}$ or

$u = 2\cos\theta$

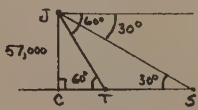
45. 

$\tan 68.17^\circ = \frac{h}{75}$

so $h = 75\tan 68.17^\circ$

$= 187.2$ meters

46. $|\overline{TS}| = |\overline{CS}| - |\overline{CT}|$



$\tan 60^\circ = \dfrac{|\overline{CJ}|}{|\overline{CT}|} = \dfrac{57{,}000}{|\overline{CT}|}$

So $|\overline{CT}| = \dfrac{57{,}000}{\tan 60^\circ} = \dfrac{57{,}000}{\sqrt{3}} = 19{,}000\sqrt{3}$

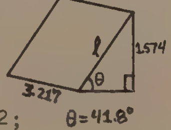
$\tan 30^\circ = \dfrac{|\overline{CJ}|}{|\overline{CS}|} = \dfrac{57{,}000}{|\overline{CS}|}$

So $|\overline{CS}| = \dfrac{57{,}000}{\tan 30^\circ} = \dfrac{57{,}000}{\frac{\sqrt{3}}{3}} = 57{,}000\sqrt{3}$

So $|\overline{TS}| = |\overline{CS}| - |\overline{CT}| = 57{,}000\sqrt{3} - 19{,}000\sqrt{3}$

$= 38{,}000\sqrt{3} \approx 65{,}820\sqrt{3}$ ft.

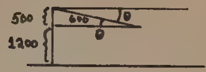
47. Surface area $= 3.217\ \ell$.



$\sin\theta = \sin 41.8^\circ = \dfrac{1.574}{\ell}$

so $\ell = \dfrac{1.574}{\sin 41.8^\circ} = 2.361475352$;

so area $= \ell(3.217) = 7.596866208 \approx 7.597\ m^2$.

48. 

$\sin\theta = \dfrac{500}{600} = \dfrac{5}{6} \approx$ so $\theta \approx 56^\circ$

49. (a) Since $0<x<\frac{\pi}{2}$, $\cos x>0$.

Hence, $-\cos x < \cos x$

or $1-\cos x < 1+\cos x$

or $\frac{1}{2}(1-\cos x) < \frac{1}{2}(1+\cos x)$,

so that $\sin^2\frac{x}{2} < \cos^2\frac{x}{2}$.

(b) Let $x = 2t$. Then $0<x<\frac{\pi}{2}$ becomes

$0 < 2t < \frac{\pi}{2}$ or $0 < t < \frac{\pi}{4}$.

$\sin^2(\frac{2t}{2}) < \cos^2(\frac{2t}{2})$

$\sin^2 t < \cos^2 t$

$|\sin t| < |\cos t|$.

But $0 < t < \frac{\pi}{4}$, so that $0 < \sin t < \cos t$.

(c) Since $\cos t > 0$, then

$0 < \dfrac{\sin t}{\cos t} < \dfrac{\cos t}{\cos t} = 1$ or

$0 < \tan t < 1$ for $0 < t < \frac{\pi}{4}$.

50. For $0<t<\frac{\pi}{4}$, from 49c, we have $0<\tan t<1$ and from theorem 3 if $0<|t|<\pi$ then

$|1-\cos t| < \frac{t^2}{2}$.

Now $|\sin t-\tan t| = \left|\sin t - \dfrac{\sin t}{\cos t}\right|$

$= \left|\sin t(1-\dfrac{1}{\cos t})\right| = |\sin t|\left|\dfrac{\cos t-1}{\cos t}\right|$

$= \dfrac{|\sin t|}{|\cos t|}\,|1-\cos t|$

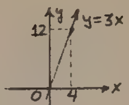
$= |\tan t|\,|1-\cos t| < 1\cdot\frac{t^2}{2} = \frac{t^2}{2}$

so $|\sin t-\cos t| < \frac{t^2}{2}$ for $0<|t|<\frac{\pi}{4}$

Thus when t is small, $|\sin t-\cos t|$ is small; hence, $\sin t \approx \cos t$.

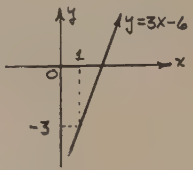
Problem Set 1.7, page 58

1. $\lim\limits_{x \to 4} 3x$



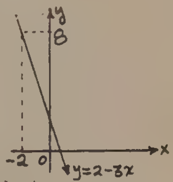
As x gets closer to 4, 3x gets closer to 12, so that $\lim\limits_{x \to 4} 3x = 12$.

2. $\lim\limits_{x \to 1} (3x-6)$



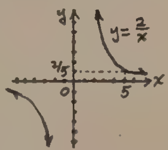
As x gets closer to 1, 3x gets closer to 3, and 3x-6 gets closer to -3, so that $\lim\limits_{x \to 1} (3x - 6) = -3$.

3. $\lim\limits_{x \to -2} (2 - 3x)$



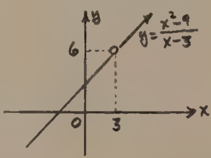
As x gets closer to -2, 3x gets closer to -6; and 2 - 3x gets closer to 8, so that $\lim\limits_{x \to -2} (2 - 3x) = 8$.

4. $\lim\limits_{x \to 5} \frac{2}{x}$



As x gets closer to 5, $\frac{2}{x}$ gets closer to $\frac{2}{5}$, so that $\lim\limits_{x \to 5} \frac{2}{x} = \frac{2}{5}$.

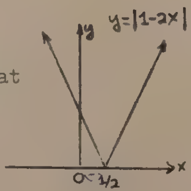
5. $\lim\limits_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim\limits_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim\limits_{x \to 3} (x + 3)$



As x gets closer to 3, x + 3 gets closer to 6, so that $\lim\limits_{x \to 3} \frac{x^2-9}{x-3} = 6$.

6. $\lim\limits_{x \to \frac{1}{2}} |1-2x|$

As x gets closer to $\frac{1}{2}$, 2x gets closer to 1, and 1-2x gets closer to zero, so that $\lim\limits_{x \to \frac{1}{2}} |1-2x| = 0$.



7. $\lim\limits_{x \to 2} \frac{x^2-5x+6}{x-2} = \lim\limits_{x \to 2} \frac{(x-3)(x-2)}{x-2} = \lim\limits_{x \to 2}(x-3) = -1$

x	1	1.9	1.99	1.999	1.9999
$\frac{x^2-5x+6}{x-2}$	-2	-1.1	-1.01	-1.001	-1.0001

x	3	2.1	2.01	2.001	2.0001
$\frac{x^2-5x+6}{x-2}$	0	-0.9	-0.99	-0.999	-0.9999

8. $\lim\limits_{x \to 1} \frac{x^3-1}{x^2-1} = \lim\limits_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

$= \lim\limits_{x \to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$

x	0	0.9	0.99	0.999	0.9999
$\frac{x^3-1}{x^2-1}$	+1	1.4263	1.4925	1.4993	1.4999

x	2	1.1	1.01	1.001	1.0001
$\frac{x^3-1}{x^2-1}$	2.3333	1.5762	1.5075	1.5008	1.5001

9. $\lim\limits_{x \to -1} \frac{x^2 - 2x - 3}{x+1} = \lim\limits_{x \to -1} \frac{(x+1)(x-3)}{x+1}$

$= \lim\limits_{x \to -1} (x - 3) = -4$

x	-2	-1.1	-1.01	-1.001	-1.0001
$\frac{x^2-2x-3}{x+1}$	-5	-4.1	-4.01	-4.001	-4.0001

x	0	-0.9	-0.99	-0.999	-0.9999
$\frac{x^2-2x-3}{x+1}$	-3	-3.9	-3.99	-3.999	-3.9999

10. $\lim\limits_{x \to -2} \frac{x+2}{x^2+3x+2} = \lim\limits_{x \to -2} \frac{x+2}{(x+2)(x+1)}$

$= \lim\limits_{x \to -2} \left(\frac{1}{x+1}\right) = -1$

x	-3	-2.1	-2.01	-2.001	-2.0001
$\frac{x+2}{x^2+3x+2}$	-0.5	-0.9091	-0.9901	-0.9990	-0.9999

x	-1	-1.9	-1.99	-1.999	-1.9999
$\frac{x+2}{x^2+3x+2}$	not defined	-1.1111	-1.0101	-1.0010	-1.0001

11. $\lim\limits_{t\to 4} \frac{\sqrt{t}-2}{t-4} = \lim\limits_{t\to 4} \frac{\sqrt{t}-2}{t-4} \cdot \frac{\sqrt{t}+2}{\sqrt{t}+2}$

$= \lim\limits_{t\to 4} \frac{t-4}{(t-4)(\sqrt{t}+2)} = \lim\limits_{t\to 4} \frac{1}{\sqrt{t}+2} = \frac{1}{4}.$

t	3	3.9	3.99	3.999	3.9999
$\frac{\sqrt{t}-2}{t-4}$	0.2679	0.2516	0.2502	0.25002	0.240002

t	5	4.1	4.01	4.001	4.0001
$\frac{\sqrt{t}-2}{t-4}$	0.2361	0.2485	0.2498	0.24998	0.249998

12. $\lim\limits_{x\to 0} \frac{|x|}{3-\sqrt{9+|x|}} = \lim\limits_{x\to 0} \frac{|x|}{3-\sqrt{9+|x|}} \cdot \frac{3+\sqrt{9+|x|}}{3+\sqrt{9+|x|}}$

$= \lim\limits_{x\to 0} \frac{|x|(3+\sqrt{9+|x|})}{9-(9+|x|)} = \lim\limits_{x\to 0} \frac{|x|(3+\sqrt{9+x})}{-|x|}$

$= \lim\limits_{x\to 0} -(3+\sqrt{9+|x|}) = -6.$

x	± 1	± 0.1	± 0.01	± 0.001	± 0.0001
$\frac{\lvert x\rvert}{3-\sqrt{9+\lvert x\rvert}}$	-6.1623	-6.0166	-6.0017	-6.0002	-6.00002

13. $\lim\limits_{h\to 1} \frac{1-h}{1-(\frac{1}{h})} = \lim\limits_{h\to 1} \frac{h(1-h)}{h-1}$

$= \lim\limits_{h\to 1} \frac{-h(h-1)}{h-1} = \lim\limits_{h\to 1} \frac{-h}{1} = -1.$

h	0	0.9	0.99	0.999	0.9999
$\frac{1-h}{1-(\frac{1}{h})}$	undefined	-0.9	-0.99	-0.9990	-0.9999

h	2	1.1	1.01	1.001	1.0001
$\frac{1-h}{1-(\frac{1}{h})}$	-2	-1.1	-1.01	-1.0010	-1.0001

14. $\lim\limits_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} \cdot \frac{(\sqrt[3]{x})^2+\sqrt[3]{x}+1}{(\sqrt[3]{x})^2+\sqrt[3]{x}+1}$

$= \lim\limits_{x\to 1} \frac{x-1}{(x-1)[(\sqrt[3]{x})^2+\sqrt[3]{x}+1]} = \lim\limits_{x\to 1} \frac{1}{(\sqrt[3]{x})^2+\sqrt[3]{x}+1} = \frac{1}{3}.$

x	0	0.9	0.99	0.999	0.9999
$\frac{\sqrt[3]{x}-1}{x-1}$	1	0.3451	0.33445	0.3334	0.33334

x	2	1.1	1.01	1.001	1.0001
$\frac{\sqrt[3]{x}-1}{x-1}$	0.2599	0.3228	0.3322	0.3332	0.33332

15. $\lim\limits_{x\to 0} \frac{\sqrt{a^2+x}-a}{x} \cdot \frac{\sqrt{a^2+x}+a}{\sqrt{a^2+x}+a} = \lim\limits_{x\to 0} \frac{\sqrt{a^2+x}-a^2}{x(\sqrt{a^2+x}+a)}$

$= \lim\limits_{x\to 0} \frac{1}{\sqrt{a^2+x}+a}$

As x gets near 0, a^2+x gets close to a^2, $\sqrt{a^2+x}$ gets close to $\sqrt{a^2} = |a| = a$ (since $a > 0$) and $\sqrt{a^2+x}+a$ gets close to $2a$; thus, $\lim\limits_{x\to 0} \frac{1}{\sqrt{a^2+x}+a} = \frac{1}{2a}.$

16. $f(x) = \frac{\sqrt{1{,}000{,}000+x}-1000}{x}$

$f(1) = 0.000500000$

$f(0.1) = 0.000500000$

$f(0.01) = 0.000500000$

$f(0.001) = 0.000000000$

Inaccuracy due to the fact that the calculator can handle only so many significant digits.

17. $f(0.001) = 0.0005$

guess: $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

18. $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$

$f(0.001) = 0.666667222$

guess: $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \frac{2}{3}$

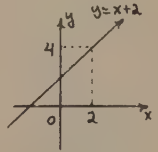
19. $f(0.001) = 1.000000333$

guess: $\lim_{t\to 0} \frac{\tan t}{t} = 1$

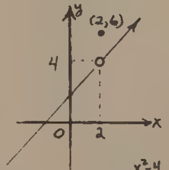
20. $f(0.001) = 0.5000000000$

guess: $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}$

21. (a) When x is close to 2, $x + 2$ is close to 4 so $\lim_{x\to 2}(x+2) = 4$.



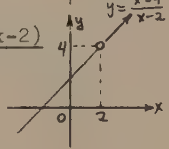
(b) When x is close to 2, $x + 2$ is close to 4 so $\lim_{x\to 2}(x+2) = 4$ and $f(2) = 6 \neq 4 = L$.



(c) $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2}$

$= \lim_{x\to 2}(x+2) = 4$

but $f(2)$ is undefined.



22. $S(x) = \lim_{h\to 0} \frac{A(x+h)^2(B-x-h) - Ax^2(B-x)}{h}$

$= \lim_{h\to 0} \frac{h(-Ax^2+2ABx-2Ax^2-2Axh+ABh-Axh-Ah^3)}{h}$

$= -Ax^2 + 2ABx - 2Ax^2$

$= -3Ax^2 + 2ABx.$

23. We want $|(4x - 1) - 11| < 0.01$, that is, $|4x - 12| < 0.01$. Now, $|4x - 12| = |4(x-3)|$ $= 4|x - 3|$; so we want $4|x - 3| < 0.01$, that is, $|x - 3| < 0.0025$. Take $\delta = 0.0025$.

24. We want $|(3 - 4x) - 7| < 0.02$, that is $|-4x - 4| < 0.02$. Now, $|-4x - 4|$ $= |(-4)(x + 1)| = |-4| \cdot |x + 1| = 4|x + 1|$; so we want $4|x + 1| < 0.02$, that is, $|x - (-1)| < 0.005$. Take $\delta = 0.005$.

25. We want $\left|\frac{x^2-25}{x-5} - 10\right| < 0.01$ when $0 < |x-5| < \delta$. When $0 < |x - 5|$, $x \neq 5$ and $\frac{x^2-25}{x-5} = \frac{(x-5)(x+5)}{x-5}$ $= x + 5$. Hence, we want $|(x + 5) - 10|$ $= |x - 5| < 0.01$. Take $\delta = 0.01$.

26. $\lim_{x\to -1}(x + 1) = 0$

Find $\delta > 0$ so that when $|x - (-1)| = |x+1| < \delta$, then $|(x + 1) - 0| = |x + 1| < 0.1$.

Choose $\delta = 0.1$.

27. We want $\left|\frac{x+1}{2} - 3\right| < 0.1$, that is, $\left|\frac{x + 1 - 6}{2}\right| < 0.1$. The latter condition is equivalent to $\left|\frac{x-5}{2}\right| < 0.1$, that is, $\frac{1}{2}|x - 5| < 0.1$. But, $\frac{1}{2}|x - 5| < 0.1$ will hold exactly when $|x - 5| < 0.2$.

Take $\delta = 0.2$.

28. Take $\delta = \frac{1}{50}$ and suppose that $|x-2| < \delta = \frac{1}{50}$. Then $-\frac{1}{50} < x - 2 < \frac{1}{50}$. Adding 4 to the latter inequality, we have,

$0 < 4 - \frac{1}{50} < x + 2 < 4 + \frac{1}{50} < 5.$

Multiplying the inequalities $0 < x+2 < 5$ and $0 \leq |x - 2| < \frac{1}{50}$, we obtain,

$|x - 2|(x + 2) < 5(\frac{1}{50}) = \frac{1}{10}$, or

$|x^2 - 4| < 0.1$, as desired.

29. We want $|(2x-5)-3| < \varepsilon$, that is, $|2x-8| < \varepsilon$. Since $|2x-8| = |2(x-4)| = 2|x-4|$, then we want $2|x-4| < \varepsilon$; that is, $|x-4| < \frac{\varepsilon}{2}$.

Take $\delta = \frac{\varepsilon}{2}$.

30. We want $|(2 - 5x) - 2| < \varepsilon$, that is, $|-5x| < \varepsilon$. Since $|-5x| = |5x| = 5|x|$, then we want $5|x| < \varepsilon$, that is $|x - 0| < \frac{\varepsilon}{5}$. Take $\delta = \frac{\varepsilon}{5}$.

31. We want $|(4x - 1) - 11| < \varepsilon$, that is, $|4x - 12| < \varepsilon$. Since $|4x - 12| = |4(x - 3)| = 4|x - 3|$, then we want $4|x - 3| < \varepsilon$, that is, $|x - 3| < \frac{\varepsilon}{4}$. Take $\delta = \frac{\varepsilon}{4}$.

32. We want $\left|\frac{x^2 - 16}{x-4} - 8\right| < \varepsilon$ whenever $0 < |x - 4| < \varepsilon$. But when $0 < |x - 4|$, $x \neq 4$ and so $\frac{x^2 - 16}{x - 4} = \frac{(x-4)(x+4)}{x-4} = x + 4$.

Thus, we want $|(x + 4) - 8| = |x - 4| < \varepsilon$. Here we simply take $\delta = \varepsilon$.

33. We want $|a - a| < \varepsilon$, that is, $0 < \varepsilon$. This will be true no matter what we take for δ, so take any $\delta > 0$.

34. We want $||x - 2| - 0| < \varepsilon$, that is, $||x - 2|| < \varepsilon$. Since $||x - 2|| = |x - 2|$, then we want $|x - 2| < \varepsilon$. Take $\delta = \varepsilon$.

35. (a) Let $\varepsilon > 0$ and choose $\delta = 1$. Then, if $0 < |x-a| < \delta$, it will follow that $|f(x) - c| < \varepsilon$ simply because $|f(x)-c| = |c - c| = 0$.

(b) Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. Then, if $0 < |x - a| < \delta = \epsilon$, it will follow that $|f(x) - a| = |x - a| < \varepsilon$.

36. Suppose that, for each $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies that $|f(x) - L_1| < \varepsilon$. Suppose also that for each $\varepsilon > 0$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ implies that $|f(x) - L_2| < \varepsilon$. Assume that $L_1 \neq L_2$, put $\varepsilon = \frac{1}{2}|L_1 - L_2|$, and select x such that both $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ hold. Then $|f(x) - L_1| < \frac{1}{2}|L_1 - L_2|$ and $|f(x) - L_2| < \frac{1}{2}|L_1 - L_2|$. Therefore, by the triangle inequality

$$\begin{aligned}|L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \le |L_1 - f(x)| + |f(x) - L_2| \\ &= |f(x) - L_1| + |f(x) - L_2| < \tfrac{1}{2}|L_1 - L_2| + \tfrac{1}{2}|L_1 - L_2| \\ &= |L_1 - L_2|\end{aligned}$$

Hence, $|L_1 - L_2| < |L_1 - L_2|$, which is absurd.

Problem Set 1.8, page 68

1. $\lim\limits_{x\to 4} 5 = 5$

2. $\lim\limits_{x\to 5} 5 = 5$

3. $\lim\limits_{x\to -4} \pi = \pi$

4. $\lim\limits_{x\to 0} \pi x = \pi \lim\limits_{x\to 0} x = \pi \cdot 0 = 0$

5. $\lim\limits_{x\to \pi} x = \pi$

6. $\lim\limits_{x\to \pi} \cos \pi = \cos \pi = -1$

7. $\lim\limits_{x\to 2} 7x = 7 \lim\limits_{x\to 2} x = 7\cdot 2 = 14$

8. $\lim\limits_{x\to -2} (x+\cos \frac{\pi}{3}) = \lim\limits_{x\to -2} x + \lim\limits_{x\to -2} \cos \frac{\pi}{3}$

$= -2+\cos\frac{\pi}{3} = -2+\frac{1}{2} = -\frac{3}{2}$

9. $\lim\limits_{x\to 4}(x-3) = \lim\limits_{x\to 4} x - \lim\limits_{x\to 4} 3 = 4 - 3 = 1$.

10. $\lim\limits_{y\to 6}(y-6) = \lim\limits_{y\to 6} y - \lim\limits_{y\to 6} 6 = 6 - 6 = 0$

11. $\lim\limits_{t\to -3}(2t+1) = \lim\limits_{t\to -3} 2t + \lim\limits_{t\to -3} 1 = 2\lim\limits_{t\to -3} t + 1$

$= 2(-3) + 1 = -6 + 1 = -5$.

12. $\lim\limits_{x\to 1}(x+1)(x-1) = \lim\limits_{x\to 1}(x+1)\cdot\lim\limits_{x\to 1}(x-1)$

$= \left[\lim\limits_{x\to 1} x + \lim\limits_{x\to 1} 1\right] \cdot \left[\lim\limits_{x\to 1} x - \lim\limits_{x\to 1} 1\right]$

$= (1+1) \cdot (1-1) = 2\cdot 0 = 0$

13. $\lim_{x\to3} x(2x-1) = \lim_{x\to3} x \left(\lim_{x\to3}(2x-1)\right)$

$= 3(\lim_{x\to3} 2x - \lim_{x\to3} 1) = 3(2 \lim_{x\to3} x - 1)$

$= 3(2 \cdot 3 - 1) = 3 \cdot 5 = 15$

14. $\lim_{x\to3} s(s-1)(s+1) = \lim_{x\to3} s \lim_{x\to3}(s-1) \cdot \lim_{x\to3}(s+1)$

$= 0 \cdot (-1)(1) = 0$

15. $\lim_{t\to-2} t\,(2t+1)\,(t-1)$

$= \lim_{t\to-2} t \cdot \lim_{t\to-2} (2t+1) \cdot \lim_{t\to-2} (t-1)$

$= -2\left[\lim_{t\to-2} 2t + \lim_{t\to-2} 1\right]\left[\lim_{t\to-2} t - \lim_{t\to-2} 1\right]$

$= -2\left[2 \lim_{t\to-2} t + 1\right]\left[-2 - 1\right]$

$= -2\left[2(-2) + 1\right] \cdot (-3) = 6(-3) = -18$

16. $\lim_{q\to2} \dfrac{48}{3q+2} = \dfrac{\lim_{q\to2} 4q}{\lim_{q\to2}(3q+2)} = \dfrac{4 \lim_{q\to2} q}{\lim_{q\to2} 3q + \lim_{q\to2} 2}$

$= \dfrac{4 \cdot 2}{3 \lim_{q\to2} q + 2} = \dfrac{8}{3 \cdot 2 + 2} = \dfrac{8}{8} = 1$

17. $\lim_{x\to3} \dfrac{3x+2}{2x+5} = \dfrac{\lim_{x\to3}(3x+2)}{\lim_{x\to3}(2x+5)} = \dfrac{\lim_{x\to3} 3x + \lim_{x\to3} 2}{\lim_{x\to3} 2x + \lim_{x\to3} 5}$

$= \dfrac{3 \lim_{x\to3} x + 2}{2 \lim_{x\to3} x + 5} = \dfrac{3 \cdot 3 + 2}{2 \cdot 3 + 5} = \dfrac{11}{11} = 1$

18. $\lim_{y\to1} \dfrac{y-1}{y+1} = \dfrac{\lim_{y\to1}(y-1)}{\lim_{y\to1}(y+1)} = \dfrac{\lim_{y\to1} y - \lim_{y\to1} 1}{\lim_{y\to1} y + \lim_{y\to1} 1}$

$= \dfrac{1-1}{1+1} = \dfrac{0}{2} = 0$

19. $\lim_{y\to5} (y^2-2y+1) = \lim_{y\to5} (y-1)^2$

$= \lim_{y\to5} (y-1) \cdot \lim_{y\to5} (y-1)$

$= (\lim_{y\to5} y - \lim_{y\to5} 1)^2 = (5-1)^2$

$= 4^2 = 16$

20. $\lim_{t\to-1} (3t^7-2t^5+4) = \lim_{t\to-1} 3t^7 - \lim_{t\to-1} 2t^5 + \lim_{t\to-1} 4$

$= 3 \lim_{t\to-1}{}^7 - 2 \lim_{t\to-1} t^5 + 4$

$= 3\left(\lim_{t\to-1} t\right)^7 - 2\left(\lim_{t\to-1} t\right)^5 + 4$

$= 3\,(-1)^7 - 2\,(-1)^5 + 4 = -3 + 2 + 4 = 3$

21. $\lim_{s\to-2} (5-3s-s^2) = \lim_{s\to-2} 5 - \lim_{s\to-2} 3s - \lim_{s\to-2} s^2$

$= 5 - 3 \lim_{s\to-2} s - (\lim_{s\to-2} s)^2$

$= 5 - 3(-2) - (-2)^2$

$= 5 + 6 - 4 = 7$

22. $\lim_{x\to-1} \left|3x^3 - 2x^2 + 5x - 1\right|$

$= \left|\lim_{x\to-1} (3x^3 - 2x^2 + 5x - 1)\right|$

$= \left|3 \lim_{x\to-1} x^3 - 2 \lim_{x\to-1} x^2 + 5 \lim_{x\to-1} x - \lim_{x\to-1} 1\right|$

$= \left|3\,(\lim_{x\to-1} x)^3 - 2\,(\lim_{x\to-1} x)^2 + 5(-1) - 1\right|$

$= \left|3 \cdot (-1) - 2(1) - 6\right| = \left|-3 - 8\right| = \left|-11\right| = 11$

23. $\lim_{u\to2} \dfrac{u^2+u+1}{u^2+2u} = \dfrac{\lim_{u\to2} (u^2+u+1)}{\lim_{u\to2} (u^2+2u)}$

$= \dfrac{\lim_{u\to2} u^2 + \lim_{u\to2} u + \lim_{u\to2} 1}{\lim_{u\to2} u^2 + 2 \lim_{u\to2} u}$

$= \dfrac{(\lim_{u\to2} u)^2 + 2 + 1}{(\lim_{u\to2} u)^2 + 2 \cdot 2} = \dfrac{4+3}{4+4} = \dfrac{7}{8}$

24. $\lim_{t\to-2} \dfrac{t^3-5t}{t+3} = \dfrac{\lim_{t\to-2} (t^3 - 5t)}{\lim_{t\to-2} (t+3)} = \dfrac{\lim_{t\to-2} t^3 - 5 \lim_{t\to-2} t}{\lim_{t\to-2} t + \lim_{t\to-2} 3}$

$= \dfrac{(\lim_{t\to-2} t)^3 - 5(-2)}{-2+3} = \dfrac{-8+10}{1} = 2$

25. $\lim_{y\to3} \sqrt{y+1}\,(2y-3) = (\lim_{y\to3} \sqrt{y+1})\left[\lim_{y\to3} (2y-3)\right]$

$= \sqrt{\lim_{y\to3} (y+1)}\;\;(2 \lim_{y\to3} y - \lim_{y\to3} 3)$

$= \sqrt{\lim_{y\to3} y + \lim_{y\to3} 1}\;\;(2 \cdot 3 - 3)$

$= \sqrt{3+1}\;\;(3) = \sqrt{4} \cdot 3 = 2 \cdot 3 = 6$

26. $\lim_{x\to 1} \frac{\sqrt{4-x^2}}{2+x} = \frac{\lim_{x\to 1} \sqrt{4-x^2}}{\lim_{x\to 1} (2+x)} = \frac{\sqrt{\lim_{x\to 1} (4-x^2)}}{\lim_{x\to 1} (2+x)}$

$= \frac{\sqrt{4-(1)^2}}{2+1} = \frac{\sqrt{3}}{3}.$

27. $\lim_{z\to 3} \sqrt{\left|\frac{2z-15}{z+1}\right|} = \sqrt{\lim_{z\to 3} \frac{|2z-15|}{|z+1|}} = \sqrt{\frac{\lim_{z\to 3} |2z-15|}{\lim_{z\to 3} |z+1|}}$

$= \sqrt{\frac{\left|\lim_{z\to 3} (2z-15)\right|}{\left|\lim_{z\to 3} (z+1)\right|}} = \sqrt{\frac{\left|2\lim_{z\to 3} z - \lim_{z\to 3} 15\right|}{\left|\lim_{z\to 3} z + \lim_{z\to 3} 1\right|}}$

$= \sqrt{\frac{|2\cdot 3-15|}{|3+1|}} = \sqrt{\frac{|-9|}{|4|}} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$

28. $\lim_{x\to \frac{8}{3}} \sqrt{\frac{9x^2-64}{3x-8}} = \sqrt{\lim_{x\to \frac{8}{3}} \frac{(3x-8)(3x+8)}{3x-8}}$

$= \sqrt{\lim_{x\to \frac{8}{3}} (3x+8)} = \sqrt{3\lim_{x\to \frac{8}{3}} x + \lim_{x\to \frac{8}{3}} 8}$

$= \sqrt{3\cdot\frac{8}{3}+8} = \sqrt{8+8} = \sqrt{16} = 4$

29. $\lim_{z\to -3} \sqrt[3]{\frac{z-4}{6z^2+2}} = \sqrt[3]{\lim_{z\to -3} \frac{z-4}{6z^2+2}} = \sqrt[3]{\frac{\lim_{z\to -3} (z-4)}{\lim_{z\to -3} (6z^2+2)}}$

$= \sqrt[3]{\frac{\lim_{z\to -3} z - \lim_{z\to -3} 4}{6\left(\lim_{z\to -3} z\right)^2 + \lim_{z\to -3} 2}} = \sqrt[3]{\frac{-3-4}{6\cdot 9+2}}$

$= \sqrt[3]{\frac{-7}{56}} = \sqrt[3]{\frac{1}{8}} = -\frac{1}{2}$

30. $\lim_{y\to 1} \sqrt[3]{\frac{27y^3+4y-4}{y^{10}+4y^2+3y}} = \sqrt[3]{\lim_{y\to 1} \frac{27y^3+4y-4}{y^{10}+4y^2+3y}}$

$= \sqrt[3]{\frac{27(1)^3+4(1)-4}{1^{10}+4(1)^2+3(1)}} = \frac{3}{2}.$

31. $\lim_{t\to \frac{1}{2}} \frac{t^2+1}{1+\sqrt{2t+8}} = \frac{\lim_{t\to \frac{1}{2}} (t^2+1)}{\lim_{t\to \frac{1}{2}} (1+\sqrt{2t+8})}$

$= \frac{\lim_{t\to \frac{1}{2}} t^2 + \lim_{t\to \frac{1}{2}} 1}{\lim_{t\to \frac{1}{2}} 1 + \lim_{t\to \frac{1}{2}} \sqrt{2t+8}}$

$= \frac{(\frac{1}{2})^2+1}{1+\lim_{t\to \frac{1}{2}} \sqrt{2t+8}}$

$= \frac{\frac{5}{4}}{1+\sqrt{\lim_{t\to \frac{1}{2}} (2t+8)}}$

$= \frac{\frac{5}{4}}{1+\sqrt{2(\frac{1}{2})+8}} = \frac{5}{16}.$

32. $\lim_{z\to 2} \frac{2z^2-5z+2}{z-2} = \lim_{z\to 2} \frac{(2z-1)(z-2)}{z-2}$

$= \lim_{z\to 2} (2z-1) = 2\lim_{z\to 2} z - 1 = 2\cdot 2-1 = 3$

33. $\lim_{t\to 3} \frac{t^2+t-12}{t-3} = \lim_{t\to 3} \frac{(t-3)(t+4)}{t-3} = \lim_{t\to 3} (t+4)$

$= \lim_{t\to 3} t + 4 = 3+4 = 7$

34. $\lim_{w\to -5} \frac{w^2-25}{w+5} = \lim_{w\to -5} \frac{(w+5)(w-5)}{w+5} = \lim_{w\to -5} (w-5)$

$= \lim_{w\to -5} w - \lim_{w\to -5} 5 = -5-5 = -10$

35. $\lim_{x\to \frac{5}{2}} \frac{4x^2-25}{2x-5} = \lim_{x\to \frac{5}{2}} \frac{(2x-5)(2x+5)}{2x-5}$

$= \lim_{x\to \frac{5}{2}} (2x+5) = 2(\frac{5}{2})+5 = 10$

36. $\lim_{x\to -3} \left|\frac{x^2+4x+3}{x+3}\right| = \lim_{x\to -3} \left|\frac{(x+3)(x+1)}{x+3}\right|$

$= \lim_{x\to -3} |x+1| = \left|\lim_{x\to -3} (x+1)\right|$

$= \left|\lim_{x\to -3} x + \lim_{x\to -3} 1\right|$

$= |-3+1| = |-2| = 2$

37. $\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h}$

$= \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} (6+h) = 6+0 = 6.$

38. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})}$

$= \lim_{x\to 0} \frac{(x+2)-2}{x(\sqrt{x+2}+\sqrt{2})} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} (\sqrt{x+2}+\sqrt{2})}$

$$= \frac{\lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0} \sqrt{x+2} + \lim\limits_{x\to 0} \sqrt{2}} = \frac{1}{\lim\limits_{x\to 0} \sqrt{x+2} + \sqrt{2}}$$

$$= \frac{1}{\sqrt{\lim\limits_{x\to 0} (x+2)} + \sqrt{2}} = \frac{1}{\sqrt{0+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

39. $$\lim_{y\to 0} \frac{2-\sqrt{4-y}}{y} \cdot \frac{2+\sqrt{4-y}}{2+\sqrt{4+y}} = \lim_{y\to 0} \frac{4-(4-y)}{y(2+\sqrt{4+y})}$$

$$= \lim_{y\to 0} \frac{y}{y(2+\sqrt{4+y})} = \lim_{y\to 0} \frac{1}{2+\sqrt{4+y}}$$

$$= \frac{\lim\limits_{y\to 0} 1}{\lim\limits_{y\to 0} (2+\sqrt{4+y})} = \frac{1}{\lim\limits_{y\to 0} 2 + \lim\limits_{y\to 0} \sqrt{4+y}}$$

$$= \frac{1}{2+\sqrt{\lim\limits_{y\to 0} (4+y)}} = \frac{1}{2+\sqrt{4+0}} = \frac{1}{2+\sqrt{4}}$$

$$= \frac{1}{2+2} = \frac{1}{4}.$$

40. $$\lim_{v\to 4} \frac{v^{\frac{5}{2}} - 16v^{\frac{1}{2}}}{v-4} = \lim_{v\to 4} \frac{v^{\frac{1}{2}}(v^2-16)}{v-4}$$

$$= \lim_{v\to 4} \frac{v^{\frac{1}{2}}(v-4)(v+4)}{v-4} = \lim_{v\to 4} v^{\frac{1}{2}}(v+4)$$

$$= (\lim_{v\to 4} v^{\frac{1}{2}})(\lim_{v\to 4}(v+4)) = (\lim_{v\to 4} v)^{\frac{1}{2}}(\lim_{v\to 4} v + \lim_{v\to 4} 4)$$

$$= 4^{\frac{1}{2}}(4+4) = 2 \cdot 8 = 16.$$

41. $$\lim_{x\to 1} \frac{\frac{1}{\sqrt{x}} - 1}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{\sqrt{x}(1-x)}$$

$$= \lim_{x\to 1} \frac{(1-\sqrt{x})}{\sqrt{x}\,(1-\sqrt{x})(1+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}\,(1+\sqrt{x})} = \frac{1}{2}.$$

42. $$\lim_{z\to -1} \frac{z^2+4z+3}{z^2-1} = \lim_{z\to -1} \frac{(z+1)(z+3)}{(z+1)(z-1)}$$

$$= \lim_{z\to -1} \frac{z+3}{z-1} = \frac{\lim\limits_{z\to -1}(z+3)}{\lim\limits_{z\to -1}(z-1)}$$

$$= \frac{-1+3}{-1-1} = \frac{2}{-2} = -1.$$

43. $$\lim_{t\to 4} \frac{\frac{1}{\sqrt{t}} - \frac{1}{2}}{t-4} = \lim_{t\to 4} \frac{2-\sqrt{t}}{2\sqrt{t}(t-4)}$$

$$= \lim_{t\to 4} \frac{2-\sqrt{t}}{2\sqrt{t}(\sqrt{t}+2)(\sqrt{t}-2)} = \lim_{t\to 4} \frac{-1}{2\sqrt{t}(\sqrt{t}+2)} = \frac{-1}{16}.$$

44. $$\lim_{t\to 0} \left(\sqrt{1+\frac{1}{|t|}} - \sqrt{\frac{1}{|t|}}\right)$$

$$= \lim_{t\to 0} \frac{\left(\sqrt{1+\frac{1}{|t|}} - \sqrt{\frac{1}{|t|}}\right)\left(\sqrt{1+\frac{1}{|t|}} + \sqrt{\frac{1}{|t|}}\right)}{\sqrt{1+\frac{1}{|t|}} + \sqrt{\frac{1}{|t|}}}$$

$$= \lim_{t\to 0} \frac{\left(1+\frac{1}{|t|}\right) - \frac{1}{|t|}}{\sqrt{1+\frac{1}{|t|}} + \sqrt{\frac{1}{|t|}}} = \lim_{t\to 0} \frac{1}{\sqrt{1+\frac{1}{|t|}} + \sqrt{\frac{1}{|t|}}}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{\frac{|t|+1}{|t|}} + \sqrt{\frac{1}{|t|}}} = \lim_{t\to 0} \frac{1}{\frac{\sqrt{|t|+1}}{\sqrt{|t|}} \quad \frac{1}{\sqrt{|t|}}}$$

$$= \lim_{t\to 0} \frac{1}{\frac{\sqrt{|t|+1}+1}{|t|}} = \lim_{t\to 0} \frac{|t|}{\sqrt{|t|+1}+1}$$

$$= \frac{\lim\limits_{t\to 0} \sqrt{|t|}}{\lim\limits_{t\to 0} (\sqrt{|t|+1}+1)} = \frac{\sqrt{\lim\limits_{t\to 0} |t|}}{\sqrt{\lim\limits_{t\to 0} |t|+1}+1}$$

$$= \frac{\sqrt{|\lim\limits_{t\to 0} t|}}{\sqrt{|\lim\limits_{t\to 0} t|+1}+1} = \frac{\sqrt{|0|}}{\sqrt{|0|+1}+1} = \frac{0}{2} = 0.$$

45. $|x \sin \frac{1}{x}| = |x||\sin \frac{1}{x}| = |x| \cdot 1 = |x|.$

Thus, $-|x| \leq x \sin \frac{1}{x} \leq |x|$

But $\lim\limits_{x\to 0} |x| = |\lim\limits_{x\to 0} x| = |0| = 0$ and

$\lim\limits_{x\to 0}(-|x|) = -\lim\limits_{x\to 0} |x| = -(0) = 0$

Thus by the squeezing property,

$\lim\limits_{x\to 0} x \sin \frac{1}{x} = 0.$

46. $$\lim_{y\to 2} \frac{\sqrt{y^3}-\sqrt{8}}{y^2-4} = \lim_{y\to 2} \frac{\sqrt{y^3}-\sqrt{8}}{(y^2-4)} \cdot \frac{\sqrt{y^3}+\sqrt{8}}{\sqrt{y^3}+\sqrt{8}}$$

$$= \lim_{y\to 2} \frac{y^3-8}{(y^2-4)\sqrt{y^3}+\sqrt{8}} = \lim_{y\to 2} \frac{(y-2)(y^2+2y+4)}{(y-2)(y+2)(\sqrt{y^3}+\sqrt{8})}$$

$$= \lim_{y\to 2} \frac{y^2+2y+4}{(y+2)\sqrt{y^3}+\sqrt{8}} = \frac{4+4+4}{4(2\sqrt{8})} = \frac{3\sqrt{8}}{16} = \frac{3\sqrt{2}}{8}.$$

47. $\lim\limits_{x\to \pi} \cos x = \cos \pi = -1.$

48. $|\sin t \cos \frac{1}{t}| = |\sin t| \cdot |\cos \frac{1}{t}|$

$\leq |\sin t| \cdot 1 = |\sin t|.$ Thus,

$-|\sin t| \leq \sin t \cos \frac{1}{t} \leq |\sin t|$

But $\lim\limits_{t\to 0} |\sin t| = |\lim\limits_{t\to 0} \sin t| = |\sin 0|$

$= |0| = 0$ and $\lim\limits_{t\to 0} (-|\sin t|) =$

$= -\lim\limits_{t\to 0} |\sin t| = -(0) = 0$

$\therefore \lim\limits_{t\to 0} \sin t \cos \frac{1}{t} = 0.$

49. $\lim\limits_{x\to\frac{\pi}{6}} \cos x = \cos \frac{\pi}{6}$ by theorem 1.

Also, $\lim\limits_{x\to\frac{\pi}{6}} \sin x = \sin \frac{\pi}{6}$ by theorem 1.

Since the limit of a quotient is the quotient of the limits

$$\lim_{x\to\frac{\pi}{6}} \frac{\cos x}{\sin x} = \frac{\lim\limits_{x\to\frac{\pi}{6}} \cos x}{\lim\limits_{x\to\frac{\pi}{6}} \sin x} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

50. $\lim\limits_{\theta\to\pi} \left|\frac{1}{\cos\theta}\right| = \left|\lim\limits_{\theta\to\pi} \frac{1}{\cos\theta}\right| = \left|\frac{1}{\cos\pi}\right|$

$= \left|\frac{1}{-1}\right| = |-1| = 1.$

51. $\lim\limits_{t\to\pi} \frac{\sin t}{t} = \frac{\lim\limits_{t\to\pi} \sin t}{\lim\limits_{t\to\pi} t} = \frac{\sin}{\pi} = \frac{0}{\pi} = 0.$

52. $\lim\limits_{x\to\frac{\pi}{4}} \sec x = \lim\limits_{x\to\frac{\pi}{4}} \frac{1}{\cos x} = \frac{\lim\limits_{x\to\frac{\pi}{4}} 1}{\lim\limits_{x\to\frac{\pi}{4}} \cos x}$

$= \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$

53. $\lim\limits_{h\to 0} \sin(\frac{\pi}{3} + h^3)$

Let $y = \frac{\pi}{3}+h^3$ so that $\sin(\frac{\pi}{3} + h^3) = \sin y$.

Then as $h\to 0$, $y\to\frac{\pi}{3}$. Hence,

$\lim\limits_{h\to 0} \sin(\frac{\pi}{3} + h^3) = \lim\limits_{y\to\frac{\pi}{3}} \sin y = \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$

54. $\lim\limits_{w\to 0} w\cdot\csc w = \lim\limits_{w\to 0} w\cdot\frac{1}{\sin w}$

$= \lim\limits_{w\to 0} \frac{1}{\frac{\sin w}{w}} = \frac{\lim\limits_{w\to 0} 1}{\lim\limits_{w\to 0} \frac{\sin w}{w}} = \frac{1}{1} = 1$

55. $\lim\limits_{x\to 0} \frac{\sin 6x}{x} = \lim\limits_{x\to 0} \frac{\sin y}{\frac{y}{6}} = 6 \lim\limits_{x\to 0} \frac{\sin y}{y}$

$= 6 \cdot 1 = 6.$

56. $\lim\limits_{x\to 0} \frac{1}{\sin 3x} = \lim\limits_{y\to 0} \frac{y/3}{\sin y} = \frac{1}{3 \lim\limits_{x\to 0} \frac{\sin y}{y}}$

$= \frac{1}{3\cdot 1} = \frac{1}{3}.$

57. $\lim\limits_{x\to 0} \frac{\sin 2x}{\sin 5x} = \lim\limits_{x\to 0} \frac{\frac{\sin 2x}{x}}{\frac{\sin 5x}{x}} = \lim\limits_{x\to 0} \frac{\frac{2\sin 2x}{2x}}{\frac{5\sin 5x}{5x}}$

$= \frac{2}{5}\left[\frac{\lim\limits_{x\to 0} \frac{\sin 2x}{2x}}{\lim\limits_{x\to 0} \frac{\sin 5x}{5x}}\right] = \frac{2}{5} \cdot \frac{1}{1} = \frac{2}{5}.$

58. $\lim\limits_{t\to 0} \frac{1-\cos 2t}{\sin t} = \lim\limits_{t\to 0} \frac{\frac{1-\cos 2t}{t}}{\frac{\sin t}{t}}$

$= \frac{\lim\limits_{t\to 0} \frac{1-\cos 2t}{t}}{\lim\limits_{t\to 0} \frac{\sin t}{t}} = \frac{\lim\limits_{t\to 0} \frac{1-\cos 2t}{t}}{1}$

$= \lim\limits_{t\to 0} \frac{2(1-\cos 2t)}{2t} = 2 \lim\limits_{t\to 0}(\frac{1-\cos 2t}{2t})$

$= 2 \cdot 0 = 0$

59. $\lim\limits_{\theta\to 0} \frac{\sin^2\theta}{\theta^2} = \lim\limits_{\theta\to 0} (\frac{\sin\theta}{\theta})^2$

$= \left[\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta}\right]^2 = 1^2 = 1.$

60. $\lim\limits_{x\to 0} \left[\frac{\sin x - \cos x \sin x}{x^2}\right]$ (- sin cos) (+ sin x)

$= \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \frac{(1-\cos x)}{x}$

$= (\lim\limits_{x\to 0} \frac{\sin x}{x}) (\lim\limits_{x\to 0} \frac{1-\cos x}{x}) = 1 \cdot 0 = 0.$

61. $\lim\limits_{u\to 0} \frac{1-\cos^2 u}{u^2} = \lim\limits_{u\to 0} \frac{1-(1-\sin^2 u)}{u^2}$

$= \lim\limits_{u\to 0} \frac{\sin^2 u}{u^2} = \lim\limits_{u\to 0} (\frac{\sin u}{u})^2 = 1^2 = 1.$

62. $\lim\limits_{x\to 3} \frac{x-3}{\sin(x-3)} = \lim\limits_{x\to 3} \frac{1}{\frac{\sin(x-3)}{x-3}}$. Let $u=x-3$,

so that as $x\to 3$, $u\to 0$. Now $\lim\limits_{x\to 3} \frac{1}{\frac{\sin(x-3)}{x-3}}$

$= \lim\limits_{u\to 0} \frac{1}{\frac{\sin u}{u}} = \frac{1}{1} = 1.$

Hence, $\lim\limits_{x\to 3} \dfrac{x-3}{\sin(x-3)} = 1.$

63. $\lim\limits_{x\to\pi} \dfrac{\cos\frac{x}{2}}{(\frac{x}{2}-\frac{\pi}{2})} = \lim\limits_{x\to\pi} \dfrac{\sin(\frac{\pi}{2}-\frac{x}{2})}{-(\frac{\pi}{2}-\frac{x}{2})}$. Let $u = \frac{\pi}{2} - \frac{x}{2}$.

When $x\to\pi$, then $u\to 0$ so $\lim\limits_{u\to 0} \dfrac{\sin u}{-u} = -1.$

So $\lim\limits_{x\to\pi} \dfrac{\cos\frac{x}{2}}{(\frac{x}{2}-\frac{\pi}{2})} = -1.$

64. $\lim\limits_{t\to 0} \dfrac{\tan 4t}{2t} = \lim\limits_{t\to 0} \dfrac{\frac{\sin 4t}{\cos 4t}}{2t}$

$= \lim\limits_{t\to 0} \dfrac{\sin 4t}{2t} \cdot \dfrac{1}{\cos 4t}$

$= (2 \lim\limits_{t\to 0} \dfrac{\sin 4t}{4t}) \lim\limits_{t\to 0} \dfrac{1}{\cos 4t}$

$= 2 \cdot 1 \cdot \lim\limits_{t\to 0} \dfrac{1}{\cos 4t}$

$= 2 \cdot 1 \cdot \lim\limits_{u\to 0} \dfrac{1}{\cos u} = 2\cdot 1\cdot 1 = 2.$

65. $\lim\limits_{\theta\to 0} \dfrac{\tan 2\theta}{\sin\theta} = \lim\limits_{\theta\to 0} \dfrac{\sin 2\theta}{\cos 2\theta} \cdot \dfrac{1}{\sin\theta}$

$= \lim\limits_{\theta\to 0} \dfrac{(\frac{\sin 2\theta}{2\theta})}{\cos 2\theta} \cdot \dfrac{2\theta}{\sin\theta}$

$= \lim\limits_{\theta\to 0} \dfrac{\frac{\sin 2\theta}{2\theta}}{\cos 2\theta} \cdot \dfrac{2}{(\frac{\sin\theta}{\theta})}$

$= \frac{1}{1} \cdot \frac{2}{1} = 2.$

66. $\lim\limits_{v\to\pi} \dfrac{1+\cos v}{(\pi-v)^2} = \lim\limits_{v\to\pi} \dfrac{1-\cos(\pi-v)}{(\pi-v)^2}.$

Let $u = \pi - v$, so that when $v\to\pi$, $u\to 0$.

$\lim\limits_{v\to\pi} \dfrac{1-\cos(\pi-v)}{(\pi-v)^2} = \lim\limits_{u\to 0} \dfrac{1-\cos u}{u^2}$

$= \lim\limits_{u\to 0} \dfrac{2(\sin^2\frac{u}{2})}{u^2} = \lim\limits_{u\to 0} \dfrac{2\sin^2\frac{u}{2}}{4(\frac{u}{2})^2}$

$= \frac{1}{2} \lim\limits_{u\to 0} \left(\dfrac{\sin\frac{u}{2}}{\frac{u}{2}}\right)^2 = \frac{1}{2}\cdot 1 = \frac{1}{2}.$ Hence,

$\lim\limits_{v\to\pi} \dfrac{1+\cos v}{(\pi-v)^2} = \frac{1}{2}.$

67. $f(x) = x^2 + 1$

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^2+1-(x^2+1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh+h^2}{h} = \lim\limits_{h\to 0} (2x+h) = 2x+0 = 2x.$

68. $f(x) = \dfrac{1}{\sqrt{x}}$

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x}\sqrt{x+h})} = \lim\limits_{h\to 0} \dfrac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$

$= \lim\limits_{h\to 0} \dfrac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$

$= \lim\limits_{h\to 0} \dfrac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$

$= \lim\limits_{h\to 0} \dfrac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$

$= \dfrac{\lim\limits_{h\to 0}(-1)}{\lim\limits_{h\to 0}\sqrt{x}\ \lim\limits_{h\to 0}\sqrt{x+h}\ \lim\limits_{h\to 0}(\sqrt{x}+\sqrt{x+h})}$

$= \dfrac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})} = \dfrac{-1}{x(2\sqrt{x})} = \dfrac{-1}{2x^{3/2}}.$

69. $f(x) = \sqrt{x}$

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim\limits_{h\to 0} \dfrac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

$= \dfrac{1}{\lim\limits_{h\to 0}(\sqrt{x+h}+\sqrt{x})} = \dfrac{1}{\sqrt{x}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}.$

70. $f(x) = \dfrac{1}{x}$

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{x-(x+h)}{hx(x+h)} = \lim\limits_{h\to 0} \dfrac{-h}{hx(x+h)} = \lim\limits_{h\to 0} \dfrac{-1}{x(x+h)}$

$= \dfrac{-1}{\lim\limits_{h\to 0} x(x+h)} = \dfrac{-1}{x^2} = -\dfrac{1}{x^2}.$

71. $\lim\limits_{h\to 0} \cos(a+h) = \lim\limits_{h\to 0} \left[\cos a \cos h - \sin a \sin h\right]$

$= \cos a \lim\limits_{h\to 0} \cos h - \sin a \lim\limits_{h\to 0} \sin h$

$= \cos a\,(1) - \sin a\,(0) = \cos a$

Thus, by property 14,

$= \lim_{x\to a} \cos x = \cos a.$

72. The proposition is certainly true for $n = 1$. Assume that it is true for a given value of n. Then

$$\lim_{x\to a} \left[f(x)\right]^{n+1} = \lim_{x\to a}\left\{ \left[f(x)\right]^n f(x)\right\}$$

$$= \lim_{x\to a} \left[f(x)\right]^n \lim_{x\to a} f(x) = L^n L = L^{n+1};$$

so it is true for n + 1. Hence for all n.

73. Property 1: $\lim_{x\to a} c = c$

Property 6: $\lim_{x\to a} \left[f(x)\cdot g(x)\right]$

$$= \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

$\lim_{x\to a} \left[c\, f(x)\right] = \lim_{x\to a} c \lim_{x\to a} f(x)$ by property 6

$= c \lim_{x\to a} f(x)$ by property 1.

But this is property 3.

74. In property 12, let $h(x) = g(x)$, since $f(x) = g(x)$ holds for all values of x except possibly for $x = a$. $g(x)=f(x)=g(x)$ is certainly true. But $\lim_{x\to a} g(x) = L$;

hence by property 12, $\lim_{x\to a} f(x) = L$ so that property 11 follows.

75. $\lim_{x\to a} \left[f(x)-g(x)\right] = \lim_{x\to a} \left[f(x)+(-1)g(x)\right]$

$= \lim_{x\to a} f(x) + \lim_{x\to a} (-1)g(x)$ by property 4

$= \lim_{x\to a} f(x) + (-1) \lim_{x\to a} g(x)$ by property 3

$= \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ which is property 5.

76. $|f(x)g(x)| = |f(x)|\,|g(x)| \le |f(x)| \cdot B$

Thus, $-B\,|f(x)| \le f(x)g(x) \le B\,|f(x)|$

But $\lim_{x\to 0} B\,|f(x)| = B\left|\lim_{x\to 0} f(x)\right| = B\,|0| = B\cdot 0 = 0$

and $\lim_{x\to 0} (-B|f(x)|) = -B \lim_{x\to 0} |f(x)| = -B\cdot 0 = 0.$

Thus by the squeezing property,

$\lim_{x\to 0} f(x)\,g(x) = 0.$

77. $p^1(x) = \lim_{h\to 0} \dfrac{P(x+h)-P(x)}{h}$

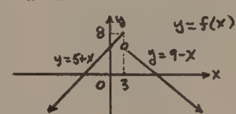
$$= \lim_{h\to 0} \frac{20(x+h)-50000-\frac{(x+h)^2}{1000} - 20x+50000+\frac{x^2}{1000}}{h}$$

$$= \lim_{h\to 0} \frac{20h - \frac{2xh+h^2}{1000}}{h} = \lim_{h\to 0} \left(20 - \frac{2x+h}{1000}\right)$$

$$= 20 - \frac{2x+0}{1000} = 20 - \frac{x}{500}.$$

Problem Set 1.9, page 76

1. (a)



(b) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (5 + x)$

$= 5 + 3 = 8$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (9 - x)$

$= 9 - 3 = 6$

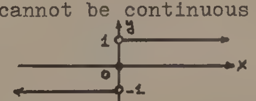
(c) $\lim_{x\to 3} f(x)$ does not exist since

$\lim_{x\to 3^-} f(x) \ne \lim_{x\to 3^+} f(x).$

(d) Since $\lim_{x\to 3} f(x)$ does not exist

f cannot be continuous at 3.

2. (a)



(b) $\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} (-1) = -1.$

$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} 1 = 1.$

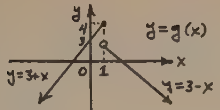
(c) $\lim_{x\to 0} F(x)$ does not exist since

$\lim_{x\to 0^-} F(x) \ne \lim_{x\to 0^+} F(x).$

(d) F is not continuous at 0 since

$\lim_{x\to 0} F(x)$ does not exist.

3. (a)



(b) $\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-} (3+x) = 3 + 1 = 4.$

$\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+} (3-x) = 3 - 1 = 2.$

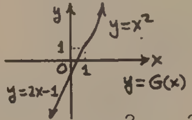
(c) $\lim\limits_{x\to 1} g(x)$ does not exist since

$\lim\limits_{x\to 1^-} g(x) \neq \lim\limits_{x\to 1^+} g(x).$

(d) Since $\lim\limits_{x\to 1} g(x)$ does not exist,

g is not continuous at 1.

4. (a)



(b) $\lim\limits_{x\to 1^+} G(x) = \lim\limits_{x\to 1^+} x^2 = 1^2 = 1.$

$\lim\limits_{x\to 1^-} G(x) = \lim\limits_{x\to 1^-} (2x-1) = 2(1)-1 = 1.$

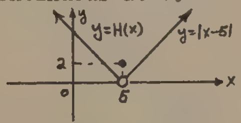
(c) Since $\lim\limits_{x\to 1^-} G(x) = \lim\limits_{x\to 1^+} G(x) = 1,$

$\lim\limits_{x\to 1} G(x) = 1.$

(d) Since $\lim\limits_{x\to 1} G(x) = 1 = G(1),$

G is continuous at 1.

5. (a)



(b) $\lim\limits_{x\to 5^-} H(x) = \lim\limits_{x\to 5^-} |x-5|$

$= \lim\limits_{x\to 5^-} (5-x) = 5 - 5 = 0.$

$\lim\limits_{x\to 5^+} H(x) = \lim\limits_{x\to 5^+} |x-5|$

$= \lim\limits_{x\to 5^+} (x-5) = 5 - 5 = 0.$

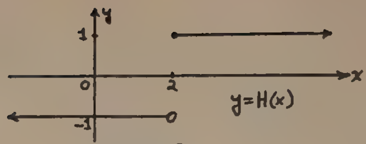
(c) Since $\lim\limits_{x\to 5^-} H(x) = \lim\limits_{x\to 5^+} H(x) = 0,$

$\lim\limits_{x\to 5} H(x) = 0.$

(d) Since $\lim\limits_{x\to 5} H(x) = 0 \neq 2 = H(5),$

H is not continuous at 5.

6. (a)



(b) $\lim\limits_{x\to 2^-} H(x) = \lim\limits_{x\to 2^-} \frac{x-2}{|x-2|}$

$= \lim\limits_{x\to 2^-} \frac{x-2}{2-x} = \lim\limits_{x\to 2^-} (-1) = -1.$

$\lim\limits_{x\to 2^+} H(x) = \lim\limits_{x\to 2^+} \frac{x-2}{|x-2|}$

$= \lim\limits_{x\to 2^+} \frac{x-2}{x-2} = \lim\limits_{x\to 2^+} 1 = 1.$

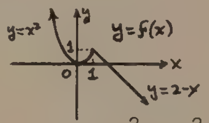
(c) Since $\lim\limits_{x\to 2^-} H(x) \neq \lim\limits_{x\to 2^+} H(x),$

$\lim\limits_{x\to 2} H(x)$ does not exist.

(d) Since $\lim\limits_{x\to 2} H(x)$ does not exist,

H is not continuous at 2.

7. (a)



(b) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^2 = 1^2 = 1.$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2-x) = 2-1 = 1.$

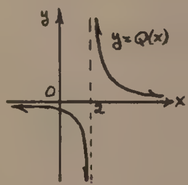
(c) Since $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = 1,$

$\lim\limits_{x\to 1} f(x) = 1.$

(d) Since $\lim\limits_{x\to 1} f(x) = 1 = f(1),$

f is continuous at 1.

8. (a)

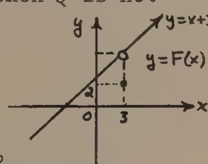


(b) $\lim\limits_{x\to 2^+} Q(x) = \lim\limits_{x\to 2^+} \frac{1}{x-2}$ does not exist (as a finite number) since as x approaches 2 through values greater than 2, $\frac{1}{x-2}$ becomes larger and larger without bound. Similarly $\lim\limits_{x\to 2^-} Q(x)$ does not exist (as a finite number).

(c) Since $\lim_{x\to 2^+} Q(x)$ does not exist (as a finite number), then $\lim_{x\to 2} Q(x)$ does not exist (as a finite number).

(d) Since $\lim_{x\to 2} Q(x)$ does not exist (as a finite number), then Q is not continuous at 2.

9. (a)



(b) $\lim_{x\to 3^-} F(x) = \lim_{x\to 3^-} \frac{x^2 - 9}{x - 3}$

$= \lim_{x\to 3^-} \frac{(x-3)(x+3)}{x - 3}$

$= \lim_{x\to 3^-} (x+3) = 3 + 3 = 6.$

$\lim_{x\to 3^+} F(x) = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{x - 3}$

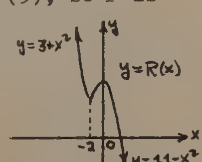
$= \lim_{x\to 3^+} (x+3) = 6.$

(c) Since $\lim_{x\to 3^-} F(x) = \lim_{x\to 3^+} F(x) = 6,$

$\lim_{x\to 3} F(x) = 6.$

(d) $\lim_{x\to 3} F(x) = 6 \neq 2 = F(3)$, so F is not continuous at 3.

10. (a)



(b) $\lim_{x\to -2^-} R(x) = \lim_{x\to -2^-} (3+x^2) = 3+(-2)^2=7.$

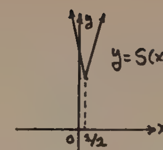
$\lim_{x\to -2^+} R(x) = \lim_{x\to -2^+} (11-x^2) = 11-(-2)^2=7.$

(c) $\lim_{x\to -2^-} R(x) = \lim_{x\to -2^+} R(x) = 7;$ hence,

$\lim_{x\to -2} R(x) = 7.$

(d) $\lim_{x\to -2} R(x) = 7 \neq 0 = R(-2)$, so R is not continuous at -2.

11. (a)



(b) $\lim_{x\to \frac{1}{2}^-} S(x) = \lim_{x\to \frac{1}{2}^-} \left[5 + |6x - 3|\right]$

$= \lim_{x\to \frac{1}{2}^-} \left[5 + (3 - 6x)\right] = 5 + 0 = 5.$

$\lim_{x\to \frac{1}{2}^+} S(x) = \lim_{x\to \frac{1}{2}^+} \left[5 + |6x - 3|\right]$

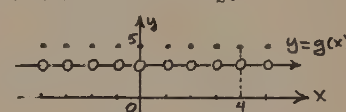
$= \lim_{x\to \frac{1}{2}^+} \left[5 + (6x - 3)\right] = 5 + 0 = 5.$

(c) Since $\lim_{x\to \frac{1}{2}^-} S(x) = \lim_{x\to \frac{1}{2}^+} S(x) = 5,$

$\lim_{x\to \frac{1}{2}} S(x) = 5.$

(d) Since $\lim_{x\to \frac{1}{2}} S(x) = 5 = S(\frac{1}{2})$, S is continuous at $\frac{1}{2}$.

12. (a)



(b) $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (⟦x⟧ + ⟦5-x⟧)$

$= \lim_{x\to 4^-} ⟦x⟧ + \lim_{x\to 4^-} ⟦5-x⟧ = 3 + 1 = 4.$

$\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} (⟦x⟧ + ⟦5-x⟧)$

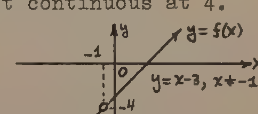
$= \lim_{x\to 4^+} ⟦x⟧ + \lim_{x\to 4^+} ⟦5-x⟧ = 4+0 = 4.$

(c) $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} (x) = 4;$

hence, $\lim_{x\to 4} g(x) = 4.$

(d) $\lim_{x\to 4} g(x) = 4 \neq 5 = g(4);$ hence, g is not continuous at 4.

13. (a)



(b) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2-2x-3}{x + 1}$

$= \lim_{x\to -1^-} \frac{(x-3)(x+1)}{x + 1}$

$= \lim_{x\to -1^-} (x-3) = -4$, so,

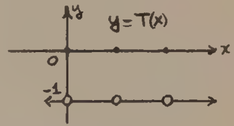
$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x-3) = -4.$

(c) Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = -4$,

$\lim_{x\to -1} f(x) = -4$.

(d) Since f is not defined at -1 (the denominator is 0 there), f is not continuous at -1.

14. (a)



(b) $\lim_{x\to 1^-} T(x) = \lim_{x\to 1^-} ([\![1-x]\!] + [\![x-1]\!])$

$= \lim_{x\to 1^-} [\![1-x]\!] + \lim_{x\to 1^-} [\![x-1]\!] = 0 + (-1) = -1.$

$\lim_{x\to 1^+} T(x) = \lim_{x\to 1^+} ([\![1-x]\!] + [\![x-1]\!])$

$= \lim_{x\to 1^+} [\![1-x]\!] + \lim_{x\to 1^+} [\![x-1]\!] = (-1) + 0 = -1.$

(c) Since $\lim_{x\to 1^-} T(x) = \lim_{x\to 1^+} T(x) = -1$,

$\lim_{x\to 1} T(x) = -1$.

(d) $\lim_{x\to 1} T(x) = -1 \neq 0 = T(1)$; hence,

T is not continuous at 1.

15. Note that $f = g \cdot h$ where $g(x) = 2$ and $h(x) = |x|$. The constant function and the absolute function are continuous at every number. By property 1, f is continuous at every number.

16. Note that $g = f \cdot h$ where $f(x) = |x|$ and $h = 1 - x$. Since absolute function is continuous at every number and a polynomial function is continuous at every number (property 3), it follows that g is continuous at every number by property 5.

17. Note that $h = f - g \cdot h$ where $f(x) = x$, $g(x) = 2$, $h(x) = |x|$. Since absolute function is continuous at each number and polynomial functions are continuous at every number (property 3), we have that h is continuous at every number by property 1.

18. F is a rational function; hence it is continuous at every number for which it is defined. Domain of F consists of all reals except $x = 1$. Hence F is continuous at every real number except $x = 1$.

19. G is a rational function; hence it is continuous at every number for which it is defined. Domain of G consists of all reals except $x = 0$. Hence, G is continuous at every real number except $x = 0$.

20. Note that $f = g \cdot h$ where $g(x) = |x|$ and $h(x) = \frac{1}{x}$, since $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, g is continuous at every number; h is continuous at every real number except 0 (Problem 19). Hence, by property 5, f is continuous at every number except 0.

21. g is a rational function; hence it is continuous at every number for which it is defined. Domain of g consists of all reals except 1. Hence g is continuous at every real number except $x = 1$.

22. By property 3 h(x) is continuous except possibly for $x = 0$. Now $\lim_{x\to 0^+} x^2 = 0$ and $\lim_{x\to 0^-} x^3 = 0$; hence $\lim_{x\to 0} h(x) = 0 = h(0)$. Thus, h is continuous at 0 and so h is continuous at every number.

23. F is the sum of two rational functions and a polynomial function. Hence F is continuous at every number for which the

rational functions are defined. F is defined at every number except for $x = 1, -1$. Thus F is continuous at every number except for $x = \pm 1$.

24. Note that $G(x) = \frac{F(x)}{H(x)}$ where $F(x) = 1$ $H(x) = |x| + 1$. $F(x)$ and $H(x)$ are continuous at every number; hence, G is continuous at every number for which $H(x) \neq 0$. But $H(x) = |x| + 1 > 0$ for all x. Thus G is continuous at every number.

25. Note that $H = F \cdot G$ where $F(x) = |x|$ and $G(x) = \frac{3}{x-2} + 4$. G is the sum of a rational function and a polynomial function, G is continuous at every number but $x = 2$. F is continuous at every number. Hence (by property 5), H is continuous at every number except $x = 2$.

26. $T = f \circ g$ where $f(x) = \sqrt{x}$, $g(x) = \frac{1}{x^2+1}$.

g is a rational function, domain g = reals. Thus g is continuous at every real number. For each value of x, $g(x) > 0$, so f is continuous at $g(x)$. Thus T is continuous at every real number.

27. $f(x) = \cot x = \frac{\cos x}{\sin x}$.

By properties 2 and 7, $f(x)$ is continuous at all numbers where $\sin x \neq 0$. But $\sin x = 0$ when $x = n\pi$, $n=0, \pm 1, \pm 2, \ldots$ Thus, cot x is continuous at every number except integer multiples of π.

28. $f(x) = \sec \frac{x}{2} = \frac{1}{\cos \frac{x}{2}}$.

By properties 2 and 7, $f(x)$ is continuous at all numbers where $\cos \frac{x}{2} \neq 0$. But $\cos \frac{x}{2} = 0$ implies $\frac{x}{2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$ or $x = \pm\pi, \pm 3\pi, \pm 5\pi \ldots$

So $\sec \frac{x}{2}$ is continuous at all numbers except odd multiples of π.

29. $g(x) = \csc x = \frac{1}{\sin x}$.

By properties 2 and 7, $g(x)$ is continuous at all numbers where $\sin x \neq 0$. But $\sin x = 0$ for $x = n\pi$, $n = 0, \pm 1, \pm 2 \ldots$ Thus csc x is continuous at all numbers except integer multiples of π.

30. $h(t) = \tan |t| = \frac{\sin |t|}{\cos |t|}$

By properties 2 and 7, h is continuous at all numbers where $\cos |t| \neq 0$ [$g(t) = \sin |t|$ is continuous using property 5]. But $\cos |t| = 0$ implies t = odd multiples of $\frac{\pi}{2}$. Thus $\tan |t|$ is continuous at all numbers except odd multiples of $\frac{\pi}{2}$.

31. $f(t) = \frac{1 - \sin t}{\cos t}$.

By properties 2 and 7, f is continuous at all numbers where $\cos t \neq 0$ [$g(t) = 1 - \sin t$ is continuous using property 1]. But $\cos t = 0$ implies t equals odd multiples of $\frac{\pi}{2}$. Thus $\frac{1-\sin t}{\cos t}$ is continuous at all numbers except odd multiples of $\frac{\pi}{2}$.

32. $$f(x) = \begin{cases} \frac{\sin 2x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

f is continuous at all values of $x \neq 0$ using properties 2 and 7. Now for $x = 0$: $\lim_{x\to 0} \frac{\sin 2x}{x} = 2 \lim_{x\to 0} \frac{\sin 2x}{2x}$ $= 2 \cdot 1 = 2$. But $f(0) = 1$ $\neq \lim_{x\to 0} f(x) = 2$. Thus f is continuous at all numbers except 0.

33. h is continuous for all $x > \frac{\pi}{4}$, since $\sin x$ is continuous. For $x < \frac{\pi}{4}$, $\tan x$ is continuous for all values of x for which $\cos x \neq 0$ $(\tan x = \frac{\sin x}{\cos x})$. But $\cos x = 0$ for odd multiples of $\frac{\pi}{2}$.

At $\frac{\pi}{4}$: $\lim_{x \to \frac{\pi}{4}^+} h(x) = \lim_{x \to \frac{\pi}{4}^+} \sqrt{2} \sin x = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1.$

$\lim_{x \to \frac{\pi}{4}^-} h(x) = \lim_{x \to \frac{\pi}{4}^-} \tan x = \tan \frac{\pi}{4} = 1.$

Now $h(\frac{\pi}{4}) = \tan \frac{\pi}{4} = 1 = \lim_{x \to \frac{\pi}{4}} h(x).$

Thus h is continuous at $\frac{\pi}{4}$. Hence h is continuous at all numbers except negative odd multiples of $\frac{\pi}{2}$.

34. $x \neq 0$, so $g(x) = \frac{1}{x}$ is continuous for all numbers in the domain of g by property 2. Now sin is continuous for all numbers, so f(where $f(x) = \sin(g(x))$) is continuous for $x \neq 0$.

Now $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin \frac{1}{x}$ = does not exist since values of $\sin \frac{1}{x}$ vacillate as $x \to 0$. Thus f is not continuous at x=0.

35. Here, f(x) is defined for all real numbers x such that $-2 \leq x \leq 2$. Thus, f is continuous on $[-2, 2]$ and $(-2, 2)$. Since 3 and 5 do not belong to the domain of f, it follows that f is discontinuous on $[2, 3]$ and $(-1, 5)$.

36. Here, g(x) is defined for all real numbers x except $x = -1$. Thus g is continuous on $(-\infty, 1)$, $(-3, -1)$, $(-\infty, -1)$, $(-1, \infty)$. Since -1 does not belong to the domain of g, then g is discontinuous on $[-1, \infty)$ and $[-2, 2]$.

37. Here, F(x) is defined for all real numbers x except for the values $x = -6$, and $x = 6$. Hence F is discontinuous on all the indicated intervals.

38. Here, f(x) is defined for all real numbers x except $x = 5$. Thus, f is continuous on the intervals $[-1, 1]$, $(-1, 1)$. Since 5 is not in the domain of f, it follows that f is discontinuous on the intervals $(-5, \infty)$, $(-\infty, 5]$ and $[-8, 6]$.

39. Here, G(x) is defined for all real numbers x except for the values $x = -\frac{3}{4}$ and $x = \frac{3}{4}$. Thus, G is continuous on the intervals $[-\frac{1}{2}, 0]$ and $(-1, -\frac{3}{4})$. Since $-\frac{3}{4}$ and $\frac{3}{4}$ are not in the domain of G, it follows that G is discontinuous on the intervals $[-\frac{3}{4}, 0]$, $(-\frac{3}{4}, \infty)$ and $[-2, \infty)$.

40. Here $f(x) = \cot x = \frac{\cos x}{\sin x}$ is defined for all real numbers x, except multiples of π. Thus f is continuous on $(0, \pi)$ and discontinuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[0, \pi]$.

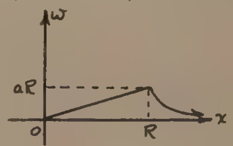
41. $\lim_{x \to R^-} w(x) = \lim_{x \to R^-} ax = aR$, whereas

$\lim_{x \to R^+} w(x) = \lim_{x \to R^+} \frac{b}{x^2} = \frac{b}{R^2}.$

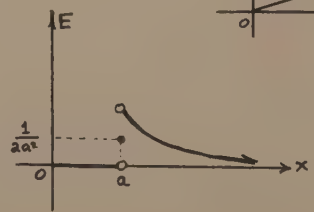
Continuity at R would require

$\lim_{x \to R^-} w(x) = \lim_{x \to R^+} w(x)$, that is,

$aR = \frac{b}{R^2}$ or $aR^3 = b$.



42. (a)



(b) $\lim_{x\to a^-} E(x) = \lim_{x\to a^-} 0 = 0$ while

$\lim_{x\to a^+} E(x) = \lim_{x\to a^+} \frac{1}{x^2} = \frac{1}{a^2}$. Hence,

E cannot be continuous at a since

$\lim E(x)$ cannot exist. Clearly,

E is continuous at every positive

number x except for x = a.

43. $\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$

$= \lim_{x\to 1} (x^2+x+1) = 1^2+1+1 = 3$; hence, put

a = 3.

44. (a) Let f be defined on some open

interval (c,a). Then $\lim_{x\to a^-} f(x) = L$

if for each $\varepsilon > 0$, there is a $\delta > 0$

such that $|f(x) - L| < \varepsilon$ holds

whenever $-\delta < x - a < 0$.

(b) Let f be defined on some open

interval (a,b). Then $\lim_{x\to a^+} f(x) = L$

if for each $\varepsilon > 0$, there is a $\delta > 0$

such that $|f(x) - L| < \varepsilon$ holds

whenever $0 < x - a < \delta$.

45. $\lim_{x\to 2^+} \sqrt{x-2} = \sqrt{\lim_{x\to 2^+} (x-2)} = \sqrt{2-2}$

$= \sqrt{0} = 0.$

Since x - 2 is not defined for values

of x that are smaller than 2, $\lim_{x\to 2^-} \sqrt{x-2}$

cannot exist. Hence $\lim_{x\to 2} \sqrt{x-2}$

cannot exist.

46. Let f(x) = sin x. Then $\lim_{x\to a} \sin x = \sin a$

by theorem 1 of section 8. Now f(a)=sin a

since f is defined for all x. Thus,

$\lim_{x\to a} \sin x = \sin a = f(a)$. Thus f is

continuous at a. Same argument applies

to f(x) = cos x, just replace sin with

cos.

The four remaining trigonometric functions

are defined in terms of sin x and cos x,

i.e., $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$

$\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sec x}$. In all

four cases, we have $\frac{f(x)}{g(x)}$ where f,g are

continuous for all values of x. Hence

by property 2, the four trigonometric

functions are continuous everywhere

where they are defined.

47. For values of x close to a (but not

equal to a) f(x) will be close to L.

Hence, in particular, for values of x

close to a, but greater than a, f(x)

will be close to L.

48. Let $\varepsilon > 0$ be given. We must find $\delta > 0$

so that $|f(x) - L| < \varepsilon$ will hold when-

ever $0 < x - a < \delta$. Since $\lim_{x\to a} f(x) = L$,

we know that there exists $\delta > 0$ so that

$|f(x) - L| < \varepsilon$ will hold whenever

$0 < |x - a| < \delta$. But, if $0 < x-a < \delta$,

then it follows that $0 < |x-a| < \delta$, so

$|f(x) - L| < \varepsilon$ as desired.

49. Since $\lim_{x\to a^+} f(x) = L$, then for values of

x close to a, but greater than a, f(x)

will be close to L. Since $\lim_{x\to a^-} f(x) = L$,

then for values of x close to a but

smaller than a, f(x) will be close to L.

Hence, if x is close to a (but different

from a), then f(x) will be close to L.

50. Let $\varepsilon > 0$ be given, Since $\lim_{x\to a^+} f(x) = L$,

there exists $\delta_1 > 0$ such that $|f(x)-L| < \varepsilon$

will hold whenever $0 < x-a < \delta_1$. Since $\lim_{x\to a^-} f(x) = L$, there exists $\delta_2 > 0$ such that $|f(x) - L| < \varepsilon$ will hold whenever $0 < a-x < \delta_2$. Let δ denote the smaller of δ_1 and δ_2 (or their common value if they are equal). Then $\delta > 0$, $\delta \le \delta_1$ and $\delta \le \delta_2$. We claim that $|f(x)-L| < \varepsilon$ will hold whenever $0 < |x-a| < \delta$. Indeed, suppose $0 < |x-a| < \delta$. Then, either $x > a$ or $x < a$. If $x > a$, we have $0 < |x-a| = x - a < \delta \le \delta_1$; hence, $|f(x) - L| < \varepsilon$. If $x < a$, we have $0 < |x-a| = a - x < \delta \le \delta_2$; hence, $|f(x) - L| < \varepsilon$. Thus, in any case, $|f(x) - L| < \varepsilon$ will hold provided $0 < |x-a| < \delta$. Thus, $\lim_{x\to a} f(x) = L$.

51. Suppose that $f(x) = g(x)$ holds for all values of x in an open interval containing a, except possibly for $x = a$. Then, if $\lim_{x\to a^+} g(x) = L$, it follows that

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = L$$

52. Suppose that $g(x) \le f(x) \le h(x)$ holds for all values of x in an open interval containing a, except possibly for $x = a$. Then, if $\lim_{x\to a^-} g(x) = L$ and $\lim_{x\to a^-} h(x) = L$ it follows that $\lim_{x\to a^-} f(x) = L$.

Review Problem Set, Chapter 1, page 77

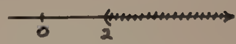
1. (a) False (b) False (c) True (d) True (e) False (f) False

2. $x - 5 \le 7$, $x \le 12$.

The solution is the interval $(-\infty, 12]$.

3. $3x + 2 > 8$, $3x > 6$, $x > 2$

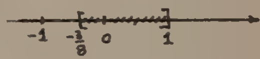
The solution is the interval $(2, \infty)$.



4. $3x - 2 \ge 1 + 2x$, $3x - 2x \ge 1 + 2$, $x \ge 3$.

The solution is the interval $[3, \infty)$.

5. $5 \ge 8x - 3 \ge -6$, $8 \ge 8x \ge -3$, $1 \ge x \ge -\frac{3}{8}$.

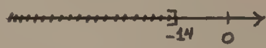


The solution is the interval $\left[-\frac{3}{8}, 1\right]$.

6. $\frac{x-1}{3} \ge 2 + \frac{x}{2}$, $2(x-1) \ge 12 + 3x$,

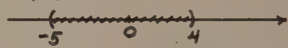
$2x - 3x \ge 14$, $-x \ge 14$, $x \le -14$.

The solution is the interval $(-\infty, -14]$.



7. $x^2 + x - 20 < 0$, $(x + 5)(x - 4) < 0$. There are two possibilities for a negative product: $x + 5 < 0$ and $x - 4 > 0$ or $x + 5 > 0$ and $x - 4 < 0$, that is, $x < -5$ and $x > 4$ or $x > -5$ and $x < 4$. From the first we have no solutions, but from the second we have the solution set $(-5, 4)$.

The solution set is $(-5, 4)$.

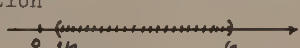


8. $x^2 - 6x - 7 \le 0$, $(x - 7)(x + 1) \le 0$. The equality yields $x = 7$ or $x = -1$. The inequality $(x - 7)(x + 1) < 0$ yields two possibilities for a negative product: $x - 7 < 0$ and $x + 1 > 0$ or $x - 7 > 0$ and $x + 1 < 0$; that is, $x < 7$ and $x > -1$ or $x > 7$ and $x < -1$. The former has a solution set $(-1, 7)$ while the latter yields no solutions. The solution set is $[-1, 7]$.

9. $2x^2 + 4 < 0$. Since $2x^2 + 4$ is always positive, we have no solutions to this inequality, that is, the solution set is empty.

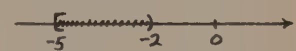
10. $\frac{x - 2}{x + 3} > 0$ provided $x - 2 > 0$ and $x + 3 > 0$ or $x - 2 < 0$ and $x + 3 < 0$, $x \neq -3$; that is, $x > 2$ and $x > -3$ or $x < 2$ and $x < -3$. The former yields the solutions $(2,\infty)$ while the latter yields the solutions $(-\infty,-3)$. So the solution consists of two intervals $(-\infty,-3)$ and $(2,\infty)$.

11. $\frac{2x - 1}{x - 6} < 0$. We exclude $x = 6$. Now, to have a negative quotient either $2x - 1 < 0$ and $x - 6 > 0$ or $2x - 1 > 0$ and $x - 6 < 0$; that is, $x < \frac{1}{2}$ and $x > 6$ or $x > \frac{1}{2}$ and $x < 6$. The former yields no solutions, while the latter gives the solutions $(\frac{1}{2},6)$. The solution set is $(\frac{1}{2},6)$.



12. $\frac{5x - 1}{x - 2} \leq 1$ is equivalent to $\frac{5x - 1}{x - 2} - 1 \leq 0$; that is, $\frac{5x - 1 - (x - 2)}{x - 2} \leq 0$. We must solve $\frac{4x + 1}{x - 2} \leq 0$. We exclude $x = 2$. For equality, $4x + 1 = 0$, $x = -\frac{1}{4}$. The inequality holds when either $4x + 1 < 0$ and $x - 2 > 0$ or $4x + 1 > 0$ and $x - 2 < 0$; that is, $x < -\frac{1}{4}$ and $x > 2$ or $x > -\frac{1}{4}$ and $x < 2$. The former yields no solutions while the latter yields the solutions $(-\frac{1}{4},2)$. The solution set is $[-\frac{1}{4},2)$.

13. $\frac{x - 4}{x + 2} \geq 3$ is equivalent to $\frac{x - 4}{x + 2} - 3 \geq 0$; that is, $\frac{x - 4 - 3x - 6}{x + 2} \geq 0$, $\frac{-2x - 10}{x + 2} \geq 0$, so $\frac{2x + 10}{x + 2} \leq 0$; that is, $\frac{x + 5}{x + 2} \leq 0$. The equality holds for $x = -5$; the inequality holds provided $x \neq -2$ and either $x + 5 < 0$ and $x + 2 > 0$ or $x + 5 > 0$ and $x + 2 < 0$; that is, $x < -5$ and $x > -2$ or $x > -5$ and $x < -2$. The former yields no solutions while the latter yields the solutions $(-5,-2)$. The solution set is $[-5,-2)$.



14. (a) If $x > 0$ then $x > 10$;
if $x < 0$ then condition is true.

(b) If $x > 0$ then $x > 100$;
if $x < 0$ then condition is true.

(c) If $x^2-1 > 0$, then $x^2-1 > 1{,}000$ or $x^2 > 1{,}001$, so that $x > \sqrt{1{,}001}$ or $x < -\sqrt{1{,}001}$. If $x^2 < 1$, then $-1 < x < 1$ makes the condition true.

(d) $x > 0$

(e) All x, except 0

15. $-4 < -2$ and $-3 < 1$, but $(-4)(-3) > (-2)(1)$, $12 > -2$.
$0 < 2$ and $-5 < -3$, but $(0)(-5) > (2)(-3)$, $0 > -6$.

16. For any a, b, $(a-b)^2 \geq 0$, so $a^2 - 2ab + b^2 \geq 0$; that is, $a^2 - 4ab + 2ab + b^2 \geq 0$, and
$a^2 + 2ab + b^2 \geq 4ab$. Since $a > 0$, $b > 0$, $a + b > 0$, and since $(a+b)^2 \geq 4ab$, we know $\sqrt{(a+b)^2} \geq 2\sqrt{ab}$, or $a+b \geq 2\sqrt{ab}$, or $\frac{a+b}{2} \geq \sqrt{ab}$. Now, from $(a+b)^2 \geq 4ab$, we can also say $(a+b)^2ab \geq 4a^2b^2$, since $a > 0$, $b > 0$. Hence, for $a > 0$, $b > 0$,

$\sqrt{(a+b)^2}\ \sqrt{ab} \geq 2ab$, so $\sqrt{ab} \geq \frac{2ab}{\sqrt{(a+b)^2}} = \frac{2ab}{a+b}$.

Putting the two inequalities together, namely $\frac{2ab}{a+b} \leq \sqrt{ab}$ and $\sqrt{ab} \leq \frac{a+b}{2}$, we have

$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2}$.

17. Let x be score on final exam. Then,

$$70 \leq \frac{3}{5} \cdot 68 + \frac{2}{5} \cdot x < 80$$
$$350 \leq 204 + 2x < 400$$
$$146 \leq 2x < 196$$
$$73 \leq x < 98$$

18. Suppose runner goes from A to B (distance d) at 8.8 mph. Let r be her rate back. Let t_1 be the time from A to B and t_2 the time from B to A. Then $8.8t_1 = rt_2$. Let R be her rate all the way, so that,

$$R\,(t_1+t_2) = 2d \text{ or } R = \frac{2d}{t_1+t_2} = \frac{\frac{2d}{t_1}}{1+\frac{t_2}{t_1}}$$

$$= \frac{2(8.8)}{1+\frac{8.8}{r}} = \frac{2r(8.8)}{r+8.8}$$

We want to find r such that

$$8 < \frac{17.6r}{r+8.8} \leq 8.5.$$

Since $r > 0$, $8(r+8.8) < 17.6r \leq 8.5(r+8.8)$

$$8r + 70.4 < 17.6r \leq 8.5r+74.8$$
$$70.4 < 9.6r \leq 0.5r+74.8$$

$70.4 < 9.6r$ implies $r > 7.\overline{3}$.

$9.6r \leq 0.5r+74.8$ implies $r \leq 8.2198$.

Thus, $7.\overline{3} < r \leq 8.2192$.

She must run faster than $7\frac{1}{3}$ miles per hour, but not faster than 8.2 miles per hour.

19. (a) 

$d = |4-(-3)| = 7$

(b) 

$d = |-5-0| = 5$

(c) 

$d = |-2.735-(-\pi)| = 0.406592654$

(d) 

$d = |4.1-(-3.2)| = 7.3$

(e) 

$d = \left|\frac{5}{2} - \left(-\frac{2}{3}\right)\right| = \frac{19}{6}$

(f) 

$d = |\sqrt{2}-1.42| = 0.005786438$

20. (a) $|-6-5| = |-11| = 11$

(b) $\left|\frac{1}{x} - \frac{1}{x+1}\right| = \left|\frac{x+1-x}{x(x+1)}\right| = \left|\frac{1}{x(x+1)}\right|$

$= \frac{1}{|x(x+1)|} = \frac{1}{|x|} \cdot \frac{1}{|x+1|}$

(c) $\left|\frac{3}{13} - \frac{4}{17}\right| = \left|\frac{3\cdot17-4\cdot13}{13\cdot17}\right|$

$= \left|\frac{1}{13\cdot17}\right| = \frac{1}{221}$

(d) $\left|\frac{x}{x+1} - \frac{x-1}{x}\right| = \left|\frac{x^2-(x^2-1)}{x(x+1)}\right| = \frac{1}{|x(x+1)|}$

$= \frac{1}{|x|} \cdot \frac{1}{|x+1|}$

21. $|x + 1| = 3$ so

$x + 1 = 3$ or $x + 1 = -3$.

Thus, $x = 2$ or $x = -4$.

22. $|2x - 3| = 5$ so

$2x - 3 = 5$ or $2x - 3 = -5$

$2x = 8$ or $2x = -2$

Hence, $x = 4$ or $x = -1$

23. $|2y + 1| = 5$ so

$2y + 1 = 5$ or $2y + 1 = -5$

$2y = 4$ or $2y = -6$

Hence, $y = 2$ or $y = -3$

24. $2t + 3 = \pm(t+2)$

$2t + 3 = t + 2$ or $2t + 3 = -(t + 2)$

so $2t - t = 2 - 3$ or $2t + 3 = - t - 2$

Hence, $t = -1$ or $3t = -5$ so $t = -\frac{5}{3}$

25. $2u^2 - u - 2 = 0$

$$u = \frac{1 \pm \sqrt{1-4(-4)}}{4} = \frac{1 \pm \sqrt{17}}{4}$$

26. $5 - 3z = \pm 2z$

$5 - 3z = 2z$ or $5 - 3z = -2z$;

so $5z = 5$ or $z = 5$

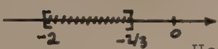
Thus, $z = 1$ or $z = 5$.

27. $|2x + 5| \leq 6$ implies $-6 \leq 2x + 5 \leq 6$

or $-11 \leq 2x \leq 1$

Thus, $-\frac{11}{2} \leq x \leq \frac{1}{2}$

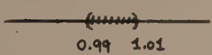
28. $|3x + 4| \leq 2$ so that $-2 \leq 3x + 4 \leq 2$



or $-6 \leq 3x \leq -2$

Hence, $-2 \leq x \leq -\frac{2}{3}$.

29. $|1-x| < 0.01$ is equivalent to $|x-1| < 0.01$



or $-0.01 < x-1 < 0.01$

Hence, $0.99 < x < 1.01$.

30. $|1 - 4x| \leq x$ (assume $x>0$, otherwise no solutions.)

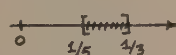
then $-x \leq 1 - 4x \leq x$

So $-x \leq 1 - 4x$ and $1 - 4x \leq x$

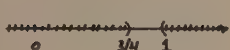
$3x \leq 1$ and $1 \leq 5x$

Thus, $x \leq \frac{1}{3}$ and $x \geq \frac{1}{5}$

implies that $\frac{1}{5} \leq x \leq \frac{1}{3}$.



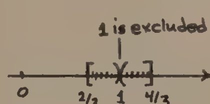
31. $|7x - 6| > x$ then $7x - 6 > x$ or $7x-6 < -x$



$6x > 6$ or $8x < 6$

Therefore, $x > 1$ or $x < \frac{6}{8} = \frac{3}{4}$

32. $\frac{1}{|x - 1|} \geq 3 \quad x \neq 1$



then $|x - 1| \leq \frac{1}{3}$

so that $-\frac{1}{3} \leq x - 1 \leq \frac{1}{3}$

Hence $\frac{2}{3} \leq x \leq \frac{4}{3}$ with $x \neq 1$.

33. $\frac{1}{|2x + 3|} \leq \frac{1}{4} \quad x \neq -\frac{3}{2}$ so $|2x + 3| \geq 4$

so $2x + 3 \geq 4$ or $2x + 3 \leq -4$;

then $2x \geq 1$ or $2x \leq -7$.

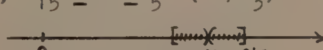
Hence, $x \geq \frac{1}{2}$ or $x \leq -\frac{1}{2}$.

34. $\frac{1}{|3x - 1|} \geq 5 \quad x \neq \frac{1}{3}$ so $|3x - 1| \leq \frac{1}{5}$

$-\frac{1}{5} \leq 3x - 1 \leq \frac{1}{5}$

$\frac{4}{5} \leq 3x \leq \frac{6}{5}$

Hence, $\frac{4}{15} \leq x \leq \frac{2}{5} \quad (x \neq \frac{1}{3})$



35. (a) True for all values since $|ab| = |a|\,|b|$

(b) Not true for all x, e.g., $x = 2$.

(c) True for all values since $|-a| = |a|$

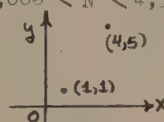
(d) True for all values since $|-a| = |a|$.

(e) True for all values by the triangle inequality.

(f) True for all values by the triangle inequality, and the fact that $|x - y| = |x + (-y)|$.

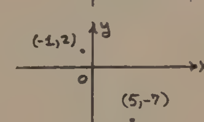
36. $|N - 4{,}000{,}000| < 500{,}000$ or $-500{,}000 < N-4{,}000{,}000 < 500{,}000$ hence, $3{,}500{,}000 < N < 4{,}500{,}000$

37.



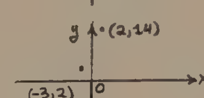
$d = \sqrt{(4-1)^2 + (5-1)^2}$
$= \sqrt{9 + 16} = \sqrt{25} = 5$

38.



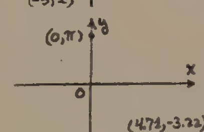
$d = \sqrt{(-1-5)^2 + (2+7)^2}$
$= \sqrt{36 + 81} = \sqrt{117}$

39.



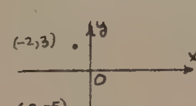
$d = \sqrt{(2+3)^2 + (14-2)^2}$
$= \sqrt{25+144} = \sqrt{169} = 13$

40.



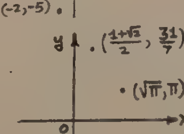
$d = \sqrt{(0-4.71)^2 + (\pi+3.22)^2}$
$= \sqrt{22.1841+40.46986109}$
$= \sqrt{62.65396109}$
≈ 7.91543

41.



$d = \sqrt{(-2+2)^2 + (3+5)^2}$
$= \sqrt{0 + 64} = \sqrt{64} = 8$

42.



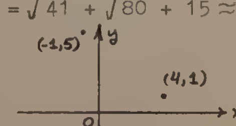
$d = \sqrt{\left[\frac{1+\sqrt{2}}{2} - \sqrt{\pi}\right]^2 + \left(\frac{31}{7} - \pi\right)^2}$
$= \sqrt{0.3196 + 1.6563}$
$= \sqrt{1.9759} \approx 1.4057$

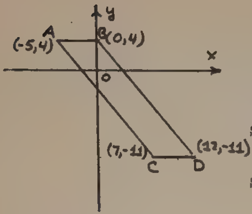
43. $d_1 = \sqrt{(-1-4)^2+(5-1)^2} = \sqrt{25+16} = \sqrt{41}$

$d_2 = \sqrt{(4-8)^2+(1+7)^2} = \sqrt{16+64} = \sqrt{80}$

$d_3 = \sqrt{(-1-8)^2+(5+7)^2} = \sqrt{81+144} = \sqrt{225} = 15$

perimeter $= \sqrt{41} + \sqrt{80} + 15 \approx 30.35$



44. 

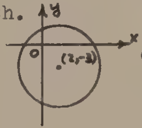
$|\overline{AB}| = |-5-0| = 5$

$|\overline{CD}| = |12-7| = 5$

slope of $\overline{AB} = \frac{4-4}{-5-0} = 0$

slope of $\overline{CD} = \frac{-11+11}{7-12} = 0$

Thus ABCD is a parallelogram because two opposite sides are parallel and equal in length.

45. 

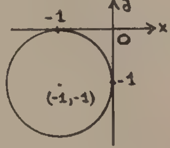
center = $(2,-3)$

$r = 5$

46. $(x^2+2x+1) + (y^2+2y+1) = 1$

$(x+1)^2 + (y+1)^2 = 1$

center = $(-1,-1)$ $r = 1$

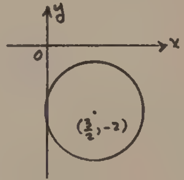


47. $(x^2-3x) + (y^2+4y+4) = 0$

$(x^2-3x+\frac{9}{4}) + (y+2)^2 = \frac{9}{4}$

$(x - \frac{3}{2})^2 + (y + 2)^2 = \frac{9}{4}$

center = $(\frac{3}{2},-2)$ $r = \frac{3}{2}$

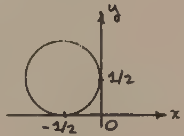


48. $x^2+y^2+x-y+\frac{1}{2} = 0$

$(x^2+x+\frac{1}{4}) + (y^2-y+\frac{1}{4}) = \frac{1}{4}$

$(x+\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{4}$

center = $(-\frac{1}{2},\frac{1}{2})$ $r = \frac{1}{2}$



49. $m = \frac{-5-2}{3-2} = -7$

$y - 2 = -7(x - 2)$

50. $m = \frac{7 - 0}{0 - 5} = \frac{7}{-5}$

$y - 0 = -\frac{7}{5}(x - 5)$

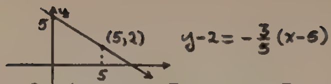
51. $m = \frac{2 + 4}{1 + 3} = \frac{6}{4} = \frac{3}{2}$

$y - 2 = \frac{3}{2}(x - 1)$

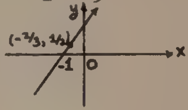
52. $m = \dfrac{\frac{2}{3} + \frac{5}{6}}{\frac{3}{2} - \frac{1}{6}} = \frac{4 + 5}{9 - 1} = \frac{9}{8}$

$y - \frac{2}{3} = \frac{9}{8}(x - \frac{3}{2})$

53. $P = (5,2)$ $m = -\frac{3}{5}$ $y-2 = -\frac{3}{5}(x-5)$



54. $P = (-\frac{2}{3}, \frac{1}{2})$ $m = \frac{3}{2}$ $y-\frac{1}{2} = \frac{3}{2}(x+\frac{2}{3})$



55. $m_{\overline{AB}} = \frac{8 - 2}{1 + 3} = \frac{6}{4} = \frac{3}{2}$

(a) $y + 5 = \frac{3}{2}(x - 7)$

(b) Here $m = -\frac{2}{3}$, so that $y+5 = -\frac{2}{3}(x-7)$.

56. $m_{\overline{AB}} = \dfrac{-\frac{3}{5} - \frac{2}{5}}{\frac{7}{3} - 1} = \dfrac{-1}{\frac{4}{3}} = -\frac{3}{4}$

(a) $y - \frac{1}{3} = -\frac{3}{4}(x - \frac{2}{5})$

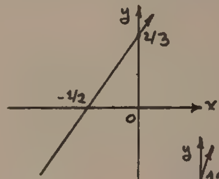
(b) Here $m = \frac{4}{3}$, so that $y-\frac{1}{3} = \frac{4}{3}(x-\frac{2}{5})$.

57. $4x - 3y + 2 = 0$

$3y = 4x + 2$

$y = \frac{4}{3}x + \frac{2}{3}$

$m = \frac{4}{3}$, $b = \frac{2}{3}$



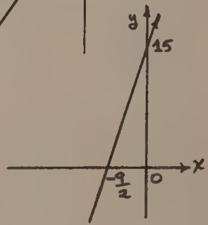
58. $\frac{2}{3}x - \frac{1}{5}y + 3 = 0$

$10x - 3y + 45 = 0$

$3y = 10x + 45$

$y = \frac{10}{3}x + 15$

$m = \frac{10}{3}$, $b = 15$



59. (a) $y - 1 = 3(x + 7)$

(b) $y - 1 = 3x + 21$ or $y = 3x + 22$

(c) $3x - y + 22 = 0$

60. $m = \frac{-3 - 5}{1 - 2} = \frac{-8}{-1} = 8$

(a) $y - 5 = 8(x - 2)$

(b) $y - 5 = 8x - 16$ or $y = 8x - 11$

(c) $8x - y - 11 = 0$

61. $7x - 3y + 2 = 0$ or $3y = 7x + 2$ or

$y = \frac{7}{3}x + \frac{2}{3}$, so that $m = \frac{7}{3}$.

(a) $y + 2 = \frac{7}{3}(x - 1)$

(b) $y + 2 = \frac{7}{3}x - \frac{7}{3}$ or $y = \frac{7}{3}x - \frac{13}{3}$

(c) $\frac{7}{3}x - y - \frac{13}{3} = 0$ or $7x - 3y - 13 = 0$

62. $2x - 5y + 4 = 0$ or $5y = 2x + 4$

or $y = \frac{2}{5}x + \frac{4}{5}$

slope of line is $\frac{2}{5}$; slope of perpendicular line L is $-\frac{5}{2}$.

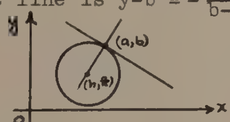
Equations for L are:

(a) $y + 4 = -\frac{5}{2}(x - 3)$

(b) $y + 4 = -\frac{5}{2}x + \frac{15}{2}$ or $y = -\frac{5}{2}x + \frac{7}{2}$

(c) $2y = -5x + 7$ or $2y + 5x - 7 = 0$

63. Slope of line through center and point (a,b): $m = \frac{b - k}{a - h}$. Thus, the slope of tangent line is $-\frac{a - h}{b - k}$, and so equation of tangent line is $y-b = -\frac{a-h}{b-k}(x-a)$.



64. $y = 20x + b$

when $x = 100$, $y = 140{,}000$ so

$140{,}000 = 20(100) + b = 2{,}000 + b$

so $b = 138{,}000$

Thus $y = 20x + 138{,}000$

Now if $x = 400$,

$y = 20(400)+138{,}000 = 8{,}000+138{,}000$

$= \$146{,}000$

65. $f(-3) = 3(-3)^2-4 = 3\cdot 9-4 = 27-4 = 23.$

66. $h(\frac{1}{2}) = \frac{1}{\frac{1}{2}} = 2.$

67. $g(\frac{6}{5}) = 6 - 5(\frac{6}{5}) = 6 - 6 = 0.$

68. $h(h(x)) = h(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x.$

69. $f(x) - f(2) = 3x^2 - 4 - \left[3\cdot 4 - 4\right]$

$= 3x^2-4-12+4 = 3x^2-12.$

70. $f(x+k)-f(x) = 3(x+k)^2-4-(3x^2-4)$

$= 3x^2+6xk+3k^2-4-3x^2+4$

$= 6xk + 3k^2.$

71. $f(g(x)) = f(6-5x) = 3(6-5x)^2-4$

$= 3(36 - 60x + 25x^2) - 4$

$= 75x^2 - 180x + 104.$

72. $g(\frac{1}{4+k}) = 6 - 5(\frac{1}{4+k}) = 6 - \frac{5}{4+k}$

$= \frac{6(4+k)-5}{4+k} = \frac{6k+19}{4+k}.$

73. $g(x) + g(-x) = 6 - 5x + 6 - 5(-x)$

$= 6 - 5x + 6 + 5x = 12.$

74. $\sqrt{f(-|x|)} = \sqrt{3(-|x|)^2-4} = \sqrt{3x^2-4}.$

75. $\frac{h(x+k) - h(x)}{k} = \frac{\frac{1}{x+k} - \frac{1}{x}}{k} = \frac{x - (x+k)}{kx(x+k)}$

$= \frac{-k}{k\cdot x(x+k)} = \frac{-1}{x(x+k)}.$

76. $\frac{1}{h(4+k)} = \frac{1}{\frac{1}{4+k}} = 4 + k.$

77. Domain equals all reals except $x = 1$.

78. $4 - x^2 > 0$ or $x^2 < 4$ or $-2 < x < 2$.

Thus, domain is $-2 < x < 2$.

79. $1 + x > 0$ or $x > -1$.

Thus, domain is $x \geq -1$.

80. Domain is all reals except $x = 1$.

81. $x^2 - 1 > 0$ or $x^2 > 1$. Thus $x > 1$ or $x < -1$. Thus domain is $x \geq 1$ or $x \leq -1$.

82. $|x| - x = 0$ if $|x| = x$ is true for $x \geq 0$.

Domain is all negative numbers.

83. (a) Graph of a function

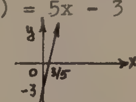
(b) Not graph of a function

84. (a) Graph of a function

(b) Not graph of a function

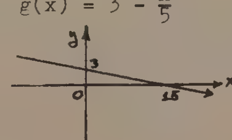
85. $f(x) = 5x - 3$ Domain: All reals

Range: All reals



86. $g(x) = 3 - \frac{x}{5}$ Domain: All reals

Range: All reals

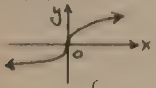


87. $F(x) = 2\sqrt{x - 2}$ Domain: $x \geq 2$

$x-2 \geq 0$ so $x \geq 2$ Range: $y \geq 0$

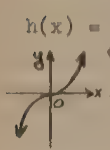
88. $G(x) = x^{\frac{1}{3}}$ Domain: All reals

Range: All reals



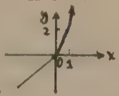
89. $h(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$ Domain: All reals

Range: All reals



90. $H(x) = \begin{cases} x & x < 0 \\ 2x & 0 \leq x \leq 1 \\ 3x^3-1 & x > 1 \end{cases}$ Domain: All reals

Range: All reals



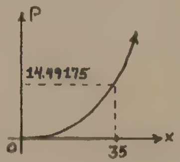
91. $P(x) = kx^3 = 3.38 \times 10^{-4}x^3$. When $x = 35$,

$y = 3.38 \times 10^{-4} \times 35^3$

$= 3.38 \times 10^{-4} \times 42875$

$= 144917.5 \times 10^{-4}$

$= 14.49175$ horsepower



92. (a) Decrease; always decreasing

(b) $F(p) = pq = p \cdot f(p)$

(c) At that price, people will not buy.

93. (a) Increase; always rising to the right.

(b) $G(p) = ps = pg(p)$

(c) At that price, producers will not supply.

94. (a) $q = f(p) = ap+b$

$\left.\begin{array}{l} p = 50 \quad q = 100{,}000 \\ p = 75 \quad q = 60{,}000 \end{array}\right\}$ so $\begin{array}{l} 100{,}000 = 50a + b \\ 60{,}000 = 75a + b \end{array}$

Subtracting we have: $40{,}000 = -25a$

so $a = -1{,}600$. Thus $b = 60{,}000 - 75a$

$= 60{,}000 - 75(-1{,}600) = 180{,}000$.

Hence, $q = -1{,}600p + 180{,}000$.

(b) $80{,}000 = -1{,}600p + 180{,}000$

$800 = -16p + 1{,}800$

or $16p = 1{,}000$

$p = 62.50$

(c) $0 = -1{,}600p + 180{,}000$

$16\ p = 1{,}800$

$p = 112.50$

95. $f(-x) = 5(-x)^5 + 3(-x)^3 + (-x)$

$= -5x^5-3x^3-x = -(5x^5+3x^3+x) = -f(x)$

Odd; symmetric with respect to the origi

96. $g(-x) = \left[(-x)^4+(-x)^2+1\right]^{-1}$

$= (x^4 + x^2 + 1)^{-1} = g(x)$

Even; symmetric with respect to the y ax

97. $h(-x) = (-x+1)(-x)^{-1} = (-x+1)(-1)x^{-1}$

$= (x - 1)x^{-1}$ Neither.

98. $F(-x) = -(-x)^3|-x| = -(-x^3)|x| = x^3|x|$

$= -F(x)$ Odd; symmetric with respect to the origin.

99. $G(-x) = (-x)^{80}-5(-x)^6+ 9$

$= x^{80} - 5x^6 + 9 = G(x)$

Even; symmetric with respect to y axis.

100 $H(x)$ not defined for negative x, so $H(x) \geq 0$ for all non-negative x. Neithe even nor odd.

101. (a) $(f+g)(x) = f(x)+g(x) = x+2+3x-4$

$= 4x -2.$

(b) $(f-g)(x) = f(x)-g(x)=x+2-(3x-4)$

$= x+2-3x+4 = -2x+6.$

(c) $(f\cdot g)(x) = f(x)\cdot g(x)$

$= (x+2)(3x-4) = 3x^2+2x-8.$

(d) $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x + 2}{3x-4}.$

102. $(f+g)(x) = f(x)+g(x) = x^2+2x+x^2-2x = 2x^2$

$(f-g)(x) = f(x)-g(x) = x^2+2x-(x^2-2x)$

$= x^2+2x-x^2+2x = 4x$

$(f\cdot g)(x) = f(x)\cdot g(x) = (x^2+2x)(x^2-2x)$

$= x^4 - 4x^2$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x^2+2x}{x^2-2x} = \frac{x+2}{x-2}$, $x \neq 0$.

103. (a) $(b+g)(x) = f(x)+g(x) = \frac{1}{x-1} + \frac{1}{x+1}$

$= \frac{x+1 + x-1}{(x-1)(x+1)} = \frac{2x}{(x-1)(x+1)}$.

(b) $(f-g)(x) = f(x)-g(x) = \frac{1}{x-1} - \frac{1}{x+1}$

$= \frac{x+1-(x-1)}{(x-1)(x+1)} = \frac{2}{(x-1)(x+1)}$.

(c) $(f\cdot g)(x) = f(x)g(x) = \frac{1}{x-1} \cdot \frac{1}{x+1}$

$= \frac{1}{(x-1)(x+1)}$.

(d) $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{x-1}}{\frac{1}{x+1}} = \frac{x+1}{x-1}$.

104. $(f+g)(x) = f(x)+g(x) = \frac{x+3}{x-2} + \frac{x}{x-2} = \frac{2x+3}{x-2}$.

$(f-g)(x) = f(x)-g(x) = \frac{x+3}{x-2} - \frac{x}{x-2} = \frac{3}{x-2}$.

$(f\cdot g)(x) = f(x)\cdot g(x) = \frac{x+3}{x-2} \cdot \frac{x}{x-2} = \frac{x(x+3)}{(x-2)^2}$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{\frac{x+3}{x-2}}{\frac{x}{x-2}} = \frac{x+3}{x}$.

105. $(f+g)(x) = f(x) + g(x) = x^4 + \sqrt{x+1}$.

$(f-g)(x) = f(x) - g(x) = x^4 - \sqrt{x+1}$.

$(f\cdot g)(x) = f(x)g(x) = x^4\sqrt{x+1}$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x^4}{\sqrt{x+1}}$.

106 $(f+g)(x) = f(x)+g(x) = x+|x-2|-x = |x-2|$.

$(f-g)(x) = f(x)-g(x) = x-|x-2|+x = 2x-|x-2|$

$(f\cdot g)(x) = f(x)\cdot g(x) = x(|x-2|-x)$

$= x|x-2|-x^2$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x}{|x-2|-x}$.

107. $(f+g)(x) = f(x)+g(x) = |x|+(-x) = |x|-x$.

$(f-g)(x) = f(x)-g(x) = |x|-(-x) = |x|+x$.

$(f\cdot g)(x) = f(x)\cdot g(x) = |x|(-x) = -x|x|$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{|x|}{-x}$.

108. $(f+g)(x) = f(x)+g(x) = \sqrt{1+x^2} + \pi|x|$.

$(f-g)(x) = f(x)-g(x) = \sqrt{1+x^2} - \pi|x|$.

$(f\cdot g)(x) = f(x)\cdot g(x) = \sqrt{1+x^2}(\pi|x|)$

$= \pi|x|\sqrt{1+x^2}$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{1+x^2}}{\pi|x|}$.

109. $(f+g)(x) = f(x)+g(x) = x^{2/3}+1+\sqrt{x}$.

$(f-g)(x) = f(x)-g(x) = x^{2/3}+1-\sqrt{x}$.

$(f\cdot g)(x) = f(x)g(x) = (x^{2/3}+1)\sqrt{x}$

$= x^{7/6}+\sqrt{x}$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x^{2/3}+1}{x} = \frac{(x^{2/3}+1)\sqrt{x}}{x}$.

110. $(f+g)(x) = f(x)+g(x) = \frac{|x|}{x} + \frac{-x}{|x|}$

$= \frac{|x|^2-x^2}{x|x|} = \frac{x^2-x^2}{x|x|} = 0$.

$(f-g)(x) = f(x)-g(x) = \frac{|x|}{x} + \frac{x}{|x|}$

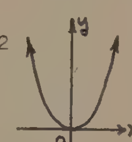
$= \frac{|x|^2+x^2}{x|x|} = \frac{x^2+x^2}{x|x|} = \frac{2x^2}{x|x|} = \frac{2x}{|x|}$.

$(f\cdot g)(x) = f(x)g(x) = (\frac{|x|}{x})(-\frac{x}{|x|}) = -1$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = (\frac{|x|}{x})/(-\frac{x}{|x|}) = \frac{|x|}{x}(-\frac{|x|}{x})$

$= -\frac{|x|^2}{x^2} = -\frac{x^2}{x^2} = -1$.

111. $y = 4x^2$



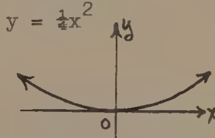
Domain: All reals

Range: $y \geq 0$

Vertex: (0,0)

Graph opens upward since $a > 0$.

112. $y = \frac{1}{4}x^2$



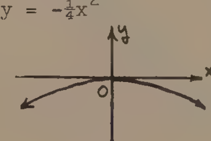
Domain: All reals

Range : $y \geq 0$

Vertex: (0,0)

Graph opens upward since $a > 0$.

113. $y = -\frac{1}{4}x^2$



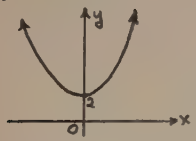
Domain: All reals

Range: $y \leq 0$

Vertex: (0,0)

Graph opens downward because $a < 0$.

114. $y = 3x^2 + 2$



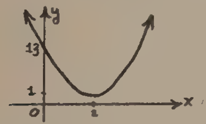
Domain: All reals

Range: $y \geq 2$

Vertex: $(0,2)$

Graph opens upward since $a > 0$.

115. $y = 3(x-2)^2 + 1$



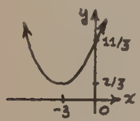
Domain: All reals

Vertex: $(2,1)$

Range: $y \geq 1$

Graph opens upward since $a > 0$.

116. $y = \frac{1}{3}\left[(x+3)^2 + 2\right] = \frac{1}{3}(x+3) + \frac{2}{3}$



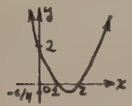
Domain: All reals

Vertex: $(-3,\frac{2}{3})$

Range: $y \geq \frac{2}{3}$

Graph opens upward since $a > 0$.

117. $y = x^2-3x+2 = x^2-3x+\frac{9}{4}-\frac{9}{4}+2 = (x-\frac{3}{2})^2 - \frac{1}{4}$



Domain: All reals

Range: $y \geq -\frac{1}{4}$

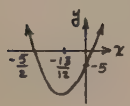
Vertex: $(\frac{3}{2}, -\frac{1}{4})$

Graph opens upward since $a > 0$.

118. $y = 6x^2 + 13x - 5 = 6(x^2 + \frac{13}{6}x) - 5$

$= 6(x^2 + \frac{13}{6}x + \frac{169}{144}) - 6 \cdot \frac{169}{144} - 5$

$= 6(x + \frac{13}{12})^2 - \frac{289}{24}$



Domain: All reals

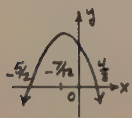
Vertex: $(-\frac{13}{12}, -\frac{289}{24})$

Range: $y \geq -\frac{289}{24}$

Graph opens upward since $a > 0$.

119. $y = -6(x^2+\frac{7}{6}x+\frac{49}{144})+20+6(\frac{49}{144})$

$= -6(x + \frac{7}{12})^2 + \frac{529}{24}$



Domain: All reals

Range: $(-\infty, \frac{529}{24})$

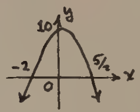
Vertex: $(-\frac{7}{12}, \frac{529}{24})$

Graph opens downward since $a < 0$.

120. $y = -2x^2 + x + 10$

$= -2(x^2 - \frac{1}{2}x + \frac{1}{16}) + \frac{2}{16} + 10$

$= -2(x - \frac{1}{4})^2 + \frac{81}{8}$



Domain: All reals

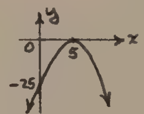
Vertex: $(\frac{1}{4}, \frac{81}{8})$

Range: $y \leq \frac{81}{8}$

Graph opens downward since $a < 0$.

121. $y = 10x - 25 - x^2 = -(x^2 - 10x + 25)$

$= -(x - 5)^2$



Domain: All reals

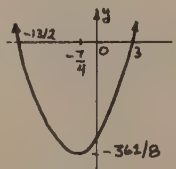
Vertex: $(5,0)$

Range: $y \leq 0$

122. $y = 7x + 2x^2 - 39 = 2x^2 + 7x - 39$

$= 2(x^2 + \frac{7}{2}x + \frac{49}{16}) - 2 \cdot \frac{49}{16} - 39$

$= 2(x + \frac{7}{4})^2 - \frac{361}{8}$



Domain: All reals

Vertex: $(-\frac{7}{4}, -\frac{361}{8})$

Range: $y \geq -\frac{361}{8}$

Graph opens upward since $a > 0$.

123. (a) $P(x) = R(x)-C(x) = 300x-4x^2-(600+60x)$

$= -4x^2+240x-600.$

(b) $p(10) = 1{,}400$ $P(35) = 2{,}900$

(c) $P(x) = -4(x-30)^2 + 3{,}000$ and its graph is downward, vertex at (30,3000

Now, solving $-4x^2+240x-600=0$, that is

$x^2-60x+150=0$, we get

$x = \frac{60\pm\sqrt{3600-600}}{2} = \frac{60\pm\sqrt{3000}}{2}$

$= 30\pm5\sqrt{30}$, so that

$x = 57.4$ or $x = 2.6$

Answer: $x = 57$, and $x = 3$.

124. Let $f(x) = ax^2 + bx + c$

We know the x coordinate of the vertex is $-\frac{b}{2a}$. Since $f(x_1) = f(x_2) = 0$, then $ax_1^2 + bx_1 + c = 0$ and $ax_2^2 + bx_2 + c = 0$.

Subtracting we have $a(x_1^2 - x_2^2) + b(x_1 - x_2) = 0$.

Assume $x_1 - x_2 \neq 0$. Then $a(x_1 + x_2) + b = 0$ or $x_1 + x_2 = -\frac{b}{a}$. Thus, we have for the x coordinate of the vertex $-\frac{b}{2a} = \frac{1}{2}(-\frac{b}{a})$

$= \frac{1}{2}(x_1 + x_2) = \frac{(x_1 + x_2)}{2}$.

125. $s = r\theta = 5(0.57) = 2.85$ meters.

126. $\theta = \frac{s}{r} = \frac{4}{40} = 0.1$ radian.

127. $r = \frac{s}{\theta} = \frac{3\pi}{\pi} = 3$ feet.

128. $s = r\theta = 13(\frac{3\pi}{7}) = \frac{39\pi}{7}$ km.

129. $\theta = \frac{s}{r} = \frac{\pi}{2}$ radians.

130. $r = \frac{s}{\theta} = \frac{17\pi}{\frac{5\pi}{6}} = 17\pi \cdot \frac{6}{5\pi} = \frac{102}{5}$ microns.

131. (a) $80° \times \frac{\pi}{180°} = \frac{4\pi}{9}$

(b) $570° \times \frac{\pi}{180°} = \frac{19\pi}{6}$

(c) $-355° \times \frac{\pi}{180°} = -\frac{71\pi}{36}$

(d) $-810° \times \frac{\pi}{180°} = -\frac{9\pi}{2}$

(e) $-310° \times \frac{\pi}{180°} = \frac{-31\pi}{18}$

(f) $765° \times \frac{\pi}{180°} = \frac{17\pi}{4}$

132. (a) 0.0873 (b) 0.4844

(c) -0.2997 (d) 0.6158

133. (a) $\frac{2\pi}{5} \cdot \frac{180°}{\pi} = 72°$

(b) $-\frac{13\pi}{4} \cdot \frac{180°}{\pi} = -585°$

(c) $-\frac{7\pi}{8} \cdot \frac{180°}{\pi} = -(\frac{315}{2})°$

(d) $\frac{35\pi}{3} \cdot \frac{180°}{\pi} = 2{,}100°$

(e) $\frac{51\pi}{4} \cdot \frac{180°}{\pi} = 2{,}295°$

(f) $\frac{18\pi}{5} \cdot \frac{180°}{\pi} = 648°$

134. (a) $286.4789°$ (b) $223.4535°$

(c) $-437.1668°$ (d) $-1226.3016°$

135. $A = \frac{1}{2}r^2\theta = \frac{1}{2}(25)^2 \cdot \frac{\pi}{6} = \frac{625\pi}{12}$.

136. $\theta = 60° = 60° \times \frac{\pi}{180°} = \frac{\pi}{3}$ radians.

$A = \frac{1}{2}r^2\theta = \frac{1}{2}(3.5)^2\frac{\pi}{3} = \frac{12.25\pi}{6}$.

137. $\theta = (\frac{4}{60})(2\pi) = \frac{2\pi}{15}$ radians.

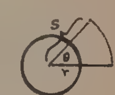
$s = r\theta = 0.6(\frac{2\pi}{15}) = \frac{1.2}{15}\pi = \frac{0.4}{5}\pi = \frac{2}{25}\pi$ meter.

138. $A = \frac{1}{2}r^2\theta = \frac{1}{2}(0.6)^2(\frac{2\pi}{15}) = \frac{0.72\pi}{30}$

$= \frac{3}{125}\pi$ sq. meter

139. $s = r\theta \quad \theta = 0.5° = 0.5° \times \frac{\pi}{180} = \frac{1}{2}(\frac{\pi}{180}) = \frac{\pi}{360}$.

$= 240{,}000\ \theta = (240{,}000)\ (\frac{\pi}{360})$



$= \frac{2{,}000}{3}\pi \approx 2094.4$ miles

140. Satellite travels an arc of 9.92(10) = 99.2 km/sec in angle $\theta = 0.75°$ $= 0.75° \times \frac{\pi}{180°}$ radians. Thus,

$r = \frac{s}{\theta} = \frac{(99.2)(180)}{0.75(\pi)} = \frac{17{,}856}{2.3562} \approx 7578$

But radius of earth = 6371; so satellite is 7578-6371 = 1207 miles high.

141. 0.4591147705 142. 0.8767267557

143. 0.5952436037 144. -0.4037049808

145. -1.625839380 146. 4.904815129

147. 1.042572391 148. -2.692748010

149. 1.701301619 150. -1.094993438

151. -26.02388181 152. -0.8202742844

153. 0.9346780153 154. -1.488669940

155. -0.9545616245 156. -0.1626668741

157. 0.4440158399 158. 0.9777795286

159. $\tan(-\theta) = -\tan\theta$.

160. $\tan(-\alpha) = -\tan\alpha$.

161. $\csc x - \cos x \cot x = \frac{1}{\sin x} - \cos x \frac{\cos x}{\sin x}$

$= \frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x$.

162. $\sec\theta - \sin\theta\tan\theta = \frac{1}{\cos\theta} - \sin\theta\frac{\sin\theta}{\cos\theta}$

$$= \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta.$$

163. $\csc^2 t\tan^2 t - 1 = \frac{1}{\sin^2 t}\cdot\frac{\sin^2 t}{\cos^2 t}$

$$= \frac{1}{\cos^2 t} - 1 = \sec^2 t - 1 = \tan^2 t.$$

164. $(\cot x+1)^2-\csc^2 x = \cot^2 x+1+2\cot x-\csc^2 x$

$$= \csc^2 x+2\cot x-\csc^2 x = 2\cot x.$$

165. $\frac{\sec^2 u + 2\tan u}{1+\tan u} = \frac{1+\tan^2 u+2\tan u}{1+\tan u}$

$$= \frac{(1+\tan u)^2}{1+\tan u} = 1+\tan u.$$

166. $\frac{\sec\beta}{\cot\beta+\tan\beta} = \frac{\frac{1}{\cos\beta}}{\frac{\cos\beta}{\sin\beta}+\frac{\sin\beta}{\cos\beta}}$

$$= \frac{\sin\beta}{\cos^2\beta+\sin^2\beta} = \frac{\sin\beta}{1} = \sin\beta.$$

167. $\frac{\sin^2\theta+2\cos^2\theta}{\sin\theta\cos\theta} - \frac{2\cos\theta}{\sin\theta}$

$$= \frac{\sin^2\theta+2\cos^2\theta-2\cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin^2 0}{\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

168. $\frac{\csc y+\cot y-(\csc y-\cot y)}{\csc^2 y-\cot^2 y} = \frac{2\cot y}{1}$

$$= 2\cot y.$$

169. $\cos(360^\circ-\theta) = \cos 360^\circ\cos\theta+\sin 360^\circ\sin\theta$

$$= 1\cdot\cos\theta+0\cdot\sin\theta = \cos\theta.$$

170. $\tan(2\pi-\beta) = \frac{\tan 2\pi-\tan\beta}{1+\tan 2\pi\tan\beta} = \frac{0-\tan\beta}{1+0}$

$$= \tan\beta.$$

171. $\sin(270+\alpha) = \sin 270^\circ\cos\quad+\cos 270^\circ\sin\alpha$

$$= (-1)\cos\alpha+0\cdot\sin\alpha = -\cos\alpha.$$

172. $\cos(270^\circ-\phi) = \cos 270^\circ\cos\phi+\sin 270^\circ\sin\phi$

$$= 0(\cos\phi)+(-1)\sin\phi = -\sin\phi.$$

173. $\sin(2\pi+t) = \sin 2\pi\cos t+\cos 2\pi\sin t$

$$= 0(\cos t)+1\cdot\sin t = \sin t.$$

174. $\cot(\frac{3\pi}{2}+x) = \frac{\cot\frac{3\pi}{2}\cot x-1}{\cot\frac{3\pi}{2}+\cot x} = \frac{0-1}{0+\cot x}$

$$= \frac{-1}{\cot x} = -\tan x.$$

175. $\sin(37^\circ+23'') = \sin 60^\circ = \frac{\sqrt{3}}{2}.$

176. $\tan(\frac{\pi}{5}+\frac{\pi}{20}) = \tan(\frac{\pi}{4}) = 1.$

177. $\sin(-y) = \sin y$ and $\sin(x+\frac{\pi}{2})$

$$= \sin x\cos\frac{\pi}{2}+\cos x\sin\frac{\pi}{2}$$

$$= \sin x\,(0)+\cos x\,(1) = \cos x,$$

so that, $\sin x\cos y - \sin(x+\frac{\pi}{2})\sin(-y)$

$$= \sin x\cos y+\cos x\sin y$$

$$= \sin(x+y).$$

178. $\cos(\pi-t) = \cos\pi\cos t+\sin\pi\sin t$

$$= -1\,(\cos t)+0\cdot\sin t$$

$$= -\cos t \quad \cos(\frac{\pi}{2}-t) = \sin t$$

Thus, $\cos(\pi-t)-\tan t\cos(\frac{\pi}{2}-t)$

$$= \cos t - \tan t\cdot\sin t$$

$$= -\cos t - \frac{\sin t}{\cos t}\cdot\sin t$$

$$= -(\frac{\cos^2 t+\sin^2 t}{\cos t}) = \frac{1}{\cos t} = -\sec t.$$

179. (a) $\sin\frac{7\pi}{12} = \sin(\frac{\pi}{4}+\frac{\pi}{3})$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{3}+\cos\frac{\pi}{4}\sin\frac{\pi}{3}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2}+\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4}(1+\sqrt{3})$$

(b) $\cos\frac{7}{12} = \cos(\frac{\pi}{4}+\frac{\pi}{3})$

$$= \cos\frac{\pi}{4}\cos\frac{\pi}{3}-\sin\frac{\pi}{4}\sin\frac{\pi}{3}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4}(1-\sqrt{3})$$

(c) $\tan\frac{7\pi}{12} = \frac{\sin\frac{7\pi}{12}}{\cos\frac{7\pi}{12}} = \frac{\frac{\sqrt{2}}{4}(1+\sqrt{3})}{\frac{\sqrt{2}}{4}(1-\sqrt{3})}$

$$= \frac{(1+\sqrt{3})(1+\sqrt{3})}{1-3} = \frac{4+2\sqrt{3}}{-2} = -(2+\sqrt{}$$

180. $\cos\alpha = \frac{3}{5}$ $\sin^2\alpha = 1-\cos^2\alpha = 1-\frac{9}{25} = \frac{16}{25}$

$\sin\alpha = \frac{\pm 4}{5}$. In Q_{IV}, $\sin\alpha < 0$,

so $\sin\alpha = -\frac{4}{5}$.

$\sin\beta = \frac{8}{17}$ $\cos^2\beta = 1-\sin^2\beta = 1-\frac{64}{289} = \frac{225}{289}$

$\cos\beta = \pm\frac{15}{17}$. In Q_I, $\cos\beta > 0$,

so $\cos\beta = \frac{15}{17}$.

$\cos\gamma = -\frac{24}{25}$ $\sin^2\gamma = 1-\cos^2\gamma = 1-\frac{576}{625} = \frac{49}{625}$

$\sin\gamma = \pm\frac{7}{25}$. In Q_{II},

$\sin\gamma > 0$, so $\sin\gamma = \frac{7}{25}$.

$\sin\theta = \frac{5}{13}$ $\cos\theta = \pm\frac{12}{13}$. In Q_{II},

$\cos\theta < 0$, so $\cos\theta = -\frac{12}{13}$.

Now,

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$= -\frac{4}{5}\left(\frac{15}{17}\right)+\frac{3}{5}\cdot\frac{8}{17} = \frac{-36}{85}$.

(b) $\cos(\gamma+\theta) = \cos\gamma\cos\theta - \sin\gamma\sin\theta$
$= -\frac{24}{25}\left(-\frac{12}{13}\right) - \frac{7}{25}\left(\frac{5}{13}\right) = \frac{253}{325}$.

(c) $\sin(\beta+\theta) = \sin\beta\cos\theta + \cos\beta\sin\theta$
$= \frac{8}{17}\left(-\frac{12}{13}\right) + \frac{15}{17}\left(\frac{5}{13}\right) = \frac{-21}{221}$.

(d) $\sin(\alpha-\gamma) = \sin\alpha\cos\gamma - \cos\alpha\sin\gamma$
$= -\frac{4}{5}\left(-\frac{24}{25}\right) - \frac{3}{5}\cdot\frac{7}{25} = \frac{75}{125} = \frac{3}{5}$.

(e) $\cos(\beta-\gamma) = \cos\beta\cos\gamma + \sin\beta\sin\gamma$
$= \frac{15}{17}\left(-\frac{24}{25}\right) + \frac{8}{17}\cdot\frac{7}{25} = \frac{-304}{425}$.

(f) $\sin(\beta-\gamma) = \sin\beta\cos\gamma - \cos\beta\sin\gamma$
$= \frac{8}{17}\left(\frac{-24}{25}\right) - \frac{15}{17}\cdot\frac{7}{25} = -\frac{297}{425}$.

(g) $\tan(\beta-\gamma) = \frac{\sin(\beta-\gamma)}{\cos(\beta-\gamma)} = \frac{-\frac{297}{425}}{\frac{-304}{425}} = \frac{297}{304}$.

(h) $\sec(\beta-\gamma) = \frac{1}{\cos(\beta-\gamma)} = -\frac{425}{304}$.

(i) $\sin(\theta-\gamma) = \sin\theta\cos\gamma - \cos\theta\sin\gamma$
$= \frac{5}{13}\left(-\frac{24}{25}\right) - \left(\frac{-12}{25}\right)\cdot\frac{7}{25} = \frac{-36}{325}$.

(j) $\cos(\beta-\theta) = \cos\beta\cos\theta + \sin\beta\sin\theta$
$= \frac{15}{17}\left(\frac{-12}{13}\right) + \frac{8}{17}\left(\frac{5}{13}\right) = \frac{-140}{221}$.

181. $\cos^2 2x - \sin^2 2x = \cos 2(2x) = \cos 4x$.

182. $1 - 2\sin^2\frac{t}{2} = \cos 2\left(\frac{t}{2}\right) = \cos t$.

183. $2\sin\frac{t}{2}\cos\frac{t}{2} = \sin 2\left(\frac{t}{2}\right) = \sin t$.

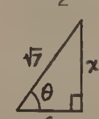
184. $\cos^4 2\theta - \sin^4 2\theta$
$= (\cos^2 2\theta + \sin^2 2\theta)(\cos^2 2\theta - \sin^2 2\theta)$
$= 1 \cdot \cos 2(2\theta) = \cos 4\theta$.

185. $2\sin^2\frac{\theta}{2} + \cos\theta = (1-\cos\theta) + \cos\theta = 1$.

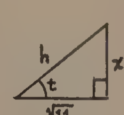
186. $\frac{\sin 4\pi t}{4\sin\pi t\cos\pi t} = \frac{\sin 2(2\pi t)}{2\cdot\sin 2\pi t}$
$= \frac{2\sin 2\pi t\cos 2\pi t}{2\sin 2\pi t} = \cos 2\pi t$.

187. $\frac{\tan\omega t}{1-\tan^2\omega t} = \frac{1}{2}\cdot\frac{2\tan\omega t}{1-\tan^2\omega t} = \frac{1}{2}\tan 2\omega t$

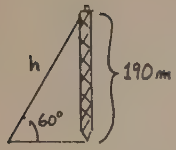
188. $\frac{\cos^2\frac{v}{2} - \cos v}{\sin^2\frac{v}{2}} = \frac{\cos^2\frac{v}{2} - (2\cos^2\frac{v}{2} - 1)}{\sin^2\frac{v}{2}}$
$= \frac{1-\cos^2\frac{v}{2}}{\sin^2\frac{v}{2}} = \frac{\sin^2\frac{v}{2}}{\sin^2\frac{v}{2}} = 1$.

189. 

$\frac{x}{\sqrt{7}} = \sin\theta$ By the Pythagorean theorem,
$x^2 + s^2 = 7$ or $s^2 = 7 - x^2$ or $s = \sqrt{7-x^2}$
So $\cot\theta = \frac{\sqrt{7-x^2}}{x}$.

190. 

$\tan t = \frac{x}{\sqrt{11}}$. By the Pythagorean theorem,
$h^2 = 11 + x^2$ or $h = \sqrt{11+x^2}$
So $\cos t = \frac{\sqrt{11}}{\sqrt{11+x^2}}$.

191. 

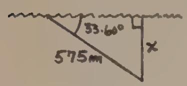
$\sin 60^\circ = \frac{190}{h}$

or $h = \frac{190}{\sin 60^\circ} = \frac{190}{\frac{\sqrt{3}}{2}}$

$= \frac{380}{\sqrt{3}}$.

Need $3 \times \frac{380}{\sqrt{3}}$ meters

$= 380\sqrt{3} \approx 658.2$ meters of cable.

192. 

$\sin 33.60^\circ = \frac{x}{575}$ or

$x = 575 \sin 33.60^\circ$

≈ 318.2 m.

193. $\lim\limits_{x\to 2} \dfrac{x^3-8}{1-\frac{2}{x}} = \lim\limits_{x\to 2} \dfrac{x^3-8}{\frac{x-2}{x}} =$

$\lim\limits_{x\to 2} \dfrac{x(x-2)(x^2+2x+4)}{x-2} = \lim\limits_{x\to 2} x(x^2+2x+4) = 24.$

x	1.9	1.99	1.999	1.9999
$\dfrac{x^3-8}{1-\frac{2}{x}}$	21.6790	23.7608	23.9760	23.9976

x	2.1	2.01	2.001	2.0001
$\dfrac{x^3-8}{1-\frac{2}{x}}$	26.4810	24.2408	24.0240	24.0024

194. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+24}-5} = \lim\limits_{x\to 1} \dfrac{(x-1)(\sqrt{x+24}+5)}{x+24-25}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)(\sqrt{x+24}+5)}{x-1} = \lim\limits_{x\to 1} (\sqrt{x+24}+5)$

$= \sqrt{25} + 5 = 5 + 5 = 10.$

x	0.9	0.99	0.999	0.9999
$\dfrac{x-1}{\sqrt{x+24}-5}$	9.98999	9.998999	9.999899	9.9999900

x	1.1	1.01	1.001	1.0001
$\dfrac{x-1}{\sqrt{x+24}-5}$	10.00999002	10.00099	10.00010	10.000010

195. Show $\lim\limits_{x\to -1} (2x-7) = -9$.

Let $\varepsilon = 0.01$. Want to find $\delta > 0$ so that when $0 < |x+1| < \delta$ then

$|2x - 7 + 9| < 0.01$; that is,

$|2x + 2| < 0.01$ or

$|x + 1| < 0.005$.

Thus, choose $\delta = 0.005$.

196. Show $\lim\limits_{x\to 3} (1-5x) = -14$.

Let $\varepsilon = 0.02$. Want to find $\delta > 0$ so that when $0 < |x-3| < \delta$, then

$|1 - 5x + 14| < 0.02$; that is,

$|-5x + 15| < 0.02$ or

$|5x - 15| < 0.02$ or

$|x - 3| < 0.004$.

Thus, choose $\delta = 0.004$.

197. Show $\lim\limits_{x\to -2} (5x+1) = -9$.

Let $\varepsilon = 0.002$. Want to find $\delta > 0$ so tha when $0 < |x+2| < \delta$, then

$|5x + 1 + 9| < \varepsilon = 0.002$; that is,

$|5x + 10| < 0.002$ or

$|x + 2| < 0.0004$.

Thus, choose $\delta = 0.0004$.

198. Show $\lim\limits_{x\to \frac{3}{2}} \dfrac{4x^2-9}{2x-3} = 6$.

Let $\varepsilon = 0.001$. Want $\delta > 0$ so that when $0 < |x - \frac{3}{2}| < \delta$, $\left|\dfrac{4x^2-9}{2x-3} - 6\right| < 0.001$;

that is, $\left|\dfrac{(2x+3)(2x-3)}{2x-3} - 6\right| = |2x + 3 - 6$

$= |2x - 3| < 0.001 = 2|x - \frac{3}{2}| < 0.001$ or

$|x - \frac{3}{2}| < 0.0005$. Thus, choose $\delta = 0.0005$.

199. Show $\lim\limits_{x\to -\frac{1}{5}} \left(\dfrac{25x^2-1}{5x+1}\right) = -2$.

Let $\varepsilon = 0.01$. Then find $\delta > 0$ so that the condition $0 < |x + \frac{1}{5}| < \delta$ implies

$$\left|\frac{25x^2-1}{5x+1}+2\right|<0.01; \text{ that is,}$$

$$\left|\frac{(5x+1)(5x-1)}{5x+1}+2\right| = |5x-1+2|$$

$$= |5x+1| < 0.01 = 5\left|x+\frac{1}{5}\right| < 0.01; \text{ so}$$

$\left|x+\frac{1}{5}\right| < 0.002$. Choose $\delta = 0.002$.

200. Show $\lim_{x\to -1} (2x-7) = -9$.

Let $\varepsilon > 0$. Want to find $\delta > 0$ so that when $0 < |x+1| < \delta$, then $|2x-7+9| < \varepsilon$, that is,

$$|2x-7+9| = |2x+2| < \varepsilon \text{ or}$$

$$|x+1| < \frac{\varepsilon}{2}.$$

Choose $\delta = \frac{\varepsilon}{2}$.

201. $\lim_{t\to 5} (6t^2+t-4) = 6\lim_{t\to 5} t^2 + \lim_{t\to 5} t - \lim_{t\to 5} 4$

$$= 6(\lim_{t\to 5} t)^2 + 5 - 4$$

$$= 6(5)^2 + 1 = 151.$$

202. $\lim_{y\to 2} \frac{3y+5}{4y^2+5y-4} = \frac{\lim_{y\to 2}(3y+5)}{\lim_{y\to 2}(4y^2+5y-4)}$

$$= \frac{3\lim_{y\to 2} y + 5}{4\lim_{y\to 2} y^2 + 5\lim_{y\to 2} y - \lim_{y\to 2} 4} = \frac{3\cdot 2+5}{4(\lim_{y\to 2} y)^2 + 5\cdot 2 - 4}$$

$$= \frac{6+5}{4(2)^2+6} = \frac{11}{22} = \frac{1}{2}.$$

203. $\lim_{t\to 1} \frac{1-t^3}{1-t^2} = \lim_{t\to 1} \frac{(1-t)(1+t+t^2)}{(1-t)(1+t)}$

$$= \lim_{t\to 1} \frac{1+t+t^2}{1+t} = \frac{\lim_{t\to 1}(1+t+t^2)}{\lim_{t\to 1}(1+t)}$$

$$= \frac{\lim_{t\to 1} 1 + \lim_{t\to 1} t + (\lim_{t\to 1} t)^2}{\lim_{t\to 1} 1 + \lim_{t\to 1} t} = \frac{1+1+1^2}{1+1} = \frac{3}{2}.$$

204. $\lim_{z\to \frac{5}{2}} \frac{4z^2-25}{2z-5} = \lim_{z\to \frac{5}{2}} \frac{(2z-5)(2z+5)}{2z-5}$

$$= \lim_{z\to 5/2}(2z+5) = 2\lim_{z\to 5/2} z + \lim_{z\to 5/2} 5$$

$$= 2\left(\frac{5}{2}\right) + 5 = 5 + 5 = 10.$$

205. $\lim_{h\to 0} \frac{1}{h}\left(\frac{6+h}{3+2h} - 2\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{6+h-6-4h}{3+2h}\right)$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{-3h}{3+2h} = \lim_{h\to 0} \frac{-3}{3+2h}$$

$$= \frac{\lim_{h\to 0}(-3)}{\lim_{h\to 0}(3+2h)} = \frac{-3}{3+2\lim_{h\to 0} h} = \frac{-3}{3+2(0)}$$

$$= \frac{-3}{3} = -1.$$

206. $\lim_{x\to 0} \frac{1}{x}\left[1-\frac{1}{(x+1)^2}\right] = \lim_{x\to 0} \frac{1}{x}\left[\frac{x^2+2x+1-1}{(x+1)^2}\right]$

$$= \lim_{x\to 0} \frac{1}{x}\cdot\frac{x^2+2x}{(x+1)^2} = \lim_{x\to 0} \frac{1}{x}\cdot\frac{x(x+2)}{(x+1)^2}$$

$$= \lim_{x\to 0} \frac{x+2}{(x+1)^2} = \frac{\lim_{x\to 0}(x+2)}{\lim_{x\to 0}(x+1)^2}$$

$$= \frac{\lim_{x\to 0} x + \lim_{x\to 0} 2}{\left[\lim_{x\to 0}(x+1)\right]^2} = \frac{0+2}{1^2} = \frac{2}{1} = 2.$$

207. $\lim_{t\to 1} \frac{\sqrt{4-t^2}}{2+t} = \frac{\lim_{t\to 1}\sqrt{4t^2}}{\lim_{t\to 1}(2+t)} = \frac{\sqrt{\lim_{t\to 1}(4-t^2)}}{\lim_{t\to 1} 2 + \lim_{t\to 1} t}$

$$= \frac{\sqrt{4-1}}{2+1} = \frac{\sqrt{3}}{3}.$$

208. $\lim_{h\to -1} \frac{3-\sqrt{h^2+h+9}}{h^3+1}\cdot\frac{3+\sqrt{h^2+h+9}}{3+\sqrt{h^2+h+9}}$

$$= \lim_{h\to -1} \frac{9-(h^2+h+9)}{(h^3+1)(3+\sqrt{h^2+h+9})}$$

$$= \lim_{h\to -1} \frac{-h(h+1)}{(h+1)(h^2+h+1)(3+\sqrt{h^2+h+9})}$$

$$= \lim_{h\to -1} \frac{-h}{(h^2+h+1)(3+\sqrt{h^2+h+9})}$$

$$= \frac{\lim_{h\to -1}(-h)}{\lim_{h\to -1}(h^2+h+1)\cdot\lim_{h\to -1}(3+\sqrt{h^2+h+9})}$$

$$= \frac{1}{(1)(3+3)} = \frac{1}{6}.$$

209. $\lim_{x\to 1} \frac{1-x}{2-\sqrt{x^2+3}} = \lim_{x\to 1} \frac{(1-x)(2+\sqrt{x^2+3})}{4-(x^2+3)}$

$= \lim_{x\to 1} \frac{(1-x)(2+\sqrt{x^2+3})}{1-x^2} = \lim_{x\to 1} \frac{(1-x)(2+\sqrt{x^2+3})}{(1-x)(1+x)}$

$= \lim_{x\to 1} \frac{2+\sqrt{x^2+3}}{1+x} = \frac{\lim_{x\to 1}(2+\sqrt{x^2+3})}{\lim_{x\to 1}(1+x)}$

$= \frac{2+\sqrt{4}}{2} = \frac{2+2}{2} = 2.$

210. $\lim_{t\to 0} \frac{\sqrt{6+t}-\sqrt{6}}{t} = \lim_{t\to 0} \frac{(6+t)-6}{t(\sqrt{6+t}+\sqrt{6})}$

$= \lim_{t\to 0} \frac{t}{t(\sqrt{6+t}+\sqrt{6})} = \lim_{t\to 0} \frac{1}{\sqrt{6+t}+\sqrt{6}}$

$= \frac{\lim_{t\to 0} 1}{\lim_{t\to 0}(\sqrt{6+t}+\sqrt{6})} = \frac{1}{\sqrt{6}+\sqrt{6}} = \frac{1}{2\sqrt{6}}.$

211. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)}$

$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{\lim_{x\to 9} 1}{\lim_{x\to 9}(\sqrt{x}+3)} = \frac{1}{\sqrt{9}+3}$

$= \frac{1}{3+3} = \frac{1}{6}.$

212. $\lim_{t\to 0} \frac{(\sqrt[3]{5+t}-\sqrt[3]{5})}{t}\left[\frac{(\sqrt[3]{(5+t)^2}+\sqrt[3]{(5+t)5}+\sqrt[3]{5^2}}{\sqrt[3]{(5+t)^2}+\sqrt[3]{(5+t)5}+\sqrt[3]{5^2}}\right]$

$= \lim_{t\to 0} \frac{(\sqrt[3]{5+t})^3-(\sqrt[3]{5})^3}{t(\sqrt[3]{5+t})^2+\sqrt[3]{(5+t)5}+\sqrt[3]{5^2})}$

$= \lim_{t\to 0} \frac{t}{t(\sqrt[3]{(5+t)^2}+\sqrt[3]{(5+t)5}+\sqrt[3]{5^2})}$

$= \lim_{t\to 0} \frac{1}{\sqrt[3]{(5+t)^2}+\sqrt[3]{(5+t)5}+\sqrt[3]{5^2}}$

$= \lim_{t\to 0} \frac{1}{\sqrt[3]{5^2}+\sqrt[3]{5\cdot 5}+\sqrt[3]{5^2}} = \frac{1}{3\sqrt[3]{25}}.$

213. $\lim_{t\to \frac{\pi}{2}} t \sin t \cos t$

$= (\lim_{t\to\frac{\pi}{2}} t)(\lim_{t\to\frac{\pi}{2}} \sin t)(\lim_{t\to\frac{\pi}{2}} \cos t)$

$= \frac{\pi}{2} \cdot 1 \cdot 0 = 0.$

214. $\lim_{y\to\frac{\pi}{4}} y \sin^4 y = \lim_{y\to\frac{\pi}{4}} y \lim_{y\to\frac{\pi}{4}} \sin^4 y$

$= \frac{\pi}{4} \cdot (\lim_{y\to\frac{\pi}{4}} \sin y)^4 = \frac{\pi}{4} \cdot (\frac{\sqrt{2}}{2})^4$

$= \frac{\pi}{4} \cdot \frac{2^2}{2^4} = \frac{\pi}{16}.$

215. $\lim_{x\to\pi/6} \sin^3 x \cos^2 x = \lim_{x\to\pi/6} \sin^3 x \lim_{x\to\pi/6} \cos^2 x$

$=(\lim_{x\to\frac{\pi}{6}} \sin x)^3 (\lim_{x\to\frac{\pi}{6}} \cos x)^2$

$= (\frac{1}{2})^3 \cdot (\frac{\sqrt{3}}{2})^3 = \frac{3\sqrt{3}}{64} = \frac{3}{32}.$

216. $\lim_{w\to 0} (w^2-\cos \pi w) = \lim_{w\to 0} w^2 - \lim_{w\to 0} \cos w$

$= 0^2 - \cos \pi\cdot 0 = 0 - \cos 0 = 0 - 1 = -1.$

217. $\lim_{t\to 0} \frac{\sin 13t}{t} = 13 \lim_{t\to 0} \frac{\sin 13t}{13t} = 13 \cdot 1 = 13$

218. $\lim_{x\to 0} \frac{x}{\sin 47x} = \lim_{x\to 0} \frac{1}{\frac{\sin 47x}{x}}$

$= \lim_{x\to 0} \frac{1}{\frac{\sin 47x}{47x}} (\frac{1}{47}) = \frac{1}{47} \cdot \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \frac{\sin 47x}{47x}}$

$= \frac{1}{47} \cdot \frac{1}{1} = \frac{1}{47}.$

219. $\lim_{u\to 0} \frac{\sin 19u}{\sin 7u} = \lim_{u\to 0} \frac{\frac{\sin 19u}{u}}{\frac{\sin 7u}{u}}$

$= \lim_{u\to 0} \frac{19\frac{\sin 19u}{19u}}{7\frac{\sin 7u}{7u}} = \frac{19}{7} \cdot \frac{\lim_{u\to 0} \frac{\sin 19u}{19u}}{\lim_{u\to 0} \frac{\sin 7u}{7u}}$

$= \frac{19}{7} \cdot \frac{1}{1} = \frac{19}{7}.$

220. Let $t = \sqrt[3]{y}$. Then as $y \to 0$, $t \to 0$.

$\lim_{y\to 0} \frac{\sin \sqrt[3]{y}}{\sqrt[3]{y}} = \lim_{t\to 0} \frac{\sin t}{t} = 1.$

221. $\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x + 1} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+1}$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{1+\cos x} = \frac{\lim\limits_{x\to\frac{\pi}{2}} \sin x}{\lim\limits_{x\to\frac{\pi}{2}} (1+\cos x)} = \frac{1}{1+0} = 1.$$

222. Let $u = \sin x$. Then as $x \to 0$, $u \to 0$.

$$\lim_{x\to 0} \frac{1 - \cos(\sin x)}{\sin x} = \lim_{u\to 0} \frac{1 - \cos u}{u} = 0.$$

223. (a) We want $|(3x-1) - (3a-1)| < \epsilon$, or, $|3x-3a| < \varepsilon$; that is, $|x-a| < \frac{\varepsilon}{3}$.

Choose x within $\frac{\varepsilon}{3}$ of a; that is,

$a - \frac{\varepsilon}{3} < x < a + \frac{\varepsilon}{3}$.

(b) Choose x within $\frac{1}{300} = \frac{0.01}{3}$ of a; that is, $a - \frac{0.01}{3} < x < a + \frac{0.01}{3}$,

$a - \frac{1}{300} < x < a + \frac{1}{300}$. No; $x = a + 0.07$ is in the interval, but $a + 0.07$ is not within $\frac{1}{300}$ of a.

224. $$\lim_{x\to 0} \frac{f(bx)}{x} = \lim_{x\to 0} \frac{bf(bx)}{bx}$$

$$= b \lim_{x\to 0} \frac{f(bx)}{bx} = b \cdot L = bL.$$

225. $$\lim_{x\to 3^-} \frac{t-3}{t^2-9} = \lim_{x\to 3^-} \frac{1}{t+3} = \frac{1}{6}.$$

226. $$\lim_{y\to 2^+} \frac{|2-y|}{y^2-4} = \lim_{y\to 2^+} \frac{y-2}{(y+2)(y-2)} = \frac{1}{4}.$$

227. $$\lim_{x\to 0^+} \frac{3}{2+\sqrt{x}} = \frac{3}{2+\lim\limits_{x\to 0^+}\sqrt{x}} = \frac{3}{2}.$$

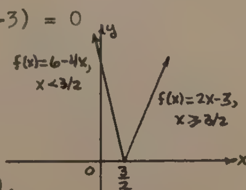
228. $$\lim_{x\to 2^-} (3 + [\![2x - 4]\!]) = 3 + (-1) = 2.$$

229. $$\lim_{x\to\frac{3}{2}^-} f(x) = \lim_{x\to\frac{3}{2}^-} (6-4x) = 0$$

$$\lim_{x\to\frac{3}{2}^+} f(x) = \lim_{x\to\frac{3}{2}^+} (2x-3) = 0$$

$$\lim_{x\to\frac{3}{2}} f(x) = 0 \text{ since}$$

$$\lim_{x\to\frac{3}{2}^-} f(x) = \lim_{x\to\frac{3}{2}^+} f(x).$$

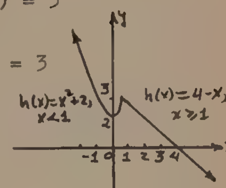


230. $$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} (x^2+2) = 3$$

$$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} (4-x) = 3$$

$$\lim_{x\to 1} h(x) = 3 \text{ since}$$

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^+} h(x).$$



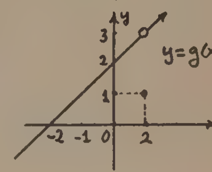
231. $$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2^-} (x+2) = 4$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{x^2-4}{x-2} = \lim_{x\to 2^+} x + 2 = 4.$$

$$\lim_{x\to 2} g(x) = 4 \text{ since}$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^-} g(x).$$

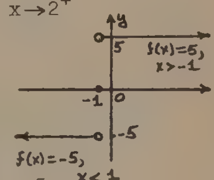


232. $$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{5x+5}{|x+1|} = 5$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{5x+5}{-|x+1|} = -5$$

$$\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x),$$

so limit does not exist.



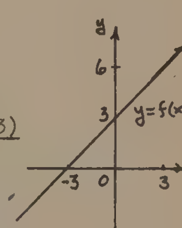
233. $$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3}$$

$$= \lim_{x\to 3} (x+3) = 6, \ x \neq 3;$$

$f(3) = 6$. Since

$$\lim_{x\to 3} f(x) = f(3),$$

f is continuous at 3.

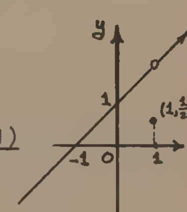


234. $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1}$$

$$= \lim_{x\to 1} (x+1) = 2, \ x \neq 1;$$

$g(1) = \frac{1}{2}$. Since

$\lim\limits_{x\to 1} g(x) \neq g(1)$, g is not continuous at 1.



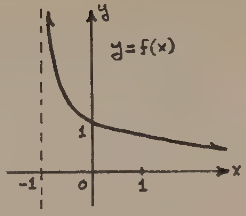
235. f is defined for $x > -1$.

$$\lim_{x\to 1}\sqrt{\frac{x-1}{x^2-1}} = \lim_{x\to 1}\sqrt{\frac{1}{x+1}}$$

$= \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$, $x \neq 1$;

$f(1) = \frac{\sqrt{2}}{2}$. f is continuous at 1 since

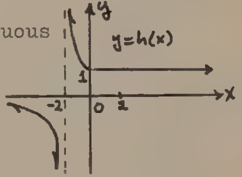
$\lim_{x\to 1} f(x) = f(1)$.



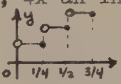
236. $\lim_{x\to 2} \frac{2-x}{2-|x|} = \lim_{x\to 2} \frac{2-x}{2-x} = 1$;

$h(2) = 1$. h is continuous at 2 since

$\lim_{x\to 2} h(x) = h(2)$.

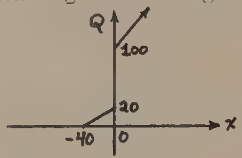


237. $f(x) = \begin{cases} 90+50\,[\![4x+1]\!], & 4x \text{ not an interger} \\ 90+50(4x), & 4x \text{ an interger.} \end{cases}$

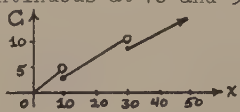


f is discontinuous at non-negative integer multiples of $\frac{1}{4}$.

238. Q is discontinuous at 0.



239. C is discontinuous at 10 and 30.



240. (a) f is continuous at 0:

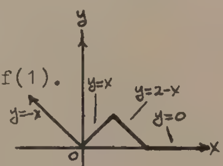
$\lim_{x\to 0^+} x = \lim_{x\to 0^-} (-x) = 0 = f(0)$.

f is continuous at 1:

$\lim_{x\to 1^-} x = \lim_{x\to 1^+} (2-x) = 1 = f(1)$.

f is continuous at 2:

$\lim_{x\to 2^-} (2-x) = \lim_{x\to 2^+} 0 = 0 = f(2)$.



(b) For $x \le 0$, $f(x) = -x$, $|x| = -x$, $|x-1| = -x+1$ and $|x-2| = -x+2$, so that $-x = Ax + B - Cx - Dx + D - Ex + 2E$, or by collecting terms, $0 = (A-C-D-E+1)x+(B+D+2E)$.

Thus, (1) $A - C - D - E + 1 = 0$ and

(2) $B + D + 2E = 0$.

For $0 \le x \le 1$, $f(x) = x$, $|x| = x$, $|x-1| = -x+1$ and $|x-2| = -x+2$, so that $x = Ax + B + Cx - Dx + D - Ex + 2E$, or, by collecting terms,

$0 = (A + C - D - E - 1)x + (B + D + 2E)$.

Thus, (3) $A + C - D - E - 1 = 0$ and

(4) $B + D + 2E = 0$.

For $1 \le x \le 2$, $f(x) = 2-x$, $|x| = x$, $|x-1| = x-1$, and $|x-2| = -x+2$, so that $2-x = Ax + B + Cx + Dx - D - Ex + 2E$; by collecting terms,

$0 = (A + C + D - E + 1)x + (B - D + 2E-2)$

Thus, (5) $A + C + D - E + 1 = 0$ and

(6) $B - D + 2E - 2 = 0$.

For $x \ge 2$, $f(x) = 0$, $|x| = x$, $|x-1| = x-1$, and $|x-2| = x - 2$, so that $0 = Ax + B + Cx + Dx - D + Ex - 2E$; by collecting terms,

$0 = (A + C + D + E)x + (B - D - 2E)$.

Thus, (7) $A + C + D + E = 0$ and

(8) $B - D - 2E = 0$.

Subtracting (1) from (3), we obtain $2C - 2 = 0$, so that $C = 1$.

Subtracting (3) from (5), we obtain $2D + 2 = 0$, so that $D = -1$.

Subtracting (8) from (6), we obtain $4E - 2 = 0$, so that $E = \frac{1}{2}$.

Substituting $D = -1$ and $E = \frac{1}{2}$ in (2), we obtain $B + 0 = 0$, so that $B = 0$.

Substituting $C = 1$, $D = -1$, and $E = \frac{1}{2}$ in (1), we obtain $A + \frac{1}{2} = 0$, so that $A = -\frac{1}{2}$. Therefore,

$f(x) = -\frac{1}{2}x + |x| - |x-1| + \frac{1}{2}|x-2|$.

241. For continuity we want $\lim_{x\to 2^-} 3x$

$= \lim\limits_{x \to 2^+} (Ax + B)$, or, $6 = \lim\limits_{x \to 2^+} (Ax + B)$;

that is, $6 = 2A + B$. In addition, for continuity we need $\lim\limits_{x \to 5^-} (Ax + B)$

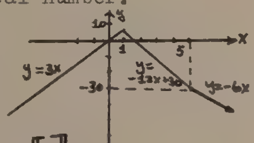
$= \lim\limits_{x \to 5^+} (-6x)$, or, $5A + B = -30$.

Solving simultaneously: $\begin{cases} 6 = 2A + B \\ -30 = 5A + B, \end{cases}$

we obtain $36 = -3A$, or, $A = -12$. So $B = 30$.

Now, $f(x) = \begin{Bmatrix} 3x \text{ if } x \leq 2 \\ -12x + 30 \text{ if } 2 < x < 5 \\ -6x \text{ if } x \geq 5. \end{Bmatrix}$ is continuous at every real number.

The graph of f shows the continuity of the function.



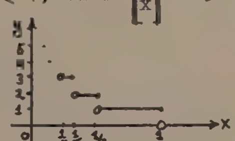
242. When $x > 1$, $\frac{1}{x} < 1$; thus $[\![\frac{1}{x}]\!] = 0$, $x > 1$.

When $\frac{1}{2} < x < 1$, $1 < \frac{1}{x} < 2$; thus $[\![\frac{1}{x}]\!] = 1$.

When $\frac{1}{2} < x < \frac{1}{3}$, $2 < \frac{1}{x} < 3$; thus $[\![\frac{1}{x}]\!] = 2$.

When $\frac{1}{3} < x < \frac{1}{4}$, $3 < \frac{1}{x} < 4$; thus $[\![\frac{1}{x}]\!] = 3$.

f is discontinuous at $x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$



243. (a) Not continuous on $[-1,1]$ since $f(\frac{1}{2})$ does not exist;

not continuous on $[\frac{1}{2},-\frac{1}{2}]$ since $f(\frac{1}{2})$ does not exist;

continuous on $(-1,\frac{1}{2})$; not continuous on $[\frac{1}{2},\infty)$ since $f(\frac{1}{2})$ does not exist.

(b) Continuous on each interval.

2 THE DERIVATIVE

Problem Set 2.1, page 91

1. The automobile travels 20 feet = $\frac{20}{5280}$ mile in $\frac{1}{4}$ second = $\frac{(\frac{1}{4})}{3600}$ hour for an average speed of $\frac{20}{5280} \div \frac{(\frac{1}{4})}{3600}$ = 54.55 miles per hour. Since the time interval is relatively short ($\frac{1}{4}$ second), it seems reasonable to use 54.55 m.p.h. as an estimate of r.

2. The automobile might have been accelerating or decelerating during the $\frac{1}{4}$ - second interval so that its average speed during this interval would differ from its instantaneous speed at the beginning of the interval.

3. (a) $x = 3$, $y = 13$. If x is increased by $\Delta x = 0.5$, then y is increased to 16.75. Thus, $\Delta y = 16.75 - 13 = 3.75$ and $\frac{\Delta y}{\Delta x} = \frac{3.75}{0.5} = 7.5$.

   (b) $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$
   $$= \lim_{\Delta x \to 0} \frac{(3 + \Delta x)^2 + (3 + \Delta x) + 1 - 13}{\Delta x}$$
   $$= \lim_{\Delta x \to 0} \frac{9 + 6\Delta x + (\Delta x)^2 + 3 + \Delta x + 1 - 13}{\Delta x}$$
   $$= \lim_{\Delta x \to 0} \frac{7\Delta x + (\Delta x)^2}{\Delta x} = 7.$$

4. (a) $$\frac{\Delta y}{\Delta x} = \frac{\frac{4}{x_1 + \Delta x} - \frac{4}{x_1}}{\Delta x} = \frac{\frac{4}{5 + 1} - \frac{4}{5}}{1} = \frac{2}{3} - \frac{4}{5} = \frac{-2}{15}.$$

   (b) $$\lim_{\Delta x \to 0} \frac{(\frac{4}{x_1 + \Delta x} - \frac{4}{x_1})}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\frac{4}{5 + \Delta x} - \frac{4}{5})}{\Delta x}$$
   $$= \lim_{\Delta x \to 0} \frac{\left[\frac{20}{5(5 + \Delta x)} - \frac{4(5 + \Delta x)}{5(5 + \Delta x)}\right]}{\Delta x}$$
   $$= \lim_{\Delta x \to 0} \frac{(\frac{20 - 20 - 4\Delta x}{5(5 + \Delta x)})}{\Delta x}$$
   $$= \lim_{\Delta x \to 0} \frac{-4}{5(5 + \Delta x)} = \frac{-4}{25}.$$

5. (a) $$\frac{\Delta s}{\Delta t} = \frac{6(t_1 + \Delta t)^2 - 6t_1^2}{\Delta t} = \frac{6(2 + 1)^2 - 24}{1}$$
   = 30 meters per second.

   (b) $$\lim_{\Delta t \to 0} \frac{6(t_1 + \Delta t)^2 - 6t_1^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{6(2 + \Delta t)^2 - 24}{\Delta t}$$
   $$= \lim_{\Delta t \to 0} \frac{6(4 + 4\Delta t + (\Delta t)^2) - 24}{\Delta t}$$
   $$= \lim_{\Delta t \to 0} \frac{24\Delta t + 6(\Delta t)^2}{\Delta t}$$
   $$= \lim_{\Delta t \to 0} (24 + 6\Delta t) = 24 \text{ meters per second.}$$

6. (a) $$\frac{\Delta s}{\Delta t} = \frac{7(t_1 + \Delta t)^3 - 7t_1^3}{\Delta t} = \frac{56 - 7}{1}$$
   = 49 meters per second.

(b) $\lim_{\Delta t \to 0} \dfrac{7(t_1 + \Delta t)^3 - 7t_1^{\,3}}{\Delta t} = \lim_{\Delta t \to 0} \dfrac{7(1 + \Delta t)^3 - 7}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{7(1 + 3\Delta t + 3(\Delta t)^2 + (\Delta t)^3) - 7}{\Delta t}$

$= \lim_{\Delta t \to 0} (21 + 21\Delta t + 7(\Delta t)^2) = 21$ meters per second.

7. (a) $\dfrac{\Delta s}{\Delta t} = \dfrac{(t_1 + \Delta t)^2 + (t_1 + \Delta t) - t_1^{\,2} - t_1}{\Delta t}$

$= 8$ meters per second.

(b) $\lim_{\Delta t \to 0} \dfrac{(t_1 + \Delta t)^2 + (t_1 + \Delta t) - t_1^{\,2} - t_1}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{(3 + \Delta t)^2 + (3 + \Delta t) - 12}{\Delta t} = 7$ meters per second.

8. (a) $\dfrac{\Delta s}{\Delta t} = \dfrac{\dfrac{2}{5 - (t_1 + \Delta t)} - \dfrac{2}{5 - t_1}}{\Delta t}$

$= \dfrac{\dfrac{2}{5 - 1.1} - \dfrac{2}{4}}{0.1} = 0.1282051280$ meter per second.

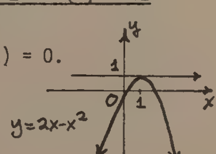
(b) $\lim_{\Delta t \to 0} \dfrac{\dfrac{2}{5 - (t_1 + \Delta t)} - \dfrac{2}{5 - t_1}}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{\dfrac{2}{5 - 1 - \Delta t} - \dfrac{2}{5 - 1}}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{1}{4 - \Delta t} = \dfrac{1}{4}$ meter per second.

9. $m = \lim_{\Delta x \to 0} \dfrac{[2(1 + \Delta x) - (1 + \Delta x)^2] - [2(1) - 1^2]}{\Delta x}$

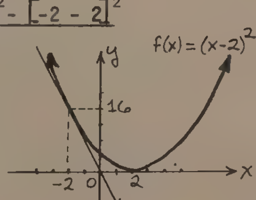
$= \lim_{\Delta x \to 0} \dfrac{-(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (-\Delta x) = 0.$



10. $m = \lim_{\Delta x \to 0} \dfrac{[(-2 + \Delta x) - 2]^2 - [-2 - 2]^2}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{-8\Delta x + (\Delta x)^2}{\Delta x}$

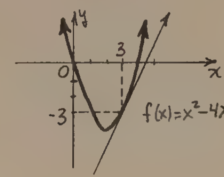
$= \lim_{\Delta x \to 0} (-8 + \Delta x) = -8.$



11. $m = \lim_{\Delta x \to 0} \dfrac{[(3 + \Delta x)^2 - 4(3 + \Delta x)] - [3^2 - 4(3)]}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{2\Delta x + (\Delta x)^2}{\Delta x}$

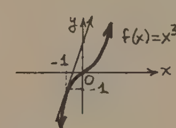
$= \lim_{\Delta x \to 0} (2 + \Delta x) = 2.$



12. $m = \lim_{\Delta x \to 0} \dfrac{(-1 + \Delta x)^3 - (-1)^3}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(-1)^3 + 3\Delta x - 3(\Delta x)^2 + (\Delta x)^3 - (-1)^3}{\Delta x}$

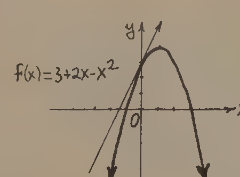
$= \lim_{\Delta x \to 0} (3 - 3\Delta x + (\Delta x)^2) = 3.$



13. $m = \lim_{\Delta x \to 0} \dfrac{[3 + 2(0 + \Delta x) - (0 + \Delta x)^2] - [3 + 2(0) - 0^2]}{\Delta x}$

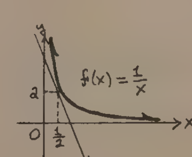
$= \lim_{\Delta x \to 0} \dfrac{2\Delta x - (\Delta x)^2}{\Delta x}$

$= \lim_{\Delta x \to 0} (2 - \Delta x) = 2.$



14. $m = \lim_{\Delta x \to 0} \dfrac{\dfrac{1}{\frac{1}{2} + \Delta x} - \dfrac{1}{\frac{1}{2}}}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\dfrac{2}{1 + 2\Delta x} - 2}{\Delta x}$

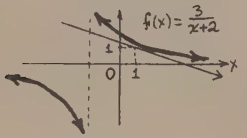
$= \lim_{\Delta x \to 0} \dfrac{-4}{1 + 2\Delta x} = -4.$



15. $m = \lim_{\Delta x \to 0} \dfrac{\dfrac{3}{1 + \Delta x + 2} - \dfrac{3}{1 + 2}}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{\dfrac{3}{3 + \Delta x} - 1}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\left(\dfrac{3 - 3 - \Delta x}{3 + \Delta x}\right)}{\Delta x}$

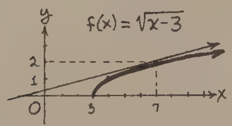
$= \lim_{\Delta x \to 0} \dfrac{-1}{3 + \Delta x} = -\dfrac{1}{3}.$



16. $m = \lim_{\Delta x \to 0} \dfrac{\sqrt{7 + \Delta x - 3} - \sqrt{7 - 3}}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\sqrt{4 + \Delta x} - 2}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(4 + \Delta x) - 4}{\Delta x(\sqrt{4 + \Delta x} + 2)}$

$= \lim_{\Delta x \to 0} \dfrac{1}{\sqrt{4 + \Delta x} + 2} = \dfrac{1}{4}.$



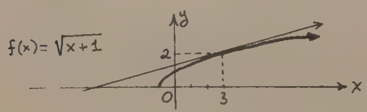
17. $m = \lim_{\Delta x \to 0} \dfrac{\sqrt{(3 + \Delta x) + 1} - \sqrt{3 + 1}}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{\sqrt{4 + \Delta x} - 2}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(\sqrt{4 + \Delta x} - 2)(\sqrt{4 + \Delta x} + 2)}{\Delta x(\sqrt{4 + \Delta x} + 2)}$

$= \lim_{\Delta x \to 0} \dfrac{(4 + \Delta x) - 4}{\Delta x(\sqrt{4 + \Delta x} + 2)}$

$= \lim_{\Delta x \to 0} \dfrac{1}{\sqrt{4 + \Delta x} + 2} = \dfrac{1}{2 + 2} = \dfrac{1}{4}.$



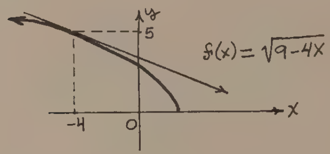
18. $m = \lim_{\Delta x \to 0} \dfrac{\sqrt{9 - 4(-4 + \Delta x)} - \sqrt{9 - 4(-4)}}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{\sqrt{25 - 4\Delta x} - 5}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(\sqrt{25 - 4\Delta x} - 5)(\sqrt{25 - 4\Delta x} + 5)}{\Delta x(\sqrt{25 - 4\Delta x} + 5)}$

$= \lim_{\Delta x \to 0} \dfrac{(25 - 4\Delta x) - 25}{\Delta x(\sqrt{25 - 4\Delta x} + 5)}$

$= \lim_{\Delta x \to 0} \dfrac{-4}{\sqrt{25 - 4\Delta x} + 5} = \dfrac{-4}{5 + 5} = \dfrac{-4}{10} = -\dfrac{2}{5}.$



19. (a) At the end of 5 seconds the object has fallen through $16(5)^2 = 400$ feet; hence, its average speed during this 5 seconds is $\dfrac{400}{5}$ = 80 feet per second.

(b) The instantaneous speed at the end of 5 seconds is

$\lim_{\Delta t \to 0} \dfrac{16(5 + \Delta t)^2 - 16(5)^2}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{160\Delta t + 16(\Delta t)^2}{\Delta t}$

$= \lim_{\Delta t \to 0} (160 + 16\Delta t) = 160$ feet per second.

20. (a) The instantaneous speed when $t = 4$ seconds is given by

$\lim_{\Delta t \to 0} \dfrac{[256(4 + \Delta t) - 16(4 + \Delta t)^2] - [256(4) - 16(4)^2]}{\Delta t}$

= 128 feet per second.

(b) The projectile will reach its maximum height at the moment when its instantaneous speed is zero. The instantaneous speed at the time t_1 is

$\lim_{\Delta t \to 0} \dfrac{[256(t_1 + \Delta t) - 16(t_1 + \Delta t)^2] - [256t_1 - 16t_1^2]}{\Delta t}$

$= 256 - 32t_1,$

so that the instantaneous speed is zero when $256 - 32t_1 = 0$ or $t_1 = 8$ seconds.

(c) The maximum height is $256(8) - 16(8)^2 = 1024$ feet.

21. The instantaneous rate of change is given by

$$\lim_{\Delta x\to 0} \frac{\frac{\sqrt{3}}{4}(x_1 + \Delta x)^2 - \frac{\sqrt{3}}{4}x_1^2}{\Delta x} = \lim_{\Delta x\to 0} \frac{\frac{\sqrt{3}}{4}\left[(10 + \Delta x)^2 - 100\right]}{\Delta x}$$

$$= \frac{\sqrt{3}}{4}\lim_{\Delta x\to 0} \frac{20(\Delta x) + (\Delta x)^2}{\Delta x} = \frac{\sqrt{3}}{4}\lim_{\Delta x\to 0} (20 + \Delta x)$$

$= 5\sqrt{3}$ cm./cm. of edge length.

22. $$\lim_{\Delta R\to 0} \frac{\frac{4}{3}\pi(5 + \Delta R)^3 - \frac{4}{3}\pi(5)^3}{\Delta R}$$

$$= \lim_{\Delta R\to 0} \frac{4}{3}\pi \frac{5^3 + 3(5)^2\Delta R + 3(5)(\Delta R)^2 + (\Delta R)^3 - 5^3}{\Delta R}$$

$$= \frac{4}{3}\pi \lim_{\Delta R\to 0} (75 + 15\Delta R + (\Delta R)^2) = \frac{4}{3}\pi(75)$$

$= 100\pi$ cubic meters per inch.

23. (a) As V increases from 100 cubic inches to 125 cubic inches, P decreases from $\frac{C}{100}$ pounds per square inch to $\frac{C}{125}$ pounds per square inch. Thus,

$$\frac{\Delta P}{\Delta V} = \frac{\frac{C}{125} - \frac{C}{100}}{125 - 100} = \frac{-C}{12,500} = -\frac{2,000}{12,500}$$

$= -0.16$(lbs. per in.^{2}) per in.^{3}.

(b) $$\lim_{\Delta V\to 0} \frac{\frac{C}{100 + \Delta V} - \frac{C}{100}}{\Delta V} = \lim_{\Delta V\to 0} \frac{C}{\Delta V} \frac{100 - (100 + \Delta V)}{100(100 + \Delta V)}$$

$$= \lim_{\Delta V\to 0} \frac{-C}{100(100 + \Delta V)} = \frac{-C}{10,000} = -\frac{2,000}{10,000}$$

$= -\frac{1}{5}$ (lbs. per in.^{2}) per in.^{3}.

24. $$\lim_{\Delta x\to 0} \frac{\Delta C}{\Delta x} = \lim_{\Delta x\to 0} \frac{\frac{800}{(10 + \Delta x)^2} - \frac{800}{10^2}}{\Delta x}$$

$$= 800 \lim_{\Delta x\to 0} \frac{100 - 100 - 20\Delta x - \Delta x^2}{\Delta x(10 + \Delta x)^2(100)}$$

$$= 800 \lim_{\Delta x\to 0} \frac{-20 - \Delta x}{(10 + \Delta x)^2(100)}$$

$$= 800\left(\frac{-20}{10,000}\right) = -\frac{8}{5} = -1.6 \text{ (parts/ml)/km.}$$

25. $R = AP - B$

$R = 0.0044\,P - 10.4$

$$\lim_{\Delta P\to 0} \frac{\Delta R}{\Delta P}$$

$$= \lim_{\Delta P\to 0} \frac{0.0044(8,000+\Delta P) - 10.4 - 0.0044(8,000) + 10.4}{\Delta P}$$

$$= \lim_{\Delta P\to 0} \frac{0.0044\,\Delta P}{\Delta P} = 0.0044$$ (breath per min.)/(newton per sq. meter).

26. $$\lim_{\Delta t\to 0} \frac{\Delta p}{\Delta t} = \lim_{\Delta t\to 0} \frac{a - b\sin c(0+\Delta t) - [a - b\sin(c)(0)]}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{-b\sin(c\Delta t)}{\Delta t} = -b\lim_{\Delta t\to 0} \frac{\sin(c\Delta t)}{\Delta t}$$

$$= -bc \lim_{\Delta t\to 0} \frac{\sin c\Delta t}{c\Delta t} = (-bc)(1)$$

$= -bc$

$= -100(1.26)$

$= -126$ (N/m^2)/sec.

27. $$\lim_{\Delta R\to 0} \frac{\Delta I}{\Delta R} = \lim_{\Delta R\to 0} \frac{\frac{100}{10 + \Delta R} - \frac{100}{10}}{\Delta R}$$

$$= 100 \lim_{\Delta R\to 0} \frac{10 - (10 + \Delta R)}{\Delta R(10)(10 + \Delta R)}$$

$$= 100 \lim_{\Delta R\to 0} \frac{-1}{10(10 + \Delta R)} = 100\left(\frac{-1}{100}\right) = -1$$

amp per ohm.

Problem Set 2.2, page 97

1. $$f'(x) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{4(x + \Delta x) + 7 - 4x - 7}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{4\Delta x}{\Delta x} = 4.$$

2. $$f'(x) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{13 - 7(x + \Delta x) - 13 + 7x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \left(\frac{-7\Delta x}{\Delta x}\right) = -7.$$

3. $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\frac{5}{11} - \frac{5}{11}}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

4. $$f'(x) = \lim_{\Delta x \to 0} \frac{3 + \sqrt{x + \Delta x} - 3 - \sqrt{x}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$
$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$
$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

5. $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 + 4(x + \Delta x)\right] - \left[x^2 + 4x\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 4x + 4\Delta x - x^2 - 4x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + 4\Delta x + (\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + 4 + \Delta x) = 2x + 4.$$

6. $$f'(x) = \lim_{\Delta x \to 0} \frac{\left[2(x + \Delta x)^3 - 1\right] - \left[2x^3 - 1\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2(x^3 + 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3) - 1 - 2x^3 + 1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{6x^2\,\Delta x + 6x(\Delta x)^2 + 2(\Delta x)^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (6x^2 + 6x\Delta x + 2\Delta x^2) = 6x^2.$$

7. $$f'(x) = \lim_{\Delta x \to 0} \frac{\left[2(x + \Delta x)^3 - 4(x + \Delta x)\right] - \left[2x^3 - 4x\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(6x^2 - 4)\Delta x + 6x(\Delta x)^2 + 2(\Delta x)^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (6x^2 - 4 + 6x\Delta x + 2(\Delta x)^2) = 6x^2 - 4.$$

8. $$f'(x) = \lim_{\Delta x \to 0} \frac{\left[\frac{(x + \Delta x)^3}{2} + \frac{3}{2}(x + \Delta x)\right] - \left[\frac{x^3}{2} + \frac{3}{2}x\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\frac{1}{2}\left[x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 3x + 3\Delta x - x^3 - 3x\right]}{\Delta x}$$
$$= \frac{1}{2} \lim_{\Delta x \to 0} \frac{(3x^2 + 3)\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$
$$= \frac{1}{2} \lim_{\Delta x \to 0} (3x^2 + 3 + 3x\Delta x + (\Delta x)^2)$$
$$= \frac{1}{2}(3x^2 + 3) = \frac{3x^2}{2} + \frac{3}{2}.$$

9. $$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{2}{x + \Delta x} - \frac{2}{x}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[\frac{2}{\Delta x} \cdot \frac{x - (x + \Delta x)}{(x + \Delta x)\,x}\right]$$
$$= \lim_{\Delta x \to 0} \frac{-2}{(x + \Delta x)x} = \frac{-2}{x^2}.$$

10. $$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{-7}{(x + \Delta x) - 3} - \frac{-7}{x - 3}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-7}{\Delta x} \frac{(x - 3) - (x + \Delta x - 3)}{(x + \Delta x - 3)(x - 3)}$$
$$= \lim_{\Delta x \to 0} \frac{7}{(x + \Delta x - 3)(x - 3)} = \frac{7}{(x - 3)^2}.$$

11. $$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\frac{3}{(t + \Delta t) - 1} - \frac{3}{t - 1}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{3}{\Delta t} \cdot \frac{(t - 1) - (t + \Delta t - 1)}{(t + \Delta t - 1)(t - 1)}$$
$$= \lim_{\Delta t \to 0} \frac{-3}{(t + \Delta t - 1)(t - 1)}$$
$$= \frac{-3}{(t - 1)^2}.$$

12. $$D_t s = \lim_{\Delta t \to 0} \frac{\frac{t + \Delta t}{(t + \Delta t) + 1} - \frac{t}{t + 1}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot \frac{(t+1)(t+\Delta t) - t(t+\Delta t + 1)}{(t+\Delta t+1)(t+1)}$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot \frac{t^2 + t\Delta t + t + \Delta t - t^2 - t\Delta t - t}{(t + \Delta t + 1)(t + 1)}$$
$$= \lim_{\Delta t \to 0} \frac{1}{(t + \Delta t + 1)(t + 1)} = \frac{1}{(t + 1)^2}.$$

13. $$f'(v) = \lim_{\Delta v \to 0} \frac{\sqrt{(v + \Delta v) - 1} - \sqrt{v - 1}}{\Delta v}$$

$$= \lim_{\Delta v \to 0} \frac{\left[\sqrt{v+\Delta v-1} - \sqrt{v-1}\right]\left[\sqrt{v+\Delta v-1} + \sqrt{v-1}\right]}{\Delta v\left[\sqrt{v+\Delta v-1} + \sqrt{v-1}\right]}$$

$$= \lim_{\Delta v \to 0} \frac{(v+\Delta v-1)-(v-1)}{\Delta v\left[\sqrt{v+\Delta v-1} + \sqrt{v-1}\right]}$$

$$= \lim_{\Delta v \to 0} \frac{1}{\sqrt{v+\Delta v-1} + \sqrt{v-1}}$$

$$= \frac{1}{\sqrt{v-1}+\sqrt{v-1}} = \frac{1}{2\sqrt{v-1}}.$$

14. $$\frac{d}{du}\sqrt{1-9u^2} = \lim_{\Delta u \to 0} \frac{\sqrt{1-9(u+\Delta u)^2} - \sqrt{1-9u^2}}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\left[\sqrt{1-9(u+\Delta u)^2} - \sqrt{1-9u^2}\right]\left[\sqrt{1-9(u+\Delta u)^2} + \sqrt{1-9u^2}\right]}{\Delta u\left[\sqrt{1-9(u+\Delta u)^2} + \sqrt{1-9u^2}\right]}$$

$$= \lim_{\Delta u \to 0} \frac{\left[1-9(u+\Delta u)^2\right]-\left[1-9u^2\right]}{\Delta u\left[\sqrt{1-9(u+\Delta u)^2} + \sqrt{1-9u^2}\right]}$$

$$= \lim_{\Delta u \to 0} \frac{1-9u^2-18u\Delta u-9(\Delta u)^2-1+9u^2}{\Delta u\left[\sqrt{1-9(u+\Delta u)^2} + \sqrt{1-9u^2}\right]}$$

$$= \lim_{\Delta u \to 0} \frac{-18u-9\Delta u}{\sqrt{1-9(u+\Delta u)^2} + \sqrt{1-9u^2}}$$

$$= \frac{-18u}{2\sqrt{1-9u^2}} = \frac{-9u}{\sqrt{1-9u^2}}.$$

15. $$D_x y = \lim_{\Delta x \to 0} \frac{\frac{2}{(x+\Delta x)+1} - \frac{2}{x+1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2}{\Delta x} \cdot \frac{(x+1)-(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}$$

$$= \lim_{\Delta x \to 0} \frac{-2}{(x+\Delta x+1)(x+1)} = \frac{-2}{(x+1)^2}.$$

16. $$h'(t) = \lim_{\Delta t \to 0} \frac{\frac{1}{\sqrt{(t+\Delta t)+1}} - \frac{1}{\sqrt{t+1}}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\sqrt{t+1} - \sqrt{t+\Delta t+1}}{\sqrt{t+\Delta t+1}\sqrt{t+1}}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\left[\sqrt{t+1} - \sqrt{t+\Delta t+1}\right]\left[\sqrt{t+1} + \sqrt{t+\Delta t+1}\right]}{\sqrt{t+\Delta t+1}\sqrt{t+1}\left[\sqrt{t+1} + \sqrt{t+\Delta t+1}\right]}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{(t+1)-(t+\Delta t+1)}{\sqrt{t+\Delta t+1}\sqrt{t+1}\left[\sqrt{t+1} + \sqrt{t+\Delta t+1}\right]}$$

$$= \lim_{\Delta t \to 0} \frac{-1}{\sqrt{t+\Delta t+1}\sqrt{t+1}\left[\sqrt{t+1} + \sqrt{t+\Delta t+1}\right]}$$

$$= \frac{-1}{\sqrt{t+1}\sqrt{t+1}\left[\sqrt{t+1} + \sqrt{t+1}\right]}$$

$$= \frac{-1}{2(t+1)\sqrt{t+1}} = -\frac{1}{2}(t+1)^{-3/2}$$

17. $$f'(-1) = \lim_{\Delta x \to 0} \frac{\left[1-2(-1+\Delta x)^2\right]-\left[1-2(-1)^2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1-2+4\Delta x-2(\Delta x)^2+1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (4-2\Delta x) = 4.$$

18. $$f'(0) = \lim_{\Delta x \to 0} \frac{\sin(0+\Delta x) - \sin 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1.$$

19. $$f'(3) = \lim_{\Delta x \to 0} \frac{\frac{7}{2(3+\Delta x)-1} - \frac{7}{2(3)-1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{7}{\Delta x} \cdot \frac{5-(5+2\Delta x)}{5(5+2\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{-14}{5(5+2\Delta x)} = \frac{-14}{25}.$$

20. $$f'(0) = \lim_{\Delta x \to 0} \frac{\cos(0+\Delta x) - \cos 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x}$$

$$= -\lim_{\Delta x \to 0} \frac{1-\cos \Delta x}{\Delta x} = 0.$$

21. $$f'(4) = \lim_{\Delta x \to 0} \frac{\frac{1}{(4+\Delta x)-1} - \frac{1}{4-1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \frac{3-(3+\Delta x)}{3(3+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{3(3+\Delta x)} = -\frac{1}{9}.$$

22. $$(D_t s)_{t=3} = \lim_{\Delta t \to 0} \frac{\sqrt{2(3+\Delta t)+3} - \sqrt{2(3)+3}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[\sqrt{9+2\Delta t} - \sqrt{9}\right]\left[\sqrt{9+2\Delta t} + \sqrt{9}\right]}{\Delta t\left[\sqrt{9+2\Delta t} + \sqrt{9}\right]}$$

$$= \lim_{\Delta t \to 0} \frac{(9+2\Delta t)-9}{\Delta t\left[\sqrt{9+2\Delta t} + \sqrt{9}\right]}$$

$$= \lim_{\Delta t \to 0} \frac{2}{\sqrt{9+2\Delta t}+3}$$

$$= \frac{2}{3+3} = \frac{1}{3}.$$

23. $\left(\frac{dy}{dx}\right)_{x=2} = \lim\limits_{\Delta x \to 0} \dfrac{\dfrac{2}{2(2 + \Delta x) + 1} - \dfrac{2}{2(2) + 1}}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{2}{\Delta x} \cdot \dfrac{5 - (5 + 2\Delta x)}{5(5 + 2\Delta x)}$

$= \lim\limits_{\Delta x \to 0} \dfrac{-4}{5(5 + 2\Delta x)} = -\dfrac{4}{25}$.

24. $D_I P = D_I(I^2 R) = \lim\limits_{\Delta I \to 0} \dfrac{(I + \Delta I)^2 R - I^2 R}{\Delta I}$

$= \lim\limits_{\Delta I \to 0} \dfrac{I^2 R + 2IR\Delta I + (\Delta I)^2 R - I^2 R}{\Delta I}$

$= \lim\limits_{\Delta I \to 0} \dfrac{2IR\Delta I + (\Delta I)^2 R}{\Delta I}$

$= \lim\limits_{\Delta I \to 0} (2IR + \Delta I R) = 2IR$

25. (#1) $\frac{d}{dx}(4x + 7) = D_x(4x + 7) = 4$.

(#3) $\frac{d}{dx}\left(\frac{5}{11}\right) = D_x\left(\frac{5}{11}\right) = 0$.

(#5) $\frac{d}{dx}(x^2 + 4x) = D_x(x^2 + 4x) = 2x + 4$.

(#7) $\frac{d}{dx}(2x^3 - 4x) = D_x(2x^3 - 4x) = 6x^2 - 4$.

(#9) $\frac{d}{dx}\left(\frac{2}{x}\right) = D_x\left(\frac{2}{x}\right) = -\frac{2}{x^2}$.

(#11) $\frac{ds}{dt} = D_t s = \dfrac{-3}{(t - 1)^2}$.

(#13) $\frac{d}{dv}(\sqrt{v - 1}) = D_v(\sqrt{v - 1}) = \dfrac{1}{2\sqrt{v - 1}}$.

(#15) $\frac{dy}{dx} = D_x y = \dfrac{-2}{(x + 1)^2}$.

26. $f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x) = \frac{df(x)}{dx} = \frac{df}{dx}$

$= \frac{d}{dx}f = D_x y = D_x(f(x)) = D_x f$.

27. $D_t s = \lim\limits_{\Delta s \to 0} \dfrac{[16(t+\Delta s)^2 + 30(t+\Delta s) + 10] - [16t^2 + 30t + 10]}{\Delta s}$

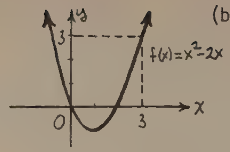
$= \lim\limits_{\Delta s \to 0} \dfrac{(32t + 30)\Delta s + 16(\Delta s)^2}{\Delta s}$

$= \lim\limits_{\Delta s \to 0} 32t + 30 + 16\Delta s$

$= 32t + 30$.

28. This problem is formally the same as Problem 23; only the symbols for the variables have been changed. Therefore, $\frac{du}{dv} = 32v + 30$.

29. (a)



(b) Since f is a polynomial function, it is continuous at any number, in particular, at 3.

(c) $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{[(x + \Delta x)^2 - 2(x + \Delta x)] - [x^2 - 2x]}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{(2x - 2)\Delta x + (\Delta x)^2}{\Delta x}$

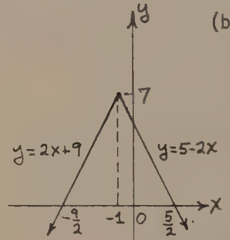
$= \lim\limits_{\Delta x \to 0} (2x - 2 + \Delta x) = 2x - 2$.

Hence, $f'_+(3) = f'_-(3) = f'(3)$

$= 2(3) - 2 = 4$.

Thus, f is differentiable at 3.

30. (a)



(b) $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+}(5-2x) = 7$.

$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-}(2x+9) = 7$.

Hence, $\lim\limits_{x \to -1} f(x) = 7 = f(-1)$, so f is continuous at -1.

(c) $f'_+(-1) = \lim\limits_{\Delta x \to 0^+} \dfrac{f(-1 + \Delta x) - f(-1)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0^+} \dfrac{[5 - 2(-1 + \Delta x)] - 7}{\Delta x}$

$= \lim\limits_{\Delta x \to 0^+} \dfrac{-2\Delta x}{\Delta x} = -2$.

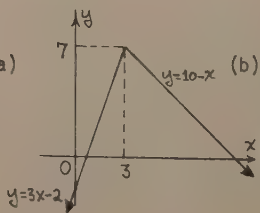
$f'_-(-1) = \lim\limits_{\Delta x \to 0^-} \dfrac{f(-1 + \Delta x) - f(-1)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0^-} \dfrac{[2(-1 + \Delta x) + 9] - 7}{\Delta x}$

$= \lim\limits_{\Delta x \to 0^-} \dfrac{2\Delta x}{\Delta x} = 2$.

Since $f'_+(-1) \neq f'_-(-1)$, then f is not differentiable at -1.

31. (a)



(b) $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+}(10-x) = 7$.

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-}(3x-2) = 7$.

Hence, $\lim\limits_{x \to 3} f(x) = 7 = f(3)$, so f is continuous at 3.

(c) $f'_+(3) = \lim_{\Delta x \to 0^+} \dfrac{f(3 + \Delta x) - f(3)}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \dfrac{[10 - (3 + \Delta x)] - 7}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \dfrac{-\Delta x}{\Delta x} = -1.$

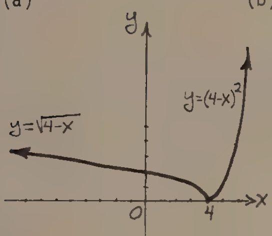
$f'_-(3) = \lim_{\Delta x \to 0^-} \dfrac{f(3 + \Delta x) - f(3)}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \dfrac{[3(3 + \Delta x) - 2] - 7}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \dfrac{3\Delta x}{\Delta x} = 3.$

Since $f'_+(3) \neq f'_-(3)$, then f is not differentiable at 3.

32. (a)



(b) $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (4-x)^2 = 0.$

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \sqrt{4-x} = 0.$

Hence, $\lim_{x \to 4} f(x) = 0 = f(0)$, so f is continuous at 4.

(c) $f'_+(4) = \lim_{\Delta x \to 0^+} \dfrac{f(4 + \Delta x) - f(4)}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \dfrac{(4 - (4 + \Delta x))^2 - 0}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \dfrac{\Delta x^2}{\Delta x} = \lim_{\Delta x \to 0^+} \Delta x = 0.$

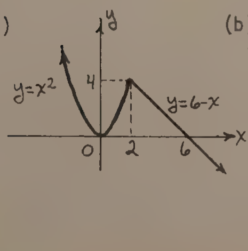
$f'_-(4) = \lim_{\Delta x \to 0^-} \dfrac{f(4 + \Delta x) - f(4)}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \dfrac{\sqrt{-\Delta x} - 0}{\Delta x} = \lim_{\Delta x \to 0^-} \dfrac{-1}{\sqrt{-\Delta x}}$

$= -\infty.$

Since $f'_+(4) \neq f'_-(4)$, then f is not differentiable at 4.

33. (a)



(b) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4.$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (6-x) = 4.$

Hence, $\lim_{x \to 2} f(x) = 4 = f(2)$, so f is continuous at 2.

(c) $f'_+(2) = \lim_{\Delta x \to 0^+} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \dfrac{[6 - (2 + \Delta x)] - 4}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \dfrac{-\Delta x}{\Delta x} = -1.$

$f'_-(2) = \lim_{\Delta x \to 0^-} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$

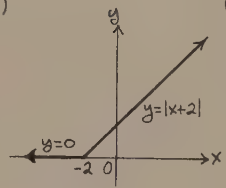
$= \lim_{\Delta x \to 0^-} \dfrac{(2 + \Delta x)^2 - 4}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \dfrac{4\Delta x + (\Delta x)^2}{\Delta x}$

$= \lim_{\Delta x \to 0^-} (4 + \Delta x) = 4.$

Since $f'_+(2) \neq f'_-(2)$, then f is not differentiable at 2.

34. (a)



(b) $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} 0 = 0.$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} |x + 2|$

$= \lim_{x \to -2^+} (x+2) = -2 + 2 = 0.$

Hence, $\lim_{x \to -2} f(x) = 0 = f(-2)$, so f is continuous at -2.

(c) $f'_+(-2) = \lim_{\Delta x \to 0^+} \dfrac{f(-2 + \Delta x) - f(-2)}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \dfrac{[(-2 + \Delta x) + 2] - 0}{\Delta x}$

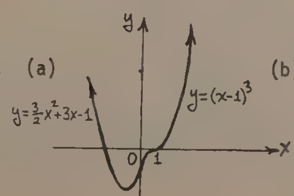
$= \lim_{\Delta x \to 0^+} \dfrac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0^+} 1 = 1.$

$f'_-(-2) = \lim_{\Delta x \to 0^-} \dfrac{f(-2 + \Delta x) - f(-2)}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \dfrac{0 - 0}{\Delta x} = \lim_{\Delta x \to 0^-} 0 = 0.$

Since $f'_+(-2) \neq f'_-(-2)$, then f is not differentiable at -2.

35. (a)



(b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (\frac{3}{2}x^2 + 3x-1) = -1.$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x-1)^3 =$

$(-1)^3 = -1$. Hence,

$\lim_{x\to 0} f(x) = -1 = f(0)$,

so f is continuous at 0.

(c) $$f'_+(0) = \lim_{\Delta x\to 0^+} \frac{f(0 + \Delta x) - f(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^+} \frac{(\Delta x - 1)^3 - (-1)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^+} \frac{(\Delta x)^3 + 3(\Delta x)^2(-1) + 3\Delta x(-1)^2}{\Delta x}$$

$$= \lim_{\Delta x\to 0^+} (\Delta x^2 + 3\Delta x(-1) + 3(-1)^2) = 3.$$

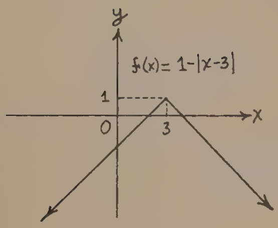
$$f'_-(0) = \lim_{\Delta x\to 0^-} \frac{f(0 + \Delta x) - f(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-} \frac{(\frac{3}{2}(\Delta x)^2 + 3\Delta x - 1) - (-1)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-} (\tfrac{3}{2}\Delta x + 3) = 3.$$

Hence, $f'_-(0) = f'_+(0) = f'(0) = 3$, and so f is differentiable at 0.

36. (a)



(b) $$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 1-|x-3|$$

$$= \lim_{x\to 3^+} [1-(x-3)]$$

$$= \lim_{x\to 3^+} (4 - x)$$

$$= 4 - 3 = 1.$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 1-|x-3|$$

$$= \lim_{x\to 3^-} [1+(x-3)]$$

$$= \lim_{x\to 3^-} (x - 2)$$

$$= 3 - 2 = 1.$$

Hence, $\lim_{x\to 3} f(x) = 1 = f(3)$, and f is continuous at 3.

(c) $$f'_+(3) = \lim_{\Delta x\to 0^+} \frac{f(3 + \Delta x) - f(3)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^+} \frac{[1 - |\Delta x|] - 1}{\Delta x}$$

$$= \lim_{\Delta x\to 0^+} -\frac{|\Delta x|}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x\to 0^+} -1 = -1.$$

$$f'_-(3) = \lim_{\Delta x\to 0^-} \frac{f(3 + \Delta x) - f(3)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-} \frac{[1 - |\Delta x|] - 1}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-} -\frac{|\Delta x|}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-(-\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-} (1) = 1.$$

Since $f'_+(3) \neq f'_-(3)$, then f is not differentiable at 3.

37. (a) If f is differentiable at x_1, then the graph of f has a tangent line at $(x_1, f(x_1))$. If the graph has a tangent line at $(x_1, f(x_1))$, then it cannot "jump" at this point, hence, f must be continuous at x_1.

(b) No. A function whose graph has a "corner" (such as the absolute-value function), can be continuous, but will not have a tangent line at the "corner".

38. $$f'_+(-1) = \lim_{\Delta x\to 0^+} \frac{f(-1 + \Delta x) - f(-1)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^+} \frac{[a(-1 + \Delta x) + b] - [a(-1) + b]}{\Delta x}$$

$$= \lim_{\Delta x\to 0^+} \frac{a\Delta x}{\Delta x} = a.$$

In order for $f'(-1)$ to exist, we must have $f'_-(-1) = f'_+(-1) = a$. This requires that

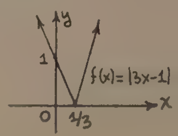
$$a = \lim_{\Delta x\to 0^-} \frac{f(-1 + \Delta x) - f(-1)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-} \frac{[(-1 + \Delta x)^2] - [a(-1) + b]}{\Delta x}$$

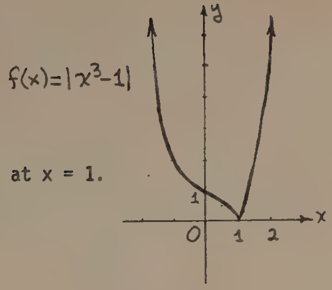
$$= \lim_{\Delta x\to 0^-} \frac{1 + a - b}{\Delta x} + 2 + \Delta x).$$

The latter limit will not be finite unless $1 + a - b = 0$. Then we will have $a = \lim_{\Delta x\to 0^-} (2 + \Delta x) = 2$, and so $1 + 2 - b = 0$ implies $b = 3$. Hence, the solution is $a = 2$, $b = 3$.

39. Not differentiable at $x = \frac{1}{3}$.



40. Not differentiable at $x = 1$.



41. Let $\Delta x = \frac{\pi}{3}(10^{-4})$.

$$f'(\tfrac{\pi}{3}) \approx \frac{\cos(\frac{\pi}{3} + \Delta x) - \cos\frac{\pi}{3}}{\Delta x}$$
$$\approx -0.8660514906.$$

42. Let $\Delta x = \frac{\pi}{4}(10^{-4})$.

$$f'(\tfrac{\pi}{4}) \approx \frac{\tan(\frac{\pi}{4} + \Delta x) - \tan(\frac{\pi}{4})}{\Delta x}$$
$$\approx 2.000157465.$$

43. Let $\Delta x = 10^{-3}$.

$$f'(10) \approx \frac{\sqrt{(10 + \Delta x) - 1} - \sqrt{9}}{\Delta x}$$
$$\approx 0.166662000.$$

44. Let $\Delta x = \frac{\pi}{6}(10^{-4})$.

$$f'(\tfrac{\pi}{6}) \approx \frac{\sqrt{\sin(\frac{\pi}{6} + \Delta x)} - \sqrt{\sin\frac{\pi}{6}}}{\Delta x}$$
$$\approx 0.6123505533.$$

Problem Set 2.3, page 108

1. $f'(x) = 6x$.
2. $g'(x) = 3x^6$.
3. $h'(x) = -20x^3$.
4. $G'(t) = 8t^{10}$.
5. $F'(y) = -4y^3$.
6. $H'(v) = -\frac{v^5}{5}$.
7. $H'(t) = 5$.
8. $G'(w) = \sqrt{3}(-8) = -8\sqrt{3}$.
9. $f'(x) = 5x^4 - 9x^2$.
10. $f'(x) = 5x^5 - 36x^3$.
11. $f'(x) = \frac{1}{2}(10x^9) + \frac{1}{5}(5x^4) = 5x^9 + x^4$.
12. $F'(x) = x^3 - x^2$.
13. $f'(t) = 8t^7 - 14t^6 + 3$.
14. $f'(t) = 6t + 7$.
15. $F'(x) = 3(-2x^{-3}) + \frac{4}{3}(-x^{-2}) = -6x^{-3} - \frac{4}{3}x^{-2} = -\frac{6}{x^3} - \frac{4}{3x^2}$.
16. $f'(t) = \frac{1}{3}(-3t^{-4}) - \frac{1}{2}(-2t^{-3}) = -t^{-4} + t^{-3} = \frac{1}{t^3} - \frac{1}{t^4}$.
17. $f'(y) = 5(-5y^{-6}) - 25(-y^{-2}) = \frac{25}{y^2} - \frac{25}{y^6}$.
18. $f'(u) = -\frac{1}{u^2} - 3(-3u^{-4}) + \frac{1}{2\sqrt{u}} = -\frac{1}{u^2} + \frac{9}{u^4} + \frac{1}{2\sqrt{u}}$.
19. $g'(x) = -6x^{-3} + 7x^{-2} + \frac{6}{2\sqrt{x}} = -\frac{6}{x^3} + \frac{7}{x^2} + \frac{3}{\sqrt{x}}$.
20. $G'(x) = -x^{-4} + x^{-3} - \frac{11}{2x\sqrt{x}} = -\frac{1}{x^4} + \frac{1}{x^3} - \frac{11}{2x\sqrt{x}}$.
21. $f'(x) = \frac{-2}{5x^2} + \frac{2\sqrt{2}}{3x^3} - \frac{1}{2x\sqrt{x}}$.
22. $f'(x) = \frac{1}{2\sqrt{x}}(x^3 - x) + \sqrt{x}(3x^2 - 1) = \frac{x^{5/2}}{2} - \frac{\sqrt{x}}{2} + 3x^{5/2}$
$- \sqrt{x} = \frac{7x^{5/2}}{2} - \frac{3\sqrt{x}}{2}$.
23. $F'(x) = x^2(9x^2) + (2x)(3x^3 - 1) = 15x^4 - 2x$.
24. $f'(x) = (x^2 + 1)(6x^2) + (2x)(2x^3 + 5) = 10x^4 + 6x^2 + 10x$.
25. $G'(x) = (x^2 + 3x)(3x^2 - 9) + (2x + 3)(x^3 - 9x)$
$= 5x^4 + 12x^3 - 27x^2 - 54x$.
26. $g'(x) = (3 - 2x)(3x^3 - 4) + (3x - x^2)(9x^2)$
$= -15x^4 + 36x^3 + 8x - 12$.
27. $f'(y) = \sqrt{y}(8y) + \frac{1}{2\sqrt{y}}(4y^2 + 7) = 8y^{3/2} + 2y^{3/2}$
$+ \frac{7}{2}y^{-1/2} = 10y^{3/2} + \frac{7}{2\sqrt{y}}$.
28. $f(t) = (6t^2 + 7)(6t^2 + 7)$ so $f'(t) = (6t^2 + 7)(12t) + (12t)$
$(6t^2 + 7) = 144t^3 + 168t$.
29. $f'(x) = (x^3 - 8)(\frac{-2}{x^2}) + (3x^2)(\frac{2}{x} - 1) = \frac{16}{x^2} + 4x - 3x^2$.
30. $f'(x) = (\frac{1}{x} + 3)(\frac{-2}{x^2}) + (\frac{-1}{x^2})(\frac{2}{x} + 7) = -\frac{4}{x^3} - \frac{13}{x^2}$.
31. $g'(x) = (\frac{1}{x^2} + 3)(-6x^{-4} + 1) + (-2x^{-3})(\frac{2}{x^3} + x) =$
$-10x^{-6} - 18x^{-4} - x^{-2} + 3$.
32. $g'(u) = (u^2 + \frac{1}{u})(1 + 3u^{-4}) + (2u - \frac{1}{u^2})(u - \frac{1}{u^3}) =$
$4u^{-5} + u^{-2} + 3u^2$.

33. $f'(x) = \dfrac{(3x - 1)(2) - (2x + 7)(3)}{(3x - 1)^2} = \dfrac{-23}{(3x - 1)^2}$.

34. $f'(x) = \dfrac{(x - 2)(6x) - 3x^2(1)}{(x - 2)^2} = \dfrac{3x^2 - 12x}{(x - 2)^2}$.

35. $g'(x) = \dfrac{(x^2 - 3x + 2)(4x + 1) - (2x^2 + x + 1)(2x - 3)}{(x^2 - 3x + 2)^2}$

$= \dfrac{-7x^2 + 6x + 5}{(x^2 - 3x + 2)^2}$.

36. $G'(t) = \dfrac{(2t^4 + 5)(3t^2) - t^3(8t^3)}{(2t^4 + 5)^2} = \dfrac{15t^2 - 2t^6}{(2t^4 + 5)^2}$.

37. $F'(t) = \dfrac{(t^2 - 1)(6t) - (3t^2 + 7)(2t)}{(t^2 - 1)^2} = \dfrac{-20t}{(t^2 - 1)^2}$.

38. $f'(x) = \dfrac{(x^2 + 19)(2x) - (x^2 - 19)(2x)}{(x^2 + 19)^2} = \dfrac{76x}{(x^2 + 19)^2}$.

39. $f'(x) = \dfrac{3x + 1}{x + 2}(1) + \dfrac{(x + 2)(3) - (3x + 1)(1)}{(x + 2)^2}(x + 7)$

$= \dfrac{(3x + 1)(x + 2)}{(x + 2)^2} + \dfrac{5x + 35}{(x + 2)^2} = \dfrac{3x^2 + 12x + 37}{(x + 2)^2}$.

40. $p'(x) = \dfrac{\sqrt{x}(1) - (x + 1)\frac{1}{2\sqrt{x}}}{x} = \dfrac{x - (x + 1)\cdot\frac{1}{2}}{x^{3/2}} =$

$\dfrac{2x - x - 1}{x^{3/2}} = \dfrac{x - 1}{x^{3/2}}$.

41. (a) $f'(x) = x^2$, so $f'(2) = 4$.

(b) $f'(x) = -3x^{-4}$, so $f'(2) = -\dfrac{3}{16}$.

(c) $f'(x) = (x^2 + 1)(-1) + 2x(1 - x)$, so $f'(2) = -9$.

(d) $f'(x) = (\frac{1}{x} + 2)(-3x^{-2}) + (-x^{-2})(\frac{3}{x} - 1)$, so

$f'(2) = -2$.

(e) $f'(x) = \dfrac{(x^2 + 2)(1) - x(2x)}{(x^2 + 2)^2}$, so $f'(2) = -\dfrac{1}{18}$.

(f) $f'(x) = \dfrac{(x + 7)(4x) - 2x^2(1)}{(x + 7)^2}$, so $f'(2) = \dfrac{64}{81}$.

42. $k = f\cdot g\cdot h = (f \cdot g)\cdot h$, so $k' = (f\cdot g)\cdot h' +$

$(f \cdot g)' \cdot h$

$= (f \cdot g) \cdot h' + (f \cdot g' + f' \cdot g)h$

$= f\cdot g\cdot h' + f\cdot g'\cdot h + f'\cdot g\cdot h$.

43. (a) $f'(x) = (2x - 5)(x + 2)(2x) + (2x - 5)(1)$

$(x^2 - 1) + 2(x + 2)(x^2 - 1)$

$= 8x^3 - 3x^2 - 24x + 1$.

(b) $f'(x) = (1 - 3x)^2(2) + (1 - 3x)(-3)(2x + 5) +$

$(-3)(1 - 3x)(2x + 5) = 54x^2 + 66x - 28$

(c) $f'(x) = (\frac{1}{x^2} + 1)(3x - 1)(2x - 3) + (\frac{1}{x^2} + 1)(3$

$(x^2 - 3x) + (\frac{-2}{x^3})(3x - 1)(x^2 - 3x)$

$= 9x^2 - 20x + 6 - 3x^{-2}$.

(d) $f'(x) = (2x^2 + 7)^2(4x) + (2x^2 + 7)(4x)(2x^2 + 7)$

$(4x)(2x^2 + 7)^2$

$= 12x(2x^2 + 7)^2$.

44. $h'(x) = D_x(h(x)) = D_x(f(x) - g(x)) = D_x(f(x) + (-1$

$g(x))$

$= D_x(f(x)) + D_x((-1)g(x)) = D_x(f(x)) + (-1)$

$D_x(g(x))$

$= D_x(f(x)) - D_x(g(x)) = f'(x) - g'(x)$.

45. (a) $(f + g)'(1) = (f' + g')(1) = f'(1) + g'(1)$

$= 2 - 3 = -1$.

(b) $(f - g)'(1) = (f' - g')(1) = f'(1) - g'(1)$

$= 2 + 3 = 5$.

(c) $(2f + 3g)'(1) = 2f'(1) + 3g'(1) = 4 - 9 = -5$.

(d) $(fg)'(1) = (fg' + f'g)(1) = f(1)g'(1) + f'(1)$

$g(1) = (1)(-3) + (2)(\frac{1}{2}) = -2$.

(e) $(\frac{f}{g})'(1) = (\dfrac{gf' - fg'}{g^2})(1) = \dfrac{g(1)f'(1) - f(1)g'(1}{(g(1))^2}$

$= \dfrac{(\frac{1}{2})(2) - (1)(-3)}{(\frac{1}{2})^2} = 16$.

(f) $(\frac{g}{f})'(1) = (\dfrac{fg' - gf'}{f^2})(1) = \dfrac{f(1)g'(1) - g(1)f'(1}{(f(1))^2}$

$= \dfrac{(1)(-3) - (\frac{1}{2})(2)}{1^2} = -4$.

46. (a) $(f + g + h)'(2) = (f' + g' + h')(2) = f'(2)$

$+ g'(2) + h'(2) = 3 + 1 + 4 = 8$.

(b) $(2f - g + 3h)'(2) = (2f' - g' + 3h')(2)$

$= 2f'(2) - g'(2) + 3h'(2) = 2(3) - 1 + 3(4) = 1$

(c) $(fgh)'(2) = (fgh' + fg'h + f'gh)(2)$

$= f(2)g(2)h'(2) + f(2)g'(2)h(2)$

$+ f'(2)g(2)h(2)$

$= (-2)(-5)(4) + (-2)(1)(2) + (3)(-5)(2$

$= 6$.

(d) $\left(\frac{fg}{h}\right)'(2) = \left(\frac{h(fg)' - (fg)h'}{h^2}\right)(2)$

$= \left(\frac{h(fg' + f'g) - fgh'}{h^2}\right)(2)$

$= \left(\frac{hfg' + hf'g - fgh'}{h^2}\right)(2)$

$= \frac{h(2)f(2)g'(2)+h(2)f'(2)g(2)-f(2)g(2)h'(2)}{(h(2))^2}$

$= \frac{(2)(-2)(1)+(2)(3)(-5)-(-2)(-5)(4)}{2^2} = -\frac{37}{2}$.

47. (a) $f'(x) = 3x^2 - 8x$, so that $f'(4) = 16$.

(b) $f'(x) = \frac{-12}{(4x - 2)^2}$, so that $f'(4) = -\frac{3}{49}$.

48. (a) $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = \frac{4}{3}\pi\frac{d}{dr}(r^3) = \frac{4}{3}\pi(3r^2) = 4\pi r^2$.

(b) $V = h\pi r^2$, so $\frac{dV}{dr} = h\pi\frac{d}{dr}(r^2) = h\pi(2r) = 2\pi hr$.

49. $f'(x) = \frac{(x^3 - 2)(1) - x(3x^2)}{(x^3 - 2)^2} = \frac{-2 - 2x^3}{(x^3 - 2)^2}$, so

$f'(1) = -4$.

The slope of the tangent line at $(1,-1)$ is -4.

50. (a) $\frac{1}{y} = \frac{1}{p} - \frac{1}{x} = \frac{x - p}{px}$, so $y = \frac{px}{x - p}$.

(b) $\frac{dy}{dx} = \frac{(x - p)\frac{d}{dx}(px) - px\frac{d}{dx}(x - p)}{(x - p)^2}$

$= \frac{(x - p)p\frac{dx}{dx} - px\left(\frac{dx}{dx} - \frac{dp}{dx}\right)}{(x - p)^2}$

$= \frac{(x - p)p - px(1 - 0)}{(x - p)^2}$

$= \frac{px - p^2 - px}{(x - p)^2} = \frac{-p^2}{(x - p)^2} = -\left(\frac{p}{x - p}\right)^2$.

51. What we want is $f'(2)$. Now, $f'(x) = 4x + 3$, so $f'(2) = 4(2) + 3 = 11$. One must first calculate the derivative f' of f, then evaluate this derivative at $x = 2$ to get $f'(2)$.

52. $D_x\left(\frac{1}{g(x)}\right) = \frac{g(x)D_x(1) - 1D_x(g(x))}{(g(x))^2}$

$= \frac{g(x)\cdot 0 - D_x(g(x))}{(g(x))^2} = -\frac{D_x(g(x))}{(g(x))^2}$.

53. speed $= \frac{ds}{dt} = \frac{d}{dt}\left(8t + \frac{2}{t}\right) = 8 - \frac{2}{t^2}$. When $t = 2$, $\frac{ds}{dt} = 8 - \frac{2}{2^2} = 7.5$ feet per second.

54. $\frac{dP}{dR} = 100\left[R(-2)(0.5 + R)^{-3} + (0.5 + R)^{-2}\right]$.

When $R = 10$,

$\frac{dP}{dR} = 100\left[-20(10.5)^{-3} + (10.5)^{-2}\right]$

$= 100(10.5)^{-2}\left[-20(10.5)^{-1} + 1\right] \approx -0.8206457189$ watt per ohm.

55. $\frac{dN}{dt} = (3t + 150)(-1) + (3)(50 - t)$

$= -6t$.

When $t = 20$,

$\frac{dN}{dt} = -6(20) = -120$ per year.

56. $\frac{dC}{dx} = 200{,}000 - 32{,}000x$

When $x = 4.5$, then $\frac{dC}{dx} = 200{,}000 - 32{,}000(4.5)$

$= 56{,}000$ dollars per million gallons.

57. $A = 14.4\left(\frac{t + 8}{(t + 8)^2}\right) = \frac{14.4}{t + 8}$

$\frac{dA}{dt} = -\frac{14.4}{(t + 8)^2}$,

When $t = 4$, $\frac{dA}{dt} = -\frac{14.4}{144} = -0.1$ (mole/m^3)/day.

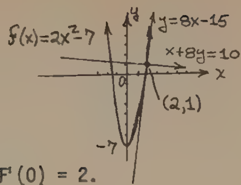
Problem Set 2.4, page 113

1. $f'(x) = 4x$, so that $f'(2) = 8$.

The equation of the tangent line is

$y - 1 = 8(x - 2)$ or $y = 8x - 15$.

The equation of the normal line is $y - 1 = -\frac{1}{8}(x - 2)$ or $x + 8y - 10 = 0$.



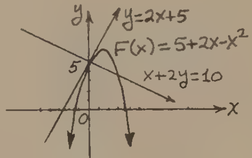
2. $F'(x) = 2 - 2x$, so that $F'(0) = 2$.

The equation of the tangent line is

$y - 5 = 2x$ or $y = 2x + 5$.

The equation of the normal line is $y - 5 = -\frac{1}{2}x$ or

$x + 2y - 10 = 0$.



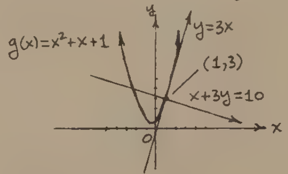
3. $g'(x) = 2x + 1$, so that $g'(1) = 3$.

The equation of the tangent line is

$y - 3 = 3(x - 1)$ or $y = 3x$.

The equation of the normal line is $y - 3 = -\frac{1}{3}(x-1)$

or $3y + x - 10 = 0$.



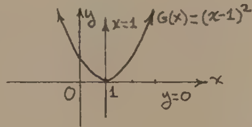
4. $G(x) = x^2 - 2x + 1$, $G'(x) = 2x - 2$; so

$G'(1) = 2 - 2 = 0$.

The equation of the tangent line is $y - 0 = 0(x-1)$

or $y = 0$.

The equation of the normal line is $x = 1$.



5. $h(x) = \frac{2}{x - 1}$, $h'(x) = -\frac{2}{(x - 1)^2}$;

so $h'(3) = -\frac{2}{(3 - 1)^2} = -\frac{1}{2}$

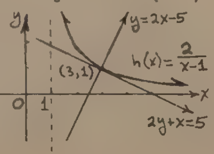
The equation of the tangent line is $y - 1 = -\frac{1}{2}(x-3)$

or $2y + x - 5 = 0$.

The equation of the normal line is $y - 1 = 2(x-3)$

or $y = 2x - 5$.

(Figure shows only one branch of curve.)



6. $H(x) = x\sqrt{x}$ and $H'(x) = \sqrt{x} + x\left(\frac{1}{2\sqrt{x}}\right) = \frac{3}{2}\sqrt{x}$.

Thus, $H'(9) = \frac{3}{2}\sqrt{9} = \frac{9}{2}$.

Thus, the equation of the tangent line is $y - 27 = \frac{9}{2}(x - 9)$ or $2y - 9x + 27 = 0$.

Thus, the equation of the normal line is $y - 27 = -\frac{2}{9}(x - 9)$ or $9y + 2x - 261 = 0$.

7. $F'(x) = 3x^2 - 16x + 9$, so that $F'(4) = -7$.

The equation of the tangent line is $y + 8 = -7(x-4)$ or $7x + y - 20 = 0$.

The equation of the normal line is $y + 8 = \frac{1}{7}(x-4)$ or $x - 7y - 60 = 0$.

8. $P(x) = x + \frac{2}{\sqrt{x}}$.

$P'(x) = 1 - \frac{2}{2x\sqrt{x}} = 1 - \frac{1}{x\sqrt{x}}$, so $P'(4) = 1 - \frac{1}{4\sqrt{4}} = 1 - \frac{1}{8} = \frac{7}{8}$.

The equation of the tangent line is $y - 5 = \frac{7}{8}(x-4)$ or $7x - 8y + 12 = 0$.

The equation of the normal line is $y - 5 = -\frac{8}{7}(x-4)$ or $8x + 7y - 67 = 0$.

9. $q'(x) = 16x^3 - 12x^2 - 50x + 1$, so that $q'(3) = 175$.

The equation of the tangent line is $y - 0 = 175(x-3)$ or $y = 175x - 525$.

The equation of the normal line is $y = -\frac{1}{175}(x-3)$ or $175y + x - 3 = 0$.

10. $Q'(x) = \frac{(x^2 + 1)(0) - 1(2x)}{(x^2 + 1)^2} = \frac{-2x}{(x^2 + 1)^2}$,

so $Q'(0) = \frac{-2.0}{(0^2 + 1)^2} = 0$.

The equation of the tangent line is $y - 1 = 0(x-0)$ or $y = 1$.

The equation of the normal line is $x = 0$.

11. $r'(x) = \frac{(x - 1) \cdot 1 - (x + 1) \cdot 1}{(x - 1)^2} = \frac{-2}{(x - 1)^2}$

so $r'(2) = -2$.

The equation of the tangent line is $y - 3 = -2(x-2)$ or $y = -2x + 7$.

The equation of the normal line is $y - 3 = \frac{1}{2}(x-2)$ or $x - 2y + 4 = 0$.

12. $R'(x) = \dfrac{(3 - x + x^2)(1 + 2x)-(1+x+x^2)(-1 + 2x)}{(3 - x + x^2)^2}$,

so $R'(1) = \dfrac{3(3) - 3(1)}{3^2} = \dfrac{9 - 3}{9} = \dfrac{6}{9} = \dfrac{2}{3}$.

The equation of the tangent line is $y - 1 = \frac{2}{3}(x-1)$ or $2x - 3y + 1 = 0$.

The equation of the normal line is $y - 1 = -\frac{3}{2}(x-1)$ or $3x + 2y - 5 = 0$.

13. $s'(x) = \dfrac{(\sqrt{x} - 1) \cdot 1 - x\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x} - 1)^2}$,

so that $s'(4) = \dfrac{1.1 - 4 \cdot \frac{1}{4}}{1^2} = 1 - 1 = 0$.

The equation of the tangent line is $y - 4 = 0(x-4)$ or $y = 4$.

The equation of the normal line is $x = 4$.

14. $S'(x) = 2ax + b$, so that $S'(0) = b$.

The equation of the tangent line is $y - c = bx$ or $y = bx + c$.

The equation of the normal line is $y - c = -\frac{1}{b}x$ or $x + by - bc = 0$.

15. $f'(x) = \frac{1}{\sqrt{x}}$, so that $f'(1) = 1$. The equation of the tangent line at $(1,2)$ is therefore $y - 2 = 1(x - 1)$, or $y = x + 1$.

(a) The point where the tangent line crosses the x axis is $(-1,0)$.

(b) The point where the tangent line crosses the y axis is $(0,1)$.

16. $f'(x) = -\frac{2}{x^2}$, so that $f'(1) = -2$. Thus, the normal line to the graph of f at $(1,2)$ has slope $\frac{1}{2}$, and its equation is $y - 2 = \frac{1}{2}(x - 1)$, or $y = \frac{x + 3}{2}$.

(a) $(-3,0)$. (b) $(0,3/2)$.

17. $f'(x) = 2x$, so that $f'(x_1) = 2x_1$. Thus, $f'(x_1) = 16$ when $x_1 = 8$. When $x_1 = 8$, we have $f(x_1) = 8^2 + 8 = 72$; hence, the equation of the tangent is $y - 72 = 16(x - 8)$, or $y = 16x - 56$.

18. The graph of the tangent line at $(a,f(a))$ is $y - f(a) = f'(a)(x - a)$.

We want the value of the x coordinate corresponding to $y = 0$.

Hence, $0 - f(a) = f'(a)(x - a)$

or $\dfrac{-f(a)}{f'(a)} = x - a$ or $x = a - \dfrac{f(a)}{f'(a)}$.

19. $f'(x) = 1 - 2x$, so that $f'(x_1) = -2x_1 + 1$. Thus, the slope m of the tangent line to the graph of f at $(x_1, f(x_1))$ is $-2x_1 + 1$. The slope of the line $x + y - 2 = 0$ is -1. Hence, we must have $-2x_1 + 1 = -1$, $x_1 = 1$. The desired tangent line passes through the point $(1,f(1)) = (1,0)$ and has slope $m = -1$, so its equation is $y - 0 = (-1)(x - 1)$, that is, $y = -x + 1$.

20. $f'(x) = 6x^2 - 2x$, so that $f'(x_1) = 6x_1^2 - 2x_1$. The slope m of the line $4x - y + 3 = 0$ is 4, so we solve $6x_1^2 - 2x_1 = 4$ to get $x_1 = 1$ or $x_1 = -\frac{2}{3}$. For $x_1 = 1$, we have $f(x_1) = f(1) = 1$ and the tangent line at $(1,1)$ is $y - 1 = 4(x - 1)$ or $y = 4x-3$. For $x_1 = -\frac{2}{3}$, we have $f(x_1) = f(-\frac{2}{3}) = -\frac{28}{27}$, and the tangent line at the point $(-\frac{2}{3}, -\frac{28}{27})$ is $y + \frac{28}{27} = 4(x + \frac{2}{3})$ or $y = 4x + \frac{44}{27}$.

21. $f'(x) = \dfrac{1}{2\sqrt{x}}$. Thus, the slope m of the tangent line to the graph of f at $(x_1,f(x_1))$ is $\dfrac{1}{2\sqrt{x}}$, and

so the slope of the normal line at $(x_1, f(x_1))$ is $-2\sqrt{x_1}$. Since the slope of the line $4x + y - 4 = 0$ is -4, it follows that $-2\sqrt{x_1} = -4$ or $x_1 = 4$ and $f(4) = \sqrt{4} = 2$. Hence, equation of the normal line is $y - 2 = -4(x - 4)$ or $y = -4x + 18$.

22. $f'(x) = 1 + \frac{1}{x^2}$ so that $f'(x_1) = 1 + \frac{1}{x_1^2}$. Thus, the slope m_1 of the normal line to the graph of f at $(x_1, f(x_1))$ is $-\frac{1}{f'(x_1)} = -\frac{1}{1 + \frac{1}{x_1^2}}$. The slope of the line $x + 2y - 3 = 0$ is $-\frac{1}{2}$, so we want $-\frac{1}{1 + \frac{1}{x_1^2}} = -\frac{1}{2}$, $1 + \frac{1}{x_1^2} = 2$, $\frac{1}{x_1^2} = 1$, $x_1^2 = 1$, $x_1 = \pm 1$. For $x_1 = 1$, $f(x_1) = f(1) = 1 - \frac{1}{1} = 0$, and the equation of the normal line at $(1,0)$ is $y - 0 = -\frac{1}{2}(x - 1)$, or $y = \frac{1 - x}{2}$. For $x_1 = -1$, $f(x_1) = f(-1) = -1 + 1 = 0$, and the equation of the normal line at $(-1,0)$ is $y - 0 = -\frac{1}{2}(x + 1)$, or $y = -\frac{1}{2}(x + 1)$.

23. $f'(x) = 2x$, so that $f'(x_1) = 2x_1$. The equation of the tangent line to the graph of f at $(x_1, f(x_1)) = (x_1, 5 + x_1^2)$ is accordingly $y - (5 + x_1^2) = 2x_1(x - x_1)$. We desire this line to pass through the point $(2,0)$, so we wish to have $0 - (5 + x_1^2) = 2x_1(2 - x_1)$, or $x_1^2 - 4x_1 - 5 = 0$; that is, $(x_1 - 5)(x_1 + 1) = 0$. Therefore, $x_1 = 5$ or $x_1 = -1$. For $x_1 = 5$, the equation of the tangent line becomes $y - (5 + 25) = 10(x - 5)$, or $y = 10x - 20$. For $x_1 = -1$, the equation of the tangent line becomes $y - (5 + 1) = -2(x + 1)$, or $y = -2x + 4$.

24. $f'(x) = 6x + 2$, so that $f'(x_1) = 6x_1 + 2$ and the slope of the normal line to the graph of f at $(x_1, f(x_1))$ is $-\frac{1}{f'(x_1)} = -\frac{1}{6x_1 + 2}$. The equation of the normal line is

$$y - (3x_1^2 + 2x_1 + 1) = -\frac{1}{6x_1 + 2}(x - x_1).$$

If $(9,5)$ is on the line, then

$$5 - (3x_1^2 + 2x_1 + 1) = \frac{-1}{6x_1 + 2}(9 - x_1)$$

or $18x_1^3 + 18x_1^2 - 19x_1 - 17 = 0$.

Since $x_1 = 1$ satisfies the equation, we have

$$(x_1 - 1)(18x_1^2 + 36x_1 + 17) = 0.$$

Using the quadratic formula, we obtain

$$x_1 = \frac{-6 \pm \sqrt{2}}{6}.$$

When $x_1 = 1$, $f(x_1) = 6$ and the equation of the normal line is $y - 6 = -\frac{1}{8}(x - 1)$

or $y = -\frac{1}{8}x + \frac{49}{8}$.

When $x_1 = \frac{-6 + \sqrt{2}}{6}$, the equation of the normal line is $42y = 3(4 + \sqrt{2})x + 102 - 27\sqrt{2}$.

When $x_1 = \frac{-6 - \sqrt{2}}{6}$, the equation of the normal line is $42y = 3(4 - \sqrt{2})x + 27\sqrt{2} + 102$.

25. $f'(x) = (x-3) \cdot 1 + (x-2) \cdot 1 = x-3 + x-2 = 2x - 5$.
$2x - 5 = 0$ when $x = \frac{5}{2}$; $f(\frac{5}{2}) = (\frac{5}{2} - 3)(\frac{5}{2} - 2) = -\frac{1}{4}$.
Tangent line is horizontal at $(\frac{5}{2}, -\frac{1}{4})$.

26. $g'(x) = 1 + (-1)x^{-2}$.
$1 - x^{-2} = 0$ when $x^{-2} = 1$ or $\frac{1}{x^2} = 1$ or $x^2 = 1$.
Thus $x = \pm 1$.
$g(1) = 1 + 1^{-1} = 2$. $g(-1) = -1 + (-1)^{-1}$
$= -1 + (-1) = -2$.
Tangent line is horizontal at $(1,2)$ and $(-1,-2)$.

27. $F'(x) = 6x + 5$.
$6x + 5 = 0$ when $x = -\frac{5}{6}$.
$F(-\frac{5}{6}) = \frac{47}{12}$.
Tangent line is horizontal at $(-\frac{5}{6}, \frac{47}{12})$.

28. $G'(x) = 3x^2 - 12x + 9$.
$3x^2 - 12x + 9 = 0$ when $x^2 - 4x + 3 = 0$;
$(x - 3)(x - 1) = 0$, so $x = 1,3$.
$G(1) = 8$, $G(3) = 4$.
Tangent line is horizontal at $(1,8)$ and $(3,4)$.

29. $h'(x) = x^2 + 4x - 5$.

$x^2 + 4x - 5 = 0$ when $(x+5)(x-1) = 0$ or $x = -5, 1$.

$h(1) = -\frac{11}{3}$, $h(-5) = \frac{97}{3}$.

Tangent line is horizontal at $(1, -\frac{11}{3})$ and $(-5, \frac{97}{3})$.

30. $H'(x) = -x^{-2} + (-2x^{-3})$.

$-x^{-2} - 2x^{-3} = 0$ when $-x - 2 = 0$ or $x = -2$.

$H(-2) = (-2)^{-1} + (-2)^{-2} = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$.

Tangent line is horizontal at $(-2, -\frac{1}{4})$.

31. $q'(x) = 2 \cdot \frac{1}{2\sqrt{x}} - 1$.

$x^{-1/2} - 1 = 0$ when $x^{-1/2} = 1$ or $x^{1/2} = 1$ or $x=1$.

$q(1) = 2\sqrt{1} - 1 + 1 = 2$.

Tangent line is horizontal at $(1,2)$.

32. $Q'(x) = \dfrac{(x - 3)(2x + 2) - (x^2 + 2x - 1) \cdot 1}{(x - 3)^2}$.

$Q'(x) = 0$ when $\dfrac{2x^2 - 4x - 6 - x^2 - 2x + 1}{(x - 3)^2} = 0$

or $x^2 - 6x - 5 = 0$.

So $x = \dfrac{6 \pm \sqrt{56}}{2} = \dfrac{6 \pm 2\sqrt{14}}{2} = 3 \pm \sqrt{14}$.

$Q(3 + \sqrt{14}) = 8 + 2\sqrt{14}$, $Q(3 - \sqrt{14}) = 8 - 2\sqrt{14}$.

Tangent line is horizontal at

$(3 + \sqrt{14}, 8 + 2\sqrt{14})$ and $(3 - \sqrt{14}, 8 - 2\sqrt{14})$.

33. $r'(x) = \dfrac{(x + 1) \cdot 2x - x^2 \cdot 1}{(x + 1)^2}$.

$r'(x) = 0$ when $\dfrac{2x^2 + 2x - x^2}{(x + 1)^2} = 0$ or $x^2 + 2x = 0$

or $x(x + 2) = 0$. So $x = 0, -2$.

$r(0) = 0$, $r(-2) = \frac{4}{-1} = -4$,

Tangent line is horizontal at $(0,0)$ and $(-2,-4)$.

34. $R'(x) = 3ax^2 + 2bx + c$.

$R'(x) = 0$ when $3ax^2 + 2bx + c = 0$;

so $x = \dfrac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \dfrac{-b \pm \sqrt{b^2 - 3ac}}{3a}$.

Tangent line is horizontal at

$\left(\dfrac{-b + \sqrt{b^2 - 3ac}}{3a}, R\dfrac{(-b + \sqrt{b^2 - 3ac})}{3a}\right)$

and at $\left(\dfrac{-b - \sqrt{b^2 - 3ac}}{3a}, R\dfrac{(-b - \sqrt{b^2 - 3ac})}{3a}\right)$.

35. $y' = 2x + b = 0$ provided $b = -2x$. Since $(2, 21+2b)$ is a point on the graph of f, then $b = -2(2) = -4$.

36. Suppose f has a relative minimum at c. Because f is differentiable at c,

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

which can be rewritten as $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

We must prove $f'(c) = 0$. If we can prove $f'(c)$ can be neither > 0 or < 0, then it must be zero. Assume $f'(c) < 0$. Since we can make $\dfrac{f(x) - f(c)}{x - c}$ as close as we please to the negative number $f'(c)$ by taking x sufficiently close to c (but not = c), there is a small open interval I containing c such that $\dfrac{f(x) - f(c)}{x - c} < 0$ if $x \neq c$ and x belongs to I. If x belongs to I and $x > c$ then $x - c > 0$, and $\dfrac{f(x) - f(c)}{x - c} < 0$. Thus, $f(x) - f(c) < 0$ or $f(x) < f(c)$. But this contradicts the fact that f has a relative minimum at c. Hence $f'(c)$ cannot be < 0. In a similar fashion, we can show $f'(c) > 0$ leads to a contradiction. Hence $f'(c) = 0$.

37. (a) $f'(x) = 3x^2$

$3x^2 = 0$ when $x = 0$; $f(0) = 0$. Hence we have a horizontal tangent at $(0,0)$.

(b) If $x > 0$, $f(x) > 0$; if $x < 0$, $f(x) < 0$. So we can never find an open interval about $x = 0$ where $f(x) > f(0) = 0$ or $f(x) < 0$ for all x in the interval, $x \neq 0$.

(c) Theorem 1 only gives a necessary condition, not a sufficient condition.

38. Clearly f is differentiable. Also, the relative extremum of a parabola is its vertex, say $(x_1, f(x_1))$, so by Theorem 1, $f'(x_1) = 0$. That is, $2ax_1 + b = 0$ and we see that $x_1 = -\frac{b}{2a}$. Hence, $(-\frac{b}{2a}, f(-\frac{b}{2a}))$ must be the coordinates of the vertex.

39. Equation of the normal line must be $x = c$.

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1. $f'(x) = 7 \cos x$.

2. $g'(x) = -x(-\sin x) + \cos x(-1)$
$= x \sin x - \cos x$.

3. $h'(t) = 4 \cos t - [t(-\sin t) + \cos t(1)]$
$= 4 \cos t + t \sin t - \cos t = 3 \cos t + t \sin t$.

4. $F'(r) = \sqrt{r}(-\sin r) + \cos r\left[\frac{1}{2} r^{-1/2}\right]$
$= -\sqrt{r} \sin r + \frac{1}{2\sqrt{r}} \cos r$.

5. $g'(x) = 3 \sec^2 x + \sec x \tan x$.

6. $G'(t) = 7t^6 - 5(-\csc^2 t) = 7t^6 + 5 \csc^2 t$.

7. $H'(y) = 8(\sec y \tan y) - \frac{1}{3} \cdot 6y^5 = 8 \sec y \tan y - 2y^5$.

8. $f'(z) = 4(-\csc z \cot z) - 3(\sec z \tan z)$
$= -4 \csc z \cot z - 3 \sec z \tan z$.

9. $g'(r) = r^4 \cos r + \sin r(4r^3) + 4(-\csc r \cot r)$
$= r^4 \cos r + 4r^3 \sin r - 4 \csc r \cot r$.

10. $H'(u) = (\sqrt{u} + 5)(-\sin u) + \cos u(\frac{1}{2} u^{-1/2})$
$= -(\sqrt{u} + 5) \sin u + \frac{1}{2\sqrt{u}} \cos u$.

11. $f'(z) = -\csc^2 z + \sqrt{z} \sec^2 z + \tan z (\frac{1}{2\sqrt{z}})$
$= -\csc^2 z + \sqrt{z} \sec^2 z + \frac{1}{2\sqrt{z}} \tan z$.

12. $f'(t) = 1 \cdot \sqrt{t} \cdot \sin t + t \cdot \frac{1}{2\sqrt{t}} \sin t + t\sqrt{t} \cos t - (-\csc^2 t)$
$= \sqrt{t} \sin t + \frac{1}{2} \sqrt{t} \sin t + t\sqrt{t} \cos t + \csc^2 t$
$= \frac{3}{2} \sqrt{t} \sin t + t\sqrt{t} \cos t + \csc^2 t$.

13. $p'(x) = \sin x(-\sin x) + \cos x(\cos x)$
$= -\sin^2 x + \cos^2 x = \cos^2 x - \sin^2 x = \cos^2 x$.

14. $F'(v) = 2 \sin v(-\sec^2 v) + (1 - \tan v) 2 \cos v$
$= -2 \sin v \sec^2 v + 2 \cos v - 2 \tan v \cos v$.

15. $g'(y) = -7\left[\cot y(-\csc y \cot y) + \csc y(-\csc^2 y)\right]$
$= -7\left[-\csc y \cot^2 y - \csc^3 y\right]$
$= 7 \csc y(\cot^2 y + \csc^2 y)$.

16. $f(\theta) = \cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \theta$
$f'(\theta) = \cos \theta(-\sin \theta) + \cos \theta(-\sin \theta) - \left[\sin \theta \cdot \cos \theta + \sin \theta \cos \theta\right]$
$= -2 \cos \theta \sin \theta - 2 \sin \theta \cos \theta$
$= -4 \sin \theta \cos \theta = -2 \sin 2\theta$.

17. $H'(x) = 3 \sec x(-\sec^2 x) + (1 - \tan x)3 \sec x \tan x$
$= -3 \sec^3 x + 3(1 - \tan x) \sec x \tan x$
$= -3 \sec x\left[\sec^2 x - \tan x + \tan^2 x\right]$.

18. $h(x) = \sin x \sin x$
$h'(x) = \sin x \cos x + \cos x \sin x$
$= 2 \sin x \cos x$
$= \sin 2x$.

19. $f'(\theta) = \frac{(\theta + 5)2 \cos \theta - 2 \sin \theta (1)}{(\theta + 5)^2}$
$= \frac{2(\theta + 5)\cos \theta - 2 \sin \theta}{(\theta + 5)^2}$

20. $G'(y) = \frac{\sqrt{y}(-7 \sin y) - 7 \cos y(\frac{1}{2\sqrt{y}})}{y}$
$= \frac{2\sqrt{y}\sqrt{y}(-7 \sin y) - 7 \cos y}{2\sqrt{y} \cdot y}$
$= -\frac{14y \sin y + 7 \cos y}{2y^{3/2}}$.

21. $p'(x) = \frac{(\cos x - 1)(2x) - (x^2 + 5)(-\sin x)}{(\cos x - 1)^2}$
$= \frac{2x(\cos x - 1) + (x^2 + 5)\sin x}{(\cos x - 1)^2}$.

22. $P'(x) = \frac{(5 + 3 \sin x)(-\sin x) - \cos x(3 \cos x)}{(5 + 3 \sin x)^2}$
$= \frac{-5 \sin x - 3 \sin^2 x - 3 \cos^2 x}{(5 + 3 \sin x)^2}$
$= \frac{-5 \sin x - 3 \cdot 1}{(5 + 3 \sin x)^2} = -\frac{3 + 5 \sin x}{(5 + 3 \sin x)^2}$.

23. $q'(t) = \frac{(\sec t + 4)(3 \sec^2 t) - 3 \tan t(\sec t \tan t)}{(\sec t + 4)^2}$

$$= \frac{3\sec^3 t + 12\sec^2 t - 3\tan^2 t\sec t}{(\sec t + 4)^2}$$

$$= \frac{3\sec t\left[\sec^2 t - \tan^2 t\right] + 12\sec^2 t}{(\sec t + 4)^2}$$

$$= \frac{12\sec^2 t + 3\sec t}{(\sec t + 4)^2}.$$

24. $Q'(u) = \dfrac{(2 + \sin u)(-\cos u) - (1 - \sin u)(\cos u)}{(2 + \sin u)^2}$

$$= \frac{-3\cos u}{(2 + \sin u)^2}$$

25. $r'(y) = \dfrac{(3 - \cos y)(-\sin y) - (3 + \cos y)(\sin y)}{(3 - \cos y)^2}$

$$= \frac{-6\sin y}{(3 - \cos y)^2}.$$

26. $R'(v) = \dfrac{(1 + \cot v)(\csc^2 v) - (1 - \cot v)(-\csc^2 v)}{(1 + \cot v)^2}$

$$= \frac{2\csc^2 v}{(1 + \cot v)^2}.$$

27. $f'(z) = \dfrac{(1 + \tan z)(-\csc z\cot z) - \csc z(\sec^2 z)}{(1 + \tan z)^2}$

$$= \frac{-\csc z\cot z - \csc z - \csc z\sec^2 z}{(1 + \tan z)^2}$$

28. $S'(w) = \dfrac{\sqrt{w}\cos w(0) - 3\left[\sqrt{w}(-\sin w) + \cos w\cdot\left(\frac{1}{2\sqrt{w}}\right)\right]}{w\cos^2 w}$

$$= \frac{3\sqrt{w}\sin w - \frac{3}{2\sqrt{w}}\cos w}{w\cos^2 w}$$

$$= \frac{6w\sin w - 3\cos w}{2w^{3/2}\cos^2 w}.$$

29. $F'(\theta) = \dfrac{(2\cos\theta - \sin\theta)(3\sec\theta\tan\theta) - 3\sec\theta(-2\sin\theta - \cos\theta)}{(2\cos\theta - \sin\theta)^2}$

$$= \frac{6\tan\theta - 3\sin\theta\cdot\frac{1}{\cos\theta}\cdot\tan\theta + 6\cdot\frac{1}{\cos\theta}\cdot\sin\theta + 3}{(2\cos\theta - \sin\theta)^2}$$

$$= \frac{12\tan\theta - 3\tan^2\theta + 3}{(2\cos\theta - \sin\theta)^2}.$$

30. $q'(t) = \dfrac{(\sin t - 2\cos t)\frac{1}{2\sqrt{t}} - \sqrt{t}(\cos t + 2\sin t)}{(\sin t - 2\cos t)^2}$

$$= \frac{(\sin t - 2\cos t) - 2\sqrt{t}\sqrt{t}(\cos t + 2\sin t)}{2\sqrt{t}(\sin t - 2\cos t)^2}$$

$$= \frac{\sin t - 2\cos t - 2t(\cos t + 2\sin t)}{2\sqrt{t}(\sin t - 2\cos t)^2}.$$

31. $f'(x) = 2\cos x$.

$f'(\frac{\pi}{6}) = 2\cos\frac{\pi}{6} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$.

Slope of the tangent line is $\sqrt{3}$; slope of the normal line is $-\frac{1}{\sqrt{3}}$.

Equation of the tangent line is $y - 1 = \sqrt{3}(x - \frac{\pi}{6})$.

Equation of the normal line is $y - 1 = -\frac{1}{\sqrt{3}}(x - \frac{\pi}{6})$.

32. $g'(x) = -4\sin x$.

$g'(\frac{\pi}{3}) = -4\sin\frac{\pi}{3} = -4(\frac{\sqrt{3}}{2}) = -2\sqrt{3}$.

Slope of the tangent line is $-2\sqrt{3}$; slope of the normal line is $\frac{1}{2\sqrt{3}}$.

Equation of the tangent line is $y - 2 = -2\sqrt{3}(x - \frac{\pi}{3})$.

Equation of the normal line is $y - 2 = \frac{1}{2\sqrt{3}}(x - \frac{\pi}{3})$.

33. $h'(x) = 3\sec^2 x$.

$h'(\frac{\pi}{4}) = 3\sec^2\frac{\pi}{4} = 3(\sqrt{2})^2 = 6$.

Slope of the tangent line is 6; slope of the normal line is $-\frac{1}{6}$.

Equation of the tangent line is $y - 3 = 6(x - \frac{\pi}{4})$.

Equation of the normal line is $y - 3 = -\frac{1}{6}(x - \frac{\pi}{4})$.

34. $F'(x) = -3(-\csc^2 x) = 3\csc^2 x$.

$F'(-\frac{\pi}{6}) = 3\left[\csc(-\frac{\pi}{6})\right]^2 = 3(-2)^2 = 12$.

Slope of the tangent line is 12; slope of the normal line is $-\frac{1}{12}$.

Equation of the tangent line is $y - 3\sqrt{3} = 12(x + \frac{\pi}{6})$.

Equation of the normal line is $y - 3\sqrt{3} = -\frac{1}{12}(x + \frac{\pi}{6})$.

35. $g'(x) = 2 - 5\cos x$.

$g'(\pi) = 2 - 5\cos\pi = 2 - 5(-1) = 7$.

Slope of the tangent line is 7; slope of the normal line is $-\frac{1}{7}$.

Equation of the tangent line is $y - 2\pi = 7(x - \pi)$.

Equation of the normal line is $y - 2\pi = -\frac{1}{7}(x - \pi)$.

36. $H'(x) = 1 - \sec^2 x$.

$H'(\pi) = 1 - \sec^2\pi = 1 - (-1)^2 = 0$.

Equation of the tangent line is $y = \pi$.

Equation of the normal line is $x = \pi$.

37. $y' = \cos x - \sin x$.

$y' = 0$ implies $\cos x - \sin x = 0$ or $\cos x = \sin x$

or $\tan x = 1$, so $x = \frac{\pi}{4}, -\frac{3\pi}{4}$.

38. $y = \cos x \cos x + 2 \sin x$.

$y' = \cos x(-\sin x) + \cos x(-\sin x) + 2\cos x$

$= -2\sin x \cos x + 2\cos x$.

$y' = 0$ implies $2\cos x(-\sin x + 1) = 0$, so:

$\cos x = 0$	$\sin x = 1$
$x = \frac{\pi}{2}, -\frac{\pi}{2}$	$x = \frac{\pi}{2}$

39. $\frac{dh}{d\theta} = \frac{v^2}{2g}(\sin\theta \cdot \cos\theta + \sin\theta \cdot \cos\theta)$

$= \frac{v^2}{2g}(2\sin\theta\cos\theta) = \frac{v^2}{g}\sin\theta\cos\theta$.

40. $\frac{dL}{d\theta} = 3\sec\theta\tan\theta + 2(-\csc\theta\cot\theta)$.

When $\theta = \frac{\pi}{6}$,

$\frac{dL}{d\theta} = 3(\frac{2}{\sqrt{3}})(\frac{1}{\sqrt{3}}) - 2(2)(\sqrt{3})$

$= 2 - 4\sqrt{3}$.

41. $\frac{dx}{dt} = 2\cos t$.

When $x = 0$, $0 = 2\sin t$ or $t = 0$.

When $t = 0$, $\frac{dx}{dt} = 2\cos 0 = 2 - 1 = 2$.

42. (a) Let $\Delta u = a\Delta x$; as $\Delta x \to 0, \Delta u \to 0$.

$\lim_{\Delta x\to 0}\frac{\sin(a\Delta x)}{\Delta x} = \lim_{\Delta u\to 0}\frac{\sin\Delta u}{\Delta u/a} = a\lim_{\Delta u\to 0}\frac{\sin\Delta u}{\Delta u}$

$= a\cdot 1 = a$.

$\lim_{\Delta x\to 0}\frac{1-\cos(a\Delta x)}{\Delta x} = \lim_{\Delta u\to 0}\frac{1-\cos\Delta u}{\Delta u/a}$

$= a\lim_{\Delta u\to 0}\frac{1-\cos\Delta u}{\Delta u} = a\cdot 0 = 0$.

(b) $D_x \sin ax = \lim_{\Delta x\to 0}\frac{\sin a(x+\Delta x) - \sin ax}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{\sin ax\cos a\Delta x + \cos ax\sin a\Delta x - \sin ax}{\Delta x}$

$= \lim_{\Delta x\to 0}\left[\cos ax\left(\frac{\sin a\Delta x}{\Delta x}\right) - \sin ax\left(\frac{1-\cos a\Delta x}{\Delta x}\right)\right]$

$= \cos ax \lim_{\Delta x\to 0}\frac{\sin a\Delta x}{\Delta x} - \sin ax\lim_{\Delta x\to 0}\frac{1-\cos a\Delta x}{\Delta x}$

$= (\cos ax)(a) - (\sin ax)(0) = a\cos ax$.

$D_x \cos ax = \lim_{\Delta x\to 0}\frac{\cos a(x+\Delta x) - \cos ax}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{\cos ax\cos a\Delta x - \sin ax\sin a\Delta x - \cos ax}{\Delta x}$

$= \lim_{\Delta x\to 0}\left[-\sin ax\left(\frac{\sin a\Delta x}{\Delta x}\right)\right] + \lim_{\Delta x\to 0}\left[-\cos ax\left(\frac{1-\cos a\Delta x}{\Delta x}\right)\right]$

$= -\sin ax\lim_{\Delta x\to 0}\frac{\sin a\Delta x}{\Delta x} - \cos ax\lim_{\Delta x\to 0}\frac{1-\cos a\Delta x}{\Delta x}$

$= (-\sin ax)(a) - (\cos ax)(0) = -a\sin ax$.

43. $\frac{dT}{d\theta} = \frac{(\cos\theta + \mu\sin\theta)0 - \mu w(-\sin\theta + \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)^2}$

$= \frac{\mu w(\sin\theta - \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)^2}$.

44. $\frac{dR}{dx} = -1800(-\sin\frac{2\pi}{365}x)\frac{2\pi}{365}$.

When $x = 152$, $\frac{dR}{dx} = 1800\cdot\frac{2\pi}{365}\sin(\frac{2\pi}{365}\cdot 152)$

$= 15.53126399$.

45. (ii) $D_x\cot x = D_x\frac{1}{\tan x} = \frac{(\tan x)(0) - (1)\sec^2 x}{\tan^2 x}$

$= -\frac{\sec^2 x}{\tan^2 x} = -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\frac{1}{\sin^2 x}$

$= -\csc^2 x$.

(iv) $D_x\csc x = D_x\frac{1}{\sin x} = \frac{(\sin x)(0) - (1)\cos x}{\sin^2 x}$

$= -\frac{\cos x}{\sin x\sin x} = -\csc x\cot x$.

Problem Set 2.6, page 123

1. $(f\circ g)(4) = f(g(4)) = f(20) = 17$.
2. $(f\circ g)(\sqrt{2}) = f(g(\sqrt{2})) = f(6) = 3$.
3. $(g\circ f)(4.73) = g(f(4.73)) = g(1.73) = 6.9929$.
4. $(g\circ f)(-2.08) = g(f(-2.08)) = g(-5.08) = 29.8064$.
5. $(f\circ f)(3) = f(f(3)) = f(0) = -3$.
6. $(g\circ g)(-3) = g(g(-3)) = g(13) = 173$.
7. $[f\circ(g\circ f)](2) = f(g(f(2))) = f(g(-1)) = f(5) = 2$.
8. $[(f\circ g)\circ f](2) = 2$ since composition is associative.
9. $(f\circ g)(x) = f(g(x)) = f(x^2+4) = x^2+4-3 = x^2+1$
10. $(g\circ f)(x) = g(f(x)) = g(x-3) = (x-3)^2+4 =$ $x^2 - 6x + 13$.
11. (a) $(f\circ g)(x) = f(g(x)) = f(x^2) = \sin(x^2)$.

    (b) $(g\circ f)(x) = g(f(x)) = g(\sin x) = \sin^2 x$.

    (c) $(g\circ g)(x) = g(g(x)) = g(x^2) = x^4$.

    (d) $[g\circ(f+h)](x) = (g(f+h)(x)) = g(f(x) +$

$h(x)) = g(\sin x + \cos x)$

$= (\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$

$= 1 + \sin 2x.$

(e) $(g \circ (\frac{f}{h}))(x) = g(\frac{f}{h}(x)) = g(\tan x) = \tan^2 x.$

(f) $((\frac{f}{h}) \circ (\frac{h}{f}))(x) = \frac{f}{h}(\frac{h}{f}(x)) = \frac{f}{h}(\frac{h(x)}{f(x)}) = \frac{f}{h}(\cot(x))$

$= \frac{f(\cot(x))}{h(\cot(x))}$

$= \frac{\sin(\cot(x))}{\cos(\cot(x))} = \tan(\cot(x)).$

(g) $[f \circ (g \circ h)](x) = f((g \circ h)(x)) = f(g(h(x)))$

$= f(g(\cos x))$

$= f(\cos^2 x) = \sin(\cos^2 x).$

(h) $((f \circ g) \circ h)(x) = ((f \circ g)(h)(x))$

$= (f \circ g)(\cos x) = f(g(\cos x)$

$= f(\cos^2 x) = \sin(\cos^2 x).$

2. $[(f \circ g) \circ h](x) = (f \circ g)(h(x)) = f(g(h(x))).$

$[f \circ (g \circ h)](x) = f((g \circ h)(x)) = f(g(h(x))).$

Therefore, $(f \circ g) \circ h(x) = f \circ (g \circ h).$

3. (a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x.$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$.

(c) $(f \circ f)(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4.$

4. (a) $(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x - 1}) = (\sqrt[3]{x - 1})^3 + 1 = x - 1 + 1 = x.$

(b) $(g \circ f)(x) = g(f(x)) = g(x^3 + 1) = \sqrt[3]{x^3 + 1 - 1}$

$= \sqrt[3]{x^3} = x.$

(c) $(f \circ f)(x) = f(f(x)) = f(x^3 + 1) = (x^3 + 1)^3 + 1$

$= x^9 + 3x^6 + 3x^3 + 2.$

5. (a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \tan \sqrt{x}.$

(b) $(g \circ f)(x) = g(f(x)) = g(\tan x) = \sqrt{\tan x}.$

(c) $(f \circ f)(x) = f(f(x)) = f(\tan x) = \tan(\tan x).$

6. (a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x - 1}) = \sqrt{x - 1} + (\sqrt{x - 1})^{-1} = \frac{x - 1 + 1}{\sqrt{x - 1}} = \frac{x}{\sqrt{x - 1}}.$

(b) $(g \circ f)(x) = g(f(x)) = g(x + x^{-1}) =$

$\sqrt{x + x^{-1} - 1} = \sqrt{x + \frac{1}{x} - 1}.$

(c) $(f \circ f)(x) = f(f(x)) = f(x + x^{-1}) = x + x^{-1} +$

$(x + x^{-1})^{-1} = x + \frac{1}{x} + \frac{1}{x + \frac{1}{x}} = x + \frac{1}{x} + \frac{x}{x^2 + 1}.$

17. (a) $(f \circ g)(x) = f(g(x)) = f(\csc x) = |\csc x|.$

(b) $(g \circ f)(x) = g(f(x)) = g(|x|) = \csc |x|.$

(c) $(f \circ f)(x) = f(f(x)) = f(|x|) = ||x|| = |x|.$

18. (a) $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x.$

(b) $(g \circ f)(x) = x.$

(c) $(f \circ f)(x) = x.$

19. (a) $(f \circ g)(x) = f(g(x)) = f(1 + \cos x) = (1+\cos x)^2 - 1 = 2 \cos x + \cos^2 x.$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 1 + \cos (x^2 - 1).$

(c) $(f \circ f)(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1$

$= x^4 - 2x^2.$

20. (a) $(f \circ g)(x) = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b.$

(b) $(g \circ f)(x) = g(f(x)) = g(ax + b) = c(ax + b) + d = acx + bc + d.$

(c) $(f \circ f)(x) = f(f(x)) = f(ax + b) = a(ax + b) + b = a^2x + ab + b.$

21. (a) $(f \circ g)(x) = f(g(x)) = f(2x - 3)$

$= \frac{1}{2(2x - 3) - 3} = \frac{1}{4x - 9}.$

(b) $(g \circ f)(x) = g(f(x)) = g(\frac{1}{2x - 3}) = 2(\frac{1}{2x - 3})$

$- 3 = \frac{11 - 6x}{2x - 3}$.

(c) $(f \circ f)(x) = f(f(x)) = f(\frac{1}{2x - 3})$

$= \frac{1}{2(\frac{1}{2x - 3}) - 3} = \frac{2x - 3}{2 - 3(2x - 3)}$

$= \frac{2x - 3}{11 - 6x}$.

22. (a) $(f \circ g)(x) = f(g(x)) = f(\frac{ax + b}{cx + d})$

$= \frac{A(\frac{ax + b}{cx + d}) + B}{C(\frac{ax + b}{cx + d}) + D}$

$= \frac{A(ax + b) + B(cx + d)}{C(ax + b) + D(cx + d)} = \frac{(Aa + Bc)x + Ab + Bd}{(aC + cD)x + bC + Dd}.$

(b) $(g \circ f)(x) = g(f(x)) = g(\frac{Ax + B}{Cx + D})$

$$= \frac{a\left(\frac{Ax + B}{Cx + D}\right) + b}{c\left(\frac{Ax + B}{Cx + D}\right) + d}$$

$$= \frac{a(Ax + B) + b(Cx + D)}{c(Ax + B) + d(Cx + D)}$$

$$= \frac{(aA + bC)x + aB + bD}{(Ac + Cd)x + Bc + dD}.$$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{Ax + B}{Cx + D}\right)$

$$= \frac{A\left(\frac{Ax + B}{Cx + D}\right) + B}{C\left(\frac{Ax + B}{Cx + D}\right) + D}$$

$$= \frac{A(Ax + B) + B(Cx + D)}{C(Ax + B) + D(Cx + D)}$$

$$= \frac{(A^2 + BC)x + AB + BD}{(AC + CD)x + BC + D^2}$$

23. (a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3 - x}\right)$

$$= \frac{3\left(\frac{1}{3 - x}\right) - 1}{\frac{1}{3 - x}} = \frac{3 - (3 - x)}{1} = x.$$

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{3x - 1}{x}\right)$

$$= \frac{1}{3 - \frac{3x - 1}{x}} = \frac{x}{3x - 3x + 1} = \frac{x}{1} = x.$$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{3x - 1}{x}\right)$

$$= \frac{3\left(\frac{3x - 1}{x}\right) - 1}{\frac{3x - 1}{x}} = \frac{9x - 3 - x}{3x - 1}$$

$$= \frac{8x - 3}{3x - 1}.$$

24. (a) $(f \circ g)(x) = f(g(x)) = f(7) = 2.$

(b) $(g \circ f)(x) = g(f(x)) = g(2) = 7.$

(c) $(f \circ f)(x) = f(f(x)) = f(2) = 2.$

25. $F(x) = \sqrt{x^2 - 3}.$

Let $F = h \circ g$. Then $(h \circ g)(x) = h(g(x)) = h(x^2 - 3)$

$= \sqrt{x^2 - 3}.$

26. $G(x) = (\sqrt{x})^2 - 3.$

Let $G = g \circ h$. Then $(g \circ h)(x) = g(f(x)) = g(\sqrt{x})$

$= (\sqrt{x})^2 - 3.$

27. $H(x) = 2\sqrt{x}.$

Let $H = h \circ f$. Then $(h \circ f)(x) = h(f(x)) = h(4x)$

$= \sqrt{4x} = 2\sqrt{x}.$

28. $K(x) = 4x^2 - 12 = 4(x^2 - 3).$

Let $K = f \circ g$. Then $(f \circ g)(x) = f(g(x)) =$

$f(x^2 - 3) = 4(x^2 - 3).$

29. $Q(x) = 4\sqrt{x},$

Let $Q = f \circ h$. Then $(f \circ h)(x) = f(h(x)) = f(\sqrt{x})$

$= 4\sqrt{x}.$

30. $q(x) = 16x^2 - 3.$

Let $q = g \circ f$. Then $(g \circ f)(x) = g(f(x)) = g(4x)$

$= (4x)^2 - 3 = 16x^2 - 3.$

31. $r(x) = \sqrt[4]{x}.$

Let $r = h \circ h$. Then $(h \circ h)(x) = h(h(x)) = h(\sqrt{x})$

$= \sqrt{\sqrt{x}} = (x^{1/2})^{1/2} = x^{1/4} = \sqrt[4]{x}.$

32. $s(x) = x^4 - 6x^2 + 6.$

Let $s = g \circ g$. Then $(g \circ g)(x) = g(g(x)) =$

$g(x^2 - 3) = (x^2 - 3)^2 - 3 = x^2 - 6x + 6.$

33. Let $f(x) = x^7$ and $g(x) = \cos x$.

Then $h(x) = (f \circ g)(x) = f(g(x)) = f(\cos(x))$

$= \cos^7 x.$

34. Let $f(x) = \sin x$ and $g(x) = x^7 + 1$.

Then $h(x) = (f \circ g)(x) = f(g(x)) = f(x^7 + 1)$

$= \sin(x^7 + 1).$

35. Let $g(x) = \tan x$ and $f(x) = 1 - x^2$.

Then $h(x) = (f \circ g)(x) = f(g(x)) = f(\tan x)$

$= 1 - \tan^2 x.$

36. Let $f(x) = 5 \csc x$ and $g(x) = |x|$.

Then $h(x) = (f \circ g)(x) = f(g(x)) = f(|x|)$

$= 5 \csc |x|.$

37. $h(x) = \sqrt{\frac{x + 1}{x - 1}}$; let $g(x) = \frac{x + 1}{x - 1}$ and $f(x) = \sqrt{x}$.

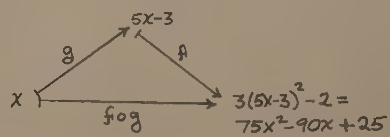
38. $h(x) = \frac{1}{(4x + 5)^2}$; let $g(x) = 4x + 5$ and $f(x) = \frac{1}{x^5}$

39. $h(x) = \frac{|x + 1|}{x + 1}$; let $g(x) = x + 1$ and $f(x) = \frac{|x|}{x}$.

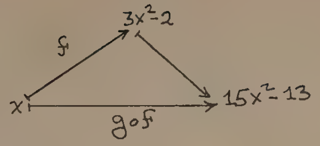
40. $h(x) = \sqrt{1 - \sqrt{x - 1}}$; let $f(x) = \sqrt{1 - x}$ and

$g(x) = \sqrt{x - 1}.$

41. (a)



(b)



42. (a) Let $f(x) = 2$ then

$f(x) \cdot f(x) = 2 \cdot 2 = 4$ whereas $(f \circ f)(x)$

$= f(f(x)) = f(2) = 2$.

(b) Let $g(x) = x^2$ then

$g(x) \cdot g(x) = x^4$ and $(g \circ g)(x) = g(g(x))$

$= g(x^2) = x^4$.

43. $f(x) = (h \circ g)(x)$, where $h(x) = \sin x$ and $g(x) = |x|$. Since g is continuous at 0 and h is continuous at $g(0) = 0$, it follows that $f = h \circ g$ is continuous at 0 by the substitution property for continuous functions.

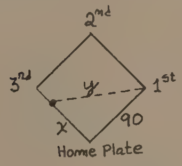
44. Suppose $\lim_{y \to c} f(y)$ exists and $\lim_{x \to a} g(x) = c$. Then provided $f \circ g$ is defined at x and $g(x) \neq c$ for all values of x in an open interval containing a, except possibly for $x = a$,

$$\lim_{x \to a} (f \circ g)(x) = \lim_{y \to c} f(y).$$

45. $g(x) = (f \circ f)(x)$, where $f(x) = \sin x$.

Using the substitution property for continuous functions, we have that g is continuous on $\mathbb{R}$.

46.



(a) $y = f(x) = \sqrt{8100 + x^2}$; $x = 50t = g(t)$.

(b) $(f \circ g)(t) = f(g(t)) = f(50t) = \sqrt{8100 + (50t)^2}$

$= 10\sqrt{81 + 25t^2}$.

(c) $y = f(x)$ and $x = g(t)$ so $y = f(g(t)) = (f \circ g)(t)$.

47. The coordinates of the intersection of g and the vertical line at x are $(x, g(x))$. Thus, the coordinates of the intersection of the horizontal line thru $(x, g(x))$ and $y = x$ are $(g(x), g(x))$. Thus, the coordinates of the intersection of the vertical line thru $(g(x), g(x))$ and the graph of f are $(g(x), f(g(x)))$. Hence, $y = f(g(x)) = (f \circ g)(x)$.

48. Choose $(x,0)$ on the x-axis, move vertically to the graph of f, then horizontally to the graph of $y = x$, then vertically to the graph of f again, and finally horizontally to the point $(0,y)$. This gives $y = (f \circ f)(x)$. To find $y = (f \circ f \circ f)(x)$ instead of moving horizontally to the point $(0,y)$, move horizontally to the graph of $y = x$, then vertically to the graph of f, then horizontally to the point $(0,y)$, and so forth.

49. $f(p) = Bp - Ap^2 = 2.6p - 0.08p^2$;

$p = 4$.

$f(4) = 2.6(4) - 0.08(4)^2 = 9.12$

1 unit of time later.

$f \circ f(4) = f(9.12) = 2.6(9.12) - 0.08(9.12)^2 = 17.06$

2 units of time later.

$f \circ f \circ f(4) = f(17.06) = 2.6(17.06) - 0.08(17.06)^2$

$= 21.07$

3 units of time later.

$f \circ f \circ f \circ f(4) = f(21.07) = 2.6(21.07) - 0.08(21.07)^2 = 19.27$

4 units of time later.

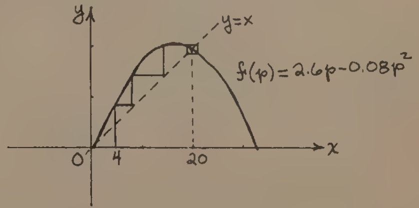
$f \circ f \circ f \circ f \circ f(4) = f(19.27) = 2.6(19.27) - 0.08(19.27)^2 = 20.40$

5 units of time later.

$f \circ f \circ f \circ f \circ f \circ f(4) = f(20.4) = 2.6(20.4) - 0.08(20.4)^2 = 19.75$

6 units of time later.

50.



(a) (Observation from graph.)

(b) No matter where you start, values of the successive iterates appear to come closer and closer to 20 as a limit.

Problem Set 2.7, page 130

1. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot (2x + 1) = \frac{2x + 1}{2\sqrt{x^2 + x + 1}}$.

2. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3u^2 - \frac{1}{\sqrt{u}})(2x + 2) = (3(x^2 + 2x)^2$

$- \frac{1}{\sqrt{x^2 + 2x}})(2x + 2)$.

3. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (-\frac{5}{u^6})(4x^3) = \frac{-20x^3}{(x^4 + 1)^6}$.

4. $D_x y = (D_u y)(D_x u) = (1)\frac{(7 + x^2)(-2x) - (7 - x^2)(2x)}{(7 + x^2)^2}$

$= \frac{-28x}{(7 + x^2)^2}$.

5. $f'(x) = 10(5 - 2x)^9(-2) = -20(5 - 2x)^9$.

6. $f'(x) = 8(2x - 3)^7(2) = 16(2x - 3)^7$.

7. $f'(y) = D_y\left[(4y + 1)^{-5}\right] = -5(4y + 1)^{-6}(4) = \frac{-20}{(4y + 1)^6}$.

8. $F'(t) = -4(2t^4 - t + 1)^{-5}(8t^3 - 1) = (4 - 32t^3)$

$(2t^4 - t + 1)^{-5}$.

9. $g'(x) = (3x^2 + 7)^2\left[3(5 - 3x)^2(-3)\right] + \left[2(3x^2 + 7)(6x)\right] \cdot$

$(5 - 3x)^3$

$= (3x^2 + 7)(5 - 3x)^2(-63x^2 + 60x - 63)$.

10. $G'(t) = (5t^2 + 1)^2\left[4(3t^4 + 2)^3(12t^3)\right] + \left[2(5t^2 + 1)(10t)\right]$

$(3t^4 + 2)^4$

$= (5t^2 + 1)(3t^4 + 2)^3(300t^5 + 48t^3 + 40t)$.

11. $f'(x) = (3x + \frac{1}{x})^2\left[5(6x - 1)^4(6)\right] + \left[2(3x + \frac{1}{x})(3 - \frac{1}{x^2})\right]$

$(6x - 1)^5$

$= (3x + \frac{1}{x})(6x - 1)^4(126x - 6 + \frac{18}{x} + \frac{2}{x^2})$.

12. $f'(t) = (3t - 1)^{-1}\left[(-3)(2t + 5)^{-4}(2)\right] + \left[(-1)\right.$

$\left.(3t - 1)^{-2}(3)\right] (2t + 5)^{-3}$

$= \frac{-24t - 9}{(2t + 5)^4(3t - 1)^2}$.

13. $g'(y) = (7y + 3)^{-2}\left[4(2y - 1)^3(2)\right] + \left[(-2)(7y + 3)^{-3}\right.$

$\left.(7)\right](2y - 1)^4$

$= \frac{(28y + 38)(2y - 1)^3}{(7y + 3)^3}$.

14. $f'(u) = (6u + \frac{1}{u})^{-5}\left[7(2u - 2)^6(2)\right] + \left[(-5)(6u + \frac{1}{u})^{-6}\right.$

$\left.(6 - \frac{1}{u^2})\right](2u - 2)^7$

$= (\frac{2u - 2}{6u + \frac{1}{u}})^6(24u + 60 + \frac{24}{u} - \frac{10}{u^2})$.

15. $f'(x) = 4(\frac{x^2 + x}{1 - 2x})^3\left[\frac{(1 - 2x)(2x + 1) - (x^2 + x)(-2)}{(1 - 2x)^2}\right]$

$= 4\frac{(x^2 + x)^3}{(1 - 2x)^5}(-2x^2 + 2x + 1)$.

16. $f'(t) = 5(\frac{1 + t^2}{1 - t^2})^4\left[\frac{(1 - t^2)(2t) - (1 + t^2)(-2t)}{(1 - t^2)^2}\right]$

$= 20t\frac{(1 + t^2)^4}{(1 - t^2)^6}$.

17. $F(x) = 3(\frac{3x + 1}{x^2})^2(-\frac{3}{x^2} - \frac{2}{x^3}) = \frac{-3(3x + 1)^2(3x + 2)}{x^7}$

18. $f(x) = (\frac{x^2 - 7}{16x})^3 = (\frac{x}{16} - \frac{7}{16x})^3$, $f'(x) = 3(\frac{x}{16} - \frac{7}{16x})$

$(\frac{1}{16} + \frac{7}{16x^2})$.

19. $f(x) = (\sqrt{x})^{-1}$, $f'(x) = (-1)(\sqrt{x})^{-2}\frac{1}{2\sqrt{x}} = \frac{-1}{2x\sqrt{x}}$.

20. $F(x) = -\frac{\left[\frac{1}{2}\sqrt{x^2 + 1}\right](2x)}{(\sqrt{x^2 + 1})^2} = \frac{-x}{(x^2 + 1)^{3/2}}$.

21. $g'(x) = \frac{1}{2\sqrt{x^2 + 2x - 1}}(2x + 2) = \frac{x + 1}{\sqrt{x^2 + 2x - 1}}$.

22. $f'(x) = \frac{1}{2\sqrt{\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4x^{3/4}}$.

23. $f'(t) = \frac{1}{2\sqrt{t^4 - t^2 + \sqrt{3}}}(4t^3 - 2t) = \frac{2t^3 - t}{\sqrt{t^4 - t^2 + \sqrt{3}}}$.

24. $g'(y) = \frac{1}{2\sqrt{y^3 - y + \sqrt{y}}}(3y^2 - 1 + \frac{1}{2\sqrt{y}})$.

25. $F'(x) = 4(x - \sqrt{x})^3(1 - \frac{1}{2\sqrt{x}})$.

26. $Q'(s) = \frac{(s)\frac{1}{2\sqrt{1 + s^3}} \cdot (3s^2) - (1 + s^3)^{1/2} \cdot 1}{s^2}$

$$= \frac{\frac{3}{2}s^3-(1+s^3)\cdot 1}{s^2(1+s^3)^{1/2}} = \frac{\frac{1}{2}s^3-1}{s^2(1+s^3)^{1/2}}$$

$$= \frac{s^3-2}{2s^2(1+s^3)^{1/2}}.$$

27. $f'(x) = 5(7)\cdot\cos 7x = 35\cos 7x.$

28. $f'(x) = -8[\sin(3x+5)](3) = -24\sin(3x+5).$

29. $g'(x) = 4(\cos 6x^2)(12x) = 48x\cos 6x^2.$

30. $g'(t) = 3[\cos(5t^2+t)](10t+1) = 3(10t+1)\cdot\cos(5t^2+t).$

31. $h'(x) = [\cos\sqrt{x}]\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}}\cos\sqrt{x}.$

32. $H'(s) = s^2(\cos s^3)(3s^2) + 2s\sin s^3 = s(3s^3\cdot\cos s^3 + 2\sin s^3).$

33. $g'(t) = 4\sin^3 3t(\cos 3t)(3) = 12\sin^3 3t\cos 3t.$

34. $g'(x) = 2\cos 5x(-\sin 5x)(5) - 2\sin 5x(\cos 5x)(5)$
$= -20\sin 5x\cos 5x(\text{or } -10\sin 10x).$

35. $H'(x) = -[\sin(\sin x)]\cos x = -\cos x\sin(\sin x).$

36. $f'(t) = 5(1-2\sin 3t)^4(-6\cos 3t) = 30(\cos 3t)(1-2\sin 3t)^4.$

37. $f'(x) = \frac{1}{2}(\cos 5x)^{-1/2}(-\sin 5x)(5) = -\frac{5}{2}\frac{\sin 5x}{\sqrt{\cos 5x}}.$

38. $G'(x) = \frac{x^2[(\sin 3x)(3)] - (4-\cos 3x)(2x)}{x^4}$
$= \frac{3x\sin 3x + 2\cos 3x - 8}{x^3}.$

39. $H'(x) = \frac{(1+\cos 5x)(\cos x) - \sin x[-(\sin 5x)(5)]}{(1+\cos 5x)^2}$
$= \frac{\cos x + \cos 5x\cos x + 5\sin x\sin 5x}{(1+\cos 5x)^2}$

40. $g'(x) =$
$\frac{\sqrt{\cos x}[\cos x-(\cos x - x\sin x)] - (\sin x - x\cos x)[\frac{1}{2}(\cos x)^{-1/2}\sin x]}{\cos x}$

$= \frac{2\cos x(\cos x - \cos x + x\sin x) - (\sin x - x\cos x)\sin x}{2(\cos x)^{3/2}}$

$= \frac{3x\sin x\cos x - \sin^2 x}{2(\cos x)^{3/2}}.$

41. $H'(t) = \frac{-27(2)\cos 2t}{\sin^2 2t} + \frac{-35(-\sin 2t)(2)}{\cos^2 2t}$
$= \frac{70\sin 2t}{\cos^2 2t} - \frac{54\cos 2t}{\sin^2 2t}$
$= 70\sec 2t\tan 2t - 54\csc 2t\cot 2t.$

42. $g'(r) = [\sec^2(5r^4)](20r^3) = 20r^3\sec^2(5r^4).$

43. $g'(t) = [-\csc^2(3t^5)](15t^4) = -15t^4\csc^2(3t^5).$

44. $h'(r) = \sec(\sqrt{r}-r)\tan(\sqrt{r}-r)\left[\frac{1}{2\sqrt{r}}-1\right]$
$= \left(\frac{1}{2\sqrt{r}}-1\right)\sec(\sqrt{r}-r)\tan(\sqrt{r}-r).$

45. $F'(u) = \frac{2u}{2\sqrt{u^2+1}}(-\csc\sqrt{u^2+1}\cot\sqrt{u^2+1})$
$= \frac{-u\csc\sqrt{u^2+1}\cot\sqrt{u^2+1}}{\sqrt{u^2+1}}.$

46. $g'(s) = -\frac{7}{s^2}\left[-\csc^2\left(\frac{7}{s}\right)\right] = \frac{7\csc^2\frac{7}{s}}{s^2}.$

47. $h'(x) = \frac{5\sec 5x\tan 5x}{2\sqrt{1+\sec 5x}}.$

48. $g'(t) = \sec^2\left(\frac{t}{t+2}\right)\left[\frac{t+2-t}{(t+2)^2}\right] = \frac{2\sec^2\frac{t}{(t+2)}}{(t+2)^2}.$

49. $h'(t) = 14\sec 7t\sec 7t\tan 7t - 14\tan 7t\sec^2 7t = 0.$

50. $g'(x) = 30\csc 15x(-\csc 15x\cot 15x) - 30\cot 15x(-\csc^2 15x) = 0.$

51. $H'(s) = 52\sec^3 13s(\sec 13s\tan 13s) - 52\tan^3 13s(\sec^2 13s)$
$= 52\sec^2 13s\tan 13s(\sec^2 13s - \tan^2 13s)$
$= 52\sec^2 13s\tan 13s$, since $\sec^2 13s - \tan^2 13s = 1$.

52. $g'(x) = 3(\tan x + \sec x)^2(\sec^2 x + \sec x\tan x)$
$= 3\sec x(\tan x + \sec x)^3.$

53. $g'(x) = x^3[5\tan^4 2x(\sec^2 2x)(2)] + 3x^2\tan^5 2x$
$= x^2\tan^4 2x(10x\sec^2 2x + 3\tan 2x).$

54. $f'(t) = \frac{(t^2+1)3(-\csc^2 3t) - \cot 3t(2t)}{(t^2+1)^2}$
$= \frac{-3(t^2+1)\csc^2 3t + 2t\cot 3t}{(t^2+1)^2}.$

55. $H'(x) = \frac{(1+\sec 5x)(2) - 2x(\sec 5x\tan 5x)(5)}{(1+\sec 5x)^2}$
$= \frac{2\sec 5x - 10x\sec 5x\tan 5x + 2}{(1+\sec 5x)^2}$

56. $g'(t) = \tan 3t[(\sec 3t\tan 3t)3] + \sec 3t[(\sec^2 3t)3]$

$= 3\sec 3t\left[\tan^2 3t + \sec^2 3t\right]$

$= 3\sec 3t(2\sec^2 3t - 1).$

57. $f'(x) = \frac{2}{3}x - 6\cot^2 2x(-\csc^2 2x) = \frac{2}{3}x + 6\cot^2 2x \csc^2 2x.$

58. $G'(r) = \frac{3}{2}r^2\left[5\csc^4 3r(-\csc 3r\cot 3r)(3)\right] + 3r\csc^5 3r$

$= \frac{3}{2}r\csc^5 3r(2 - 15r^2\cot 3r).$

59. $g'(t) = \dfrac{t^3\left[6\sec 3t(\sec 3t\cdot\tan 3t)\right] - (\sec^2 3t)3t^2}{t^6}$

$= \dfrac{3t^2\sec^2 3t(2t\tan 3t - 1)}{t^6}$

$= \dfrac{3\sec^2 3t(2t\tan 3t - 1)}{t^4}.$

60. $f'(\theta) = 3\left(\frac{\theta}{\tan\theta}\right)^2\left[\dfrac{\tan\theta - \theta\sec^2\theta}{\tan^2\theta}\right]$

$= \dfrac{3\theta^2(\tan\theta - \theta\sec^2\theta)}{\tan^4\theta}.$

61. $f'(x) = \cos(\tan 5x^2)\left[\sec^2 5x^2(10x)\right]$

$= 10x\sec^2 5x^2\cdot\cos(\tan 5x^2).$

62. $g'(x) = \sec(\csc^2 7x)\tan(\csc^2 7x)\left[2\csc 7x(-\csc 7x)(\cot 7x)7\right]$

$= -14\csc^2 7x\cot 7x\sec(\csc^2 7x)\tan(\csc^2 7x).$

63. $(f\circ g)'(2) = f'(g(2))g'(2) = f'(5)\cdot(-4) = (-3)(-4) = 12.$

64. $(f\circ g)'(5) = f'(g(5))g'(5) = f'(2)\cdot 6 = 4\cdot 6 = 24.$

65. $(g\circ f)'(5) = g'(f(5))f'(5) = g'(5)(-3) = 6(-3) = -18.$

66. $(g\circ g)'(5) = g'(g(5))g'(5) = g'(2)\cdot 6 = -4(6) = -24.$

67. $h'(x) = 2f(x)f'(x),$

so $h'(5) = 2f(5)f'(5) = 2(5)(-3) = -30.$

68. $H'(x) = f'(g(6x-7))g'(6x-7)\cdot 6,$

so

$H'(2) = f'(g(5))g'(5)\cdot 6 = f'(2)(6)(6) = 4(36) = 144.$

69. $F'(x) = (g(x))^4 f'(x) + f(x)\cdot 4\left[g(x)\right]^3 g'(x),$

so

$F'(2) = \left[g(2)\right]^4 f'(2) + f(2)\cdot 4\cdot\left[g(2)\right]^3 g'(2)$

$= 5^4\cdot 4 + 2\cdot 4\cdot(5)^3(-4) = 5^3(20 - 32)$

$= -1500.$

70. $G'(x) = g(x)\dfrac{1}{2\sqrt{f(x)}}f'(x) + \sqrt{f(x)}\cdot g'(x),$

so

$G'(5) = \dfrac{g(5)f'(5)}{2\sqrt{f(5)}} + \sqrt{f(5)}g'(5)$

$= \dfrac{2(-3)}{2\sqrt{5}} + \sqrt{5}(6) = \dfrac{-3}{\sqrt{5}} + 6\sqrt{5} = \dfrac{27}{\sqrt{5}}.$

71. $(f\circ g)'(x) = f'(g(x))g'(x) = \left[(f'\circ g)(x)\right]g'(x)$

$= \left[(f'\circ g)g'\right](x),$

so $(f\circ g)' = (f'\circ g)g'.$

72. $f'(7x+3)$ – find $f'(x)$, then find $f'(7x+3)$

$D_x f(7x+3)$ – find $f(7x+3)$, then find derivative of that result. In fact, $D_x\left[f(7x+3)\right] = 7f'(7x+3)$ by the chain rule.

73. $(f\circ g)'(5) = f'(g(5))g'(5) = f'(7)\cdot\frac{1}{4} = 20\cdot\frac{1}{4}$

$= 5.$

74. $\frac{dy}{dt} = y_2\left[10\cdot\cos(10^5\pi t)\cdot 10^5\pi\right] + y_1\left[\cos(2\pi 10^4 t)\cdot 2\pi 10^4\right]$

$= 10^6\pi\sin(2\pi 10^4 t)\cdot\cos(10^5\pi t) + 2\pi 10^5\sin(10^5\pi t)\cdot\cos(2\pi 10^4 t).$

75. $\frac{dI}{dt} = 30\cos 120\pi t(120\pi).$

When $t = 0.97,$

$\frac{dI}{dt} = 3600\pi\cos\left[120\cdot\pi\cdot(0.97)\right]$

≈ 3495 amp/sec.

76. $\frac{dy}{dt} = -8000\sin\left(\frac{\pi t}{40} - \frac{2\pi}{9}\right)\cdot\left[\frac{\pi}{40}\right] = -200\pi\sin\left(\frac{\pi t}{40} - \frac{2\pi}{9}\right)$

When $t = 10,$

$\frac{dy}{dt} = -200\pi\sin\left(\frac{\pi}{4} - \frac{2\pi}{9}\right)$

≈ -54.8 km/min.

77. $\frac{dN}{dt} = 1000\cos(0.25t)(0.25).$

When $t = 10,$

$\frac{dN}{dt} = 1000\cos(2.5)(0.25)$

$= 250\cos(2.5)$

≈ -200 moose/year.

78. $\frac{dA}{dt} = 18\cos\left(\frac{\pi t}{4}\right)\left(\frac{\pi}{4}\right).$

When $t = 9,$

$\frac{dA}{dt} = \frac{9\pi}{2}\cos\left(\frac{9\pi}{4}\right)$ (gallon/day)/month.

79. (a) $u(t) = 40{,}000 + 10{,}000t$

(b) $(F\circ u)(t) = F(u(t)) = F(40{,}000 + 10{,}000t)$

$= 6\sqrt{40{,}000 + 10{,}000t}$

$= 600\sqrt{4 + t}.$

This gives labor force that will be required t years from now.

(c) $(F \circ u)'(t) = F'(u(t)) \cdot u'(t) = \frac{3}{\sqrt{u(t)}}(10{,}000)$

$$= \frac{30{,}000}{\sqrt{40{,}000 + 10{,}000t}} = \frac{300}{\sqrt{4 + t}}$$

This gives the rate at which the required labor force will be changing t years from now.

(d) $(F \circ u)'(5) = \frac{300}{\sqrt{4 + 5}} = \frac{300}{3} = 100$ persons/yr.

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$18x + 8y\frac{dy}{dx} = 0,\ \frac{dy}{dx} = -\frac{9x}{4y}$.

$8xy\frac{dy}{dx} + 4y^2 + 3x^2\frac{dy}{dx} + 6xy = 0,\ \frac{dy}{dx} = \frac{-4y^2 - 6xy}{8xy + 3x^2}$.

$x^2\frac{dy}{dx} + 2xy - 2xy\frac{dy}{dx} - y^2 + 2x = 0,\ \frac{dy}{dx} = \frac{y^2 - 2xy - 2x}{x^2 - 2xy}$.

$2xy\frac{dy}{dx} + y^2 + 3x^2 + 3y^2\frac{dy}{dx} = 0,\ \frac{dy}{dx} = \frac{-y^2 - 3x^2}{2xy + 3y^2}$.

$2x - 3x\frac{dy}{dx} - 3y + 2y\frac{dy}{dx} = 3,\ \frac{dy}{dx} = \frac{3y - 2x + 3}{2y - 3x}$.

$3xy^2\frac{dy}{dx} + y^3 + 6y^2\frac{dy}{dx} = 2x - 8y\frac{dy}{dx},\ \frac{dy}{dx} = \frac{2x - y^3}{(3x + 6)y^2 + 8y}$.

$(-1)x^{-2} + (-1)y^{-2}\frac{dy}{dx} = 8\frac{dy}{dx}$,

so $\frac{dy}{dx} = \frac{-x^{-2}}{8 + y^{-2}} = \frac{-y^2}{8x^2y^2 + x^2}$.

$2x - \frac{x\frac{dy}{dx} + y}{2\sqrt{xy}} - \frac{dy}{dx} = 0,\ \frac{dy}{dx} = \frac{4x\sqrt{xy} - y}{2\sqrt{xy} + x}$.

$x^4\frac{dy}{dx} + 4x^3y + \frac{x\frac{dy}{dx} + y}{2\sqrt{xy}} = 0,\ \frac{dy}{dx} = -\frac{8x^3y\sqrt{xy} + y}{2x^4\sqrt{xy} + x}$.

$y\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x}y' + \frac{1}{2\sqrt{y}}y' = 0$,

$$y' = \frac{\frac{-y}{2\sqrt{x}}}{\sqrt{x} + \frac{1}{2\sqrt{y}}} = \frac{-y\sqrt{y}}{2x\sqrt{y} + \sqrt{x}}.$$

11. $y \cdot 3(x^2 - y^2)^2(2x - 2yy') + (x^2 - y^2)^3y' = 1$,
$6xy(x^2 - y^2)^2 - 1 = 6y^2(x^2 - y^2)^2y' - (x^2 - y^2)^3y'$,
so
$$y' = \frac{6xy(x^2 - y^2)^2 - 1}{6y^2(x^2 - y^2)^2 - (x^2 - y^2)^3} = \frac{6xy(x^2 - y^2)^2 - 1}{(x^2 - y^2)^2(7y^2 - x^2)}.$$

12. $3(4x - 1)^2(4) = 15y^2\frac{dy}{dx},\ \frac{dy}{dx} = \frac{12(4x - 1)^2}{15y^2}$.

13. $\frac{y \cdot 2x - x^2y'}{y^2} - y' = \frac{1}{2} + 8y^{-3}y'$,
$2xy - x^2y' - y^2y' = \frac{y^2}{2} + 8y^{-1}y'$,
$$y' = \frac{2xy - \frac{y^2}{2}}{8y^{-1} + x^2 + y^2} = \frac{4xy^2 - y^3}{16 + x^2y + y^3}.$$

14. $7(5x^2y + 4)^6(5x^2y' + 10xy) = 3x^2$,
$$y' = \frac{3x^2 - 70xy(5x^2y + 4)^6}{35x^2(5x^2y + 4)^6}.$$

15. $\frac{1 + \frac{dy}{dx}}{2\sqrt{x + y}} + \frac{1 - \frac{dy}{dx}}{2\sqrt{x - y}} = 0,\ \frac{dy}{dx} = \frac{\sqrt{x + y} + \sqrt{x - y}}{\sqrt{x + y} - \sqrt{x - y}}$.

16. $\frac{(x - y) - x(1 - \frac{dy}{dx})}{(x - y)^2} + \frac{x\frac{dy}{dx} - y}{x^2} = 0,\ \frac{dy}{dx} = \frac{y}{x}$.

17. $\frac{dy}{dx} = \cos(2x + y)\left[2 + \frac{dy}{dx}\right] = 2\cos(2x + y) + \frac{dy}{dx}\cos(2x + y)$,
$\frac{dy}{dx}\left[1 - \cos(2x + y)\right] = 2\cos(2x + y)$,
$$\frac{dy}{dx} = \frac{2\cos(2x + y)}{1 - \cos(2x + y)}.$$

18. $\cos y - x\sin y\left(\frac{dy}{dx}\right) = 2(x + y)\left(1 + \frac{dy}{dx}\right);\ -x\sin y\left(\frac{dy}{dx}\right) - 2(x + y)\frac{dy}{dx} = -\cos y + 2(x + y)$,
$\frac{dy}{dx}\left[x\sin y + 2(x + y)\right] = \cos y - 2(x + y)$,
$$\frac{dy}{dx} = \frac{\cos y - 2(x + y)}{x\sin y + 2(x + y)}.$$

19. $\sec^2(xy)\left[y + x\frac{dy}{dx}\right] + y + x\frac{dy}{dx} = 0,\ x\sec^2(xy)\frac{dy}{dx} + x\frac{dy}{dx} = -y\sec^2(xy) - y$,
$$\frac{dy}{dx} = \frac{-y\left[\sec^2(xy) + 1\right]}{x\left[\sec^2(xy) + 1\right]} = -\frac{y}{x}.$$

20. $2\tan x\sec^2 x + 2\tan y\sec^2 y\frac{dy}{dx} = 0$,
$$\frac{dy}{dx} = \frac{-\tan x\sec^2 x}{\tan y\sec^2 y}.$$

21. $2 \sin x \cos x + 2 \cos y(-\sin y) \frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{\sin x \cos x}{\sin y \cos y}$.

22. $\sec(x + y)\tan(x + y)\left[1 + \frac{dy}{dx}\right] - \csc(x + y)\cot(x+y)\left[1 + \frac{dy}{dx}\right] = 0$,
$\frac{dy}{dx}\left[\sec(x + y)\tan(x + y) - \csc(x + y)\cot(x + y)\right] = \csc(x + y)\cot(x + y) - \sec(x + y)\tan(x + y)$,
$\frac{dy}{dx} = \frac{\csc(x + y)\cot(x + y) - \sec(x + y)\tan(x + y)}{-[\csc(x + y)\cot(x + y) - \sec(x + y)\tan(x + y)]} = -1$.

23. $-\sin xy\left[xy' + y\right] + 2y\, y' = 0$
$y' = \frac{y \sin xy}{-x \sin xy + 2y}$.

24. $\sin x \sec^2 y(y') + \tan y \cos x = 3y^2 y'$
$y' = \frac{\tan y \cos x}{3y^2 - \sin x \sec^2 y}$.

25. $-\csc^2(3x + y)\left[3 + y'\right] = 5xy' + 5y$
$y' = -\frac{5y + 3\csc^2(3x + y)}{\csc^2(3x + y) + 5x}$.

26. $\sec(x^2 + y^2)\tan(x^2 + y^2)\left[2x + 2y\, y'\right] = 15x^2$
$y' = \frac{15x^2 - 2x \sec(x^2 + y^2)\tan(x^2 + y^2)}{2y \sec(x^2 + y^2)\tan(x^2 + y^2)}$.

27. $2x + x\frac{dy}{dx} + y + 4y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{-2x - y}{x + 4y}$, so when $x = 2$ and $y = 3$, $\frac{dy}{dx} = \frac{-4 - 3}{2 + 12} = -\frac{1}{2}$. Tangent line has equation $(y - 3) = -\frac{1}{2}(x - 2)$, or $y = -\frac{1}{2}x + 4$. Normal line has equation $(y - 3) = 2(x - 2)$, or $y = 2x - 1$.

28. $3x^2 - 6xy\frac{dy}{dx} = 3y^2 + 3y^2\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 - 3xy}$; so when $x = 2$ and $y = -1$, $\frac{dy}{dx} = \frac{(-1)^2 - (2)^2}{(-1)^2 - 3(2)(-1)} = -\frac{3}{7}$. Tangent line has equation $(y + 1) = -\frac{3}{7}(x-2)$, or $y = -\frac{3}{7}x - \frac{1}{7}$. Normal line has equation $(y+1) = \frac{7}{3}(x - 2)$, or $y = \frac{7}{3}x - \frac{17}{3}$.

29. $\frac{1}{\sqrt{2x}} + \frac{3}{2\sqrt{3y}}\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{2\sqrt{3y}}{3\sqrt{2x}}$. When $x = 2$ and $y = 3$, $\frac{dy}{dx} = -\frac{2\sqrt{9}}{3\sqrt{4}} = -1$. Equation of tangent line is $y - 3 = (-1)(x - 2)$, or $y = -x + 5$. Equation of normal line is $y - 3 = x - 2$, or $y = x + 1$.

30. $2x - \frac{x\frac{dy}{dx} + y}{\sqrt{xy}} - 2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{2x\sqrt{xy} - y}{2y\sqrt{xy} + x}$. When $x = 8$ and $y = 2$, $\frac{dy}{dx} = \frac{16\sqrt{16} - 2}{4\sqrt{16} + 8} = \frac{31}{12}$. Equation of tangent line is $y - 2 = \frac{31}{12}(x - 8)$, or $y = \frac{31}{12}x - \frac{56}{3}$. Equation of normal line is $y - 2 = -\frac{12}{31}(x - 8)$, or $y = -\frac{12}{31}x + \frac{158}{31}$.

31. $\cos xy\left[xy' + y\right] = y'$
$y' = \frac{y \cos xy}{1 - x \cos xy}$. When $x = \frac{\pi}{2}$, $y = 1$,
$y' = \frac{1 \cos\frac{\pi}{2}}{1 - \frac{\pi}{2}\cos\frac{\pi}{2}} = \frac{0}{1 - 0} = 0$.
Equation of tangent line is $y = 1$.
Equation of normal line is $x = \frac{\pi}{2}$.

32. $2(5 + \tan xy)\sec^2 xy\left[xy' + y\right] = 0$,
$xy' + y = 0$,
so $y' = -\frac{y}{x}$; when $x = \frac{\pi}{12}$, $y = 3$,
$y' = \frac{-3}{\frac{\pi}{12}} = -\frac{36}{\pi}$.
Equation of tangent line is $y - 3 = -\frac{36}{\pi}(x - \frac{\pi}{12})$.
Equation of normal line is $y - 3 = \frac{\pi}{36}(x - \frac{\pi}{12})$.

33. $6x\frac{dx}{dy} + 5x + 5\frac{dx}{dy}y = 0$, $\frac{dx}{dy} = -\frac{5x}{6x + 5y}$.

34. $2x^2 y + 2x\frac{dx}{dy}y^2 = 2x\frac{dx}{dy} + 2y$, $\frac{dx}{dy} = \frac{y}{x} \cdot \frac{x^2 - 1}{1 - y^2}$.

35. $2x\frac{dx}{dy} = 2y - 1$, $\frac{dx}{dy} = \frac{2y - 1}{2x}$.

36. $\frac{x + \frac{dx}{dy}y}{2\sqrt{xy}} + 4xy^3 + \frac{dx}{dy}y^4 = 0$, $\frac{dx}{dy} = \frac{-x - 8xy^3\sqrt{xy}}{y + 2y^4\sqrt{xy}}$.

37. $f(x) = \frac{5x - 6}{4}$.

38. $f(x) = \frac{1}{2}\sqrt{5x^2 - 6}$ and $g(x) = -\frac{1}{2}\sqrt{5x^2 - 6}$.

39. $f(x) = \frac{-x + \sqrt{x^2 + 4x}}{2}$ and $g(x) = \frac{-x - \sqrt{x^2 + 4x}}{2}$.

40. $f(x) = \frac{-4 + \sqrt{16 - 12x^2}}{6x}$ and $g(x) = \frac{-4 - \sqrt{16 - 12x^2}}{6x}$.

41. $f(x) = \frac{x + 1}{2 - 3x}$.

42. $f(x) = 1$ and $g(x) = x$ and $h(x) = -x$.

43. $f(x) = \sqrt[4]{x}$ and $g(x) = -\sqrt[4]{x}$.

44. Let $g(y) = y^3(y^2 + 4) = y^5 + 4y^3$. Then $g'(y) = 5y^4 +$

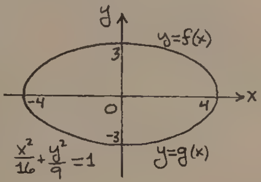
$12y^2$ is positive for $y \neq 0$; hence, g is increasing on the interval $(-\infty,\infty)$. Let $f = g^{-1}$. Then f is implicit in the given equation.

5. $y^3 - 3y^2 + 3y - 3x - 3 = 0$; $(y^3 - 3y^2 + 3y - 1) - 3x - 2 = 0$; $(y-1)^3 = 3x + 2$; so $f(x) = 1 + \sqrt[3]{3x+2}$.

6. $f(x) = x$.

7. Let $y = f(x) = \frac{3}{4}\sqrt{16 - x^2}$. Then $y^2 = \frac{9}{16}(16 - x^2)$, $\frac{y^2}{9} = 1 - \frac{x^2}{16}$, so $\frac{x^2}{16} + \frac{y^2}{9} = 1$ as desired. Every point on the graph of f is on the graph of $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

8. Let $y = g(x) = -\frac{3}{4}\sqrt{16 - x^2}$. Then $y^2 = \frac{9}{16}(16 - x^2)$ $\frac{y^2}{9} = 1 - \frac{x^2}{16}$, so $\frac{x^2}{16} + \frac{y^2}{9} = 1$ as desired. Every point on the graph of g is on the graph of $\frac{x^2}{16} + \frac{y^2}{9} = 1$. The graph of $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (an ellipse) is shown in the accompanying figure. The graph of f is the "top half" of the ellipse, while the graph of g is the "bottom half."



9. (a) $f'(x) = \frac{3}{4}\left(\frac{-2x}{2\sqrt{16 - x^2}}\right) = -\frac{3x}{4\sqrt{16 - x^2}}$.

(b) $g'(x) = -\frac{3}{4}\left(\frac{-2x}{2\sqrt{16 - x^2}}\right) = \frac{3x}{4\sqrt{16 - x^2}}$.

(c) $\frac{2x}{16} + \frac{2y}{9}\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{9x}{16y}$.

(d) Let $y = f(x) = \frac{3}{4}\sqrt{16 - x^2}$. By (c), $\frac{dy}{dx} = -\frac{9x}{16y} = -\frac{9x}{16\left(\frac{3}{4}\sqrt{16 - x^2}\right)}$

$= -\frac{3x}{4\sqrt{16 - x^2}}$ as in (a).

(e) Let $y = g(x) = -\frac{3}{4}\sqrt{16 - x^2}$. By (c), $\frac{dy}{dx} = -\frac{9x}{16y} = -\frac{9x}{16\left(-\frac{3}{4}\sqrt{16 - x^2}\right)} = \frac{3x}{4\sqrt{16 - x^2}}$

as in (b).

0. The equation $f'(x) = -\frac{3x}{4\sqrt{16 - x^2}}$ gives the slope of the tangent line to the graph of f (hence, to the graph of $\frac{x^2}{16} + \frac{y^2}{9} = 1$) at the point $(x, f(x))$. The equation $g'(x) = \frac{3x}{4\sqrt{16 - x^2}}$ gives the slope of the tangent line to the graph of g(hence, to the graph of $\frac{x^2}{16} + \frac{y^2}{9} = 1$) at the point $(x, g(x))$. But, in any case, the equation $\frac{dy}{dx} = -\frac{9x}{16y}$ gives the slope of the tangent line to the graph of $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point (x,y).

51. (a) $\frac{2x}{30} - \frac{2y}{20}\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{2x}{3y}$. Thus, the slope of the tangent line to the graph of $\frac{x^2}{30} - \frac{y^2}{20} = 1$ at the point $(6,-2)$ is $\frac{2(6)}{3(-2)} = -2$.

(b) $y^2 = 20\left(\frac{x^2}{30} - 1\right)$, so $y = \pm 2\sqrt{5}\sqrt{\frac{x^2}{30} - 1}$. Since we want the tangent line at the point $(6,-2)$, we use the minus sign in the solution to get $y = -2\sqrt{5}\sqrt{\frac{x^2}{30} - 1}$. Thus,

$$\frac{dy}{dx} = -2\sqrt{5}\,\frac{\frac{2x}{30}}{2\sqrt{\frac{x^2}{30} - 1}} = \frac{-2\sqrt{5}\,x}{30\sqrt{\frac{x^2}{30} - 1}};$$ so when $x = 6$ (and $y = -2$), we have $\frac{dy}{dx} = \frac{-2\sqrt{5}\,(6)}{30\sqrt{\frac{36}{30} - 1}} = -2$.

52. $8x + 6(y - 200)y' = 0$,

$y' = \frac{-4x}{3(y - 200)}$. When $x = 150$, $y = 100$, so that the slope of the tangent line $m = \frac{-4(150)}{3(100 - 200)} = 2$.

Equation of tangent line is $y - 100 = 2(x - 150)$

When $y = 0$,

$-100 = 2(x - 150)$,

so $x = 100$.

53. $140x + 2y\,y' - 2000 - 150y' = 0$

When $x = 10$, $y = 80$;

$1400 + 160y' - 2000 - 150y' = 0$

or

$y' = \frac{600}{10} = 60$.

Problem Set 2.9, page 142

1. $f'(x) = \frac{\sqrt[5]{x}}{5x}$.

2. $f'(x) = \frac{\sqrt[7]{x^2}}{7x^2}(2x) = \frac{2\sqrt[7]{x^2}}{7x}$

3. $g'(x) = -16x^{-13/9}$. 4. $f'(x) = 15x^{-2/7}$.

5. $h'(t) = -\frac{2}{3}(1-t)^{-5/3}(-1) = \frac{2}{3}(1-t)^{-5/3}$.

6. $f(x) = x^{-4/5}$, so that $f'(x) = -\frac{4}{5}x^{-9/5}$.

7. $f'(u) = \frac{3}{4}(1+\frac{2}{u})^{-1/4}(\frac{-2}{u^2}) = \frac{-3}{2u^2}(1+\frac{2}{u})^{-1/4}$.

8. $g(s) = (9-s^2)^{1/3}(9+s^2)^{-1/3}$

$g'(s) = (9-s^2)^{-1/3}(-\frac{1}{3})(9+s^2)^{-4/3}(25)+(9+s^2)^{-1/3}(\frac{1}{3})\cdot$

$(9-s^2)^{-2/3}(-25)$

$= (9-s^2)^{-2/3}(9+s^2)^{-4/3}(-\frac{25}{3})\cdot$

$[(9-s^2)+(9+s^2)]$

$= -150(9-s^2)^{-2/3}(9+s^2)^{-4/3}$.

9. $g'(x) = -\frac{1}{2}x^{-3/2} - \frac{1}{3}x^{-4/3} - \frac{1}{4}x^{-5/4}$.

10. $f(x) = x^{1/2} + x^{1/3} + x^{1/4}$, so that $f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4}$.

11. $f(t) = (t^3 - t^{1/4})^{1/5}$, so that $f'(t) = \frac{1}{5}(t^3 - t^{1/4})^{-4/5}(3t^2 - \frac{1}{4}t^{-3/4})$.

12. $g(y) = (y^4 - y + y^{1/3})^{1/2}$, so that

$g'(y) = \frac{1}{2}(y^4 - y + y^{1/3})^{-1/2}(4y^3 - 1 + \frac{1}{3}y^{-2/3})$.

13. $g'(x) = \dfrac{\sqrt[10]{\frac{x}{x+1}}}{10(\frac{x}{x+1})}\left(\dfrac{(x+1)-x}{(x+1)^2}\right) = \dfrac{\sqrt[10]{\frac{x}{x+1}}}{10x(x+1)}$.

14. $f'(x) = (x + x^{1/5})(1 - 2\cdot\frac{1}{3}x^{-2/3}) + (x - 2x^{1/3})(1 + \frac{1}{5}x^{-4/5})$

$= 2x + \frac{6}{5}x^{1/5} - \frac{8}{3}x^{1/3} - \frac{16}{15}x^{-7/15}$.

15. $h'(x) = (1+x)^{-3/4}(\frac{1}{2})(2x+1)^{-1/2}(2) + (-\frac{3}{4})(1+x)^{-7/4}(2x+1)^{1/2}$

$= (1+x)^{-3/4}(2x+1)^{-1/2} - \frac{3}{4}(1+x)^{-7/4}(2x+1)^{1/2}$

$= (1+x)^{-7/4}(2x+1)^{-1/2}[(1+x) - \frac{3}{4}(2x+1)]$

$= \frac{1-2x}{4}(1+x)^{-7/4}(2x+1)^{-1/2}$.

16. $f(t) = t(36-t^2)^{-1/4}$

$f'(t) = t(-\frac{1}{4})(36-t^2)^{-5/4}(-2t) + (36-t^2)^{-1/4}$

$= (36-t^2)^{-5/4}\left[\frac{t^2}{2} + (36-t^2)\right]$

$= (36 - \frac{t^2}{2})(36-t^2)^{-5/4}$.

17. $f'(t) = \sqrt[4]{t+2}\,\dfrac{\sqrt[5]{t+5}}{5(t+5)} + \dfrac{\sqrt[4]{t+2}}{4(t+2)}\sqrt[5]{t+5}$

$= \sqrt[4]{t+2}\,\sqrt[5]{t+5}\left[\dfrac{1}{5(t+5)} + \dfrac{1}{4(t+2)}\right]$

$= \dfrac{9t+33}{20(t+5)(t+2)}\sqrt[4]{t+2}\,\sqrt[5]{t+5}$.

18. $g(x) = x^{1/3}(1+2\sqrt{x}) = x^{1/3} + 2x^{5/6}$, so that

$g'(x) = \frac{1}{3}x^{-2/3} + \frac{5}{3}x^{-1/6}$.

19. $f(t) = \sqrt[5]{\sin t} = (\sin t)^{1/5}$, so that $f'(t)$

$= \frac{1}{5}\sin^{-4/5}t\cos t$.

20. $g(x) = \sqrt[7]{\cos 3x} = (\cos 3x)^{1/7}$, so that

$g'(x) = \frac{1}{7}(\cos 3x)^{-6/7}(-\sin 3x)(3)$

$= -\frac{3}{7}\cos^{-6/7}3x\sin 3x$.

21. $g(x) = \cos^{3/4}x$, so that

$g'(x) = \frac{3}{4}\cos^{-1/4}x(-\sin x) = -\frac{3}{4}\cos^{-1/4}x\sin x$.

22. $h(t) = \sin^{5/7}(4t-1)$, so that

$h'(t) = \frac{5}{7}\sin^{-2/7}(4t-1)(\cos(4t-1))(4)$

$= \frac{20}{7}\sin^{-2/7}(4t-1)\cos(4t-1)$.

23. $F'(t) = \frac{3}{2}\sec^{1/2}(4t^2+1)\sec(4t^2+1)\tan(4t^2+1)\cdot 8$

$= 12t\sec^{3/2}(4t^2+1)\tan(4t^2+1)$.

24. $G(s) = 3\cot(4s^2)^{1/3}$

$G'(s) = -3\csc^2\sqrt[3]{4s^2}\left[\frac{1}{3}(4s^2)^{-2/3}(8s)\right]$

$= -8s(4s^2)^{-2/3}\csc^2\sqrt[3]{4s^2} = -2^{23/3}\cdot s^{2/3}\csc^2\sqrt[3]{4s^2}$.

25. $H'(y) = \tan\sqrt[3]{y}(-\csc\sqrt[4]{y}\cot\sqrt[4]{y})\frac{y^{1/4}}{4y} + \csc\sqrt[4]{y}(\sec^2\sqrt[3]{y}$ $\frac{y^{1/3}}{3y}$

$= -\frac{1}{4}y^{-3/4}\tan\sqrt[3]{y}\csc\sqrt[4]{y}\cot\sqrt[4]{y} + \frac{1}{3}y^{-2/3}\csc\sqrt[4]{y}\sec^2\sqrt[3]{y}$

$= \frac{y^{-3/4}\csc\sqrt[4]{y}}{12}(4y^{1/12}\sec^2\sqrt[3]{y} - 3\tan\sqrt[3]{y}\cot\sqrt[4]{y})$.

26. $F'(\theta) = \frac{1}{3}(\frac{\sin\theta - 1}{\cos\theta})^{-2/3}\left[\frac{\cos\theta(\cos\theta)-(\sin\theta-1)(-\sin\theta)}{\cos^2\theta}\right]$

$= \frac{1}{3}(\frac{\sin\theta - 1}{\cos\theta})^{-2/3}\,\frac{1-\sin\theta}{\cos^2\theta}$

$$= -\frac{1}{3}\,\frac{(\sin\theta - 1)^{1/3}}{\cos^{4/3}\theta}.$$

7. $P'(z) = z^{2/3}(-\sin z^{-2/3})(-\frac{2}{3}z^{-5/3}) + \cos z^{-2/3}\left[\frac{2}{3}z^{-1/3}\right]$

$= \frac{2}{3}z^{-1}\sin z^{-2/3} + \frac{2}{3}z^{-1/3}\cos z^{-2/3}$

$= \frac{2}{3}z^{-1}(\sin z^{-2/3} + z^{2/3}\cos z^{-2/3})$.

8. $Q'(\phi) = \frac{4}{3}\left(\frac{\cos\phi + 1}{1 - \cos\phi}\right)^{1/3}\left[\frac{(1-\cos\phi)(-\sin\phi)-(\cos\phi+1)(\sin\phi)}{(1-\cos\phi)^2}\right]$

$= \frac{4}{3}\left(\frac{\cos\phi + 1}{1 - \cos\phi}\right)^{+1/3}\left[\frac{-2\sin\phi}{(1-\cos\phi)^2}\right]$

$= -\frac{8}{3}\,\frac{(\cos\phi + 1)^{1/3}\sin\phi}{(1-\cos\phi)^{7/3}}$.

9. $f'(\theta) = \frac{1}{3}(3 + \cos^4 3\theta)^{-2/3}\left[4\cos^3 3\theta\right]\left[-\sin 3\theta\right]\left[3\right]$

$= -4\sin 3\theta\cos^3 3\theta(3 + \cos^4 3\theta)^{-2/3}$.

0. $g'(x) = -\frac{4}{5}(7 - \sec^2 x)^{-9/5}\left[-2\sec x(\sec x\tan x)\right]$

$= \frac{8}{5}\sec^2 x\tan x(7 - \sec^2 x)^{-9/5}$.

1. $y' = \frac{1}{3}(4x^2 + 23)^{-2/3}(8x)$.

When $x = 1$,

$y' = \frac{1}{3}(4 + 23)^{-2/3}(8) = \frac{8}{3}\cdot\frac{1}{9} = \frac{8}{27}$.

Equation of the tangent line is $y - 3 = \frac{8}{27}(x - 1)$.

Equation of the normal line is $y - 3 = -\frac{27}{8}(x - 1)$.

2. $y' = \frac{1}{5}\left(\frac{1 - x}{1 + x}\right)^{-4/5}\left[\frac{(1 + x)(-1) - (1 - x)}{(1 + x)}\right]$.

When $x = 1$, y' is not defined.

So equation of the tangent line is $x = 1$.

Equation of the normal line is $y = 0$.

3. $y' = \frac{1}{7}\left(\frac{3x^3 + 1}{x^3 - 1}\right)^{-6/7}\left[\frac{(x^3 - 1)(9x^2) - (3x^3 + 1)(3x^2)}{(x^3 - 1)^2}\right]$.

When $x = 0$,

$y' = \frac{1}{7}(-1)^{-6/7}\left[\frac{0 - 0}{(-1)^2}\right] = 0$.

So equation of the tangent line is $y = -1$.

Equation of the normal line is $x = 0$.

4. $y' = \frac{3}{2}\left(\frac{x}{5x^2 + 3}\right)^{1/2}\left[\frac{5x^2 + 3 - x(10x)}{(5x^2 + 3)^2}\right]$.

When $x = 3$,

$y' = \frac{3}{2}\left(\frac{1}{16}\right)^{1/2}\left[\frac{48 - 90}{(48)^2}\right] = \frac{-7}{1024}$.

So equation of the tangent line is $y - \frac{1}{64} - \frac{7}{1024}(x - 3)$.

Equation of the normal line is $y - \frac{1}{64} = \frac{1024}{7}(x - 3)$.

35. $2\cdot\frac{1}{3}x^{-2/3} - \frac{2}{3}y^{-1/3}y' = 1$

When $x = 1$, $y = 1$; then $\frac{2}{3} - \frac{2}{3}y' = 1$ or $y' = -\frac{1}{2}$.

So equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1)$.

Equation of the normal line is $y - 1 = 2(x - 1)$.

36. $y' + \frac{1}{4}(15 + 2\sin xy)^{-3/4}\left[(2\cos xy)(xy' + y)\right] = 0$.

When $x = \frac{\pi}{3}$, $y = \frac{1}{2}$;

$y' + \frac{1}{4}(15 + 2\sin\frac{\pi}{6})^{-3/4}\left[(2\cos\frac{\pi}{6})(\frac{\pi}{3}y' + \frac{1}{2})\right] = 0$,

$y' + \frac{\sqrt{3}}{32}(\frac{\pi}{3}y' + \frac{1}{2}) = 0$, $y' = \dfrac{-\frac{\sqrt{3}}{64}}{1 + \frac{\sqrt{3}\pi}{96}}$.

So equation of the tangent line is $y - \frac{1}{2} =$

$$\frac{-\frac{\sqrt{3}}{64}}{1 + \frac{\sqrt{3}\pi}{96}}\left(x - \frac{\pi}{3}\right).$$

Equation of the normal line is $y - \frac{1}{2} =$

$$\frac{1 + \frac{\sqrt{3}\pi}{96}}{\frac{\sqrt{3}}{64}}\left(x - \frac{\pi}{3}\right).$$

37. (a) $D_x\sqrt[4]{x} = D_x(\sqrt{\sqrt{x}}) = \frac{1}{2\sqrt{\sqrt{x}}}\,\frac{1}{2\sqrt{x}} = \frac{1}{4x^{3/4}} = \frac{1}{4}x^{3/4}$, for $x > 0$.

(b) $D_x(\sqrt[4]{x}) = D_x x^{1/4} = \frac{1}{4}x^{1/4 - 1} = \frac{1}{4}x^{-3/4}$.

38. (a) We used the assumption than n is odd when we wrote Δx as $\left[(\Delta x)^{1/n}\right]^n$. (If n had been <u>even</u> and Δx had been <u>negative</u>, this would not have worked.)

(b) We used the assumption that $m > n$, that is, $m - n > 0$, when we wrote $0^{m-n} = 0$.

39. $\frac{dN}{dt} = 1200\cdot\frac{3}{2}t^{1/2} = 1800t^{1/2}$.

When $N = 25$, $\frac{dN}{dt} = 1800(25)^{1/2} = 1800(5) = 9000$.

40. $\frac{dP}{dt} = 1000 \cdot \frac{1}{4}(t^5 + 10t^2 + 9)^{-3/4}(5y^4 + 20t)$.
When $t = 2$, $\frac{dP}{dt} = 250(81)^{-3/4}(120) = 250(\frac{1}{27})(120) =$
$\frac{10{,}000}{9} = 1111.1$.

Problem Set 2.10, page 148

1. $v = 3t^2 + 4t + 3$ ft/sec, $a = 6t + 4$ ft/sec^2.

2. $v = \dfrac{-2t}{(t^2+1)^2}$ cm/sec, $a = \dfrac{6t^2 - 2}{(t^2+1)^3}$ cm/sec^2.

3. $v = 3\pi \cos \pi t + 4\pi \sin 2\pi t$ m/sec,
$a = -3\pi^2 \sin \pi t + 8\pi^2 \cos 2\pi t$ m/sec^2.

4. $v = \dfrac{2t^2 + 4}{\sqrt{t^2+4}}$ mi/hr, $a = \dfrac{2t^3 + 12t}{(t^2+4)\sqrt{t^2+4}}$ mi/hr^2.

5. $v = \frac{25}{4} t^{3/2} + t^{1/2}$ km/hr, $a = \frac{75}{8} t^{1/2} + \frac{1}{2} t^{-1/2}$ km/hr^2.

6. $v = gt + v_0$, $a = g$.

7.

Problem #	v	a
1	10 m/sec	10 m/sec^2
3	-3π m/sec	$-8\pi^2$ m/sec^2
5	$\frac{29}{4}$ km/hr	$\frac{79}{8}$ km/hr^2

8. (a) $v = 9.8t$ m/sec^2.
(b) $s = 4.9t^2$, so $t = \dfrac{\sqrt{s}}{\sqrt{4.9}}$;
$v = 9.8 \dfrac{\sqrt{s}}{\sqrt{4.9}} = \dfrac{9.8\sqrt{s}\sqrt{4.9}}{4.9} = 2\sqrt{4.9s}$.

9. $f'(x) = 15x^2 + 4$, $f''(x) = 30x$.

10. $g(x) = x^4 + 7x^2$, $g'(x) = 4x^3 + 14x$, $g''(x) = 12x^2 + 14$.

11. $f'(t) = 35t^4 - 46t + 1$, $f''(t) = 140t^3 - 46$.

12. $F'(x) = x^3(2(x+2)) + 3x^2(x+2)^2 = x^2(x+2)(5x+6)$.
$F''(x) = 2x^3 + 12x^2(x+2) + 6x(x+2)^2 = 4x(5x^2 + 12x + 6)$.

13. $G(x) = x^6 - 27$, $G'(x) = 6x^5$, $G''(x) = 30x^4$.

14. $f'(u) = 6u(u^2+1)^2$, $f''(u) = 6(u^2+1)(5u^2+1)$.

15. $g(t) = t^{7/2} - 5t$, $g'(t) = \frac{7}{2}t^{5/2} - 5$, $g''(t) = \frac{35}{4}t^{3/}$

16. $f'(x) = 1 + \frac{3}{x^2}$, $f''(x) = -\frac{6}{x^3}$.

17. $f(x) = x^2 - x^{-3}$, $f'(x) = 2x + 3x^{-4}$, $f''(x) = 2 - 12x^{-5}$.

18. $g'(x) = 2(x + \frac{1}{x})(1 - \frac{1}{x^2}) = 2(x - \frac{1}{x^3})$, $g''(x) = 2(1 + \frac{3}{x^4})$.

19. $f'(u) = \dfrac{4}{(2-u)^2}$, $f''(u) = \dfrac{8}{(2-u)^3}$.

20. $F'(v) = \frac{1}{2}v^{-1/2} - \frac{1}{2}v^{-3/2}$, $F''(v) = -\frac{1}{4}v^{-3/2} + \frac{3}{4}v^{-5/2}$.

21. $f'(t) = \dfrac{t}{\sqrt{t^2+1}}$, $f''(t) = \dfrac{1}{(t^2+1)^{3/2}}$.

22. $g'(y) = \dfrac{3}{2\sqrt{3y+1}}$. $g''(y) = \dfrac{-9}{4(\sqrt{3y+1})^3}$.

23. $F'(r) = 2(1 - \sqrt{r})(\frac{-1}{2\sqrt{r}}) = 1 - \frac{1}{\sqrt{r}}$, $F''(r) = \dfrac{1}{2(\sqrt{r})^3}$

24. $h'(x) = (x^2+1)^{-3/2}$, $h''(x) = -3x(x^2+1)^{-5/2}$.

25. $f'(x) = -77 \sin 11x$,
$f''(x) = -847 \cos 11x$.

26. $f'(t) = 12 \cos(5 - 2t)$,
$f''(t) = 24 \sin(5 - 2t)$.

27. $F'(\theta) = 2 \cos 2\theta - 3 \sin 3\theta$,
$F''(\theta) = -4 \sin 2\theta - 9 \cos 3\theta$.

28. $h'(x) = 4(x + \sin x)^3(1 + \cos x)$,
$h''(x) = 4(x \sin x)^3(-\sin x) + (1 + \cos x) \cdot 12 \cdot (x + \sin x)^2(1 + \cos x)$
$= 4(x + \sin x)^2\left[(x + \sin x)(-\sin x) + (1 + \cos x)^2 \cdot 3\right]$.

29. $H'(t) = -14 \csc 7t \cot 7t$,
$H''(t) = -14 \csc 7t(-7\csc^2 7t) + \cot 7t\left[98 \csc 7t \cot 7t\right]$
$= 98 \csc 7t(\csc^2 7t + \cot^2 7t)$.

. $p'(y) = 15 \tan^2 4y(\sec^2 4y)(4) = 60 \tan^2 4y \sec^2 4y,$

$p''(y) = 60 \tan^2 4y \left[2 \sec 4y \sec 4y \tan 4y \cdot 4\right] + \sec^2 4y \left[120 \tan 4y \sec^2 4y \cdot 4\right]$
$= 480 \tan 4y \sec^2 4y(\tan^2 4y + \sec^2 4y).$

. $Q'(\theta) = \theta\left[-\csc^2 3\theta \cdot 3\right] + \cot 3\theta = -3\theta \csc^2 3\theta + \cot 3\theta,$

$Q''(\theta) = -3\theta\left[2 \csc 3\theta(-\csc 3\theta \cot 3\theta)(3)\right] - 3 \csc^2 3\theta + (-\csc^2 3\theta)(3)$
$= 6 \csc^2 3\theta\left[3\theta \cot 3\theta - 1\right].$

. $S'(x) = \dfrac{5 \cos 5x}{2\sqrt{1 + \sin 5x}},$

$S''(x) = \dfrac{2\sqrt{1 + \sin 5x}(-25 \sin 5x) - 5 \cos 5x\left(\dfrac{5 \cos 5x}{\sqrt{1 + \sin 5x}}\right)}{4(1 + \sin 5x)}$

$= \dfrac{-50(\sin 5x)(1 + \sin 5x) - 25 \cos^2 5x}{4(1 + \sin 5x)^{3/2}}$

$= \dfrac{-50 \sin 5x - 50 \sin^2 5x - 25 \cos^2 5x}{4(1 + \sin 5x)^{3/2}}$

$= \dfrac{-50 \sin 5x - 25 \sin^2 5x - 25}{4(1 + \sin 5x)^{3/2}}$

$= \dfrac{-25(\sin 5x + 1)^2}{4(1 + \sin 5x)^{3/2}} = -\dfrac{25}{4}\sqrt{(\sin 5x) + 1}.$

. $G'(x) = \left[\cos\left(\dfrac{x}{x + 1}\right)\right]\left[\dfrac{(x + 1) - x}{(x + 1)^2}\right] = \dfrac{1}{(x + 1)^2}\cos\left(\dfrac{x}{x + 1}\right),$

$G''(x) = \dfrac{1}{(x + 1)^2}\left(-\sin\left(\dfrac{x}{x + 1}\right)\left[\dfrac{1}{(x + 1)^2}\right]\right) + \cos\left(\dfrac{x}{x + 1}\right)\left[\dfrac{-2}{(x + 1)^3}\right]$

$= \dfrac{-1}{(x + 1)^3}\left[\dfrac{1}{x + 1}\sin\left(\dfrac{x}{x + 1}\right) + 2 \cos\left(\dfrac{x}{x + 1}\right)\right].$

. $y' = \frac{1}{2}(1 + \sec x)^{-1/2}(\sec x \tan x) = \dfrac{\sec x \tan x}{2\sqrt{1 + \sec x}},$

$y'' = \dfrac{2\sqrt{1 + \sec x}(\sec x \sec^2 x + \tan x \sec x \tan x)}{4(1 + \sec x)} - \dfrac{\sec x \tan x(1 + \sec x)^{-1/2}(\sec x \tan x)}{4(1 + \sec x)}$

$= \dfrac{2(1 + \sec x)(\sec^3 x + \sec x \tan^2 x) - \sec^2 x \tan^2 x}{4(1 + \sec x)^{3/2}}$

$= \dfrac{2 \sec^4 x + 2 \sec^3 x + 2 \sec x \tan^2 x + \sec^2 x \tan^2 x}{4(1 + \sec x)^{3/2}}.$

35. $y' = -20 \sec 5x \tan 5x,$

$y'' = -20 \sec 5x(5 \sec^2 5x) + \tan 5x(-100 \sec 5x \tan 5x),$
$= -100 \sec 5x(\sec^2 5x + \tan^2 5x).$

36. $y' = x^2\left[\left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right] + \sin \frac{1}{x}(2x)$
$= -\cos \frac{1}{x} + 2x \sin \frac{1}{x},$

$y'' = \sin \frac{1}{x}\left(-\frac{1}{x^2}\right) + 2x \cos \frac{1}{x}\left(-\frac{1}{x^2}\right) + 2 \sin \frac{1}{x}$
$= -\frac{1}{x^2} \sin \frac{1}{x} - \frac{2}{x} \cos \frac{1}{x} + 2 \sin \frac{1}{x}.$

37. $f'(-1) = 2(-1) + 3(-1)^{-4} = -2 + 3 = 1,$

$f''(-1) = 2 - 12(-1)^{-5} = 2 + 12 = 14.$

38. $h'(\sqrt{2}) = (2 + 1)^{-3/2} = 3^{-3/2} = \frac{\sqrt{3}}{9},$

$h''(\sqrt{2}) = -3\sqrt{2}(2 + 1)^{-5/2} - 3\sqrt{2} \cdot 3 = -\sqrt{2}\, 3^{-3/2} = -\frac{\sqrt{6}}{9}$

39. $F'(\frac{\pi}{6}) = 2 \cos \frac{\pi}{3} - 3 \sin \frac{\pi}{2} = 2(\frac{1}{2}) - 3(1) = -2,$

$F''(\frac{\pi}{6}) = -4 \sin \frac{\pi}{3} - 9 \cos \frac{\pi}{2} = -4(\frac{\sqrt{3}}{2}) - 9(0) = -2\sqrt{3}.$

40. When $x = \frac{\pi}{4}$,

$\dfrac{dy}{dx} = y' = \dfrac{\sec \frac{\pi}{4} \tan \frac{\pi}{4}}{2\sqrt{1 + \sec \frac{\pi}{4}}} = \dfrac{\sqrt{2} \cdot 1}{2\sqrt{1 + \sqrt{2}}} = \dfrac{\sqrt{2}}{2\sqrt{1 + \sqrt{2}}}$

$\dfrac{d^2y}{dx^2} = y'' = \dfrac{2 \sec^3 \frac{\pi}{4} + 2 \sec^3 \frac{\pi}{4} + 2 \sec \frac{\pi}{4} \tan^2 \frac{\pi}{4}}{4(1 + \sec \frac{\pi}{4})^{3/2}} + \dfrac{\sec^2 \frac{\pi}{4} \tan^2 \frac{\pi}{4}}{4(1 + \sec \frac{\pi}{4})^{3/2}}$

$= \dfrac{6(1 + \sqrt{2})}{4(1 + \sqrt{2})^{3/2}} = \dfrac{3}{2\sqrt{1 + \sqrt{2}}}.$

41. $f'(x) = 28x^3 - 15x^2 + 16x - 3;\ f''(x) = 84x^2 - 30x + 16$

$f'''(x) = 168x - 30 \quad ;\ f''''(x) = 168$

$f^{(n)}(x) = 0,\ n \geq 5.$

42. $f(t) = (t + 1)^{-1/2},$

$f'(t) = -\frac{1}{2}(t + 1)^{-3/2};\ f''(t) = (-\frac{3}{2})(-\frac{1}{2})(t + 1)^{-5/2};$

$f'''(t) = (-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})(t + 1)^{-7/2};$

$f''''(t) = (-\frac{7}{2})(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})(t + 1)^{-9/2}.$

So

$f^{(10)}(t) = (-\frac{19}{2})(-\frac{17}{2})(-\frac{15}{2})\cdots(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})(t+1)^{-21/2}.$

43. $y = (x^2 - 1)^{1/2}$.

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}};$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{x^2 - 1} - x \cdot \frac{1}{2} \cdot (x^2 - 1)^{-1/2}(2x)}{x^2 - 1}$$

$$= \frac{x^2 - 1 - x^2}{(x^2 - 1)^{3/2}} = \frac{-1}{(x^2 - 1)^{3/2}} = -(x^2 - 1)^{-3/2};$$

$$\frac{d^3y}{dx^3} = \frac{3}{2}(x^2 - 1)^{-5/2}(2x) = 3x(x^2 - 1)^{-5/2}.$$

44. $y = x^{1/3}$.

$y' = \frac{1}{3}x^{-2/3}$; $y'' = \frac{1}{3}(-\frac{2}{3})x^{-5/3}$; $y''' = \frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})^{-8/3}$;

$y^{IV} = \frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})x^{-11/3}$.

So we have

$$D_x^n(\sqrt[3]{x}) = \frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})\cdots(-\frac{3n-4}{3})x^{\frac{-3n-1}{3}}.$$

45. $y = x^{-1}$. $y' = -1x^{-2}$; $y'' = (-1)(-2)x^{-3}$; $y''' = (-1)(-2)(-3)x^{-4}$; $y^{IV} = (-1)(-2)(-3)(-4)x^{-5}$.

So

$$D_x^n(\tfrac{1}{x}) = (-1)^n n!\, x^{-(n+1)}.$$

46. $y = \sin x$. $y' = \cos x$; $y'' = -\sin x$; $y''' = -\cos x$; $y^{IV} = \sin x$.

So $D_x^{70}\sin x = D_x^{17\cdot 4+2}\sin x = D_x^2\left[D_x^{17\cdot 4}\sin x\right] \cdot D_x^2(\sin x) = -\sin x$

47. $\frac{dy}{dx} = 6x + 2$; $\frac{d^2y}{dx^2} = 6$.

So

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^2(6) - 2x(6x + 2) + 2(3x^2 + 2x)$$

$= 0$.

48. (a) $(f \cdot g)''' = (f \cdot g'' + 2f' \cdot g' + f'' \cdot g)'$

$= f \cdot g''' + f' \cdot g'' + 2f' \cdot g'' + 2f'' \cdot g' + f'' \cdot g' + f''' \cdot g$

$= f \cdot g''' + 3f' \cdot g'' + 3f'' \cdot g' + f''' \cdot g$.

(b) $(f \cdot g)''(1) = f(1) \cdot g''(1) + 2f'(1)g'(1) + f''(1) \cdot g(1)$

$= (-3)(\frac{2}{3}) + 2(-1)(4) + (16)(\frac{1}{2}) = -2$.

49. Using the Leibniz rule for second derivatives, we have

$$f''(x) = x^4 \cdot g''(x) + 2(4x^3 \cdot g'(x)) + 12x^2 \cdot g(x),$$

so

$$f''(2) = 16g''(2) + 64g'(2) + 48g(2) = \frac{736}{3}.$$

50. (a) We have $f'(x) = 2x$ for $x < 1$ and $f'(x) = 2$ for $x > 1$. We must calculate $f'(1)$ from the definition:

$f'(1) = \lim_{\Delta x \to 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x}$. But

$$\lim_{\Delta x \to 0^+} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{[2(1 + \Delta x) - 1] - 1}{\Delta x} = 2 \text{ and}$$

$$\lim_{\Delta x \to 0^-} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(1 + \Delta x)^2 - 1}{\Delta x} = 2, \text{ so } f'(1) = 2.$$

This gives

$$f'(x) = \begin{cases} 2x & \text{for } x \le 1 \\ 2 & \text{for } x > 1. \end{cases}$$

Thus, $f''(x) = 2$ for $x < 1$, $f''(x) = 0$ for $x > 1$, and f' is not differentiable at 1 (since it is not continuous at 1).

(b) We have $f'(x) = x$ for $x > 0$ and $f'(x) = -x$ for $x < 0$. Thus, $f'(x) = |x|$ for $x \neq 0$. Notice that $f(x) = \frac{x|x|}{2}$. By definition,

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x|\Delta x|}{2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{2} = 0.$$

Consequently, $f'(x) = |x|$ holds for all values of x. Therefore, $f''(x) = 1$ if $x > 0$ and $f''(x) = -1$ if $x < 0$, so $f''(x) = \frac{x}{|x|}$ for $x \neq 0$. Of course f' is not differentiable at 0.

1. (a) $v = 3t^2 - 12t + 12$, $a = 6t - 12$; so when $a = 0$, $t = 2$.

(b) $v = \dfrac{1}{2\sqrt{1+t}}$, $a = -\dfrac{1}{4(\sqrt{1+t})^3}$; so a is never zero.

(c) $v = 5 - \dfrac{2}{(t+1)^2}$, $a = \dfrac{4}{(t+1)^3}$; so a is never zero.

2. Take the y axis to be the number scale along which P moves, so that $s = y = f(x) = f(g(t)) = (f \circ g)(t)$. Thus, by the chain rule,

$v = \dfrac{ds}{dt} = f'(g(t))g'(t) = (f' \circ g)(t)g'(t)$, so that

$a = \dfrac{d^2s}{dt^2} = f'(g(t))g''(t) + f''(g(t))g'(t)g'(t) = f'(g(t))g''(t) + f''(g(t))\left[g'(t)\right]^2$.

3. Let f be a rational function, so that $f = \dfrac{p}{q}$, where p and q are polynomial functions and q is not identically zero. Then $f' = \dfrac{qp' - pq'}{q^2}$. Since the derivative of a polynomial is again a polynomial, p' and q' are polynomials. It follows that qp' - pq' is a polynomial, so f' is a ratio of polynomials; that is, f' is a rational function. Since the derivative of a rational function remains rational, the successive differentiations of a rational function can produce only rational functions. Therefore, the n^{th} derivative of a rational function will still be a rational function.

4. $(f \circ g)'(x) = f'(g(x))g'(x)$, and

$$(f \circ g)''(x) = f'(g(x))g''(x) + f''(g(x))g'(x)g'(x) = f'(g(x))g''(x) + f''(g(x))\left[g'(x)\right]^2.$$

Hence,

$$(f \circ g)''(-1) = f'(g(-1))g''(-1) + f''(g(-1))\left[g'(-1)\right]^2 = f'(27)\left(\tfrac{1}{4}\right) + f''(27)(-1)^2 = (-4)\left(\tfrac{1}{4}\right) + 1 = 0.$$

55. $4x^3 + 4y^3\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^3$,

$$\frac{d^2y}{dx^2} = -3\left(\frac{x}{y}\right)^2 \frac{y - x\frac{dy}{dx}}{y^2} = -3\left(\frac{x}{y}\right)^2 \frac{y + x\left(\frac{x}{y}\right)^3}{y^2} = -3\left(\frac{x}{y}\right)^2 \frac{y^4 + x^4}{y^5} = \frac{-3x^2(64)}{y^7} = -\frac{192x^2}{y^7}.$$

56. $3x^2 + 3y^2\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^2$, $\dfrac{d^2y}{dx^2} = -2\left(\dfrac{x}{y}\right)\dfrac{y - x\frac{dy}{dx}}{y^2}$

$$= -2\left(\frac{x}{y}\right)\frac{y + x\left(\frac{x}{y}\right)^2}{y^2} = -2\left(\frac{x}{y}\right)\frac{y^3 + x^3}{y^4} = -\frac{32x}{y^5}.$$

57. $x^2 \cdot 3\cos 3y\dfrac{dy}{dx} + \sin 3y(2x) = 0$,

$$\frac{dy}{dx} = \frac{-2\sin 3y}{3x\cos 3y} = \frac{-2\tan 3y}{3x};$$

$$\frac{d^2y}{dx^2} = \frac{3x(-6\sec^2 3y)\frac{dy}{dx} + 2\tan 3y(3)}{9x^2} = \frac{-6x\sec^2 3y\left(\frac{-2\tan 3y}{3x}\right) + 2\tan 3y}{3x^2} = \frac{2\tan 3y(2\sec^2 3y + 1)}{3x^2}.$$

58. $x(-\csc^2 y)\dfrac{dy}{dx} + \cot y = \dfrac{dy}{dx}$;

$$\frac{dy}{dx} = \frac{\cot y}{1 + x\csc^2 y};$$

$$\frac{d^2y}{dx^2} = \frac{(1+x\csc^2 y)(-\csc^2 y)\frac{dy}{dx} - \cot y\left[2x\csc y(-\csc y\cot y)\frac{dy}{dx} + \csc^2 y\right]}{(1 + x\csc^2 y)^2}$$

$$= \frac{(1 + x\csc^2 y)(-\csc^2 y)\frac{\cot y}{1 + x\csc^2 y}}{(1 + x\csc^2 y)^2} - \frac{\cot y\left[-2x\csc^2 y\cot y\left(\frac{\cot y}{1 + x\csc^2 y}\right) + \csc^2 y\right]}{(1 + x\csc^2 y)^2}$$

$$= \frac{(1 + x\csc^2 y)(-\csc^2 y\cot y) + 2x\csc^2 y\cot^3 y}{(1 + x\csc^2 y)^3} - \frac{\cot y\csc^2 y(1 + x\csc^2 y)}{(1 + x\csc^2 y)^3}$$

$$= \frac{2\csc^2 y\cot y(y\cot y - x\csc^2 y - 1)}{(1 + x\csc^2 y)^3}$$

59. $v = \dfrac{ds}{dt} = 2kt$.

When $t = 2$, $v = 500$.

So $500 = 2 \cdot k \cdot 2$ or $k = 125$.

So $s = 125(2^2) = 125(4) = 500$m.

60. $v = \frac{ds}{dt}$ $s = f(t)$ So $\frac{dv}{ds}$

$= \frac{d}{ds}\left(\frac{ds}{dt}\right) = \frac{d}{ds}(f'(t)) = f''(t)\frac{dt}{ds}$

or $\frac{dv}{ds} = a \cdot \frac{dt}{ds}$ or $a = \frac{ds}{dt} \cdot \frac{dv}{ds} = v\frac{dv}{ds}$

61. $R = \frac{dN}{dt} = 500(20 + 36t - 3t^2)$.

So $\frac{dR}{dt} = 500(36 - 6t)$.

Thus, $\frac{dR}{dt} = 0$ when $36 - 6t = 0$, that is, when $t = 6$.

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1. The polynomial function $f(x) = 4x^3 - x^2$ is continuous on the interval $[0,1]$. $f(0) = 0$, $f(1) = 3$. Because $0 < 2 < 3$, Theorem 1 guarantees the existence of at least one value c between 0 and 1 such that $f(c) = 2$, i.e., $4c^3 - c^2 = 2$.

2. The polynomial function $f(x) = 2x^4 - 4x^3 + 8x$ is continuous on the interval $[1,2]$. $f(1) = 6$, $f(2) = 16$. Because $6 < 7.07 < 16$, Theorem 1 guarantees the existence of at least one value c between 1 and 2 such that $f(c) = 7.07$, i.e., $2c^4 - 4c^3 + 8c \doteq 7.07$.

3. The polynomial function $f(x) = x^3 + 3x^2 - 9x$ is continuous on the interval $[2,3]$. $f(2) = 2$, $f(3) = 27$. Because $2 < 10 < 27$, Theorem 1 guarantees the existence of at least one value c between 2 and 3 such that $f(c) = 10$, i.e., $c^3 + 3c^2 - 9c = 10$.

4. The function $f(x) = \frac{\sqrt{8x - 15}}{x}$ is continuous on the interval $[2,3]$. $f(2) = \frac{1}{2}$, $f(3) = 1$. Because $\frac{1}{2} < \frac{2}{3} < 1$, Theorem 1 guarantees the existence of at least one value c between 2 and 3 such that $f(c) = \frac{2}{3}$, i.e., $\frac{\sqrt{8c - 15}}{c} = \frac{2}{3}$.

5. The function $f(x) = \frac{x^3 + 5}{\sqrt{x} + 1}$ is continuous on the interval $[0,1]$. $f(0) = 5$, $f(1) = 3$. Because $3 < 4 < 5$, Theorem 1 guarantees the existence of at least one value c between 0 and 1 such that $f(c) = 4$, i.e., $\frac{c^3 + 5}{\sqrt{c} + 1} = 4$.

6. The polynomial function $f(x) = x^4 - 8x^2 + x$ is continuous on the interval $[2.5,2.6]$. $f(2.5) = -8.4375$ and $f(2.6) = -5.7824$. Because $-8.4375 < -6 < -5.7824$, Theorem 1 guarantees the existence of at least one value c between 2.5 and 2.6 such that $f(c) = -6$, i.e., $c^4 - 8c^2 + c = -6$.

7. The polynomial function $f(x) = 2x^3 - 3x^2 - 12x$ is continuous on the interval $[-2,-1]$. $f(-2) = -4$, $f(-1) = 7$. Because $-4 < 1 < 7$, Theorem 1 guarantees the existence of at least one value c between -2 and -1 such that $f(c) = 1$, i.e., $2c^3 - 3c^2 - 12c = 1$.

8. The function $f(x) = \frac{1}{x^4 - 4x^3 + 4x^2}$ is continuous on the interval $[1.4,1.5]$. $f(1.4) = 1.4172$, $f(1.5) = 1.\overline{7}$. Because $1.41723 < \sqrt{3} < 1.\overline{7}$, Theorem 1 guarantees the existence of at least one value c between 1.4 and 1.5 such that $f(c) = \sqrt{3}$, i.e., $\frac{1}{c^4 - 4c^3 + 4c^2} = \sqrt{3}$.

9. The function $f(x) = \sin x + 2 \cos 2x$ is continuous on the interval $\left[\frac{3\pi}{4}, \pi\right]$. $f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $f(\pi) = 2$. Because $\frac{\sqrt{2}}{2} < 1 < 2$, Theorem 1 guarantees the existence of at least one value c between $\frac{3\pi}{4}$ and π such that $f(c) = 1$, i.e., $\sin c + 2 \cos 2c = 1$.

10. The function $f(x) = 2 \csc x + \cot x$ is continuous on the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$. $f\left(\frac{\pi}{6}\right) = 4 + \sqrt{3}$, $f\left(\frac{\pi}{3}\right) = \frac{5\sqrt{3}}{3} \approx 2.89$. Because $\frac{5\sqrt{3}}{3} < 4 < 4 + \sqrt{3}$, Theorem 1 guarantees the existence of at least one value c between $\frac{\pi}{6}$ and $\frac{\pi}{3}$ such that $f(c) = 4$, i.e., $2 \csc c + \cot c = 4$.

11. The function $f(x) = x + \sin x$ is continuous on the interval $\left[0, \frac{\pi}{6}\right]$. $f(0) = 0$, $f(\frac{\pi}{6}) = \frac{\pi}{6} + \frac{1}{2}$. Because $0 < 1 < \frac{\pi}{6} + \frac{1}{2} \approx 1.02$, Theorem 1 guarantees the existence of at least one value c between 0 and $\frac{\pi}{6}$ such that $f(c) = 1$, i.e., $c + \sin c = 1$.

12. The function $f(x) = \dfrac{\sin x}{2 + \cos x}$ is continuous on the interval $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. $f(\frac{\pi}{6}) = \dfrac{4 - \sqrt{3}}{13}$, $f(\frac{\pi}{2}) = \frac{1}{2}$. Because $\dfrac{4 - \sqrt{13}}{13} < \frac{1}{4} < \frac{1}{2}$, Theorem 1 guarantees the existence of at least one value c between $\frac{\pi}{6}$ and $\frac{\pi}{2}$ such that $f(c) = 1$, i.e., $\dfrac{\sin c}{2 + \cos c} = \frac{1}{4}$.

13. f is continuous on $[1,2]$. $f(1) = -2$, $f(2) = 15$, f changes sign on $[1,2]$; so by the change-of-sign property, f has a zero on $[0,1]$.

14. g is continuous on $[2.1,2.2]$. $f(2.1) = -4.3659$, $f(2.2) = 0.1936$, g changes sign on $[2.1,2.2]$; so by the change-of-sign property, g has a zero on $[2.1,2.2]$.

15. f is continuous on $[1.5,1.6]$. $f(1.5) = -0.156$, $f(1.6) = 1.29376$, f changes sign on $[1.5,1.6]$; so by the change-of-sign property, f has a zero on $[1.5,1.6]$.

16. g is continuous on $[-8.2,-8.1]$. $f(-8.2) = 2.6896$, $f(-8.1) = -64.9539$, g changes sign on $[-8.2,-8.1]$; so by the change-of-sign property, g has a zero on $[-8.2,-8.1]$.

17. h is continuous on $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, $h(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $h(\frac{\pi}{3}) = \frac{\sqrt{3}-2}{2} \approx \frac{-0.3}{2}$, h changes sign on $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$; so by the change-of-sign property, h has a zero on $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$.

18. F is continuous on $[1.9,2]$. $F(1.9) = 0.2489$, $F(2) = -0.0055$, F changes sign on $[1.9,2]$; so by the change-of-sign property, F has a zero on $[1.9,2]$.

19. $f(x) = x^2 - 6x + 8 = (x - 4)(x - 2)$

$f(x) = 0$ when $x = 2,4$

$f(1) = 3$, $f(3) = -1$, $f(5) = 3$.

20. $g(x) = 25x^2 - 20x + 4$

$= (5x - 2)^2 \geq 0$ for all x.

$g(x) = 0$ when $x = \frac{2}{5}$; otherwise, $g(x) > 0$.

21. $F(x) = 2x(x - 3)(x + 5)$.

$F(x) = 0$ when $x = 0, 3, -5$.

$F(-6) = -108$, $F(-1) = 32$,

$F(1) = -24$, $F(4) = 72$.

22. $G(x) = x^4 - 7x^2 + 12 = (x^2 - 4)(x^2 - 3)$.

$G(x) = 0$ when $x = \pm 2, \pm\sqrt{3}$.

$G(-3) = 30$, $G(-1.9) = -0.2379$,

$G(0) = 12$, $G(1.9) = -0.2379$,

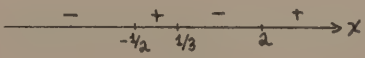
$G(3) = 30$.

23. $h(x) = (2x + 1)(3x - 1)(x - 2)$.

$h(x) = 0$ when $x = -\frac{1}{2}, \frac{1}{3}, 2$.

$h(-1) = -12$, $h(0) = 2$, $h(1) = -6$,

$h(3) = 56$.



24. $H(x) = \sqrt{x - 1}\,(x - 2)|x - 3|$.

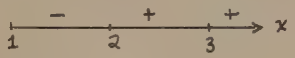
$H(x) = 0$ when $x = 1$, $x = 2$, $x = 3$.

$H(x)$ is defined only when $x \geq 1$.

$H(1.5) = \sqrt{0.5}\,(-0.5)(1.5) < 0,$

$H(2.5) = \sqrt{1.5}\,(0.5)(0.5) > 0,$

$H(3.5) = \sqrt{2.5}\,(1.5)(0.5) > 0.$



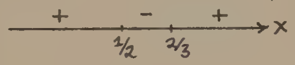
$H(1) = 0$

25. $q(x) = \dfrac{2x - 1}{3x - 2}.$

$q(x) = 0$ when $x = \frac{1}{2}$; $q(x)$ not defined when $x = \frac{2}{3}.$

$q(0) = \frac{1}{2}$, $q(\frac{3}{5}) = -1,$

$q(1) = 1.$



26. $Q(x) = \dfrac{(3x + 4)(2x - 1)}{(x + 2)(x - 1)}.$

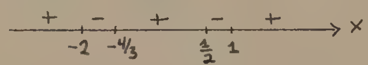
$Q(x) = 0$ when $x = -\frac{4}{3}, \frac{1}{2};$

$Q(x)$ is not defined when $x = -2, 1.$

$Q(-3) = \frac{35}{4}$, $Q(-1.5) = -\frac{8}{5},$

$Q(0) = 2$, $Q(0.6) = -1.11538,$

$Q(2) = \frac{15}{2}.$



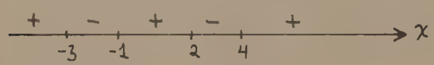
27. $r(x) = \dfrac{(x + 1)(x - 2)}{(x + 3)(x - 4)}.$

$r(x) = 0$ when $x = -1, 2$; $r(x)$ is not defined when $x = -3, 4.$

$r(-4) = \frac{9}{4}$, $r(-2) = -\frac{2}{3},$

$r(0) = \frac{1}{6}$, $r(3) = -\frac{2}{3},$

$r(5) = \frac{9}{4}.$



28. $R(x) = \dfrac{x^2 - 2x - 15}{x^2 - 2x + 10} = \dfrac{(x - 5)(x + 3)}{(x^2 - 2x + 1) + 9} = \dfrac{(x - 5)(x + 3)}{(x - 1)^2 + 9}.$

$R(x) = 0$ when $x = 5, -3$; $R(x)$ defined for all values of x.

$R(-4) = \frac{9}{34}$, $R(0) = -\frac{3}{2},$

$R(6) = \frac{9}{34}.$

29. $S(x) = \dfrac{(x - 2)^2(x + 1)(2x - 1)}{(x + 3)(x - 1)^2(x + 4)}.$

$S(x) = 0$ when $x = 2, -1, \frac{1}{2}$; $S(x)$ is not defined when $x = -3, 1, -4.$

$S(-5) = \dfrac{49(-4)(-11)}{-2(36)(-1)} > 0,$

$S(-3.5) = \dfrac{(30.25)(-2.5)(-8)}{(-0.5)(20.25)(0.5)} < 0,$

$S(-2) = \dfrac{16(-1)(-5)}{1(9)(2)} > 0,$

$S(\frac{3}{4}) = \dfrac{\frac{25}{16}(\frac{7}{4})(\frac{1}{2})}{\frac{15}{4}(\frac{1}{16})(\frac{19}{4})} > 0,$

$S(0) = \dfrac{4(1)(-1)}{3(1)(4)} < 0,$

$S(1.5) = \dfrac{(0.25)(2.5)2}{4.5(0.25)(5.5)} > 0,$

$S(3) = \dfrac{1(4)(5)}{6(4)(7)} > 0.$

30. $S(x) = \dfrac{1 + x}{(1 - x^2)^2}.$

$S(x) = 0$ for no values of x; $S(x)$ is undefined when $x = 1, -1.$

$S(-2) = -\frac{1}{9}$, $S(0) = 1,$

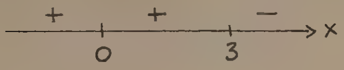
$S(2) = \frac{3}{9}.$

31. $f(x) = 3x^{2/3} - x^{5/3}.$

$3x^{2/3} - x^{5/3} = 0$ or $x^{2/3}(3 - x) = 0$

for $x = 0$, $x = 3.$

$f(-1) = 4$, $f(1) = 2$, $f(8) = -20$.



32. $F(x) = (x - 1)^{1/3}x^{-2/3}$.

$F(x) = 0$ when $x = 1$; $F(x)$ is undefined when $x = 0$.

$F(-1) = (-2)^{1/3} \cdot 1 < 0$,

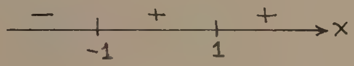
$F(\frac{1}{2}) = (-\frac{1}{2})^{1/3} \cdot 2^{2/3} < 0$,

$F(8) = 7^{1/3} \cdot \frac{1}{4} > 0$.

33. $g(x) = 0$ when $x = \pm 1$.

$g(-2) = 1 - (-2)^2 = 1 - 4 < 0$,

$g(0) = 1$, $g(2) = 2^2 - 1 = 3$.



34. $G(x) = 0$ when $x = \pm 5, 7$.

$G(x) = \sqrt{25 - x^2}$ implies $-5 \le x \le 5$.

$G(0) = \sqrt{25} = 5$,

$G(6) = 7 - 6 = 1$,

$G(8) = 7 - 8 = -1$.

$G(-5) = 0$

35. Midpoint of interval $[1.5, 1.6]$ is 1.55. Now $f(1.55) = 0.498859688$. Since $f(1.5) < 0$, there is a zero between 1.5 and 1.55. Midpoint of interval $[1.5, 1.55]$ is 1.525. Now $f(1.525) = 0.154854503$. Since $f(1.5) < 0$, there is a zero between 1.5 and 1.525.

36. Midpoint of interval $[-8.2, -8.1]$ is -8.15. Now $g(-8.15) = -31.717$. Since $g(-8.15) < 0$ and $f(-8.2) > 0$, there is a zero between -8.15 and -8.2. Midpoint of interval $[-8.15, -8.2]$ is -8.175. Now $g(-8.175) = -14.661$. Since $g(-8.2) > 0$, there is a zero between -8.175 and -8.2.

37. Midpoint of interval $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ is $\frac{7\pi}{24}$. Now $h(\frac{7\pi}{24}) = 0.276$. Since $h(\frac{\pi}{3}) < 0$, there is a zero between $\frac{7\pi}{24}$ and $\frac{\pi}{3}$. Midpoint of interval $\left[\frac{7\pi}{24}, \frac{\pi}{3}\right]$ is $\frac{5\pi}{16}$. $h(\frac{5\pi}{16}) = 0.066$. Since $h(\frac{\pi}{3}) < 0$, there is a zero between $\frac{5\pi}{16}$ and $\frac{\pi}{3}$.

38. Midpoint of interval $[1.9, 2]$ is 1.95.

Now $F(1.95) = 1.226$. Since $F(2) < 0$, there is a zero between 1.95 and 2.

Midpoint of interval $[1.95, 2]$ is 1.975.

Now $F(1.975) = 0.0588$. Since $F(2) < 0$, there is a zero between 1.975 and 2.

Midpoint of interval $[1.975, 2]$ is 1.9875. Now $F(1.9875) = 0.0267$. Since $F(2) < 0$, there is a zero between 1.9875 and 2.

Midpoint of interval $[1.9875, 2]$ is 1.99375. Now $F(1.99375) = 0.0106$. Since $F(2) < 0$, there is a zero between 1.99375 and 2.

39. $f'(x) = 5x^4 - 6x^2$.

So $$x_{n+1} = x_n - \frac{x_n^5 - 2x_n^3 - 1}{5x_n^4 - 6x_n^2} = \frac{4x_n^5 - 4x_n^3 + 1}{5x_n^4 - 6x_n^2}.$$

Let $x_1 = \dfrac{1.5 + 1.6}{2} = 1.55$.

So $$x_2 = 1.55 - \frac{(1.55)^5 - 2(1.55)^3 - 1}{5(1.55)^4 - 6(1.55)^2} = 1.515464963,$$

$$x_3 = \frac{4x_2^5 - 4x_2^3 + 1}{5x_2^4 - 6x_2^2} = 1.512890049,$$

$$x_4 = \frac{4x_3^5 - 4x_3^3 + 1}{5x_3^4 - 6x_3^2} = 1.512876397,$$

$$x_5 = \frac{4x_4^5 - 4x_4^3 + 1}{5x_4^4 - 6x_4^2} = 1.512876397.$$

So desired zero $z \approx 1.512876397$.

40. $f'(x) = 4x^3 + 18x^2 - 36x$.

So $x_{n+1} = x_n - \dfrac{x_n^4 + 6x_n^3 - 18x_n^2}{4x_n^3 + 18x_n^2 - 36x_n}$

$= \dfrac{3x_n^4 + 12x_n^3 - 18x_n^2}{4x_n^3 + 18x_n^2 - 36x_n}$.

Let $x_1 = \dfrac{-8.2 + (-8.1)}{2} = -8.15$.

So $x_2 = \dfrac{3x_1^4 + 12x_1^3 - 18x_1^2}{4x_1^3 + 18x_1^2 - 36x_1} = -8.196892699$,

$x_3 = \dfrac{3x_2^4 + 12x_2^3 - 18x_2^2}{4x_2^3 + 18x_2^2 - 36x_2} = -8.196152618$,

$x_4 = \dfrac{3x_3^4 + 12x_3^3 - 18x_3^2}{4x_3^3 + 18x_3^2 - 36x_3} = -8.196152417$,

$x_5 = -8.196152417$.

So desired zero $z \approx -8.196152417$.

41. $f'(x) = \cos x - 4 \sin 2x$.

So $x_{n+1} = x_n - \dfrac{\sin x_n + 2\cos 2x_n}{\cos x_n - 4\sin 2x_n}$.

Let $x_1 = \dfrac{\frac{\pi}{4} + \frac{\pi}{3}}{2} = \dfrac{7\pi}{24}$.

So $x_2 = x_1 - \dfrac{\sin x_1 + 2\cos 2x_1}{\cos x_1 - 4\sin 2x_1} = 1.001004516$,

$x_3 = x_2 - \dfrac{\sin x_2 + 2\cos 2x_2}{\cos x_2 - 4\sin 2x_2} = 1.002965391$,

$x_4 = x_3 - \dfrac{\sin x_3 + 2\cos 2x_3}{\cos x_3 - 4\sin 2x_3} = 1.002966954$,

$x_5 = 1.002966954$.

So the desired zero $z \approx 1.002966954$.

42. $f'(x) = 2\sin x \cos x - 2\sin x = \sin 2x - 2\sin x$.

$x_{n+1} = x_n - \dfrac{\sin^2 x_n + 2\cos x_n}{\sin 2x_n - 2\sin x_n}$.

Let $x_1 = \dfrac{1.9 + 2}{5} = 1.95$.

So

$x_2 = x_1 - \dfrac{\sin^2 x_1 + 2\cos x_1}{\sin 2x_1 - 2\sin x_1} = 1.998161678$,

$x_3 = x_2 - \dfrac{\sin^2 x_2 + 2\cos x_2}{\sin 2x_2 - 2\sin x_2} = 1.997874921$,

$x_4 = x_3 - \dfrac{\sin^2 x_3 + 2\cos x_3}{\sin 2x_3 - 2\sin x_3} = 1.997874921$.

So the desired zero $z \approx 1.997874921$.

43. $G'(x) = 3x^2 - 7$.

$x_{n+1} = x_n - \dfrac{x_n^3 - 7x_n + 7}{3x_n^2 - 7}$.

Let $x_1 = -3.5$.

So $x_2 = x_1 - \dfrac{x_1^3 - 7x_1 + 7}{3x_1^2 - 7} = -3.117647059$,

$x_3 = x_2 - \dfrac{x_2^3 - 7x_2 + 7}{3x_2^2 - 7} = -3.050896499$,

$x_4 = x_3 - \dfrac{x_3^3 - 7x_3 + 7}{3x_3^2 - 7} = -3.048919053$,

$x_5 = x_4 - \dfrac{x_4^3 - 7x_4 + 7}{3x_4^2 - 7} = -3.048917340$,

$x_6 = -3.048917340$.

So the desired zero $z \approx -3.048917340$.

44. $H'(x) = 3x^2 - 8x - 2$.

$x_{n+1} = x_n - \dfrac{x_n^3 - 4x_n^2 - 2x_n + 4}{3x_n^2 - 8x_n - 2}$.

Let $x_1 = 4.5$.

$x_2 = x_1 - \dfrac{x_1^3 - 4x_1^2 - 2x_1 + 4}{3x_1^2 - 8x_1 - 2} = 4.274725275$,

$x_3 = x_2 - \dfrac{x_2^3 - 4x_2^2 - 2x_2 + 4}{3x_2^2 - 8x_2 - 2} = 4.249449816$,

$x_4 = x_3 - \dfrac{x_3^3 - 4x_3^2 - 2x_3 + 4}{3x_3^2 - 8x_3 - 2} = 4.249140584$,

$x_5 = x_4 - \dfrac{x_4^3 - 4x_4^2 - 2x_4 + 4}{3x_4^2 - 8x_4 - 2} = 4.249140538$,

$x_6 = 4.249140538$.

So the desired zero $z \approx 4.249140538$.

45. $y = 2x^3 - 4x^2 + 5x$ and $y = 7$ intersect at a point where x is close to 1.5.

Let $f(x) = 2x^3 - 4x^2 + 5x - 7$. Then $f'(x) = 6x^2 - 8x + 5$. Want to find a zero of f. Let $x_1 = 1.5$.

$$x_{n+1} = x_n - \frac{2x_n^3 - 4x_n^2 + 5x_n - 7}{6x_n^2 - 8x_n + 5}.$$

So $x_2 = x_1 - \dfrac{2x_1^3 - 4x_1^2 + 5x_1 - 7}{6x_1^2 - 8x_1 + 5} = 1.769230769$,

$x_3 = x_2 - \dfrac{2x_2^3 - 4x_2^2 + 5x_2 - 7}{6x_2^2 - 8x_2 + 5} = 1.727530613$,

$x_4 = x_3 - \dfrac{2x_3^3 - 4x_3^2 + 5x_3 - 7}{6x_3^2 - 8x_3 + 5} = 1.726280494$,

$x_5 = x_4 - \dfrac{2x_4^3 - 4x_4^2 + 5x_4 - 7}{6x_4^2 - 8x_4 + 5} = 1.726279398$,

$x_6 = 1.7262794$.

So the desired zero $z \approx 1.7262794$.

46. $y = 15x^5 + 13x^3$ and $y = 1$ intersect at a point where x is close to 0.5.

Let $f(x) = 15x^5 + 13x^3 - 1$. Then $f'(x) = 75x^4 + 39x^2$. Want to find a zero of f. Let $x_1 = 0.5$.

$$x_{n+1} = x_n - \frac{15x_n^5 + 13x_n^3 - 1}{75x_n^4 + 39x_n^2}.$$

$x_2 = x_1 - \dfrac{15x_1^5 + 13x_1^3 - 1}{75x_1^4 + 39x_1^2} = 0.424242424$,

$x_3 = x_2 - \dfrac{15x_2^5 + 13x_2^3 - 1}{75x_2^4 + 39x_2^2} = 0.403206315$,

$x_4 = x_3 - \dfrac{15x_3^5 + 13x_3^3 - 1}{75x_3^4 + 39x_3^2} = 0.401761621$,

$x_5 = x_4 - \dfrac{15x_4^5 + 13x_4^3 - 1}{75x_4^4 + 39x_4^2} = 0.401755168$,

$x_6 = 0.401755168$.

So the desired zero $z \approx 0.401755168$.

47. $y = x$ and $y = \dfrac{\sin x}{x}$ intersect at a point where x is close to 0.7.

Let $f(x) = x - \dfrac{\sin x}{x}$. Then $f'(x) = 1 - \dfrac{\cos x}{x} + \dfrac{\sin x}{x^2}$. Want to find a zero of f. Let $x_1 = 0.7$.

$$x_{n+1} = x_n - \frac{x_n - \dfrac{\sin x_n}{x_n}}{1 - \dfrac{\cos x_n}{x_n} + \dfrac{\sin x_n}{x_n^2}}$$

$$x_2 = x_1 - \frac{x_1 - \dfrac{\sin x_1}{x_1}}{1 - \dfrac{\cos x_1}{x_1} + \dfrac{\sin x_1}{x_1^2}}$$

$= 0.880272721$,

$$x_3 = x_2 - \frac{x_2 - \dfrac{\sin x_2}{x_2}}{1 - \dfrac{\cos x_2}{x_2} + \dfrac{\sin x_2}{x_2^2}}$$

$= 0.876727499$,

$$x_4 = x_3 - \frac{x_3 - \dfrac{\sin x_3}{x_3}}{1 - \dfrac{\cos x_3}{x_3} + \dfrac{\sin x_3}{x_3^2}}$$

$= 0.876726215$,

$x_5 = 0.876726215$.

So the desired zero $z \approx 0.876726215$.

48. $y = x^{3/2} + x$ and $y = 1 - 2x^{1/2}$ intersect at a point where x is close to 0.5.

Let $f(x) = x^{3/2} + x - 1 + 2x^{1/2}$. Then $f'(x) = \frac{3}{2}x^{1/2} + 1 + x^{-1/2}$.

Want to find a zero of f. Let $x_1 = 0.5$.

$$x_{n+1} = x_n - \frac{x_n^{1.5} + x_n - 1 + 2x_n^{1.5}}{1.5x_n^{0.5} + 1 + x_n^{-0.5}}.$$

$x_2 = x_1 - \dfrac{x_1^{1.5} + x_1 - 1 + 2x_1^{0.5}}{1.5x_1^{0.5} + 1 + x_1^{-0.5}} = 0.135161721$,

$x_3 = x_2 - \dfrac{x_2^{1.5} + x_2 - 1 + 2x_2^{0.5}}{1.5x_2^{0.5} + 1 + x_2^{-0.5}} = 0.153857769$,

$x_4 = x_3 - \dfrac{x_3^{1.5} + x_3 - 1 + 2x_3^{0.5}}{1.5x_3^{0.5} + 1 + x_3^{-0.5}} = 0.154171420$,

$x_5 = x_4 - \dfrac{x_4^{1.5} + x_4 - 1 + 2x_4^{0.5}}{1.5x_4^{0.5} + 1 + x_4^{-0.5}} = 0.154171495$,

$x_6 = 0.154171495$.

So the desired zero $z \approx 0.154171495$.

49. (a) $a = 1$.

Then $b = \frac{1}{2}(a + \frac{2}{a}) = 1.5$.

Then $c = \frac{1}{2}(b + \frac{2}{b}) = 1.416666667$.

Then $d = \frac{1}{2}(c + \frac{2}{c}) = 1.414215687$.

Then $e = \frac{1}{2}(d + \frac{2}{d}) = 1.414213563$.

Then $f = \frac{1}{2}(e + \frac{2}{e}) = 1.414213563$.

So $\sqrt{2} \approx 1.414213563$

(b) Let $f(x) = x^2 - k$. Then a zero of f is $\sqrt{k}$, and $f'(x) = 2x$.

Thus $x_{n+1} = x_n - \dfrac{x_n^2 - k}{2x_n} = x_n - \dfrac{x_n}{2} + \dfrac{k}{2x_n} = \dfrac{x_n}{2} + \dfrac{k}{2x_n}$

$= \frac{1}{2}(x_n + \frac{k}{x_n}) = \frac{1}{2}(a + \frac{k}{a})$.

50. $G(x) = x^3 - 7x + 7$; $G'(x) = 3x^2 - 7$.

$G(1.3) = 0.097$, $G(1.5) = -0.125$,

$G(1.7) = 0.013$

Thus there is one root between 1.3 and 1.5 and another root between 1.5 and 1.7.

For the interval $[1.3, 1.5]$, let $x_1 = 1.4$.

$$x_{n+1} = x_n - \frac{x_n^3 - 7x_n + 7}{3x_n^2 - 7}.$$

$$x_2 = x_1 - \frac{x_1^3 - 7x_1 + 7}{3x_1^2 - 7} = 1.350000000,$$

$$x_3 = x_2 - \frac{x_2^3 - 7x_2 + 7}{3x_2^2 - 7} = 1.356769984,$$

$$x_4 = x_3 - \frac{x_3^3 - 7x_3 + 7}{3x_3^2 - 7} = 1.356895824,$$

$$x_5 = x_4 - \frac{x_4^3 - 7x_4 + 7}{3x_4^2 - 7} = 1.356895868,$$

$x_6 = x_5$.

So the desired zero $z \approx 1.356895868$.

For the interval $[1.5, 1.7]$, let $x_1 = 1.6$.

$$x_{n+1} = x_n - \frac{x_n^3 - 7x_n + 7}{3x_n^2 - 7}.$$

$$x_2 = x_1 - \frac{x_1^3 - 7x_1 + 7}{3x_1^2 - 7} = 1.752941177,$$

$$x_3 = x_2 - \frac{x_2^3 - 7x_2 + 7}{3x_2^2 - 7} = 1.700717128,$$

$$x_4 = x_3 - \frac{x_3^3 - 7x_3 + 7}{3x_3^2 - 7} = 1.692251090,$$

$$x_5 = x_4 - \frac{x_4^3 - 7x_4 + 7}{3x_4^2 - 7} = 1.692021640,$$

$$x_6 = x_5 - \frac{x_5^3 - 7x_5 + 7}{3x_5^2 - 7} = 1.692021472,$$

$$x_7 = x_6 - \frac{x_6^3 - 7x_6 + 7}{3x_6^2 - 7} = 1.692021469,$$

$x_8 = 1.692021470$,

$x_9 = 1.692021472$,

$x_{10} = 1.692021469$,

$x_{11} = 1.692021470$,

$x_{12} = 1.692021472$,

etc.

The desired zero is $z \approx 1.69202147$.

51. Let $f(x) = x^n - k$. Then a zero of f is $\sqrt[n]{k}$, $k > 0$

$f'(x) = nx^{n-1}$.

Thus, if a is an approximation to $\sqrt[n]{k}$, then $b =$

$$a - \frac{a^n - k}{na^{n-1}} = \frac{na^n - a^n + k}{na^{n-1}} = \frac{(n-1)a^n + k}{na^{n-1}}$$

is often a better approximation to $\sqrt[n]{k}$.

52. (a) Let $F(t) = (t - a)\sqrt{t + a} - b$; $a = 324$; $b = 4.32 \times 10^5$.

Then $F(5000) = (5000 - a)\sqrt{5000 + a} - b = -90812.17460$;

$F(6000) = (6000 - a)\sqrt{6000 + a} - b = 19375.84810$.

Since we have a change of sign, by the change-of-sign property, there is a solution of $F(t) = 0$ on the interval $[5000, 6000]$.

(b) $F'(t) = (t - a)\frac{1}{2}(t + a)^{-1/2} + \sqrt{t + a} = \dfrac{3t + a}{2\sqrt{t + a}}$

$$\text{So } t_{n+1} = t_n - \frac{(t_n - a)\sqrt{t_n + a} - b}{\left(\dfrac{3t_n + a}{2\sqrt{t_n + a}}\right)}$$

Let $t_1 = \dfrac{5000 + 6000}{2} = 5500$,

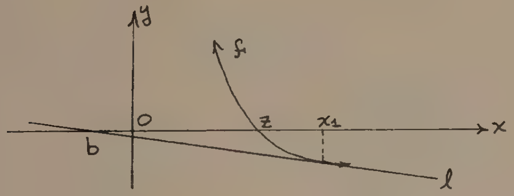
$t_2 = 5835.605659$,

$t_3 = 5830.603719$,

$t_4 = 5830.602629$,

$t_5 = t_4$.

So desired zero $z \approx 5830.602629$.

53. 

The tangent line ℓ intersects the x axis at b which is farther from z than x is. Since $x_2 = b$, x_2 will be a rather poor estimate for z.

54. $f(x) = \dfrac{1 - 4x}{1 + 4x}$. $f'(x) = \dfrac{-8}{(1 + 4x)^2} < 0$ for $x \neq -\frac{1}{4}$.

Let $x_1 = 1$,

then $x_2 = x_1 - \dfrac{\left(\dfrac{1 - 4x_1}{1 + 4x_1}\right)}{\dfrac{-8}{(1 + 4x_1)^2}} = 1 - \dfrac{15}{8} = -\dfrac{7}{8}$.

$(-\frac{7}{8}, 0)$ is farther from $(\frac{1}{4}, 0)$ than $(1,0)$ is. So this approximation is not improving. Here we have a decreasing graph to the right of the asymptote $x = -\frac{1}{4}$ which is concave upward, hence we are in the situation of Problem 53.

55. $f'(x) = 6x - 4x^3$.

$$x_{n+1} = x_n - \frac{2 + 3x_n^2 - x_n^4}{6x_n - 4x_n^3}.$$

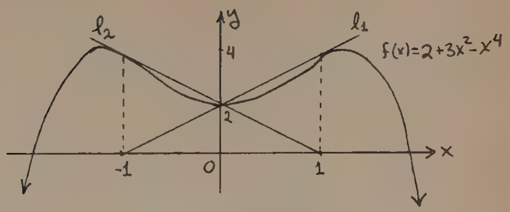
$x_1 = 1$,

$$x_2 = x_1 - \frac{2 + 3x_1^2 - x_1^4}{6x_1 - 4x_1^3} = -1,$$

$$x_3 = x_2 - \frac{2 + 3x_2^2 - x_2^4}{6x_2 - 4x_2^3} = 1,$$

$$x_4 = x_3 - \frac{2 + 3x_3^2 - x_3^4}{6x_3 - 4x_3^3} = -1,$$

$x_5 = 1$, $x_6 = -1$, $x_7 = 1$, $x_8 = -1$, etc.

56. 

Notice that the tangent line ℓ_1, corresponding to $x = 1$ intersects the x axis at -1; the tangent line ℓ_2 corresponding to $x = -1$, intersects the x axis at 1; hence, the alternating pattern of Problem 55.

57. Let $f(x) = 8x^3 - 6x - 1$.

(a) $f(-1) = -3$ $f(-\frac{1}{2}) = 1$;

so there is a root between -1 and $-\frac{1}{2}$.

$f(0) = -1$;

so there is a root between $-\frac{1}{2}$ and 0.

$f(\frac{1}{2}) = -3$, $f(1) = 1$;

so there is a root between $\frac{1}{2}$ and 1.

(b) Let $x_1 = -0.75$.

Then $x_2 = x_1 - \dfrac{8x_1^3 - 6x_1 - 1}{24x_1^2 - 6} = -0.766666667$,

$x_3 = -0.766045322$,

$x_4 = -0.766044443$,

$x_5 = x_4$.

So desired zero $z \approx -0.766044443$

Let $x_1 = 0.25$.

Then $x_2 = -0.166666667$,

$x_3 = -0.173611111$,

$x_4 = -0.173648177$,

$x_5 = -0.173648178$,

$x_6 = x_5$.

So desired zero $z \approx -0.173648177$

Let $x_1 = 0.75$.

Then $x_2 = 1.033333333$,

$x_3 = 0.950437802$,

$x_4 = 0.939859952$,

$x_5 = 0.939692662$,

$x_6 = 0.939692621$,

$x_7 = x_6$

So desired zero $z \approx 0.939692621$

(c) $\cos 20^\circ = 0.939692621$.

58. Let $f(x) = x^3 - 3rx^2 + 4r^3s = x^3 - 3(0.4)x^2 + 4(0.4)^3(0.174)$

$= x^3 - 1.2x^2 + 0.198144$

$f(0) = 0.198144$,

$f(1) = -0.001856$.

Thus, by the change-of-sign property, there is a root between 0 and 1.

$f'(x) = 3x^2 - 2.4x$.

Let $x_1 = 0.5$.

Then $x_2 = 0.551431111$,

$x_3 = 0.553691758$,

$x_4 = 0.553697461$,

$x_5 = x_4$.

So desired zero $z \approx 0.553697461$.

The depth is about 0.554 meter.

59. $f(x) = x^{-1} - k$, so $f'(x) = -x^{-2}$.

Thus,

$$x_{n+1} = x_n - \frac{x_n^{-1} - k}{-x_n^{-2}} = x_n + \frac{x_n^{-1}}{x_n^{-2}} - \frac{k}{x_n^{-2}}$$

$$= x_n + x_n - kx_n^2 = 2x_n - kx_n^2.$$

Let $x_n = a$ and $x_{n+1} = b$; then

$b = 2a - ka^2$.

60. Suppose $0 < a < \frac{2}{k}$, $k > 0$. Then, adding $-\frac{1}{k}$ to all members, we obtain $-\frac{1}{k} < a - \frac{1}{k} < \frac{1}{k}$ or $\left|a - \frac{1}{k}\right| < \frac{1}{k}$. Multiplying both sides of the last inequality by $|1 - ak|$, we have $\left|a - \frac{1}{k}\right| \cdot |1 - ak| < \frac{1}{k}|1 - ak|$, or

since $\frac{1}{k} > 0$,

$\left|(a - \frac{1}{k})(1 - ak)\right| < \left|\frac{1}{k}(1 - ak)\right|$ or

$\left|a - a^2k - \frac{1}{k} + a\right| < \left|\frac{1}{k} - a\right|$.

Therefore, $\left|2a - ka^2 - \frac{1}{k}\right| < \left|a - \frac{1}{k}\right|$;

that is, $\left|b - \frac{1}{k}\right| < \left|a - \frac{1}{k}\right|$.

61. $f(v) = v^3 - av^2 + bv - c$

$f'(v) = 3v^2 - 2av + b$, $a = 2.28 \times 10^{-2}$

$b = 3.60 \times 10^{-6}$, $c = 1.51 \times 10^{-10}$

$f(0) = -c < 0$ and $f(1) > 0$.

Let $v_1 = 0.5$.

$$v_2 = v_1 - \frac{v_1^3 - av_1^2 + bv_1 - c}{3v_1^2 - 2av_1 + b} = 0.3359$$

$v_3 = 0.2266$	$v_9 = 0.0296$
$v_4 = 0.1538$	$v_{10} = 0.0249$
$v_5 = 0.1053$	$v_{11} = 0.0230$
$v_6 = 0.0732$	$v_{12} = 0.0227$
$v_7 = 0.0520$	$v_{13} = 0.0226$
$v_8 = 0.0382$	$v_{14} = v_{13}$

Hence, $v = 0.023$ cubic meter.

62. x_1, $F(x_1)$, $F \circ F(x_1) = F(F(x_1)) = F(x_2)$,

$F \circ F \circ F(x_1) = F(F(F(x_1))) = F(F(x_2)) = F(x_3)$, et

Review Problem Set, Chapter 2, page 159

1. (a) To find average speed during 3^{rd} second, put $\Delta s = s(3) - s(2)$, so that $\frac{\Delta s}{\Delta t} = \frac{456 - 336}{\Delta t} = \frac{120}{1}$ $= 120$ feet/sec.

(b) Instantaneous speed $\frac{ds}{dt} = 200 - 32t$.

When $t = 2$, instantaneous speed is $200 - 64 =$ 136 feet/sec.

When $t = 3$, instantaneous speed is $200 - 96 =$ 104 feet/sec.

(c) When the object reaches its highest point, its instantaneous speed will be zero; hence, by (b), $200 - 32t = 0$, or

$t = \frac{25}{4}$ seconds. When $t = \frac{25}{4}$, we have

$s = 200(\frac{25}{4}) - 16(\frac{25}{4})^2 = 625$ feet.

2. (a) $x'(t) = 2 - t$. $x'(0) = 2 > 0$, so the particle is moving in the positive direction.

(b) $x'(t) = 2 - t.$ $x'(1) = 1$ foot/sec.

(c) $x'(t) = 0$, so $2 - t = 0$ when $t = 2$. After 2 seconds, direction changes.

3. (a) $\dfrac{A(x_2) - A(x_1)}{x_2 - x_1} = \dfrac{(20.2)^2 - 20^2}{0.2} = \dfrac{408.04 - 400}{0.2}$

$= \dfrac{8.04}{0.2}$

$= 40.2$ sq. inches per unit change in side, where x_2 is 20.2 inches, x_1 is 20 inches.

(b) $\dfrac{A(T_2) - A(T_1)}{T_2 - T_1} = \dfrac{408.04 - 400}{25} = \dfrac{8.04}{25}$

$= 0.3216$ sq. inch per degree change in temp. where $T_2 = 75°$, $T_1 = 50°$.

4. $y' = 3x^2$

when $x = 2$. Magnification is $3(2)^2 = 12$.

5. Tangent line: $y - 2 = f'(4)(x - 4)$. $f'(x) = 2x - 4$

So $f'(4) = 4$. $y - 2 = 4(x - 4)$. So $4x - y = 14$.

Normal line: $y - 2 = -\frac{1}{4}(x - 4)$; $4y - 8 = -x + 4$; $4y + x = 12$.

6. Tangent line: $y - \frac{4}{3} = g'(2)(x - 2)$. $g'(x) = \dfrac{(x + 1)\cdot 0 - 4}{(x + 1)^2} = \dfrac{-4}{(x + 1)^2}$. So $g'(2) = -\frac{4}{9}$,

and $y - \frac{4}{3} = -\frac{4}{9}(x - 2)$; that is, $9y - 12 = -4x + 8$, or, $9y + 4x = 20$.

Normal line: $y - \frac{4}{3} = \frac{9}{4}(x - 2)$, or, $12y - 16 = 27x - 54$, or, $12y - 27x = -38$, or, $27x - 12y = 38$.

7. Tangent Line: $y - \frac{27}{16} = f'(\frac{3}{2})(x - \frac{3}{2})$, $f'(x) = \frac{4}{3}x^3$, so that $f'(\frac{3}{2}) = \frac{4}{3}(\frac{27}{8}) = \frac{9}{2}$, $y - \frac{27}{16} = \frac{9}{2}(x - \frac{3}{2})$, $16y - 27 = 72x - 108$, $72x - 16y = 81$.

Normal line: $y - \frac{27}{16} = -\frac{2}{9}(x - \frac{3}{2})$, $y + \frac{2}{9}x = \frac{97}{48}$, $144y + 32x = 291$.

8. Tangent line: $y = f'(0)x$, $f'(x) = 96 - \frac{3}{2}x^2$, so that $f'(0) = 96$, $y = 96x$.

Normal line: $y = -\frac{1}{96}x$, $x + 96y = 0$.

9. (i) All are continuous on (a,b) except for Figure (d), which shows a discontinuity at 0, for example.

(ii) None of the functions are differentiable on (a,b).

(iii) None of them.

10. $\lim\limits_{h\to 0} \dfrac{f(a + h) - f(a - h)}{2h}$

$= \lim\limits_{h\to 0} \dfrac{f(a + h) - f(a) + f(a) - f(a + h)}{2h}$

$= \lim\limits_{h\to 0} \dfrac{1}{2}\dfrac{f(a + h) - f(a)}{h} + \lim\limits_{(-h)\to 0} \dfrac{1}{2}\dfrac{f(a-(-h))-f(a)}{(-h)}$

$= \frac{1}{2}f'(a) + \frac{1}{2}f'(a) = f'(a).$

11. $f(x + \Delta x) = 3(x + \Delta x) - 2 = 3x + 3\Delta x - 2$

(a) $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x}$

$= \dfrac{3\Delta x}{\Delta x} = 3.$

(b) $f'(x) = \lim\limits_{\Delta x\to 0} (3) = 3.$

12. (a) $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{(x + \Delta x) - (x + \Delta x)^2 - x + x^2}{\Delta x}$

$= \dfrac{\Delta x - 2x\Delta x - \Delta x^2}{\Delta x}$

$= 1 - 2x - \Delta x.$

(b) $f'(x) = \lim\limits_{\Delta x\to 0} (1 - 2x - \Delta x) = 1 - 2x.$

13. (a) $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{(x + \Delta x)^2 + x + \Delta x + 1 - x^2 - x - 1}{\Delta x}$

$= \dfrac{2x\Delta x + \Delta x^2 + \Delta x}{\Delta x} = 2x + \Delta x + 1.$

(b) $f'(x) = \lim\limits_{\Delta x\to 0} (2x + \Delta x + 1) = 2x + 1.$

14. (a) $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{2(x + \Delta x)^3 - 1 - 2x^3 + 1}{\Delta x}$

$= \dfrac{6x^2\Delta x + 6x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$

$= 6x^2 + 6x\Delta x + (\Delta x)^2.$

(b) $f'(x) = \lim\limits_{\Delta x\to 0} (6x^2 + 6x\Delta x + (\Delta x)^2) = 6x^2.$

15. (a) $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$$= \frac{\frac{1}{2}\left[(x+\Delta x)^2 - 4(x+\Delta x) + 3\right] - \frac{1}{2}(x^2 - 4x + 3)}{\Delta x}$$

$$= \frac{1}{2}\left(\frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x}\right) = \frac{1}{2}(2x + \Delta x - 4).$$

(b) $f'(x) = \lim\limits_{\Delta x \to 0}\left[\frac{1}{2}(2x + \Delta x - 4)\right] = \frac{1}{2}(2x - 4) = x-2$

16. (a) $\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \dfrac{\frac{1}{x+\Delta x - 1} - \frac{1}{x-1}}{\Delta x}$

$$= \frac{-\Delta x}{\Delta x(x-1)(x+\Delta x - 1)}$$

$$= \frac{-1}{(x-1)(x+\Delta x-1)}$$

(b) $f'(x) = \lim\limits_{\Delta x \to 0}\left[\dfrac{-1}{(x-1)(x+\Delta x - 1)}\right]$

$$= \frac{-1}{(x-1)(x-1)} = \frac{-1}{(x-1)^2}.$$

17. (a) $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \frac{1 + \frac{2}{x+\Delta x} - 1 - \frac{2}{x}}{\Delta x} = \frac{-2\Delta x}{\Delta x(x)(x+\Delta x)}$$

$$= \frac{-2}{x(x+\Delta x)}.$$

(b) $f'(x) = \lim\limits_{\Delta x \to 0}\left(\dfrac{-2}{x(x+\Delta x)}\right) = \dfrac{-2}{x^2}.$

18. (a) $\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \dfrac{-\pi^2 + \pi^2}{\Delta x} = \dfrac{0}{\Delta x} = 0.$

(b) $f'(x) = \lim\limits_{\Delta x \to 0}(0) = 0.$

19. Problem 11:

$D_x(3x - 2) = 3\, D_x x - D_x 2 = 3(1) - 0 = 3$

Problem 13:

$D_x(x^2 + x + 1) = D_x x^2 + D_x x + D_x 1 = 2x + 1 + 0 =$
$2x + 1$

Problem 15:

$D_x \frac{1}{2}(x^2 - 4x + 3) = \frac{1}{2} D_x(x^2 - 4x + 3) = \frac{1}{2}\Big[D_x x^2 -$
$4 D_x x + D_x 3\Big] = \frac{1}{2}\left[2x - 4 + 0\right] = x - 2$

Problem 17:

$D_x(1 + \frac{2}{x}) = D_x(1 + 2x^{-1}) = D_x(1) + D_x 2x^{-1} = 0 + 2$
$D_x x^{-1} = 2(-1)x^{-2} = -2x^{-2} = -\frac{2}{x^2}$

20. When is $D_x(x^n \cdot x^m) = D_x x^n \cdot D_x x^m$

or $D_x x^{n+m} = D_x x^n \cdot D_x x^m$;

i.e., $(n+m)x^{n+m-1} = nx^{n-1} \cdot m \cdot x^{m-1}$

or $(n+m)x^{n+m-1} = nmx^{m+n-2}$

or $(n+m)x = nm$?

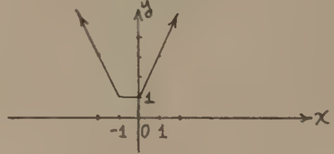
True if $m = n = 0$.

21. (a) For $x \geq 0$, $f(x) = x + x + 1 = 2x + 1$.

For $-1 \leq x \leq 0$, $f(x) = -x + x + 1 = 1$.

For $x \leq -1$, $f(x) = -x - x - 1 = -2x - 1$.

(b) f is not differentiable at 0 or at -1.



22. Yes, $\lim\limits_{\Delta x \to 0} \dfrac{f(0+\Delta x) - f(0)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x g(\Delta x) - 0 \cdot g(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} g(\Delta x) = g(0), \text{ by continuity of } g.$$

$g(0)$ exists since g is continuous at 0. So $f'(0)$
$= g(0)$.

23. (a) $(f+g)'(7) = f'(7) + g'(7) = 3 + (-\frac{1}{30}) =$

(b) $(f-g)'(7) = f'(7) - g'(7) = 3 + \frac{1}{30} = \frac{91}{30}$.

(c) $(fg)'(7) = (fg')(7) + (f'g)(7) = f(7)g'(7) +$
$f'(7)g(7)$

$$= 10(-\tfrac{1}{30}) + 3(5) = -\tfrac{1}{3} + 15 = \tfrac{44}{3}.$$

(d) $(\frac{f}{g})'(7) = \dfrac{g(7)f'(7) - f(7)g'(7)}{[g(7)]^2}$

$$= \frac{5 \cdot 3 - 10(-\frac{1}{30})}{(5)^2}$$

$$= \frac{15 + \frac{1}{3}}{25} = \frac{46}{75}.$$

(e) $\dfrac{f(7)\left[f + 3g\right]'(7) - (f+3g)(7)f'(7)}{[f(7)]^2}$

$$= \frac{10\left[3 + 3(-\frac{1}{30})\right] - \left[10 + 3 \cdot 5\right](3)}{100}$$

$$= \frac{10\left[\frac{29}{10}\right] - 75}{100} = \frac{29 - 75}{100} = -\frac{46}{100} = -\frac{23}{50}$$

(f) $\left[f' + 2g'\right](7) = f'(7) + 2g'(7) = 3 + 2(-\frac{1}{30})$

$3 - \frac{1}{15} = \frac{44}{15}$.

(g) $\left(\frac{f}{f + g}\right)'(7) = \frac{(f + g)(7)\left[f'(7)\right] - f(7)\left[f+g\right]'(7)}{\left[(f + g)(7)\right]^2}$

$= \frac{\left[f(7) + g(7)\right](3) - 10\left[f'(7) + g'(7)\right]}{(10 + 5)^2}$

$= \frac{15(3) - 10\left[3 + \left(-\frac{1}{30}\right)\right]}{225}$

$= \frac{45 - 10\left(\frac{89}{30}\right)}{225} = \frac{45 - \frac{89}{3}}{225}$

$= \frac{46}{675}$.

24. (a) $f(x + y) = f(x) = f(y)$ for all x,y. Let $x = y = 0$. Then $f(0) = f(0) + f(0)$; hence, $f(0) = 0$.

(b) $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f(x) + f(\Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x}$ [see (a)]

$= f'(0)$.

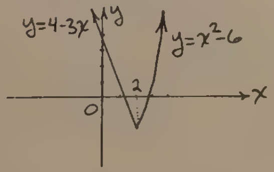
25. (a) $\lim_{x \to 2^-} (4 - 3x) = -2$; $\lim_{x \to 2^+} (x^2 - 6) = -2$;

$f(2) = -2$; f is continuous at 2.

(b) $f'_-(2) = -3$; $f'_+(2) = 2(2) = 4$.

(c) f is not differentiable at 2.

(d)

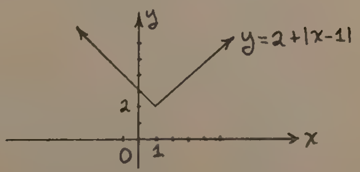


26. (a) f is continuous at 1.

(b) $f'_-(1) = -1$; $f'_+(1) = 1$.

(c) f is not differentiable at 1.

(d)

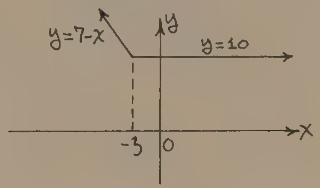


27. (a) f is continuous at -3.

(b) $f'_-(-3) = -1$; $f'_+(-3) = 0$.

(c) f is not differentiable at -3.

(d)



28. (a) $\lim_{x \to 2^-} [\![x - 2]\!] = -1$; $\lim_{x \to 2^+} [\![x - 2]\!] = 0$

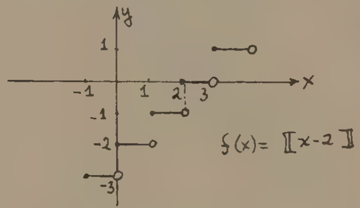
So f is not continuous at $x = 2$.

(b) $f'_+(2) = \lim_{h \to 0^+} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[\![h]\!]}{h} = 0$;

$f'_-(2) = \lim_{h \to 0^-} \frac{[\![h]\!]}{h} = \lim_{h \to 0^-} \frac{-1}{h}$ doesn't exist.

(c) f is not differentiable at 2.

(d)



29. $f'(3) = \lim_{\Delta x \to 0} \frac{f(3 + \Delta x) - f(3)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\sqrt{(3 + \Delta x)^2 + 1} - \sqrt{10}}{\Delta x}$

If $x = 3(10^{-4})$,

then $\frac{\sqrt{(3 + \Delta x)^2 + 1} - \sqrt{10}}{\Delta x} \approx 0.94869$;

so $f'(3) \approx 0.94869$.

30. For $\Delta x = (0.6169)10^{-4}$, $\frac{g(a + \Delta x) - g(a)}{\Delta x}$

$= \frac{\cot\sqrt{0.6169 + \Delta x} - \cot\sqrt{0.6169}}{\Delta x}$

≈ -1.273024801;

so $g'(0.6169) \approx -1.273024801$.

31. $f'(x) = 35x^4$.

32. $g'(x) = 8x^{-3}$.

33. $h'(x) = -\frac{3}{2}x^{-4}$.

34. $F'(x) = \frac{3\pi}{2}x^{1/2}$.

35. $G'(x) = \frac{2\sqrt{2}}{3}x^{-1/3}$.

36. $H(x) = -2x^{-3/4}$ so, $H'(x) = \frac{3}{2}x^{-7/4}$.

37. $f'(x) = 10x - 7.$ 38. $g'(x) = 0.$

39. $h'(x) = -27x^2 + 6x - 1.$ 40. $F'(x) = -\frac{14}{3}x^{-1/3} + \frac{14}{3}x^{-8}.$

41. $G'(t) = 105t^{20} + 60t^3 + 10t^{-3}.$

42. $H'(u) = 7.44u^{1.4} - 5.04x^{-5.2}.$

43. $f'(x) = (x^2 + 2x + 1)(6x^2) + (2x^3 + 5)(2x + 2)$
$= 10x^4 + 16x^3 + 6x^2 + 10x + 10.$

44. $g'(t) = (\sqrt{t} + 1)(2 + \frac{1}{2\sqrt{t}}) + (2t + \sqrt{t} - 2)\frac{1}{2\sqrt{t}}$
$= 3\sqrt{t} + 3 - \frac{1}{2\sqrt{t}}.$

45. $h'(s) = 27(6s^4 + 5s^2 - s^{-1})^{26}(24s^3 + 10s + s^{-2}).$

46. $F'(v) = 99(2\sqrt{v} - 3v^{-2})^{98}(\frac{1}{\sqrt{v}} + 6v^{-3}).$

47. $G'(x) = \frac{(x + 2)(2x + 3) - (x^2 + 3x - 1)}{(x + 2)^2}$
$= \frac{x^2 + 4x + 7}{(x + 2)^2}.$

48. $H'(t) = \frac{(t - 1)\frac{1}{2\sqrt{t}} - (\sqrt{t} + 3)}{(t - 1)^2}$
$= \frac{-t - 1 - 6\sqrt{t}}{2\sqrt{t}(t - 1)^2}.$

49. $f(x) = x^{5/2} + 2x^{3/2} + x^{1/2}$ so $f'(x) = \frac{5}{2}x^{3/2} + 3x^{1/2}$
$+ \frac{1}{2}x^{-1/2}.$

50. $g'(t) = 2\sqrt{t + 1}(2) + (2t + 1) \cdot 2 \cdot \frac{1}{2}(t + 1)^{-1/2}$
$= \frac{6t + 5}{\sqrt{t + 1}}.$

51. $h(x) = 1 + 3x^{-2} - x^{-1} - 3x^{-3}$ so $h'(x) = -6x^{-3} + x^{-2}$
$+ 9x^{-4} = (\frac{1}{x} - \frac{3}{x^2})^2.$

52. $F'(y) = \frac{\sqrt{y + 1}(2y - 5) - (y^2 - 5y + 4) \cdot \frac{1}{2}(y + 1)^{-1/2}}{(\sqrt{y + 1})^2}$
$= \frac{2(y + 1)(2y - 5) - (y^2 - 5y + 4)}{2(y + 1)^{5/2}}$
$= \frac{3y^2 - y - 14}{2(y + 1)^{5/2}}.$

53. $G'(x) = \frac{1}{2}(x^2 + 12)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 12}}.$

54. $H'(z) = \frac{1}{2}\left[z^2 + \sqrt{z^2 + 1}\right]^{-1/2}\left[2z + \frac{1}{2}(z^2 + 1)^{-1/2} \cdot 2z\right]$
$= \frac{3}{2}(z^2 + \sqrt{z^2 + 1})^{-1/2}\left[2 + (z^2 + 1)^{-1/2}\right].$

55. $f'(x) = 8 \cdot \frac{1}{3}x^{-2/3} - \left[\frac{\sqrt{x} - 1 - x(\frac{1}{2\sqrt{x}})}{(\sqrt{x} - 1)^2}\right]$
$= \frac{8}{3}x^{-2/3} - \frac{\sqrt{x} - 2}{2(\sqrt{x} - 1)^2} = \frac{8\sqrt[3]{x}}{3x} - \frac{\frac{1}{2}\sqrt{x} - 1}{(\sqrt{x} - 1)^2}$

56. $g'(x) = \frac{x - \sqrt{x}(1 - \frac{1}{2\sqrt{x}})}{(x - \sqrt{x})^2} = -\frac{\sqrt{x}}{2(x - \sqrt{x})^2}.$

57. $h'(x) = (x^2 + 7)^3 \cdot \frac{1}{2}(x - 7)^{-1/2} +$
$\sqrt{x - 7}(3)\ (x^2 + 7)^2\ (2x)$
$= \frac{1}{2}(x - 7)^{-1/2}(x^2 + 7)^2\left[x^2 + 7 + 12x(x - 7)\right]$
$= \frac{1}{2}(x - 7)^{-1/2}(x^2 + 7)^2(13x^2 - 84x + 7).$

58. $f(y) = \frac{y + 1}{y - 1}$, $y \neq 0$, so $f'(y) = \frac{y - 1 - (y + 1)}{(y - 1)^2}$
$= \frac{-2}{(y - 1)^2}$, $y \neq 0.$

59. $g'(t) = \frac{3}{2}(\frac{t^2 - 3t + 2}{t^2 + 2t + 5})^{1/2}.$
$\left[\frac{(t^2 + 2t + 5)(2t - 3) - (t^2 - 3t + 2)(2t + 2)}{(t^2 + 2t + 5)^2}\right]$
$= \frac{3}{2}\ \frac{(t^2 - 3t + 2)^{1/2}}{(t^2 + 2t + 5)^{5/2}}\ (5t^2 + 6t - 19).$

60. $h'(q) = \frac{1}{2}(q + \sqrt{q + \sqrt{q}})^{-1/2}\left[1 + \frac{1}{2}(q + \sqrt{q})^{-1/2} \cdot (1 + \frac{1}{2\sqrt{q}})\right].$

61. $f'(x) = \frac{(cx + d)a - (ax + b)c}{(cx + d)^2} = \frac{ad - bc}{(cx + d)^2}$

62. $q'(x) = r(\frac{ax + b}{cx + d})^{r-1}(\frac{ad - bc}{(cx + d)^2})$ (by Problem 61)
$= \frac{r(ax + b)^{r-1}(ad - bc)}{(cx + d)^{r+1}}$

63. $f'(x) = \frac{1}{2} \cdot \cos 2x(2) = \cos 2x.$

64. $g'(t) = t(\cos\frac{1}{t})(-\frac{1}{t^2}) + \sin\frac{1}{t} = -\frac{1}{t}\cos\frac{1}{t} + \sin\frac{1}{t}$

65. $h'(t) = \frac{3}{2}\sin 4t(4) = 6\sin 4t.$

66. $F'(x) = \sqrt{x + 1}\sec^2(x + 1) + \left[\tan(x + 1)\right]\frac{1}{2\sqrt{x + 1}}$
$= \sqrt{x + 1}\sec^2(x + 1) + \frac{1}{2\sqrt{x + 1}}\tan(x + 1).$

67. $G'(x) = \frac{5}{3}\cos(3x - 1)\left[3\right] = 5\cos(3x - 1).$

68. $H'(\theta) = \left[-\csc^2(\sqrt{\theta} + \theta)\right]\left(\frac{1}{2\sqrt{\theta}} + 1\right) = -\left[\frac{1}{2\sqrt{\theta}} + 1\right]\csc^2$
$(\sqrt{\theta} + \theta)$.

69. $f'(t) = \dfrac{(1 + \csc t)(-\sin t) - \cos t(-\csc t \cot t)}{(1 + \csc t)^2}$

$= \dfrac{-\sin t - 1 + \cot^2 t}{(1 + \csc t)^2}$.

70. $g'(y) = \frac{1}{3}(\cos 5y)^{-2/3}\left[(-\sin 5y)(5)\right] =$
$-\frac{5}{3}\sin 5y\,(\cos 5y)^{-2/3}$.

71. $h'(\theta) = (3\sin\theta)(-2\sin 2\theta) + (\cos 2\theta)(3\cos\theta)$

$= -6\sin\theta\sin 2\theta + 3\cos 2\theta\cos\theta$.

72. $F'(x) = x(\sec^2\sqrt{3}\,x)(\sqrt{3}) + \tan\sqrt{3}\,x$

$= \sqrt{3}\,x\sec^2\sqrt{3}\,x + \tan\sqrt{3}\,x$.

73. $G'(s) = \frac{2}{3}(\sec^2\frac{3}{2}s)(\frac{3}{2}) - \frac{3}{4}(-\csc^2\frac{4}{3}s)(\frac{4}{3})$

$= \sec^2\frac{3}{2}s + \csc^2\frac{4}{3}s$.

74. $H'(x) = \left[2\cos(\csc x)\right]\left[-\csc x\cot x\right]$

$= -2\csc x\cot x\cos(\csc x)$.

75. $f'(x) = \dfrac{(x + 1)(-2\csc x\cot x + 3\csc^2 x)}{(x + 1)^2}$

$- \dfrac{(2\csc x - 3\cot x)}{(x + 1)^2}$.

76. $g'(x) = \frac{3}{2}\cot^{1/2}5x(-\csc^2 5x)(5) = -\frac{15}{2}\cot^{1/2}5x\csc^2 5x$.

77. $h'(\theta) = \dfrac{(\sec\theta + \tan\theta)(1) - \theta(\sec\theta\tan\theta + \sec^2\theta)}{(\sec\theta + \tan\theta)^2}$

78. $q'(y) = \frac{1}{2}(\sin\sqrt{y})^{-1/2}(\cos\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right) = \frac{1}{4\sqrt{y}}(\sin\sqrt{y})^{-1/2}\cos\sqrt{y}$,
$\cos\sqrt{y}$

79. $f'(t) = -a\sin(\omega t - \phi)\left[\omega\right] = -a\,\omega\sin(\omega t - \phi)$.

80. $g'(x) = \begin{cases} -3\sin x & \text{if } x > 0 \\ 8x & \text{if } x < 0. \end{cases}$

$g'(0) = \lim\limits_{h\to 0}\dfrac{g(0 + h) - g(0)}{h} = \lim\limits_{h\to 0}\dfrac{g(h) - 3}{h}$.

$\lim\limits_{h\to 0^-}\dfrac{g(h) - 3}{h} = \lim\limits_{h\to 0^-}\dfrac{4h^2 + 3 - 3}{h}$

$= \lim\limits_{h\to 0}(4h) = 0$.

$\lim\limits_{h\to 0^+}\dfrac{g(h) - 3}{h} = \lim\limits_{h\to 0^+}\dfrac{3\cos h - 3}{h}$

$= 3\lim\limits_{h\to 0^+}\left(\dfrac{\cos h - 1}{h}\right) = 3(0) = 0$.

So $g'(0) = 0$.

So $g'(x) = \begin{cases} -3\sin x & \text{if } x \geq 0 \\ 8x & \text{if } x < 0. \end{cases}$

81. $\dfrac{dP}{dt} = -3\sin\left[(6.5564)t - 5.804\right]\left[6.5564\right]$

$= -19.6692\sin\left[(6.5564)t - 5.804\right]$.

When $t = 21$,

$\dfrac{dP}{dt} = -19.6692\sin\left[(6.5564)21 - 5.804\right]$

$= 1.3069$.

82. $f'(t) = -A\sin\left[\omega_c t + bt\cos\omega_m t\right]$
$\left[\omega_c + bt(-\sin\omega_m t)(\omega_m) + (\cos\omega_m t)b\right]$
$= -A\sin\left[\omega_c t + bt\cos\omega_m t\right]$
$\left[\omega_c - bt\,\omega_m\sin\omega_m t + b\cos\omega_m t\right]$

83. $f'(x) = kx^{k-1}$, $f'(1) = k \cdot 1^{k-1} = k$. Therefore, the equation of the desired tangent line is $y - 1 = k(x - 1)$. To find the y intercept, we put $x = 0$ and solve for y to get $y = 1 - k$. The distance between $(0,1-k)$ and $(0,0)$ is $|1 - k| = |k - 1|$.

84. Let $f(x) = x^n$, so that $f'(x) = nx^{n-1}$. Since n is odd, $n - 1$ is even, so that $1^{n-1} = (-1)^{n-1} = 1$. Hence, $f'(-1) = f'(1) = n$; so the tangent lines are parallel since they have the same slopes.

85. $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{(1 + u)(-1) - (1 - u)(1)}{(1 + u)^2}\cdot$

$\dfrac{(2 + x)(-1) - (3 - x)(1)}{(2 + x)^2}$.

$= \dfrac{-2}{(1 + u)^2}\cdot\dfrac{-5}{(2 + x)^2}$.

$= \dfrac{10}{\left[(1 + u)(2 + x)\right]^2}$

$= \dfrac{10}{\left[\left(1 + \frac{3 - x}{2 + x}\right)(2 + x)\right]^2}$

$= \dfrac{10}{\left[(2 + x) + (3 - x)\right]^2} = \dfrac{10}{25} = \dfrac{2}{5}$.

86. $\dfrac{dE}{dx} = \dfrac{(a^2 + x^2)^{3/2}Q - Qx\left(\frac{3}{2}\right)(a^2 + x^2)^{1/2}(2x)}{(a^2 + x^2)^3}$
$= \dfrac{Q(a^2 + x^2)^{1/2}\left[a^2 + x^2 - 2x^2\right]}{(a^2 + x^2)^3} = \dfrac{Q(a^2 - x^2)}{(a^2 + x^2)^{5/2}}$.

87. $f'(x) = 6x - 2$.

Slope of $x + 4y - 1 = 0$ is $-\frac{1}{4}$ since $y = -\frac{1}{4}x + \frac{1}{4}$.

Thus, want $6x - 2 = 4$.

So $6x = 6$ or $x = 1$.

$f(1) = 3 - 2 + 1 = 2$.

So $(1,2)$ satisfies the condition; $y - 2 = 4(x - 1)$.

88. $g'(x) = \dfrac{(x - 1) - x}{(x - 1)^2} = \dfrac{-1}{(x - 1)^2}$.

Suppose a is the abcissa of the desired point.

Then we want the line $y - g(a) = -\dfrac{1}{(a - 1)^2}(x - a)$.

When $x = 0$, $y = 4$. Thus, $4 - \dfrac{a}{a - 1} = -\dfrac{1}{(a - 1)^2}(0 - a)$

or $4(a - 1)^2 - a(a - 1) = a$

or $3a^2 - 8a + 4 = 0$

or $(3a - 2)(a - 2) = 0$; so

$a = \frac{2}{3}$ (reject since $x > 1$)	$a = 2$

Thus, $x = 2$; now $g(2) = \dfrac{2}{2 - 1} = 2$. Thus, $(2,2)$ satisfies the condition; $y - 2 = -1(x - 2)$.

89. $h'(x) = -\sin x$. Thus, the slope of the normal line is $\dfrac{1}{\sin x}$. The slope of $2x - y + 4 = 0$ is 2 since $y = 2x + 4$. Thus, $\dfrac{1}{\sin x} = 2$, or $\sin x = \frac{1}{2}$. Thus, $x = \frac{\pi}{6}$ $(0 \le x \le \frac{\pi}{2})$.

$h(\frac{\pi}{6}) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

Thus $(\frac{\pi}{6}, \frac{\sqrt{3}}{2})$ satisfies the condition; $y - \frac{\sqrt{3}}{2} = 2(x - \frac{\pi}{6})$.

90. $F'(x) = \dfrac{\sqrt{x^2 + 1} - x \cdot \frac{1}{2}(x^2 + 1)^{-1/2}(2x)}{x^2 + 1}$

$= \dfrac{\sqrt{x^2 + 1} - x^2(x^2 + 1)^{-1/2}}{x^2 + 1}$

$= \dfrac{1}{(x^2 + 1)^{3/2}}$.

Slope of normal line is $-(x^2 + 1)^{3/2}$ and is equal to -1 since $y = -x + 1$; so $(x^2 + 1)^{3/2} = 1$.

Hence, $x^2 + 1 = 1$ or $x^2 = 0$, so $x = 0$.

$F(0) = \dfrac{0}{\sqrt{0^2 + 1}} = 0$. Thus, $(0,0)$ satisfies the condition; the desired line is $y = -x$.

91. $f'(x) = 3x^2 - 3 = 0$. $x^2 = 1$, so $x = \pm 1$. $f(1) = -4$, $f(-1) = 0$; points are $(1,-4)$ and $(-1,0)$.

92. $g'(x) = 4x^{1/3} - 4x^{-2/3} = 0$; $4x^{-2/3}(x - 1) = 0$; Since $x \ne 0$, $x - 1 = 0$ or $x = 1$. Now $g(1) = 3 - 12 + 2 = -7$; $(1,-7)$ is a point.

93. $h'(x) = \dfrac{(x^2 + 4) - x(2x)}{(x^2 + 4)^2} = 0$ So $x^2 + 4 - 2x^2 = 0$ or $x^2 = 4$; thus, $x = \pm 2$. $h(2) = \dfrac{2}{4 + 4} = \dfrac{1}{4}$, $(-2) = \dfrac{-2}{4 + 4} = -\frac{1}{4}$; points are $(2, \frac{1}{4})$ and $(-2, -\frac{1}{4})$.

94. $F'(x) = x \cdot \frac{1}{2}(3 - x)^{-1/2}(-1) + \sqrt{3 - x} = 0$ provided $-\frac{x}{2} + 3 - x = 0$.

So $3x = 6$ or $x = 2$. $F(2) = 2\sqrt{1} = 2$. Thus, $(2,2)$ is a point.

95. $G'(x) = 2\sin x\cos x - \sin x = 0$

$= \sin x(2\cos x - 1) = 0$, so

$\sin x = 0$,	$\cos x = \frac{1}{2}$,
$x = 0, \pi, -\pi$.	$x = \frac{\pi}{3}, -\frac{\pi}{3}$.
$G(0) = 1$,	$G(\frac{\pi}{3}) = \frac{5}{4} = G(-\frac{\pi}{3})$.
$G(\pi) = G(-\pi) = -1$.	

So points are $(0,1)$, $(\pi,-1)$, $(-\pi,-1)$, $(\frac{\pi}{3},\frac{5}{4})$, $(\frac{-\pi}{3},\frac{5}{4})$

96. $H'(x) = 1 - \csc^2 x = 0$.

Thus, $\csc^2 x = 1$ or $\sin^2 x = 1$; so

$\sin x = 1$,	$\sin x = -1$,
$x = \frac{\pi}{2}$	$x = \frac{3\pi}{2}$

$H(\frac{\pi}{2}) = \frac{\pi}{2} + \cot\frac{\pi}{2} = \frac{\pi}{2}$, $H(\frac{3\pi}{2}) = \frac{3\pi}{2} + \cot\frac{3\pi}{2} = \frac{3\pi}{2}$.

Points are $(\frac{\pi}{2},\frac{\pi}{2})$ and $(\frac{3\pi}{2},\frac{3\pi}{2})$.

97. Necessary condition for relative extrema. If the function f has a relative extremum at the number c, and if f is differentiable at c, then $f'(c) = 0$.

98. (a) $g'(x) = 1 - \cos x = 0$ or $\cos x = 1$ so $x = 0$; $g(0) = 0 - \sin 0 = 0$, thus g has a horizontal tangent at (0,0).

(b) For $x > 0$ and small, we have $g(x) > 0$; whereas for $x < 0$ and close to zero, $g(x) < 0$. Thus, no relative extremum at (0,0).

99. $h'(x) = f'(g(x))g'(x)$.
So $h'(2) = f'(g(2))g'(2) = f'(0)(-1) = 12(-1) = -12$.

100. $h'(x) = f'(g(x))g'(x)$.
So $h'(\pi) = f'(g(\pi))g'(\pi) = f'(\pi)(\pi) = \frac{1}{\pi}(\pi) = 1$.

101. $h'(x) = f'(g(x))g'(x)$.
So $h'(\sqrt{2}) = f'(g(\sqrt{2}))g'(\sqrt{2}) = \sqrt{2}\cdot\sqrt{2} = 2$.

102. $h'(x) = f'(g(x))g'(x)$.
So $h'(1.732) = f'(g(1.732))g'(1.732) = f'(7.007)(5) = 0.2(5) = 1$.

103. If s is the side of the square, then $s = \frac{d}{\sqrt{2}}$, and so the area $A = s^2 = f(d) = \frac{d^2}{2}$.
Let $g(d) = \frac{1}{2}d$ and $h(d) = d^2$;
then $g(h(d)) = g(d^2) = \frac{1}{2}d^2$.
Thus, $f = g \circ h$.

104. (a) Orbit gives successive new prices.

(b) $Po = f(Po)$ then
$Po = Po + k(q - s)$
$0 = 0 + k(q - s)$
or $q = s$, i.e., $APo - B = b - aPo$, so
$$Po = \frac{B + b}{A + a}.$$
Point where supply meets demand.

105. $f(x) = \alpha + \beta x$, $\beta \neq 1$

$x_2 = f(x_1) = \alpha + \beta x$

$x_3 = f \circ f(x_1) = f(f(x_1)) = f(\alpha + \beta x) = \alpha + \beta(\alpha + \beta x) = \alpha + \alpha\beta + \beta^2 x$,

$x_4 = f \circ f \circ f(x_1) = f(f(f(x_1))) = f(x_3) = f(\alpha + \alpha\beta + \beta^2 x) = d + \beta(\alpha + \alpha\beta + \beta^2 x) = \alpha + \alpha\beta + \alpha\beta^2 + \beta^3 x$,

$x_5 = \alpha + \alpha\beta + \alpha\beta^2 + \alpha\beta^3 + \beta^4 x$,

etc.

(a) Let $x_0 = f(x_0) = \alpha + \beta x_0$
or $x_0(1 - \beta) = \alpha$.
Thus, $x_0 = \frac{\alpha}{1 - \beta}$ $(\beta \neq 1)$.

(b) $|f(x) - x_0| = |\alpha + \beta x - x_0| = |\alpha - x_0 + \beta x|$
$= |-\beta x_0 + \beta x| = |\beta||x - x_0|$.

(c) Consider the statement P(n):
P(n): $|x_n + 1 - x_0| = |\beta|^n |x - x_0|$, where n is a positive integer. By part (b), P(1): $|x_2 - x_0| = |\beta||x_1 - x_0|$ is true since $x_2 = f(x_1)$.
Assume P(k) is true, i.e., $|x_{k+1} - x_0| = |P|^k |x_1 - x_0|$
Show P(k + 1) is true, i.e., $|x_{k+2} - x_0| = |\beta|^{k+1} |x_1 - x_0|$.
Now,
$$|x_{k+2} - x_0| = |\alpha + \alpha\beta + \alpha\beta^2 + \cdots + \alpha\beta^k + \beta^{k+1} x - x_0|$$
$$= |-\beta x_0 + \alpha\beta + \alpha\beta^2 + \cdots + \alpha\beta^k + \beta^{k+1} x|$$
$$= |\beta|\,|-x_0 + \alpha + \alpha\beta + \cdots + \alpha\beta^{k-1} + \beta^k x|$$
$$= |\beta||x_{k+1} - x_0| = |\beta|\cdot|\beta|^k|x_1 - x_0|$$
$$= |\beta|^{k+1}|x_1 - x_0|$$
Hence, P(k + 1) is true, and so P(n) is true for all positive integers.

(d) If $|\beta| < 1$, then $|\beta^n| < 1$.
Thus, $|x_n + 1 - x_0| = |\beta^n|\,|x_1 - x_0| < |x_1 - x_0|$.
Thus, $x_n + 1$ is closer to x_0 than x_1.

106. By the triangle inequality,

$|x_{n+1} - x_0| = |x_{n+1} + (-x_0)| \le |x_{n+1}| + |-x_0| =$
$|x_{n+1}| + |x_0|$.

Therefore, $|x_{n+1}| \ge |x_{n+1} - x_0| - |x_0|$, and it follows from part (c) of Problem 105 that

$$|x_{n+1}| \ge |\beta|^n |x_1 - x_0| - |x_0| .$$

If $|\beta| > 1$ and $x_1 \ne x_0$, then $|x_1 - x_0| > 0$ and $|\beta|^n |x_1 - x_0|$ can be made arbitrarily large by choosing n large enough. Hence, $|x_{n+1}|$ can be made arbitrarily large by choosing n large enough.

If $\beta = -1$, then $f(x) = \alpha - x$, $x_2 = f(x_1) = \alpha - x_1$, $x_3 = f(x_2) = \alpha - x_2 = \alpha - (\alpha - x_1) = x_1$, $x_4 = f(x_3) = f(x_1) = \alpha - x_1$, and so forth. Thus, if $\beta = -1$, the values of x_n oscillate back and forth between x_1 and $\alpha - x_1$.

107. $\alpha = 2.7$, $x_1 = 5$.

(a) $\beta = 0.9$.

$x_1 = 5.0$	$x_2 = 7.2$
$x_3 = 9.18$	$x_4 = 10.962$
$x_5 = 12.56580$	$x_6 = 14.00922$
$x_7 = 15.308298$	$x_8 = 16.4774682$
$x_9 = 17.52972138$	$x_{10} = 18.47674924$

(b) $\beta = 1.1$.

$x_1 = 5.0$	$x_2 = 8.2$
$x_3 = 11.72$	$x_4 = 15.592$
$x_5 = 19.8512$	$x_6 = 24.53632$
$x_7 = 29.689952$	$x_8 = 35.3589472$
$x_9 = 41.59484192$	$x_{10} = 48.45432611$

108. $f(p) = p + k(q - s)$

$= p + k(b - ap - Ap + B)$

$= k(b + B) + (1 + ak - Ak)p = \alpha + \beta p$ where

$\alpha = k(b + B)$ and $\beta = 1 - ak - Ak$.

From Problem 105, if $|\beta| < 1$ then successive prices approach a number $\dfrac{\alpha}{1-\beta} = \dfrac{k(b+B)}{1-(1-ak-Ak)} = \dfrac{k(b+B)}{ak+Ak}$. $|B| < 1$ means $|1 - ak - Ak| < 1$. Now, $\beta = 1 - ak - Ak$, so the recursive pricing model is stable if and only if $|1 - ak - Ak| < 1$ and the equilibrium price is

$$\frac{k(b + B)}{ak + Ak}.$$

109. $9x^2 + 6y^2 D_x y = 0$ so $D_x y = \dfrac{-9x^2}{6y^2} = \dfrac{-3x^2}{2y^2}$

110. $4x^3 + 4x^2(2y\, D_x y) + y^2(8x) = 0$

or

$x^2 + 2xy\, D_x y + 2y^2 = 0$.

So $D_x y = -\dfrac{x^2 + 2y^2}{2xy}$.

111. $10x + 24y^2 D_x y = 16 \cdot \frac{1}{2}(x + 1)^{-1/2}$.

So $D_x y = \dfrac{8(x + 1)^{-1/2} - 10x}{24y^2}$

$= \dfrac{4(x + 1)^{-1/2} - 5x}{12y^2}$.

112. $12x^2 - 5x(2y\, D_x y) - 5y^2 + 3y^2 D_x y = 0$.

So $D_y x = \dfrac{5y^2 - 12x^2}{3y^2 - 10xy}$.

113. $1 - \cos y\, D_x y = 0$ or $D_y x = \dfrac{1}{\cos y} = \sec y$.

114. $\cos y\, D_x y = -\sin x$ or $D_x y = -\dfrac{\sin x}{\cos y}$.

115. $D_x y = x(-\sin y)\, D_x y + \cos y$.

So $D_x y = \dfrac{\cos y}{1 + x \sin y}$.

116. $y \sec^2 x^2 (2x) + \tan x^2\, D_x y - x \cdot 4y^3\, D_x y - y^4 = 0$

So $D_x y = \dfrac{y^4 - 2xy \sec^2 x^2}{\tan x^2 - 4xy^3}$.

117. $18x + 6y^2 D_x^2 y + 6\, D_x y(2y\, D_x y) = 0$.

So $D_x^2 y = -\dfrac{18x + 12y(D_x y)^2}{6y^2}$

$= -\dfrac{18x + 12y\left(\frac{-3x^2}{2y^2}\right)^2}{6y^2}$

$= -\dfrac{18x + 12y\left(\frac{9x^4}{4y^4}\right)}{6y^2}$

$= -\dfrac{18xy^3 + 27x^4}{6y^5}$

$$= -\frac{6xy^3 + 9x^4}{2y^5}$$

$$= -\frac{3x(2y^3 + 3x^3)}{2y^5}$$

$$= -\frac{3x(1)}{2y^5} = -\frac{3x}{2y^5}.$$

18. $$D_x^2 y = -\left[\frac{2xy(2x+4y\,D_x y)-(x^2+2y^2)(2x\,D_x y + 2y)}{4x^2y^2}\right]$$

$$= -\left[\frac{2xy\left[2x + 4y\left(-\frac{x^2+2y^2}{2xy}\right)\right]}{4x^2y^2} - \frac{(x^2+2y^2)\left[2x\left(-\frac{x^2+2y^2}{2xy}\right) + 2y\right]}{4x^2y^2}\right]$$

$$= -\frac{x^4 + 2x^2y^2 - 8y^4}{4x^2y^3}$$

$$= \frac{8y^4 - 2x^2y^2 - x^4}{4x^2y^3}$$

19. $2y\,y' + 2xy' + 2y = 0$; at $(3,2)$,

$4y' + 6y' + 4 = 0.$

So $y' = -\frac{4}{10} = -\frac{2}{5}.$

Equation of tangent line : $y - 2 = -\frac{2}{5}(x - 3)$

Equation of normal line : $y - 2 = \frac{5}{2}(x - 3).$

20. $2x + 4xy' + 4y + 2y\,y' = 0$; at $(2,-1)$,

$4 + 8y' - 4 - 2y' = 0.$

So $y' = 0.$

Thus equation of tangent line is : $y = -1$

Thus equation of normal line is : $x = 2.$

21. $3x^2 - x(2y\,y') - y^2 + 3y^2y' = 0$; at $(2,2)$,

$12 - 8y' - 4 + 12y' = 0$

or $y' = -2.$

Equation of tangent line : $y - 2 = -2(x - 2)$

Equation of normal line : $y - 2 = \frac{1}{2}(x - 2).$

22. $y \cdot \frac{1}{2}(2x + 1)^{-1/2}(2) + \sqrt{2x+1}\,y' = y'$; at $(4,\frac{1}{2})$,

$\frac{1}{2}(9)^{-1/2} + \sqrt{9}\;y' = y',$

so $y' = -\frac{1}{12}.$

Equation of tangent line : $y - \frac{1}{2} = -\frac{1}{12}(x - 4).$

Equation of normal line : $y - \frac{1}{2} = 12(x - 4).$

123. $1 + \sin y\,y' = 0$; at $(\frac{1}{2},\frac{\pi}{3})$,

$1 + (\sin\frac{\pi}{3})y' = 0.$

or $y' = -\frac{1}{\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}.$

Equation of tangent line : $y - \frac{\pi}{3} = -\frac{2}{\sqrt{3}}(x - \frac{1}{2})$

$$= -\frac{2\sqrt{3}}{3}(x - \frac{1}{2}).$$

Equation of normal line : $y - \frac{\pi}{3} = \frac{\sqrt{3}}{2}(x - \frac{1}{2}).$

124. $\pi\left[\sec^2(x - y)\right](1 - y') = 2$; at $(\frac{\pi}{2},\frac{\pi}{4})$,

$\pi\sec^2(\frac{\pi}{4})\left[1 - y'\right] = 2$ or $\pi(2)(1 - y') = 2$

or $y' = 1 - \frac{1}{\pi} = \frac{\pi - 1}{\pi}.$

Equation of tangent line : $y - \frac{\pi}{4} = \frac{\pi - 1}{\pi}(x - \frac{\pi}{2})$

Equation of normal line : $y - \frac{\pi}{4} = \frac{\pi}{1 - \pi}(x - \frac{\pi}{2}).$

125. $2x + 2y\,y' = 0$ or $y' = -\frac{x}{y}$. Thus, the slope of tangent line at (a,b) is $-\frac{a}{b}$. Slope of line through $(0,0)$ and (a,b) is $\frac{b - 0}{a - 0} = \frac{b}{a}$. But $(-\frac{a}{b})(\frac{b}{a}) = -1$. Thus, the 2 lines are perpendicular.

126. $2(x^2 + y^2)(2x + 2y\,y') = 2x - 2y\,y'.$

So

$$y' = \frac{x - 2x^3 - 2xy^2}{2x^2y + 2y^3 + y} = 0.$$

$x - 2x^3 - 2xy^2 = 0$ gives $x = 0$ (reject), or $1 - 2x^2 - 2y^2 = 0$ or $x^2 + y^2 = \frac{1}{2}$. Substituting in the original equation, we have $x^2 - y^2 = \frac{1}{4}$. Adding,we get $2x^2 = \frac{3}{4}$ or $x^2 = \frac{3}{8}$. Thus, $x = \pm\sqrt{\frac{3}{8}} = \frac{\pm\sqrt{6}}{4}$. Substituting into above, we have $2y^2 = \frac{1}{4}$ or $y^2 = \frac{1}{8}$. Thus, $y = \pm\sqrt{\frac{1}{8}} = \frac{\pm\sqrt{2}}{4}$. Thus, four points are $(\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4})$, $(\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4})$, $(-\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4})$, $(-\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4})$.

127. $3x^2 + 3y^2y' = 3xy' + 3y$ or $x^2 + y^2y' = xy' + y$

or $y' = \dfrac{y - x^2}{y^2 - x} = 0$, so $y = x^2$.

Substituting in the original equation, we have

$x^3 + x^6 = 3x^3$. Reject $x = 0$, so

$1 + x^3 = 3$ or $x^3 = 2$, so $x = \sqrt[3]{2}$.

Thus, $y = \sqrt[3]{4}$ so point is $(\sqrt[3]{2}, \sqrt[3]{4})$.

128. Ellipse intersects y axis when $x = 0$ so $y^2 = 4$

or $y = \pm 2$.

Now $2x + xy' + y + 2y\,y' = 0$.

So $y' = -\dfrac{2x + y}{x + 2y}$.

At point $(0,2)$, $y' = -\frac{1}{2}$; at $(0,-2)$, $y' = -\frac{1}{2}$.

So tangent lines at these points are parallel.

129. $f'(x) = 20x^4 + 6x - 1$ so $f''(x) = 80x^3 + 6$.

130. $g(x) = x - x^{-1}$ so $g'(x) = 1 + x^{-2}$ and $g''(x) = -2x^{-3}$.

131. $h'(t) = \dfrac{t + 1 - t}{(t + 1)^2} = \dfrac{1}{(t + 1)^2} = (t + 1)^{-2}$

So $h''(t) = -2(t + 1)^{-3}$.

132. $F'(s) = \frac{1}{2}(2s + 1)^{-1/2}(2) = (2s + 1)^{-1/2}$

$F''(s) = -\frac{1}{2}(2s + 1)^{-3/2}(2) = -(2s + 1)^{-3/2}$.

133. $G'(u) = 7(u^{1/4} + 3)^6(\frac{1}{4}u^{-3/4}) = \frac{7}{4}u^{-3/4}(u^{1/4} + 3)^6$

$G''(u) = \frac{7}{4}u^{-3/4} \cdot 6(u^{1/4} + 3)^5 \cdot \frac{1}{4}u^{-3/4} + (u^{1/4} + 3)^6$

$\cdot \frac{7}{4}(-\frac{3}{4})u^{-7/4}$

$= \frac{21}{8}u^{-3/2}(u^{1/4} + 3)^5 - \frac{21}{16}u^{-7/4}(u^{1/4} + 3)^6$

$= \frac{21}{16}u^{-7/4}(u^{1/4} + 3)^5(u^{1/4} - 3)$.

134. $H(w) = w^{1/3} - 4(1 + w)^{1/3}$ so $H'(w) = \frac{1}{3}w^{-2/3}$

$-\frac{4}{3}(1 + w)^{-2/3}$

and $H''(w) = -\frac{2}{9}w^{-5/3} + \frac{8}{9}(1 + w)^{-5/3}$.

135. $f'(\theta) = \theta^2\left[-3 \sin 3\theta\right] + \cos 3\theta \left[2\theta\right]$

$f''(\theta) = \theta^2\left[-9 \cos 3\theta\right] + (3 \sin 3\theta)\left[2\theta\right]$

$+ (\cos 3\theta)(2) + 2\theta(-3 \sin 3\theta)$

$= -9\theta^2 \cos 3\theta - 12\theta \sin 3\theta + 2 \cos 3\theta$.

136. $g'(x) = (x^2 + 7)(\frac{1}{7}) \cos \frac{x}{7} + (\sin \frac{x}{7})(2x)$

$g''(x) = \frac{1}{7}(x^2 + 7)(-\frac{1}{7} \sin \frac{x}{7}) + \frac{1}{7} \cos \frac{x}{7}(2x)$

$+ 2 \sin \frac{x}{7} + 2x \cdot \frac{1}{7} \cos \frac{x}{7}$

$= -\frac{1}{49}(x^2 + 7) \sin \frac{x}{7} + \frac{2x}{7} \cos \frac{x}{7} + 2 \sin \frac{x}{7}$

$+ \frac{2x}{7} \cos \frac{x}{7}$

$= \left(\dfrac{98 - x^2 - 7}{49}\right)(\sin \frac{x}{7}) + \frac{4x}{7} \cos \frac{x}{7}$.

137. $f'(x) = 12x^{1/2} - 12x^{1/3}$,

$f''(x) = 6x^{-1/2} - 4x^{-2/3}$,

$f'''(x) = -3x^{-3/2} + \frac{8}{3}x^{-5/3}$.

138. $\dfrac{dy}{dx} = 12x^3 - 6x^2 + 14x - 5$,

$\dfrac{d^2y}{dx^2} = 36x^2 - 12x + 14$,

$\dfrac{d^3y}{dx^3} = 72x - 12$,

$\dfrac{d^4y}{dx^4} = 72$.

139. $y' = \dfrac{x + 1 - (x - 1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2} = 2(x + 1)^{-2}$,

$y'' = -4(x + 1)^{-3}$, $y''' = 12(x + 1)^{-4}$.

140. $f'(x) = a_n \cdot n \cdot x^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_1$,

$f''(x) = a_n \cdot n \cdot (n-1)x^{n-2} + a_{n-1}(n-1)(n-2)x^{n-2} +$

$\cdots + 2a_2$.

But $D_x^k(x^j)$, where $j < h = 0$.

So $f^{(n)}(x) = a_n \cdot n(n-1)(n-2)\cdots 2 - 1 = n!\,a_n$.

141. $y = \cos ax$ $\qquad y^{IV} = a^4 \cos ax$

$y' = -a \sin ax$ $\qquad y^V = -a^5 \sin ax$

$y'' = -a^2 \cos ax$ $\qquad y^{VI} = -a^6 \cos ax$

$y''' = a^3 \sin ax$

So $\dfrac{d^6}{dx^6} \cos ax = -a^6 \cos ax$.

142. $g(\theta) = a \sin b\theta$ $\qquad g^V(\theta) = ab^5 \sin b\theta$ n=1

$g'(\theta) = ab \cos b\theta$ $\qquad g^{VI}(\theta) = -ab^6 \sin b\theta$

$g''(\theta) = -ab^2 \sin b\theta$ $\qquad g^{VII}(\theta) = -ab^7 \cos b\theta$

$g'''(\theta) = -ab^3 \cos b\theta$ $\qquad g^{VIII}(\theta) = ab^8 \sin b\theta$

$g^{IV}(\theta) = ab^4 \sin b\theta$ $\qquad g^{IX}(\theta) = ab^9 \cos b\theta$ n=2

etc

$g^{(4n+1)}(\theta) = ab^{4n+1} \cos b\theta$.

143. $f'(x) = 30x^4 + 28x^3 - 24x^2 - 18x + 10$

$f''(x) = 120x^3 + 84x^2 - 48x - 18$

$f'''(x) = 360x^2 + 168x - 48$

$f^{(iv)}(x) = 720x + 168$

$f^{(v)}(x) = 720$

$f^{(n)}(x) = 0\ ,\ n \geq 6\ .$

144. For $x \neq 0$, $f'(x) = 3|x|^2 \cdot \frac{x}{|x|} = 3x|x|$. Also, $f'(0) = \lim\limits_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{|\Delta x|^3}{\Delta x}$ $= \lim\limits_{\Delta x \to 0} \Delta x|\Delta x| = 0$. Therefore, $f'(x) = 3x|x|$ holds for all values of x. Now, $f''(x) = 3x \frac{x}{|x|} + 3|x| = 3|x| + 3|x| = 6|x|$ for $x \neq 0$. Also, $f''(0) = \lim\limits_{\Delta x \to 0} \frac{f'(\Delta x) - f'(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{3\,\Delta x\,|\Delta x|}{\Delta x}$ $= \lim\limits_{\Delta x \to 0} 3|\Delta x| = 0$. Therefore, $f''(x) = 6|x|$ for all values of x.

145. (a) $(fg)''(-2) = f''(-2)g(-2) + 2f'(-2)g'(-2) + g''(-2)f(-2)$

$= (-4)(4) + 2(-3)(-\frac{1}{2}) + (-3)(1)$

$= -16.$

(b) $(fh)''(-2) = f''(-2)h(-2) + 2f'(-2)h'(-2) + h''(-2)f(-2)$

$= (-4)(6) + 2(-3)(-8) + 7(1) = 31.$

(c) $(f + g)''(-2) = f''(-2) + g''(-2) = (-4) + (-3)$

$= -7.$

(d) $(g - h)''(-2) = g''(-2) - h''(-2) = (-3) - (7)$

$= -10.$

(e) $[fgh]''(-2) = ([fgh]')'(-2) = (f'gh + fg'h + fgh')'(-2)$

$= (f''gh + f'g'h + f'gh' + f'g'h + fg''h + fg'h' + f'gh' + fg'h' + fgh'')(-2)$

$= (f''gh + fg''h + fgh'' + 2f'g'h + 2f'gh' + 2fg'h')(-2)$

$= (-4)(4)(6) + (1)(-3)(6)$

$+ (1)(4)(7) + 2(-3)(-\frac{1}{2})(6)$

$+ 2(-3)(4)(-8) + 2(1)(-\frac{1}{2})(-8)$

$= -96 - 18 + 28 + 18 + 192 + 8$

$= 132.$

(f) $(\frac{f}{g})''(-2) = \left[(\frac{f}{g})'\right]'(-2) = \left[\frac{gf' - fg'}{g^2}\right]'(-2)$

$= \left[\frac{g^2(g'f' + gf'' - f'g' - fg'')}{g^4} - \frac{(gf' - fg')2gg'}{g^4}\right](-2)$

$= \frac{16(\frac{3}{2} + (-16) - \frac{3}{2} + 3)}{4^4} - \frac{(-12 + \frac{1}{2})(2)(-2)}{4^4}$

$= \frac{-208 - 46}{4^4} = -\frac{254}{256} = -\frac{127}{128}\ .$

146. Let f be the function defined by $f(x) = \sqrt{r^2 - x^2}$ for $-r < x < r$. Then $f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}$ and $f''(x)$ $= \frac{-\sqrt{r^2 - x^2} - x\,\frac{x}{\sqrt{r^2 - x^2}}}{r^2 - x^2} = \frac{-r^2 + x^2 - x^2}{(r^2 - x^2)^{3/2}} =$ $\frac{r^2}{(r^2 - x^2)^{3/2}}$. Now, $\left[1 + (f'(x))^2\right]^{3/2} = \left[1 + \frac{x^2}{r^2 - x^2}\right]^{3/2} =$

$\left[\frac{r^2 - x^2 + x^2}{r^2 - x^2}\right]^{3/2} = \frac{r^3}{(r^2 - x^2)^{3/2}}$. Hence,

$$k = \frac{\left[-\frac{r^2}{(r^2 - x^2)^{3/2}}\right]}{\left[\frac{r^3}{(r^2 - x^2)^{3/2}}\right]} = -\frac{1}{r}\ .$$

147. (a) $D_x y$: $2(x - a) + 2y\,D_x y = 0.$

$D_x^2 y$: $2 + 2y\,D_x^2 y + 2(D_x y)^2 = 0$; $1 + y\,D_x^2 y + (D_x y)^2 = 0$;

so $D_x^2 y = \frac{-(D_x y)^2 - 1}{y}$.

Therefore, $y\,D_x^2 y + 1 + (D_x y)^2 = y\left[\frac{-(D_x y)^2 - 1}{y}\right] + 1 + (D_x y)^2 = 0.$

(b) $\left[1 + (y')^2\right]^3$: First, $2x + 2y\,y' = 0$; $y' = -\frac{x}{y}$. So, $\left[1 + (y')^2\right]^3 = (1 + \frac{x^2}{y^2})^3 = (\frac{y^2 + x^2}{y^2})^3$ $= (\frac{a^2}{y^2})^3$. Now, $2 + 2(y')^2 + 2y\,y'' = 0$ and y''

$$= \frac{-(1 + (y')^2)}{y}. \quad \text{So} \quad a^2(y'')^2 = a^2$$

$$\left(-\frac{(1 + (y')^2)}{y}\right)^2 = a^2\left(\frac{\left[1 + (y')^2\right]^2}{y^2}\right) = \frac{a^2}{y^2}$$

$$\left[1 + \frac{x^2}{y^2}\right]^2 = \frac{a^2}{y^2}\left[\frac{y^2 + x^2}{y^2}\right]^2 = \frac{a^2}{y^2} \cdot \frac{a^4}{y^4} = \frac{a^6}{y^6}$$

$$= \left(\frac{a^2}{y^2}\right)^3. \quad \text{Therefore, } \left[1 + (y')^2\right]^3 = a^2(y'')^2.$$

148. $g'(t) = \frac{-f'(t)}{2\sqrt{1 - f(t)}}$, $g''(t) =$

$$\frac{2\sqrt{1 - f(t)}(-f''(t)) - (-f'(t))\frac{-f'(t)}{\sqrt{1 - f(t)}}}{4(1 - f(t))}$$

$$= \frac{2(1 - f(t))(-f''(t)) - (f'(t))^2}{4(1 - f(t))^{3/2}};$$

$$g''(-2) = \frac{2\left[1 - (-3)\right](-5) - 9}{4(1 + 3)^{3/2}} = \frac{-40 - 9}{4(8)} = \frac{-49}{32}.$$

149. $v(t) = s'(t) = 12t - 6t^2$,

$a(t) = s''(t) = 12 - 12t$.

150. $v(t) = s'(t) = 128t - 64t^3$,

$a(t) = s''(t) = 128 - 192t^2$.

151. $v(t) = s'(t) = 6\pi \cos 2\pi t$,

$a(t) = s''(t) = -12\pi^2 \sin 2\pi t$.

152. $s(t) = \cos^2 t$,

$v(t) = s'(t) = 2 \cos t(-\sin t)$

$= -\sin 2t$,

$a(t) = s''(t) = -2 \cos 2t$.

153. For $x \neq 1$, $f'(x) = \begin{cases} 3x^2 & \text{if } x < 1 \\ 3 & \text{if } x > 1 \end{cases}$. For $x = 1$,

$f'_+(1) = \lim_{\Delta x \to 0^+} \frac{\left[3(1 + \Delta x) - 2\right] - 1}{\Delta x} = \lim_{\Delta x \to 0^+} 3 = 3$,

$f'_-(1) = \lim_{\Delta x \to 0^-} \frac{(1 + \Delta x)^3 - 1}{\Delta x} = \lim_{\Delta x \to 0^-} (3 + 3\Delta x + (\Delta x)^2) = 3$. Thus, $f'(1) = 3$. Therefore,

$f'(x) = \begin{cases} 3x^2 & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases}$; so, for $x \neq 1$,

$f''(x) = \begin{cases} 6x & \text{if } x < 1 \\ 0 & \text{if } x > 1 \end{cases}$. Notice that $f''(1)$ does not exist.

154. $\frac{ds}{dt} = -A \cdot 2\pi\nu \sin(2\pi\nu t - \phi)$,

$\frac{d^2s}{dt^2} = -4\pi^2\nu^2 A \cos(2\pi\nu t - \phi)$.

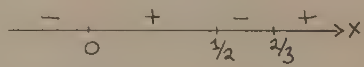
So $\frac{d^2s}{dt^2} + 4\pi^2\nu^2 s = -4\pi^2\nu^2 A \cos(2\pi\nu t - \phi) + 4\pi^2\nu^2\left[A \cos(2\pi\nu t - \phi) + k\right]$

$= 4\pi^2\nu^2 k$.

155. The polynomial function $f(x) = 4x^3 - 7x^2 + 2x$ is continuous on $[1,2]$. $f(1) = -1$ and $f(2) = 8$, and $-1 < \sqrt{5} < 8$. Thus, the intermediate value theorem states there is a number c, $1 < c < 2$, such that $4c^3 - 7c^2 + 2c = \sqrt{5}$.

156. The function $f(x) = 5 \sin 2\pi x - 4 \tan \pi x$ is continuous on $\left[0,\frac{1}{4}\right]$. $f(0) = 0$, $f(\frac{1}{4}) = 1$ and $0 < 0.707 < 1$. Hence, the intermediate value theorem states there is a number c, $0 < c < \frac{1}{4}$, such that $5 \sin 2\pi c - 4 \tan \pi c = 0.707$.

157. $f(x) = 3x(2x - 1)(3x - 2)$.



$f(-1) = -45 < 0$,

$f(\frac{1}{4}) > 0$,

$f(\frac{7}{12}) < 0$,

$f(1) = 3 > 0$.

158. $g(x) = x^3 + 8x^2 + 11x - 20 = (x - 1)(x + 4)(x +5$

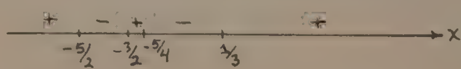
$g(-6) < 0$,

$g(-4.5) > 0$,

$g(0) = -20 < 0$,

$g(2) = 42 > 0$.

159. $h(x) = \frac{8x^2 + 22x + 15}{6x^2 + 13x - 5} = \frac{(4x + 5)(2x + 3)}{(3x - 1)(2x + 5)}$.



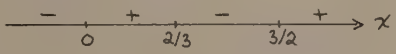
$f(-3) > 0$, $f(0) = -3 < 0$,

$f(-2) < 0$, $f(1) = \frac{45}{14} > 0$.

$f(-\frac{11}{8}) > 0$,

160. $F(x) = 6x^{11/5} - 13x^{6/5} + 6x^{1/5} = x^{1/5}(6x^2 - 13x + 6)$

$= x^{1/5}(2x - 3)(3x - 2)$.

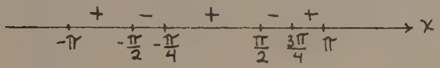


$F(-1) < 0$, $F(1) = -1 < 0$,

$F(\frac{1}{4}) > 0$, $F(2) > 0$.

161. $1 - \sin x = 0$, $\sin x = 1$, $x = \frac{\pi}{2}$; G undefined at $-\frac{\pi}{2}$ and at $\frac{\pi}{2}$.

$1 + \tan x = 0$, $\tan x = -1$, $x = \frac{3\pi}{4}, -\frac{\pi}{4}$.



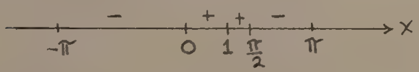
$G(-\frac{\pi}{3}) = \dfrac{1 - \sin(-\frac{\pi}{3})}{1 + \tan(-\frac{\pi}{3})} = \dfrac{1 + \sin\frac{\pi}{3}}{1 - \tan\frac{\pi}{3}} = \dfrac{1 + \frac{\sqrt{3}}{2}}{1 - \sqrt{3}} < 0$,

$G(0) = 1 > 0$,

$G(\frac{2\pi}{3}) = \dfrac{1 - \sin\frac{2\pi}{3}}{1 + \tan\frac{2\pi}{3}} = \dfrac{1 - \frac{\sqrt{3}}{2}}{1 - \sqrt{3}} < 0$,

$G(\frac{5\pi}{6}) = \dfrac{1 - \sin\frac{5\pi}{6}}{1 + \tan\frac{5\pi}{6}} = \dfrac{1 - \frac{1}{2}}{1 - \frac{\sqrt{3}}{3}} > 0$.

162. $\tan x = 0$, $x = 0, \pi, -\pi$; $(x - 1)^{2/3} = 0$ for $x = 1$; $\tan x$ is undefined for $x = \frac{\pi}{2}$.



$H(-\frac{\pi}{4}) = (-\frac{\pi}{4} - 1)^{2/3}\tan(-\frac{\pi}{4}) < 0$,

$H(\frac{1}{2}) > 0$,

$H(\frac{\pi}{3}) = (\frac{\pi}{3} - 1)^{2/3}\tan\frac{\pi}{3} > 0$,

$H(\frac{5\pi}{6}) = (\frac{5\pi}{6} - 1)^{2/3}(\tan\frac{5\pi}{6}) < 0$.

163. f is continuous on $[-2,-1]$.

$f(-2) = -1$, $f(-1) = 4$; so f changes sign on $[-2,-1]$. Hence, by the change-of-sign property, f has a zero in $[-2,-1]$.

164. g is continuous on $[4,5]$.

$g(4) = -6.56$, $g(5) = 1$; so g changes sign on $[4,5]$. Hence, by the change-of-sign property, g has a zero in $[4,5]$.

165. h is continuous on $[0,1]$.

$h(0) = 1$, $h(1) = -0.45969...$; so h changes sign on $[0,1]$. Hence, by the change-of-sign property, h has a zero on $[0,1]$.

166. F is continuous on $[0.5,0.6]$.

$F(0.5) = -0.0565313025...$,

$F(0.6) = 0.1028870067...$; so F changes sign on $[0.5,0.6]$. Hence, by the change-of-sign property, F has a zero in $[0.5,0.6]$.

167. Let $x = \frac{-2 + (-1)}{2} = -1.5$; $f(-2) < 0$, $f(-1) > 0$.

$f(-1.5) = 2.375 > 0$, so the zero is in the interval $(-2,-1.5)$.

Let $x = \frac{-2 + (-1.5)}{2} = -1.75$.

$f(-1.75) = 0.953125 > 0$, so the zero is between -2 and -1.75.

168. Let $x = \frac{0.5 + 0.6}{2} = 0.55$; $F(0.5) < 0$ and $F(0.6) > 0$.

$F(0.55) = 0.024967... > 0$, so the zero is between 0.5 and 0.55.

Let $x = \frac{0.5 + 0.55}{2} = 0.525$.

$F(0.525) = -0.01530... < 0$, so the zero is between 0.525 and 0.55.

Let $x = \frac{0.525 + 0.55}{2} = 0.5375$.

$F(0.5375) = 0.004917... > 0$, so the zero is between 0.525 and 0.5375.

Let $x = \frac{0.525 + 0.5375}{2} = 0.53125$.

$F(0.53125) = -0.00519... < 0$, so the zero is between 0.53125 and 0.5375.

169. $f(x)\ x^3 + x^2 + x + 5$; $f'(x) = 3x^2 + 2x + 1$; x_{n+1}

$= x_n - \dfrac{x_n^3 + x_n^2 + x_n + 5}{3x_n^2 + 2x_n + 1}$.

$x_1 = -1.5$ $x_5 = -1.881239402$

$x_2 = -2$ $x_6 = -1.881239401$

$x_3 = -1.888888889$ $x_7 = x_6$

$x_4 = -1.881273797$

So desired zero $z \approx -1.881239401$.

170. $g(x) = 1 + 3x^{5/3} - 15x^{2/3}$; $g'(x) = 5x^{2/3} - 10x^{-1/3}$;

$x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$.

$x_1 = 4.5$ $x_6 = 4.884206891$

$x_2 = 4.907922907$ $x_7 = x_4$

$x_3 = 4.884284561$ $x_8 = x_5$

$x_4 = 4.884206892$ $x_9 = x_6$

$x_5 = 4.884206890$ $x_{10} = x_4$ etc.

So desired zero $z \approx 4.88420689$.

171. $h(x) = \cos x - \sqrt{x}$; $h'(x) = -\sin x - \dfrac{1}{2\sqrt{x}}$;

$x_{n+1} = x_n - \dfrac{h(x_n)}{h'(x_n)}$.

$x_1 = 0.5$ $x_4 = 0.641714371$

$x_2 = 0.643675632$ $x_5 = x_4$

$x_3 = 0.641714867$

So desired zero $z \approx 0.641714371$.

172. $F(x) = 3 \sin x - 2 \sin \frac{x}{2} - 1$;

$F'(x) = 3 \cos x - \cos \frac{x}{2}$

$x_1 = 0.55$ $x_4 = 0.534455228$

$x_2 = 0.534347655$ $x_5 = x_4$

$x_3 = 0.534455223$

So desired zero $z \approx 0.534455228$.

173. Let $f(x) = x^5 + x - 17$; find a zero of $f(x)$. A quick graph of $y = x^5 + x$ and $y = 17$ shows a point of intersection with abscissa x about 1.5. $f'(x) = 5x^4 + 1$; $x_{n+1} = x_n - \dfrac{x_n^5 + x_n - 17}{5x_n^4 + 1}$.

$x_1 = 1.5$ $x_5 = 1.725027753$

$x_2 = 1.800475059$ $x_6 = 1.725027751$

$x_3 = 1.730978149$ $x_7 = x_6$

$x_4 = 1.725067632$

So desired zero $z \approx 1.725027751$.

174. Let $f(x) = x^3 + x - 3 + \sin x$; find a zero of $f(x)$. A quick graph of $y = x^3 + x$ and $y = 3 - \sin x$ shows a point of intersection with abscissa x about 2π. $f'(x) = 3x^2 + 1 + \cos x$.

$x_1 = 2\pi$ $x_7 = 1.034932108$

$x_2 = 4.196309639$ $x_8 = 1.034242095$

$x_3 = 2.804701561$ $x_9 = 1.034241825$

$x_4 = 1.866304887$ $x_{10} = 1.034241826$

$x_5 = 1.299586579$ $x_{11} = x_9$

$x_6 = 1.069434412$ $x_{12} = x_{10}$ etc.

So desired zero $z \approx 1.03424183$.

175. Slope of tangent line given by $y' = 4x^3 + 4x - 4 = 0$, or $x^3 + x - 1 = 0$. Let $g(x) = x^3 + x - 1$. $g(0) < 0$ while $g(1) > 0$. There is a zero for g in $(0,1)$.

Guess $x_1 = 0.5$ $x_4 = 0.682328423$

Then $x_2 = 0.714285714$ $x_5 = 0.682327804$

$x_3 = 0.683179724$ $x_6 = x_5$

So desired x coordinate is ≈ 0.682327804.

176. Let $f(x) = \frac{1}{2} - \left(\frac{\sin x}{x}\right)^2$; $f'(x) = -\dfrac{2 \sin x}{x^3}$ $(x \cos x - \sin x)$.

A quick sketch of $y = \frac{1}{2}$ and $y = \left(\frac{\sin x}{x}\right)^2$ gives an $x \approx \frac{\pi}{4}$,

So $x_1 = \frac{\pi}{4}$ $x_5 = 1.391557379$

$x_2 = 1.486522383$ $x_6 = 1.391557378$

$x_3 = 1.390677830$ $x_7 = x_6$

$x_4 = 1.391557335$

So desired zero $z \approx 1.391557378$.

177. f is not continuous on $[0,3]$ since $f(2)$ is not

defined.

78. Suppose f is not constant on I. Then it would take on 2 different integral values. By the intermediate value theorem, it would have to take on all values between these 2 integers, hence, it would have to take on a non-integral value. This is a contradiction. Thus, f is constant on I.

3 APPLICATIONS OF THE DERIVATIVE

Problem Set 3.1, page 170

1. No. The mean value theorem with its conditions of continuity and differentiability guarantees a tangent line parallel to the secant line. If the function were continuous (but not necessarily differentiable), there still would be a tangent line parallel to the secant line.
2. Let $f(x) = 2$, $a \le x \le b$. f is differentiable on (a,b) and continuous on $[a,b]$.
3. $f(a) = 4$, $f(b) = 16$, $f'(c) = 2c$; $\frac{16 - 4}{4 - 2} = 2c$, $c = 3$.
4. $f(a) = 2$, $f(b) = 3$, $f'(c) = \frac{1}{2\sqrt{c}}$; $\frac{3 - 2}{9 - 4} = \frac{1}{2\sqrt{c}}$, $c = \frac{25}{4}$.
5. $f(a) = 1$, $f(b) = 27$, $f'(c) = 3c^2$; $\frac{27 - 1}{3 - 1} = 3c^2$, $c = \sqrt{\frac{13}{3}}$.
6. $f(a) = 2$, $f(b) = \frac{5}{3}$, $f'(c) = \frac{-1}{(c - 1)^2}$; $\frac{\frac{5}{3} - 2}{1.6 - 1.5} = \frac{-1}{(c - 1)^2}$, $c = 1 + \sqrt{\frac{3}{10}}$.
7. $f(0) = 0$, $f(\pi) = 0$, $f'(c) = \cos c$; $\frac{0 - 0}{\pi - 0} = \cos c$, $0 = \cos c$, $c = \frac{\pi}{2}$.
8. $f(-1) = -3$, $f(1) = 3$, $f'(c) = 2c + 3$; $\frac{3 + 3}{1 + 1} = 2c + 3$, $c = 0$.
9. $f(x) = x^2 - 3x$, $[a,b] = [0,3]$. f is continuous on $[0,3]$ and differentiable on $(0,3)$. $f(0) = f(3) = 0$. So there is a c in $(0,3)$ such that $f'(c) = 2c - 3 = 0$. Thus, $c = \frac{3}{2}$.
10. $f(x) = x^2 - 5x + 6$, $[a,b] = [2,3]$. f is continuous on $[2,3]$ and differentiable on $(2,3)$. $f(2) = f(3) = 0$. So there is a c in $(2,3)$ such that $f'(c) = 2c - 5 = 0$. Thus, $c = \frac{5}{2}$.
11. $f(x) = x^3 - 3x^2 - x + 3$, $[a,b] = [-1,3]$. f is continuous on $[-1,3]$ and differentiable on $(-1,3)$. $f(-1) = f(3) = 0$. So there is a c in $(-1,3)$ such that $f'(c) = 3c^2 - 6c - 1 = 0$. Thus, $c = 1 \pm \frac{2}{3}\sqrt{3}$.
12. $f(x) = \sqrt{x}(x^3 - 1)$, $[a,b] = [0,1]$. f is continuous on $[0,1]$ and differentiable on $(0,1)$. $f(0) = f(1) = 0$. So there is a c in $(0,1)$ such that $f'(c) = \sqrt{c}\,(3c^2) + \frac{c^3 - 1}{2\sqrt{c}} = 0$. Thus, $2c(3c^2) + c^3 - 1 = 0$, $7c^3 = 1$, $c = \sqrt[3]{\frac{1}{7}}$.
13. $f(x) = x^{3/4} - 2x^{1/4}$, $[a,b] = [0,4]$. f is continuous on $[0,4]$ and differentiable on $(0,4)$. $f(0) = f(4) = 0$. So there is a c in $(0,4)$ such that $f'(c) = \frac{3}{4}c^{-1/4} - \frac{2}{4}c^{-3/4} = \frac{3}{4\sqrt[4]{c}} - \frac{2}{4\sqrt[4]{c^3}} = 0$. Thus, $c = \frac{4}{9}$.
14. $f(x) = x^3 - 3x$, $[a,b] = [-\sqrt{3}, \sqrt{3}]$. f is continuous on $[-\sqrt{3}, \sqrt{3}]$ and differentiable on $[-\sqrt{3}, \sqrt{3}]$. $f(-\sqrt{3}) = f(\sqrt{3}) = 0$. So there is a c in $(-\sqrt{3}, \sqrt{3})$ such that $f'(c) = 3c^2 - 3 = 0$. Thus, $c = \pm 1$.
15. $f(x) = \sin x$, $[a,b] = [0,4\pi]$. f is continuous on $[0,4\pi]$ and differentiable on $(0,4\pi)$. $f(0) = f(4\pi) = 0$. So there is a c in $(0,4\pi)$ such that $f'(c) =$

$\cos c = 0$. Thus, $c = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

. $f(x) = \sqrt{1 - \cos x}$, $[a,b] = [-\frac{\pi}{2}, \frac{\pi}{2}]$. f is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$. $f(-\frac{\pi}{2}) = f(\frac{\pi}{2}) = 0$. So there is a c in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $f'(c) = \frac{1}{2}(1 - \cos c)^{-\frac{1}{2}}(\sin c) = \frac{\sin c}{\sqrt{1 - \cos c}} = 0$. Thus, $c = 0$.

7. $f(x) = \frac{1}{6}x^2$, $[a,b] = [2,6]$. f is defined and continuous on $[2,6]$ and differentiable on $(2,6)$. So there is a c on $(2,6)$ such that $f'(c) = \frac{f(a) - f(b)}{a - b}$. $f(a) = f(2) = \frac{2}{3}$, $f(b) = f(6) = 6$, $f'(c) = \frac{c}{3}$. $\frac{c}{3} = \frac{6 - 2/3}{6 - 2}$, so $c = 4$.

8. $f(x) = x^3 + x - 1$, $[a,b] = [0,2]$. f is defined and continuous on $[2,6]$ and differentiable on $(2,6)$. $f(a) = f(0) = -1$, $f(b) = f(2) = 9$, $f'(c) = 3c^2 + 1$. $3c^2 + 1 = \frac{9 - (-1)}{2 - 0}$, so $c = \frac{2}{\sqrt{3}}$. Rejected $c = -\frac{2}{\sqrt{3}}$ since it's not in the interval $(0,2)$.

9. $f(a) = f(0) = 0$, $f(b) = f(2) = 16$, $f'(c) = 6c^2$. $6c^2 = \frac{16 - 0}{2 - 0}$; so $c = \frac{2}{3}\sqrt{3}$ for $0 < c < 2$.

0. $f(a) = 1$, $f(b) = 2$, $f'(c) = \frac{1}{2\sqrt{c}}$. $\frac{1}{2\sqrt{c}} = \frac{2 - 1}{4 - 1}$; so $c = \frac{9}{4}$.

1. $f(a) = -1$, $f(b) = \frac{1}{2}$, $f'(c) = \frac{2}{(c + 1)^2}$. $\frac{2}{(c + 1)^2} = \frac{\frac{1}{2} - (-1)}{3 - 0}$ or $(c + 1)^2 = 4$; so $c = 1$. Rejected -3 since c must be in the interval $(0,3)$.

2. $f(a) = 2$, $f(b) = 3$, $f'(c) = \frac{1}{2\sqrt{c + 1}}$. $\frac{1}{2\sqrt{c + 1}} = \frac{3 - 2}{8 - 3}$ or $5(\frac{1}{2\sqrt{c + 1}}) = 1$. Thus, $c = \frac{21}{4}$.

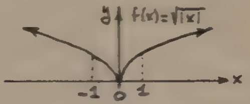
3. $f(a) = 4$, $f(b) = 3$, $f'(c) = \frac{-c}{\sqrt{25 - c^2}}$. $\frac{3 - 4}{4 - (-3)} = \frac{-c}{\sqrt{25 - c^2}}$, $-1 = 7(\frac{-c}{\sqrt{25 - c^2}})$, $\sqrt{25 - c^2} = 7c$, $25 - c^2 = 49c^2$, $c = \pm\frac{\sqrt{2}}{2}$.

4. $f(a) = 0$, $f(b) = 0$, $f'(c) = \frac{c^2 + 8c - 5}{(c + 4)^2}$. $0 = 4(\frac{c^2 + 8c - 5}{(c + 4)^2})$, $c^2 + 8c - 5 = 0$, $c = \frac{-8 \pm \sqrt{84}}{2} = \sqrt{21} - 4$. Rejected $-\sqrt{21} - 4$ since $-1 < c < 3$.

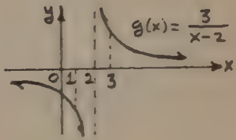
25. $f(\frac{\pi}{2}) = \frac{\pi}{2}$, $f(\frac{5\pi}{2}) = \frac{5\pi}{2}$, $f'(c) = 1 + \sin c$. $\frac{5\pi}{2} - \frac{\pi}{2} = (\frac{5\pi}{2} - \frac{\pi}{2})(1 + \sin c)$, $1 = 1(1 + \sin c)$, $\sin c = 0$, $c = \pi, 2\pi$ for $\frac{\pi}{2} < c < \frac{5\pi}{2}$.

26. $f(0) = 0$, $f(\pi) = 2\pi$, $f'(c) = 2 + 20 \sin 2c \cos 2c$. $2\pi - 0 = (\pi - 0)(2 + 20 \sin 2c \cos 2c)$, $0 = 2 \sin 2c \cos 2c = \sin 2(2c) = \sin 4c$, $4c = \pi, 2\pi, 3\pi$. Thus, $c = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ for $0 < c < \pi$.

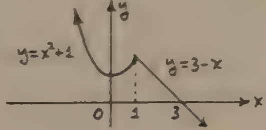
27. f is not differentiable at 0.



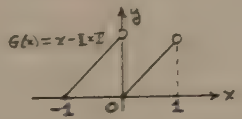
28. g is not defined at 2.



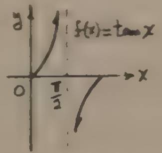
29. f is not differentiable at 1.



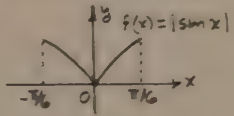
30. f is not continuous at 0.



31. f is not defined at $x = \frac{\pi}{2}$.



32. f is not differentiable at $x = 0$.



33. (a) f is not differentiable on (a,b).

(b) f is not differentiable on (a,b).

(c) f is not defined on [a,b].

(d) f is not continuous on [a,b].

(e) f is not continuous on [a,b].

34. $|f(b) - f(a)| = |(b - a)f'(c)| = |b - a|\cdot|f'(c)| \leq |b - a| \cdot 1 = |b - a|$.

35. Assume f takes on a negative value somewhere on (a,b). By the extreme value theorem, f takes on a minimum value, say, c, in [a,b]. Thus, $f(c) < 0$; so c is in (a,b). By the necessary condition for relative extrema, we have $f'(c) = 0$.

36. $|\sin x - \sin y| \leq |x - y|$ by Problem 34 since $D_x \sin x = |\cos x| \leq 1$ holds for all x.

37. The slope of the secant line is $\frac{f(b) - f(a)}{b - a}$ so the equation of this line is $y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$ or $y = g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$.

38. Let x and y be points in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then applying the mean value theorem [x,y], we have $|\tan x - \tan y| = |x - y||\sec^2 c|$, $x < c < y$. But in $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\sec c \geq 1$; so $|\tan x - \tan y| \geq |x - y|$.

39. $f(\frac{\pi}{6}) = \frac{1}{2}$, $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $f'(c) = \cos c$. $\frac{\sqrt{3}}{2} - \frac{1}{2} = (\frac{\pi}{3} - \frac{\pi}{6}) \cos c$, or $\cos c - \frac{3}{\pi}(3 - 1) = 0$. Let $f(x) = \cos x - \frac{3}{\pi}(\sqrt{3} - 1)$. Want to find a zero of f(x). $f'(x) = -\sin x$. Let $x_1 = .5$. Then $x_2 = .87234315$, $x_3 = .79917111$, $x_4 = .79670080$, $x_5 = .79669782$, $x_6 = x_5$. So the desired zero $z \approx .79669782$.

40. $f(-1) = 3$, $f(1) = -1$, $f'(x) = 10x^4 + 4x^3 - 9x^2 + 2x - 1$. $-1 - 3 = (1 + 1)(10c^4 + 4c^3 - 9c^2 + 2c - 1)$ or $10c^4 + 4c^3 - 9c^2 + 2c + 1 = 0$. Let $f(x) = 10x^4 + 4x^3 - 9x^2 + 2x + 1$. Want to find a zero of f(x). $f'(x) = 40x^3 + 12x^2 - 18x + 2$. Let $x_1 = 0$. Then $x_2 = .5$, $x_3 = .26388889$, $x_4 = .23768291$, $x_5 = .23681061$, $x_6 = .23680961$, $x_7 = x_6$. So the desired zero $z \approx .23680961$.

41. (a) $f(1) = -12$ and $f(2) = 28$, so f(1) and f(2) hav opposite signs.

(b) If f had two zeros between 1 and 2, then by Rolle's Theorem, $f'(c) = 5c^4 + 6c^2 - 5 = 0$ fo some c between 1 and 2. But, if $1 < c < 2$, then $1 < c^4 < 16$ and $1 < c^2 < 4$, so $11 < 5c^4 + 6c^2 < 104$ and $6 < 5c^4 + 6c^2 - 5 <$ In particular, we cannot have $5c^4 + 6c^2 - 5 =$

42. Let $A = \lim_{x\to a^+} f(x)$ and $B = \lim_{x\to b^-} f(x)$ and define the function g by the equation

$$g(x) = \begin{cases} A & x \leq a \\ f(x) & a < x < b \\ B & b \leq x \end{cases}$$

Then g is differentiable on the open interval (a, and $g'(x) = f'(x)$ for $a < x < b$. Also, g is continuous on the closed interval [a,b]. By the mea value theorem applied to g, $g(b) - g(a) = (b - a)g'(c)$ for some c, $a < c < b$; that is, $\lim_{x\to a^+} f(x) - \lim_{x\to b^-} f(x) = B - A = g(b) - g(a) = (b - a)f'(c)$.

43. $(Ab^2 + Bb + C) - (Aa^2 + Ba + C) = (b - a)(2Ac + B$ $A(b^2 - a^2) + B(b - a) = (b - a)(2Ac + B)$, $A(b - a)(b + a) + B(b - a) = (b - a)(2Ac + B)$, $A(b + a) + B = 2Ac + B$, $2Ac = A(b + a)$, $c = \frac{a + b}{2}$

44. $f'(x) = [G(a) - G(b)]$, $F'(x) - [F(a) - F(b)]$ $G'($ $f(a) = f(b) = F(b)G(a) - F(a)G(b)$. There is a c, $a < c < b$, such that $f'(c) = 0$. Hence, $[G(a) - G(b)]\ F'(c) = [F(a) - F(b)]\ G'(c)$. If $G(a) \neq G(b)$ and $G'(c) \neq 0$, we can conclude $\frac{F'(c)}{G'(c)}$ $\frac{F(a) - F(b)}{G(a) - G(b)}$.

45. ψ is continuous on [a,b] and differentiable on (a,b).

$\psi(a) = 0 \cdot f(b) - (a - b)f(a) - (b - a)f(a) = 0$

$\psi(b) = (b - a)f(b) - 0 \cdot f(a) - (b - a)f(b) = 0$.

By Rolle's Theorem, there is a c, a < c < b, such that $\psi'(c) = 0$. Now, $\psi'(x) = f(b) - f(a) - (b - a)f'(x)$; so $f(b) - f(a) - (b - a)f'(c) = 0$. Thus, $f'(c) = \frac{f(b) - f(a)}{b - a}$.

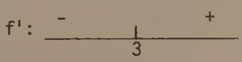
46. By the mean value theorem, there is a c between x and Δx such that $f(x + \Delta x) - f(x) = f'(c)(x + \Delta x - x) = f'(c)\Delta x$ or $f(a + \Delta x) = f(x) + f'(c)\Delta x$.

Problem Set 3.2, page 179

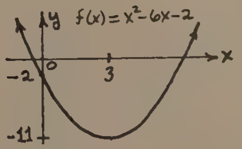
1. (a) f is constant on $(-\infty,-2]$.
   f is decreasing on $[-2,0]$.
   f is increasing on $[0,2]$.
   f is constant on $[2,\infty)$.
   (b) g is decreasing on $(-\infty,-3]$.
   g is increasing on $[-3,-2/3]$.
   g is decreasing on $[-2/3,2]$.
   g is constant on $[2,4]$.
   g is increasing on $[4,\infty)$.
   (c) h is increasing on $(-\infty,-5]$.
   h is decreasing on $[-5,-2]$.
   h is constant on $[-2,2]$.
   h is increasing on $[2,4]$.
   h is decreasing on $[4,\infty)$.

2. Read definition again. Results are consistent with this.

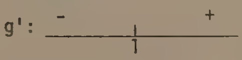
3. $f'(x) = 2x - 6 = 2(x - 3)$. f':



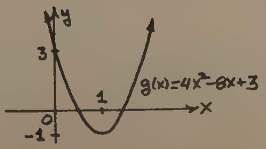
Decreasing on $(-\infty,3]$.
Increasing on $[3,\infty)$.



4. $g'(x) = 8x - 8 = 8(x - 1)$. g':

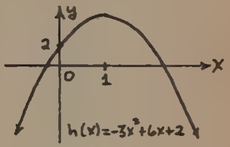


Decreasing on $(-\infty,1]$.
Increasing on $[1,\infty)$.

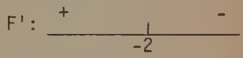


5. $h'(x) = -6x + 6 = -6(x - 1)$. h':

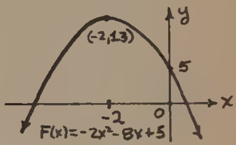
Decreasing on $(-\infty,1]$.
Increasing on $[1,\infty)$.



6. $F'(x) = -4x - 8 = -4(x + 2)$. F':

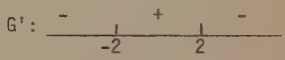


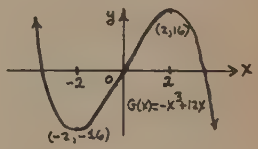
Increasing on $(-\infty,-2]$.
Decreasing on $[-2,\infty)$.



7. $G'(x) = -3x^2 + 12 = -3(x^2 - 4) = -3(x + 2)(x - 2)$.
Decreasing on $(-\infty,-2]$.
Increasing on $[-2,2]$.
Decreasing on $[2,\infty)$.

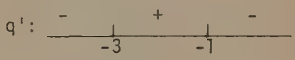
G':

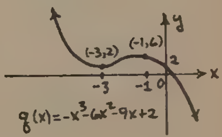




8. $q'(x) = -3x^2 - 12x - 9 = -3(x^2 + 4x + 3) = -3(x + 1)(x + 3)$.
Decreasing on $(-\infty,-3]$.
Increasing on $[-3,-1]$.
Decreasing on $[-1,\infty)$.

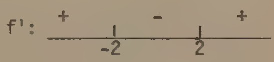
q':





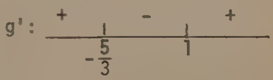
9. $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$.
Increasing on $(-\infty,-2]$.
Decreasing on $[-2,2]$.
Increasing on $[2,\infty)$.

f':



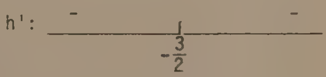
10. $g'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1)$.
Increasing on $(-\infty,-5/3]$.
Decreasing on $[-5/3,1]$.
Increasing on $[1,\infty)$.

g':



11. $h'(x) = -12x^2 - 36x - 27 = -3(4x^2 + 12x + 9) = -3(2x + 3)^2$.
Decreasing on $\mathbb{R}$.

h':



12. $F'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$.

F':

Increasing on $(-\infty,1]$ and $[1,\infty)$.

13. $G'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) =$
$3(x - 1)(x - 3)$.

G': $+$ | 1 | $-$ | 3 | $+$

Increasing on $(-\infty,1]$.
Decreasing on $[1,3]$.
Increasing on $[3,\infty)$.

14. $H'(x) = 12x^3 + 24x^2 - 36x = 12x(x^2 + 2x - 3) =$
$12x(x + 3)(x - 1)$.

H': $-$ | -3 | $+$ | 0 | $-$ | 1 | $+$

Decreasing on $(-\infty,-3]$.
Increasing on $[-3,0]$.
Decreasing on $[0,1]$.
Increasing on $[1,\infty)$.

15. $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2)$.

f': $-$ | -2 | $+$ | 0 | $-$ | 2 | $+$

Decreasing on $(-\infty,-2]$.
Increasing on $[-2,0]$.
Decreasing on $[0,2]$.
Increasing on $[2,\infty)$.

16. $g'(x) = 4x^3 + 12x^2 + 12x + 4 = 4(x^3 + 3x^2 + 3x + 1) =$
$4(x + 1)^3$.

Decreasing on $(-\infty,-1]$. g': $-$ | -1 | $+$

Increasing on $[-1,\infty)$.

17. $h'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) =$
$15x^2(x + 1)(x - 1)$.

h': $+$ | -1 | $-$ | 0 | $-$ | 1 | $+$

Increasing on $(-\infty,-1]$.
Decreasing on $[-1,1]$.
Increasing on $[1,\infty)$.

18. $F'(x) = 20x^4 - 100x^3 + 120x^2 = 20x^2(x^2 - 5x + 6) =$
$20x^2(x - 2)(x - 3)$.

F': $+$ | 0 | $+$ | 2 | $-$ | 3 | $+$

Increasing on $(-\infty,2]$.
Decreasing on $[2,2]$.
Increasing on $[3,\infty)$.

19. $G'(x) = -\frac{1}{3}x^{-2/3}$.

Graph is decreasing on $\mathbb{R}$. G': $-$ | 0 | $-$

20. $H(x) = \frac{8}{5}x^{5/3} - \frac{32}{3}x^{-1/3} = \frac{8}{3}x^{-1/3}(x^2 - 4) =$
$\frac{8}{3}x^{-1/3}(x + 2)(x - 2)$.

H': $-$ | -2 | $+$ | 0 | $-$ | 2 | $+$

Decreasing on $(-\infty,-2]$; increasing on $[-2,0]$;
decreasing on $[0,2]$; increasing on $[2,\infty)$.

21. $p'(x) = 1 + 8x^{-3} = 1 + \frac{8}{x^3} = \frac{x^3 + 8}{x^3} =$
$\frac{(x + 2)(x^2 - 2x + 4)}{x^3}$. p': $+$ | -2 | $-$ | 0 | $+$

Increasing on $(-\infty,-2]$; decreasing on $[-2,0)$;
increasing on $(0,\infty)$.

22. $q'(x) = \frac{1}{2\sqrt{x}} + 9x^{-2} = \frac{1}{2\sqrt{x}} + \frac{9}{x^2} > 0$.

Increasing on $(0,\infty)$.

23. $r'(x) = \frac{1}{2\sqrt{x}} - 4x^{-2} = \frac{1}{2\sqrt{x}} - \frac{4}{x^2} = \frac{x^2 - 8x^{1/2}}{2x^{5/2}} =$
$\frac{x^{1/2}(x^{3/2} - 8)}{2x^{5/2}}$. r': not in domain | 0 | $-$ | 4 | $+$

Decreasing on $(0,4]$; increasing on $[4,\infty)$.

24. $P'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3} = \frac{2x^4 + 6}{x^3}$.

P': $-$ | 0 | $+$

Decreasing on $(-\infty,0)$; increasing on $(0,\infty)$.

25. $Q'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x + 1)$.

Q': $-$ | -1 | $+$ | 0 | $+$

Decreasing on $(-\infty,-1]$; increasing on $[-1,\infty)$.

26. $R'(x) = \frac{7}{5}x^{2/5} - \frac{24}{5}x^{-2/5} = \frac{1}{5}x^{-2/5}(7x^{4/5} - 24)$.

R': $+$ | $-(\frac{24}{7})^{5/4}$ | $-$ | 0 | $-$ | $(\frac{24}{7})^{5/4}$ | $+$

Increasing on $(-\infty, -(\frac{24}{7})^{5/4}]$.
Decreasing on $[-(\frac{24}{7})^{5/4},0)$.

Decreasing on $(0, (\frac{24}{7})^{5/4}]$.

Increasing on $(\frac{24}{7})^{5/4}, \infty)$.

27. $U'(x) = \frac{-(x^2 + 1)}{(x^2 - 1)^2} < 0$.

Decreasing on $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$.

28. $V'(x) = \frac{-x(x + 2)}{(x^2 + 2x + 2)^2}$.

V': $-\ \underset{-2}{|}\ +\ \underset{0}{|}\ -$

Decreasing on $(-\infty,-2]$; increasing on $[-2,0]$; decreasing on $[0,\infty)$.

29. $S'(x) = \cos x$.

S': $+\ \underset{-\frac{3\pi}{2}}{|}\ -\ \underset{-\frac{\pi}{2}}{|}\ +\ \underset{\frac{\pi}{2}}{|}\ -\ \underset{\frac{3\pi}{2}}{|}\ +\ \underset{\frac{5\pi}{2}}{|}\ -$

Increasing on $[2\pi k - \frac{\pi}{2}, 2\pi k + \frac{\pi}{2}]$, k an integer.

Decreasing on $[2\pi k + \frac{\pi}{2}, 2\pi k + \frac{3\pi}{2}]$, k an integer.

30. $s'(x) = -6 \sin 2x$.

s': $-\ \underset{-\frac{3\pi}{2}}{|}\ +\ \underset{-\pi}{|}\ -\ \underset{-\frac{\pi}{2}}{|}\ +\ \underset{0}{|}\ -\ \underset{\frac{\pi}{2}}{|}\ +\ \underset{\pi}{|}\ -\ \underset{\frac{3\pi}{2}}{|}\ +$

s is increasing and decreasing as indicated on chart above.

31. $T'(x) = \sec^2 x > 0$.

Increasing on $(\pi k - \frac{\pi}{2}, \pi k + \frac{\pi}{2})$, k an integer.

32. $t'(x) = \cos x - \sin x$.

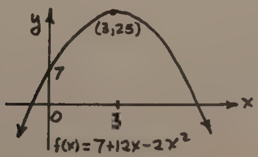
$+\ \underset{-\frac{7\pi}{4}}{|}\ -\ \underset{-\frac{3\pi}{4}}{|}\ +\ \underset{\frac{\pi}{4}}{|}\ -\ \underset{\frac{5\pi}{4}}{|}\ +\ \underset{\frac{9\pi}{4}}{|}\ -$

t is increasing and decreasing as indicated on chart above.

33. $f'(x) = 12 - 4x = 4(3 - x)$.

Critical number: 3. f': $+\ \underset{3}{|}\ -$

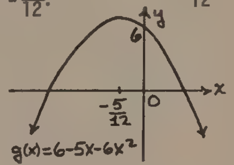
Relative maximum at $x = 3$.



34. $g'(x) = -5 - 12x = -(5 + 12x)$.

Critical number: $-\frac{5}{12}$. g': $+\ \underset{-\frac{5}{12}}{|}\ -$

Relative maximum at $x = -\frac{5}{12}$.

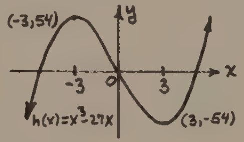


35. $h'(x) = 3x^2 - 27 = 3(x^2 - 9) = 3(x + 3)(x - 3)$.

Critical numbers: 3, -3.

h': $+\ \underset{-3}{|}\ -\ \underset{3}{|}\ +$

Relative maximum at $x = -3$; relative minimum at $x = 3$.

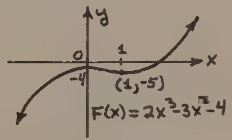


36. $F'(x) = 6x^2 - 6x = 6x(x - 1)$.

Critical numbers: 0, 1.

F': $+\ \underset{0}{|}\ -\ \underset{1}{|}\ +$

Relative maximum at $x = 0$; relative minimum at $x = 1$.

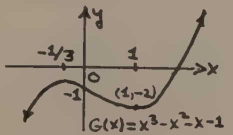


37. $G'(x) = 3x^2 - 2x - 1 = (3x + 1)(x - 1)$.

Critical numbers: $-\frac{1}{3}$, 1.

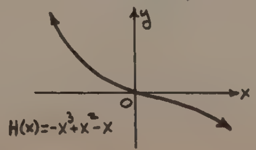
G': $+\ \underset{-\frac{1}{3}}{|}\ -\ \underset{1}{|}\ +$

Relative maximum at $x = -\frac{1}{3}$; relative minimum at $x = 1$.



38. $H'(x) = -3x^2 + 2x - 1 = -3(x - \frac{1}{3})^2 - \frac{2}{3} < 0$.

No critical numbers; H always decreasing.



39. $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) =$
$6(x + 2)(x - 1)$.
Critical numbers: $-2, 1$.

f': $+$ -2 $-$ 1 $+$

Relative maximum at $x = -2$; relative minimum at $x = 1$.

40. $g'(x) = 6x^2 + 2x - 20 = 2(3x^2 + x - 10) =$
$2(3x - 5)(x + 2)$.
Critical numbers: $\frac{5}{3}, -2$.

g': $+$ -2 $-$ $\frac{5}{3}$ $+$

Relative maximum at $x = -2$; relative minimum at $x = \frac{5}{3}$.

41. $h'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1)$.
Critical number: 1.
h': $-$ 1 $+$
Relative minimum at $x = 1$.

42. $F'(x) = (x - 1)^2 2(x - 2) + (x - 2)^2 2(x - 1) =$
$2(x - 1)(x - 2)(2x - 3)$.
Critical numbers: $1, 2, \frac{3}{2}$.

F': $-$ 1 $+$ $\frac{3}{2}$ $-$ 2 $+$

Relative minimum at $x = 1$;
relative maximum at $x = \frac{3}{2}$; relative minimum at $x = 2$.

43. $G'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) =$
$4x(x - 1)(x - 2)$.
Critical numbers: $0, 1, 2$.

G': $-$ 0 $+$ 1 $-$ 2 $+$

Relative minimum at $x = 0$; relative maximum at $x = 1$;
relative minimum at $x = 2$.

44. $H'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) =$
$12x(x + 2)(x - 1)$.
Critical numbers: $0, -2, 1$.

H': $-$ -2 $+$ 0 $-$ 1 $+$

Relative minimum at $x = -2$; relative maximum at $x = 0$; relative minimum at $x = 1$.

45. $f'(x) = \frac{1}{\sqrt{x}} - 1 = \frac{1 - \sqrt{x}}{\sqrt{x}}$.
Critical numbers: 0, 1.

f': not in domain 0 $+$ 1 $-$

Relative maximum at $x = 1$.

46. $g'(x) = 1 - 3(\frac{1}{3}x^{-2/3}) = 1 - \frac{1}{x^{2/3}} = \frac{x^{2/3} - 1}{x^{2/3}}$.
Critical numbers: 0, 1.
Relative maximum at $x = 0$; g': $+$ 0 $-$ 1 $+$
relative minimum at $x = 1$.

47. $h'(x) = \frac{-3}{(x - 2)^2}$.
No critical numbers; no relative extrema.

48. $r'(x) = \frac{-x^2 + 2x - 2}{(x^2 - 2x)^2} = \frac{-[(x - 1)^2 + 1]}{[x(x - 2)]^2}$.
No critical numbers; no relative extrema.

49.
$$f'(x) = \begin{cases} 2x & x < 4 \\ -3 & x > 4 \end{cases} \qquad f'(4) \text{ doesn't exist}$$
Critical numbers: 0, 4.

f': $-$ 0 $+$ 4 $-$

Relative minimum at $x = 0$; neither relative maximum nor relative minimum at $x = 4$.

50.
$$g'(x) = \begin{cases} \frac{-x}{25 - x^2} \\ -1 \end{cases} \qquad g'(4) \text{ doesn't exist}$$
Critical numbers: 0, 4.

g': -5 $+$ 0 $-$ 4 $-$

Relative maximum at $x = 0$.

51.
$$h'(x) = \begin{cases} 2x & x > 1 \\ -3 & x < 1 \end{cases} \qquad h'(1) \text{ doesn't exist}$$
Critical number: 1.

h': $-$ 0 $-$ 1 $+$

Relative minimum at $x = 1$.

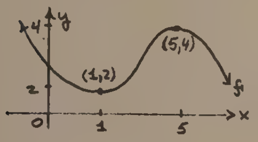
52. $q'(x) = \begin{cases} 2x + 10 & x < -4 \\ 2x + 2 & x > -4 \end{cases}$ $q'(-4)$ doesn't exist

Critical numbers: $-5, -1, -4$.

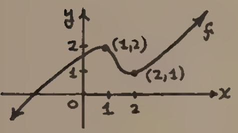
q': $-$ | -5 | $+$ | -4 | $-$ | -1 | $+$

Relative minimum at $x = -5$; relative maximum at $x = -4$; relative minimum at $x = -1$.

53.



54.



55. If f has the property that $f'(c)$ is not defined for some c in (a,b), then c is a critical point. Hence, assume there is no such c; i.e. f is differentiable on (a,b). Then by the mean value theorem, there is a c $(a < c < b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$ since $f(a) = f(b)$.

56. We show that $f'(x) \geq 0$ holds for all x in I by showing that the contrary case -- that there is at least one number c in I such that $f'(c) < 0$ -- leads to a contradiction. But, if such a c existed, then since f' is continuous, there would be a small open interval J contained in I such that $c \in J$ and $f'(x) < 0$ for every x in J. By the test for increasing and decreasing functions, f is decreasing in J, which is a contradiction since J is contained in I and f is increasing on I.

57. $f'(x) = 3ax^2 + 2bx + c$. If f is increasing, then $3ax^2 + 2bx + c \geq 0$. This implies that the discriminant $(2b)^2 - 4(3a)c \leq 0$, $4b^2 - 12ac \leq 0$, $b^2 - 3ac \leq 0$, or $b^2 \leq 3ac$.

58. For definiteness, suppose f is increasing on (a,b). We must prove that for $a < c < b$, $f(a) < f(c)$ and $f(c) < f(b)$. We prove just that $a < c < b$ implies $f(a) < f(c)$, since the proof that $f(c) < f(b)$ is similar. The proof is a result of showing that a contradiction occurs if $f(a) < f(c)$ fails; that is, that a contradiction follows from the supposition that $f(c) \leq f(a)$. Thus, suppose $f(c) \leq f(a)$. Define the number c_1 as follows:

$$c_1 = \begin{cases} \frac{a + c}{2} & \text{if } f(c) = f(a) \\ c & \text{if } f(c) < f(a) \end{cases}.$$

Notice that $a < c_1 < c < b$ and that, if $f(c) < f(a)$ then $f(c_1) = f(c) < f(a)$. On the other hand, if $f(c) = f(a)$, then $c_1 = \frac{a + c}{2} < c$ so $f(c_1) < f(c) \leq f(a)$. In either case, $f(c_1) < f(a)$ and $a < c_1 < b$. Let $y = \frac{f(a) + f(c_1)}{2}$ so that $f(c_1) < y < f(a)$. By the intermediate value theorem, there is a number x with $a < x < c_1$ such that $f(x) = y$. But $a < x < c_1 < b$ implies $y = f(x) < f(c_1)$, and this contradicts $f(c_1) < y$.

59. $f'(x) = -2 \sin 2x + 2 \cos x = -2(2 \sin x \cos x) + 2 \cos x = 2 \cos x (-2 \sin x + 1)$, $2\pi \leq x \leq 2\pi$.

(a) The critical numbers for f are those for which $\cos x = 0$ and $-2 \sin x + 1 = 0$ or $\sin x = \frac{1}{2}$.

$\cos x = 0$ for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$.

$\sin x = \frac{1}{2}$ for $x = \frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$.

(b) f is increasing (+) and decreasing (-) on the closed intervals indicated on the chart below.

f': -2π | $+$ | $\frac{-11\pi}{6}$ | $-$ | $\frac{-9\pi}{6}$ | $+$ | $\frac{-7\pi}{6}$ | $-$ | $\frac{-3\pi}{6}$ | $+$ | $\frac{\pi}{6}$ | $-$ | $\frac{\pi}{2}$ | $+$ | $\frac{5\pi}{6}$ | $-$ | $\frac{3\pi}{2}$ | $+$ | 2π

(c) Relative maxima at $\frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$;

relative minima at $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.

60. $\frac{dp}{dv} = -RT(v - b)^{-2} + 2av^{-3} = 0$.

$R = 8.206 \times 10^{-2}$, $a = 3.59$, $b = .0427$, $T = 260°$.

The derivative is not defined when v = (b,0); however, since v > b = .0427, we can ignore this value of v.

(a) Critical numbers: .032568291, .086728106, .21723034.

p': + (0.0325...) − (0.0427) − (0.0867...) + (0.217230...) −

(b) Decreasing on (0.0427, 0.086728106) and (0.21723034,∞); increasing on (0.086728106, 0.021723034).

(c) Relative minimum at 0.086728106; relative maximum at 0.21723034.

61. $C'(t) = \dfrac{(t^2 + 3t + 4)7 - 7t(2t + 3)}{(t^2 + 3t + 4)^2}$, $t \geq 0$.

If $C'(t) = 0$, then $7t^2 = 28$, $t^2 = 4$, $t = 2$.

(a) Increasing on [0,2]; decreasing on [2,∞).

0 + 2 −

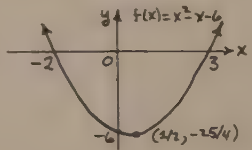
(b) Relative maximum at $t = 2$.

(c) $C(2) = \dfrac{7(2)}{2^2 + 3(2) + 4} = 1$ milligram per liter.

Problem Set 3.3, page 186

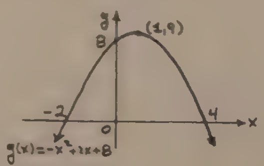
1. $f'(x) = 2x - 1$.

$f''(x) = 2 > 0$. Since $f''(x)$ is always positive, graph is concave upward over ℝ. No points of inflection.



2. $f'(x) = -2x + 2$.

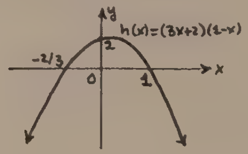
$f''(x) = -2$. Since $f''(x)$ is always negative, graph is concave downward over ℝ. No points of inflection.



3. $h(x) = (3x + 2)(1 - x) = -3x^2 + x + 2$.

$h'(x) = -6x + 1$.

$h''(x) = -6$. Since $f''(x)$ is always negative, graph is concave downward over ℝ. No points of inflection

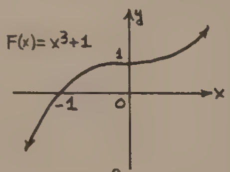


4. $F'(x) = 3x^2$.

$F''(x) = 6x$. Possible inflection point at $x = 0$.

F": − 0 +

Concave downward on (−∞,0); concave upward on (0,∞); (0,1) is a point of inflection.

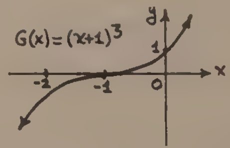


5. $G'(x) = 3(x + 1)^2$.

$G''(x) = 6(x + 1)$.

G": − 0 +

Possible inflection point at $x = -1$; concave downward on (−∞,−1); concave upward on (−1,∞); (−1,0) is an inflection point.

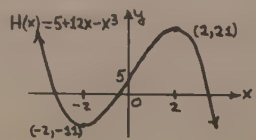


6. $H'(x) = 12 - 3x^2$.

$H''(x) = -6x$. Possible inflection point at $x = 0$.

H": + 0 −

Concave upward (−∞,0); concave downward on (0,∞); (0,5) is an inflection point; relative extrema at $x = \pm 2$.



7. $f'(x) = 3x^2 + 12x.$

$f''(x) = 6x + 12.$ Possible inflection point at $x = -2.$

f'': $-$ $\underset{-2}{|}$ $+$

Concave downward $(-\infty,-2)$; concave upward $(-2,\infty)$; $(-2,2)$ is an inflection point.

8. $g'(x) = x^2 + x - 2.$

$g''(x) = 2x + 1.$

g'': $-$ $\underset{-\frac{1}{2}}{|}$ $+$

Concave downward on $(-\infty,-\frac{1}{2})$; concave upward on $(-\frac{1}{2},\infty)$; $(-\frac{1}{2}, \frac{5}{6})$ is a point of inflection.

9. $h'(x) = 6x^2 - x - 7.$

$h''(x) = 12x - 1.$

h'': $-$ $\underset{\frac{1}{12}}{|}$ $+$

Concave downward $(-\infty, \frac{1}{12})$; concave upward $(\frac{1}{12},\infty)$; point of inflection is $(\frac{1}{12}, \frac{611}{432})$.

10. $P'(x) = 12x^2 - 36x + 15.$

$P''(x) = 24x - 36.$

P'': $-$ $\underset{\frac{3}{2}}{|}$ $+$

Concave downward $(-\infty,\frac{3}{2})$; concave upward $(\frac{3}{2},\infty)$; $(\frac{3}{2}, \frac{1}{2})$ is a point of inflection.

11. $Q'(x) = 3x^2 - 12x + 9.$

$Q''(x) = 6x - 12.$

Q'': $-$ $\underset{2}{|}$ $+$

Concave downward $(-\infty,2)$; concave upward $(2,\infty)$; $(2,3)$ is a point of inflection.

12. $R'(x) = 3x^2 + 6x.$

$R''(x) = 6x + 6.$

R'': $-$ $\underset{-1}{|}$ $+$

Concave downward $(-\infty,-1)$; concave upward $(-1,\infty)$; $(-1,7)$ is a point of inflection.

13. $p'(x) = 4x^3 + 8x.$

$p''(x) = 12x^2 + 8 > 0.$

Since $p''(x)$ is always positive, graph is concave upward on $\mathbb{R}$. No points of inflection.

14. $q'(x) = 8 - 4x - 4x^3.$

$q''(x) = -4 - 12x^2 < 0.$

Since $q''(x)$ is always negative, graph is concave downward on $\mathbb{R}$. No points of inflection.

15. $r'(x) = -4x^3.$

$r''(x) = -12x^2 < 0.$

Since $r''(x)$ is always negative, graph is concave downward on $\mathbb{R}$. No points of inflection.

16. $u'(x) = 4x^3 + 6x^2 - 24x.$

$u''(x) = 12x^2 + 12x - 24.$ If $u''(x) = 0$, then $x^2 + x - 2 = 0$ or $(x + 2)(x - 1) = 0.$

u'': $+$ $\underset{-2}{|}$ $-$ $\underset{1}{|}$ $+$

Concave upward on $(-\infty,-2)$; concave downward on $(-2,1)$; concave upward on $(1,\infty)$; $(-2, -48)$ and $(1,-9)$ are points of inflection.

17. $V'(x) = 4x^3 - 12x^2 - 36x.$

$V''(x) = 12x^2 - 24x - 36.$

If $V''(x) = 0$, then $x^2 - 2x - 3 = 0$ or $(x - 3)(x + 1) = 0.$

V'': $+$ $\underset{-1}{|}$ $-$ $\underset{3}{|}$ $+$

Concave upward on $(-\infty,-1)$; concave downward on $(-1,3)$; concave upward on $(3,\infty)$; $(-1,-13)$ and $(3,-189)$ are points of inflection.

18. $w'(x) = \frac{4}{3}x^3 - x.$

$w''(x) = 4x^2 - 1.$

If $w''(x) = 0$, then $x^2 - 1 = 0$ or $x = \pm\frac{1}{2}.$

w'': $+$ $\underset{-\frac{1}{2}}{|}$ $-$ $\underset{\frac{1}{2}}{|}$ $+$

Concave upward on $(-\infty,-\frac{1}{2})$; concave downward on $(-\frac{1}{2},\frac{1}{2})$; concave upward on $(\frac{1}{2},\infty)$; $(-\frac{1}{2}, \frac{-5}{48})$ and $(\frac{1}{2}, \frac{-5}{48})$ are points of inflection.

19. $f'(x) = 10x^4 - 60x^2.$

$f''(x) = 40x^3 - 120x = 40x(x^2 - 3).$

If $f''(x) = 0$, then $x = 0, \pm\sqrt{3}.$

f'': $-$ $\underset{-\sqrt{3}}{|}$ $+$ $\underset{0}{|}$ $-$ $\underset{\sqrt{3}}{|}$ $+$

Concave downward $(-\infty,-\sqrt{3})$; concave upward $(-\sqrt{3},0)$; concave downward $(\sqrt{3},\infty)$; $(-\sqrt{3}, 42\sqrt{3})$, $(0,0)$,

$(\sqrt{3}, -42\sqrt{3})$ are points of inflection.

20. $g'(x) = 15x^4 - 15x^2$.

$g''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$.

$g''(x) = 0$; $x = \pm\sqrt{2}/2$.

g'': $- \;\big|\; + \;\big|\; - \;\big|\; +$ at $-\frac{\sqrt{2}}{2}$, 0, $\frac{\sqrt{2}}{2}$

Concave downward on $(-\infty, -\frac{\sqrt{2}}{2})$; concave upward on $(-\frac{\sqrt{2}}{2}, 0)$; concave downward on $(0, \frac{\sqrt{2}}{2})$; concave upward on $(\frac{\sqrt{2}}{2}, 0)$; $(-\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{8})$, $(0,0)$, and $(\frac{\sqrt{2}}{2}, \frac{13\sqrt{2}}{8})$ are points of inflection.

21. $h(x) = 2x + 2x^{-1}$.

$h'(x) = 2 - 2x^{-2}$.

$h''(x) = 4x^{-3}$; $h''(x) = 0$ when $x = 0$.

Concave downward on $(-\infty, 0)$.

Concave upward on $(0, \infty)$.

h'': $- \;\big|\; +$ at 0

No point of inflection at $x = 0$ since $h'(0)$ does not exist.

22. $F(x) = x + x^{-\frac{1}{2}}$. Note: $x > 0$.

$F'(x) = 1 - \frac{1}{2}x^{-3/2}$.

$F''(x) = \frac{3}{4}x^{-5/2}$.

F'': $\big|\; +$ at 0

Concave upward on $(0, \infty)$. No points of inflection.

23. $G'(x) = 2x - 10x^{-3}$.

$G''(x) = 2 + 30x^{-4} > 0$.

Not defined at $x = 0$.

G'': $+ \;\big|\; +$ at 0

Concave upward on $(-\infty, 0)$ and on $(0, \infty)$. No points of inflection.

24. $H'(x) = \frac{8}{3}(x - 1)^{5/3} + 2(x - 1)$.

$H''(x) = \frac{40}{9}(x - 1)^{2/3} + 2 > 0$.

Concave upward on $\mathbb{R}$; no points of inflection.

25. $p'(x) = \frac{5}{3}x^{2/3}$.

$p''(x) = \frac{10}{9}x^{-1/3}$; $p''(x) = 0$ when $x = 0$.

p'': $- \;\big|\; +$ at 0

Concave downward on $(-\infty, 0)$; concave upward on $(0, \infty)$; $(0,0)$ is a point of inflection.

26. $q(x) = (x + 2)^{1/3}x^{-2/3} = \left[\frac{x + 2}{x^2}\right]^{1/3} = \left[\frac{1}{x} + 2x^{-2}\right]^{1/3}$.

$q'(x) = \frac{1}{3}(\frac{1}{x} + 2x^{-2})^{-2/3}(-\frac{1}{x^2} - 4x^{-3}) =$

$-\frac{x + 4}{3[(x + 2)^2x^5]^{1/3}}$.

$q''(x) = -\frac{3[(x + 2)^2x^5]^{1/3}}{9[(x + 2)^2x^5]^{2/3}} -$

$\frac{(x + 4)[(x + 2)^2x^5]^{-2/3}(7x^6 + 24x^5 + 20x^4)}{9[(x + 2)^2x^5]^{2/3}}$.

If $q''(x) = 0$, then $x^3 + 10x^2 + 26x + 20 = 0$ or $x = -2, -4 \pm \sqrt{6}$.

q'': $+ \;\big|\; - \;\big|\; + \;\big|$ at $-4 - \sqrt{6}$, -2, $-4 + \sqrt{6}$

Concave upward on $(-\infty, -4 - \sqrt{6})$; concave downward on $(-4 - \sqrt{6}, -2)$; concave upward on $(-2, -4 + \sqrt{6})$ concave downward on $(-4 + \sqrt{6}, \infty)$; points of inflection are: $(-4 - \sqrt{6}, -(2 + \sqrt{6})^{1/3}(4 + \sqrt{6})^{-2/3})$ and $(-4 + \sqrt{6}, (-2 + \sqrt{6})^{1/3}(-4 + \sqrt{6})^{-2/3})$. No point of inflection at $x = -2$ since $q'(-2)$ does not exist.

27. $r'(x) = \frac{-(2x + 1)}{(x^2 + x)^2}$.

$r''(x) = \frac{6x^2 + 6x + 2}{(x^2 + x)^3}$.

(Note: $6x^2 + 6x + 2 > 0$.) Possible points of inflection at $x = 0, -1$.

r'': $+ \;\big|\; - \;\big|\; +$ at -1, 0

Concave upward on $(-\infty, -1)$; concave downward on $(-1, 0)$; concave upward on $(0, \infty)$; no points of inflection since $r'(0)$ and $r'(-1)$ do not exist.

28. $s'(x) = \frac{2}{3}(x - 2)^{-1/3}$.

$s''(x) = -\frac{2}{9}(x - 2)^{-4/3}$.

Possible inflection point at $x = 2$.

s'': $- \;\big|\; -$ at 2

Concave downward on $(-\infty, 2)$; concave upward on $(2, \infty$ no point of inflection.

29. $p'(x) = \frac{(x^2 + 4)5 - 5x(2x)}{(x^2 + 4)^2} = \frac{-5x^2 + 20}{(x^2 + 4)^2}$.

$p''(x) = \frac{10x^3 - 120x}{(x^2 + 4)^3}$.

Concave downward on $(-\infty, -2\sqrt{3})$; concave upward on $(-2\sqrt{3},0)$; concave downward on $(0,2\sqrt{3})$; concave upward on $(2\sqrt{3},\infty)$. Points of inflection are $(-2\sqrt{3}, -\frac{5}{8}\sqrt{3})$, $(0,0)$, $(2\sqrt{3}, \frac{5}{8}\sqrt{3})$.

P": $-$ | $+$ | $-$ | $+$ at $-2\sqrt{3}$, 0, $2\sqrt{3}$

30. $Q'(x) = \frac{1}{3}\left(\frac{x}{x-1}\right)^{-2/3}\left(\frac{-1}{(x-1)^2}\right) = -\frac{1}{3}x^{-2/3}(x-1)^{-4/3}$.

$Q''(x) = \frac{2}{9}(3x-1)x^{-5/3}(x-1)^{-7/3}$.

Q": $-$ | $+$ | $-$ | $+$ at 0, $\frac{1}{3}$, 1

Concave downward on $(-\infty,0)$; concave upward on $(0,\frac{1}{3})$; concave downward on $(\frac{1}{3},1)$; concave upward on $(1,\infty)$; points of inflection at $(\frac{1}{3}, 1-\sqrt[3]{\frac{1}{2}})$. Note $Q'(0)$ and $Q'(1)$ do not exist.

31. $R'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}}$. Note: $x \geq 0$.

$R''(x) = \frac{1}{4}x^{-3/2}$. R": $+$ after 0

Concave upward on $(0,\infty)$. No point of inflection.

32. $f'(x) = \frac{|x^2-1|}{x^2-1}(2x)$.

$f''(x) = \frac{2|x^2-1|}{x^2-1}$.

f": $+$ | $-$ | $+$ at -1, 1

Concave upward on $(-\infty,-1)$; concave downward on $(-1,1)$; concave upward on $(1,\infty)$; no points of inflection since $f'(1)$ and $f'(-1)$ do not exist.

33. $g'(x) = -\sin x$.

$g''(x) = -\cos x$.

If $g''(x)$, then $\cos x = 0$ so $x =$ odd multiples of $\frac{\pi}{2}$. Concave upward on $(\frac{\pi}{2}+2\pi k, \frac{3\pi}{2}+2\pi k)$, k an integer. Concave downward on $(-\frac{\pi}{2}+2\pi k, \frac{\pi}{2}+2\pi k)$, k an integer. Points of inflection are at $x =$ odd multiples of $\frac{\pi}{2}$.

$-$ | $+$ | $-$ | $+$ | $-$ | $+$ | $-$ at $-\frac{5\pi}{2}$, $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$

34. $h'(x) = 2\cos 2x$.

$h''(x) = -2\sin 2x$.

If $h''(x) = 0$, then $\sin 2x = 0$ for $2x = n\pi$, n an integer; thus $x = n(\frac{\pi}{2})$, n an integer. h is concave upward (+) or concave downward (-) on the intervals indicated on the chart below. Points of inflection at $x = n(\frac{\pi}{2})$, n an integer.

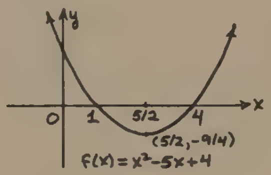
h": $+$ | $-$ | $+$ | $-$ | $+$ | $-$ | $+$ at $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$

35. $f(x) = x^2 - 5x + 4$.

$f'(x) = 2x - 5 = 0$; $x = \frac{5}{2}$.

$f''(x) = 2$; $f''(\frac{5}{2}) = 2 > 0$.

Relative minimum at $x = \frac{5}{2}$.

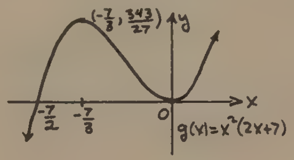


36. $g(x) = 2x^3 + 7x^2$.

$g'(x) = 6x^2 + 14x = 0$; $x = 0, -\frac{7}{3}$.

$g''(x) = 12x + 14$; $g''(0) = 14 > 0$, $g''(-\frac{7}{3}) = -14 < 0$.

Relative minimum at $x = 0$; relative maximum at $x = -\frac{7}{3}$.

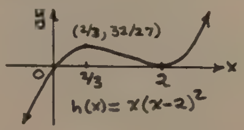


37. $h(x) = x^3 - 4x^2 + 4x$.

$h'(x) = 3x^2 - 8x + 4 = 0$; $x = \frac{2}{3}$, x.

$h''(x) = 6x - 8$; $h''(2) = 4 > 0$, $h''(\frac{2}{3}) = -4 < 0$.

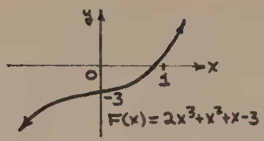
Relative minimum at $x = 2$; relative maximum at $x = \frac{2}{3}$.



38. $F'(x) = 6x^2 + 2x + 1 = 0$; no real x satisfies equation.

$F''(x) = 12x + 2.$

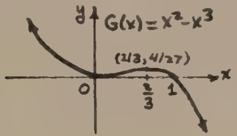
No relative extrema.



39. $G'(x) = 2x - 3x^2 = 0;\ x = 0, \frac{2}{3}.$

$G''(x) = 2 - 6x;\ G''(0) = 2 > 0,\ G''(\frac{2}{3}) = -2 < 0.$

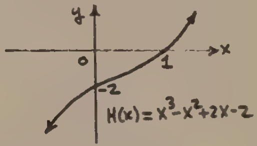
Relative minimum at $x = 0$; relative maximum at $x = \frac{2}{3}$.



40. $H'(x) = 3x^2 - 2x + 2 = 0$; no real x satisfies equation.

$H''(x) = 6x - 2.$

No relative extrema.



41. $f'(x) = 4x^3 - 12x = 0;\ x = 0, \pm\sqrt{3}.$

$f''(x) = 12x^2 - 12;\ f''(0) = -12 < 0,$

$f''(\sqrt{3}) = f''(-\sqrt{3}) = 2x.$

Relative maximum at $x = 0$; relative minima at $x = \sqrt{3}$ and $x = -\sqrt{3}$.

42. $g'(x) = 12x^3 + 24x^2 - 36x = 0;\ x = 0,1,-3.$

$g''(x) = 36x^2 + 48x - 36.$

$g''(0) = -36 < 0$; relative maximum at $x = 0$.

$g''(1) = 48 > 0$; relative minimum at $x = 1$.

$g''(-3) = 144 > 0$; relative minimum at $x = -3$.

43. $h'(x) = \dfrac{x^2 - 2x}{(x - 1)^2} = 0;\ x = 0,2.$

$h''(c) = \dfrac{2}{(x - 1)^3}.$

$h''(0) = -2 < 0$; relative maximum at $x = 0$.

$h''(2) = 2 > 0$; relative minimum at $x = 2$.

44. $F(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2} = 1 + x^{-1} + x^{-2}.$

$F'(x) = -x^{-2} - 2x^{-3};\ x = -2.$

$F''(x) = 2x^{-3} + 6x^{-4};\ F''(-2) = \frac{1}{8} > 0.$

Relative minimum at $x = -2$.

45. $G'(x) = \cos x + \sin x = 0;\ \tan x = -1$ for $x = \frac{3\pi}{4} + 2k\pi$, k an integer or $x = -\frac{\pi}{4} + 2k\pi$, k an integer.

$G''(x) = -\sin x + \cos x;\ G''(\frac{3\pi}{4} + 2k\pi) < 0,$

$G''(-\frac{\pi}{4} + 2k\pi) > 0.$

Relative maxima at $x = \frac{3\pi}{4} + 2k\pi$, k an integer; relative minima at $x = -\frac{\pi}{4} + 2k\pi$, k an integer.

46. $H'(x) = 1 - \cos x = 0;\ \cos x = 1$ for $x = k(2\pi)$, k an integer.

$H''(x) = \sin x;\ H''(2k\pi) = \sin 2k\pi = 0$; so test fails

47. (a) (i) Increasing on [a,b] and [c,d].

(ii) Decreasing on [b,c] and [d,e].

(iii) Never concave upward.

(iv) Never concave downward.

(v) No inflection points.

(b) (i) Increasing on [b,e]; hence on [b,c], [c,d] and [d,e].

(ii) Not decreasing on any displayed subinterval.

(iii) Concave upward on (c,d).

(iv) Concave downward on (a,b), (b,c) and (d,e

(v) Points of inflection at (c,f(c)) and (d,f(d)).

(c) (i) Increasing on [a,b] and [d,e].

(ii) Decreasing on [b,c].

(iii) Concave upward on (c,d).

(iv) Concave downward on (a,c); therefore on (a,b) and (b,c).

(v) Points of inflection (c,f(c)), (d,f(d)), (e,f(e)).

(d) (i) Increasing on [c,e]; hence on [c,d] and [d,e].

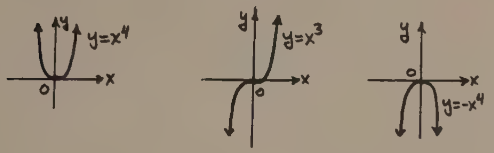
(ii) Decreasing on [a,c]; hence on [a,b] and [b,c].

(iii) Concave upward on (b,d); hence on (b,c) and (c,d).

(iv) Concave downward on (a,b) and (d,e).

(v) Points of inflection are $(b,f(b))$ and $(d,f(d))$.

48. All three functions, f, g, and h, have zero first and second derivatives at 0. Notice, however, that f has a relative minimum at 0, that h has a relative maximum at 0, and that g has neither a relative maximum nor a relative minimum at 0.



49. $f'(x) = 6x^2 - 6x = 6x(x - 1)$.

Critical points: 0, 1.

f': + (0) − (1) +

Relative maximum at $x = 0$; relative minimum at $x = 1$.

50. $g'(x) = 5x^2 - 7x - 6 = (5x + 3)(x - 2)$.

Critical points: $2, -\frac{3}{5}$.

g': + $(-\frac{3}{5})$ − (2) +

Relative maximum at $x = -\frac{3}{5}$; relative minimum at $x = 2$.

51. $h'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1)$. Critical point: 1.

h': − (1) +

Relative minimum at $x = 1$.

52. $F'(x) = 4x^3 + 12x^2 + 12x + 4 = 4((x^3 + 3x^2 + 3x + 1) = 4(x + 1)^3$.

Critical point: -1.

F': − (−1) +

Relative minimum at $x = -1$.

53. $G'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)$.

Critical points: 0, 2, −1.

G': − (−1) + (0) − (2) +

Relative minimum at $x = -1$; relative maximum at $x = 0$; relative minimum at $x = 2$.

54. $H'(x) = \dfrac{1 - x}{(x + 1)^3}$.

Critical point: 1.

H': + (1) −

Relative maximum at $x = 1$.

55. $p'(x) = \dfrac{4}{(x + 2)^2}$.

No critical numbers; no relative extrema.

$p'(x)$ is always increasing.

56. $g'(x) = 2x + 6x^{-3} = \dfrac{2x^4 + 6}{x^3}$.

No critical numbers; no relative extrema.

57. $r'(x) = \frac{4}{5} x^{-1/5}$.

Critical number: 0. r': − (0) +

Relative minimum at $x = 0$.

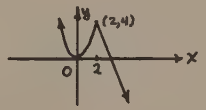
58. $p'(x) = 2x^{-1/3} - \frac{5}{3}x^{2/3} = \dfrac{6 - 5x}{3x^{1/3}}$.

Critical numbers: $0, \frac{6}{5}$.

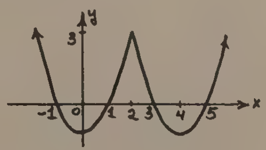
p': − (0) + $(\frac{6}{5})$ −

Relative maximum at $x = 0$; relative minimum at $x = \frac{6}{5}$.

59. Relative minimum at $x = 0$; relative maximum at $x = 2$.



60. Relative minimum at $x = 0$; relative maximum at $x = 2$; relative minimum at $x = 4$.



61.
$$h'(x) = \begin{cases} 3x^2 & x \le 1 \\ 2(x - 2) & x > 1 \end{cases}.$$

$h'_-(1) = 3$ and $h'_+(1) = -2$, so $f'(1)$ does not exist.

Critical numbers: 0, 1, 2.

Relative maximum at $x = 1$; relative minimum at $x = 2$.

62. $F'(x) = -\frac{2}{x^3}$, $x \neq 0$; $F'(0)$ does not exist, so neither first- nor second-derivative test applies. But for $x \neq 0$, $F(x) > 0 = F(0)$. Thus, F has a relative minimum at $x = 0$.

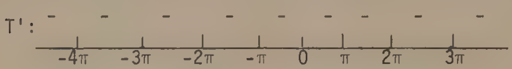
63. $S'(x) = \cos x + (-\csc x \cot x) = \cos x(\sin^2 x - 1) = \cos x(-\cos^2 x) = -\cos^3 x$.

$S'(x) = \cos x = 0$ when $x = -\frac{\pi}{2} + 2\pi k$ or $x = \frac{\pi}{2} + 2\pi k$, k an integer. The second-derivative test fails since $S''(x) = -\sin x + \csc x(\csc^2 x + \cot^2 x) = 0$ at critical numbers. Since $S'(x) < 0$ for $-\frac{\pi}{2} + 2\pi k < x < 2\pi k$ and $S'(x) > 0$ for $-\pi + 2\pi k < \frac{x}{\pi} < -\frac{\pi}{2} + 2\pi k$, there is a relative maximum at $-\frac{\pi}{2} + 2\pi k$, k an integer. Similarly, by first derivative test, there is a relative minimum at $\frac{\pi}{2} + 2\pi k$, k an integer.

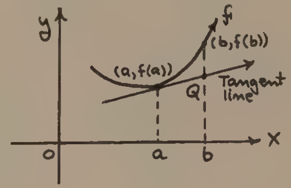
64. $T(x) = x - \tan x$. Note: $x \neq$ odd multiples of $\frac{\pi}{2}$.

$T'(x) = 1 - \sec^2 x$.

$T'(x) = 1 - \sec^2 x = 0$, when $\sec^2 x = 1$ or $\cos^2 x = 1$ or $\cos x = \pm 1$. So $\cos x = 1$ when $x = k(2\pi)$, k an integer; and $\cos x = -1$ when $x = k(2k + 1)$, k an integer; that is, $x = k\pi$, k an integer. No extrema.

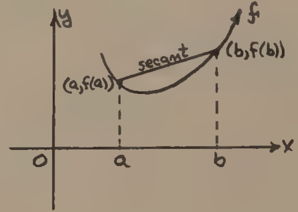


65. The tangent line to the graph of the function f at the point $(a,f(a))$ has the equation $y = f(a) + f'(a)(x - a)$; hence, the point Q on the tangent line with abscissa b is given by $Q = (b, f(a) + f'(a)(b - a))$. Hence, if $(b,f(b))$ lies strictly above the tangent line to the graph of f at $(a,f(a))$, then $f(b) > f(a) + f'(a)(b - a)$. On the other hand, if for every pair of distinct numbers a and b in I, we have $f(b) > f(a) + f'(a)(b - a)$, then the ordinate of a point other than $(a,f(a))$ on the graph of f lies strictly above the ordinate of a point on the tangent line of f at $(a,f(a))$. Hence, the graph of f lies strictly above the tangent to the graph of f at $(a,f(a))$.



66. Let a and b denote any two distinct numbers in I. To prove that $f(b) > f(a) + f'(a)(b - a)$ is equivalent to proving that $f(b) - f(a) > f'(a)(b - a)$. We assume that $a < b$ (a similar argument holds if $b < a$). Then, by the mean value theorem, there is a number c with $a < c < b$ such that $f(b) - f(a) = f'(c)(b - a)$. But, since f is concave upward on I, by definition f' is increasing on I, so that $f'(c) > f'(a)$. Thus, $f(b) - f(a) = f'(c)(b - a) > f'(a)(b - a)$ and we are done.

67.



68. For definiteness, suppose $a < b$. Let $x = ta + (1 - t)b$, where $0 < t < 1$. It is easy to see that $a < x < b$. Since the graph of f is concave upward on I and a, x, and b belong to I, by Problem 66, $f(a) - f(x) + f'(x)(x - a) > 0$ and $f(b) - f(x) + f'(x)(x - b) > 0$. Since $a < x < b$, it follows that $\frac{b - x}{b - a} > 0$ and $\frac{x - a}{b - a} > 0$. Therefore, if we define a quantity q by $q = \frac{b - x}{b - a}[f(a) - f(x) +$

$f'(x)(x - a)] + \frac{x - a}{b - a}[f(b) - f(x) + f'(x)(x - b)]$,

we have $q > 0$. Algebraic manipulation gives $0 < q = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) - f(x)$; hence, $f(x) < f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$. Therefore,

$f(ta + (1 - t)b) < f(a) + \frac{f(b) - f(a)}{b - a}[(ta + (1 - t)b - a] = f(a) + \frac{f(b) - f(a)}{b - a}(1 - t)(b - a) = f(a) + [f(b) - f(a)](1 - t) = tf(a) + (1 - t)f(b)$, as required.

69. Let $P = f(t)$, where P is performance and t is study time. Then $f'(t)$ is the rate of increase of performance per unit of study time. At the point of diminishing returns, $f''(t) = 0$ and $f'(t)$ is a maximum.

70. (a) $p(4) - p(0) = 240$ units.

(b) $p'(t) = 72 + 18t - 9t^2$.

$p''(t) = 18 - 18t = 18(1 - t)$.

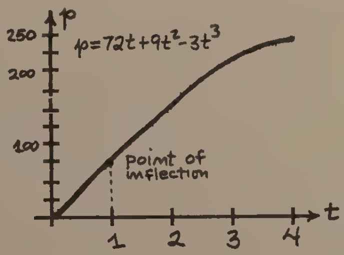
An inflection point occurs when $t = 1$, $p = 78$.

(c) $t = 1$ corresponds to 9:00 a.m.

(d) $p'(1) = 72 + 18(1) - 9(1)^2 = 81$ units per hour.

(e)

t	p
0	0
1	78
2	156
3	216
4	240



Problem Set 3.4, page 193

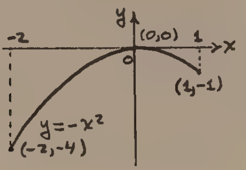
1. $f'(x) = -2x$.

Critical number: 0.

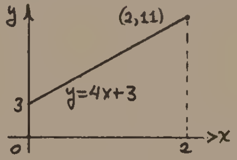
f': + | − (at 0)

$f(0) = 0$; $f(-2) = -4$; $f(1) = -1$.

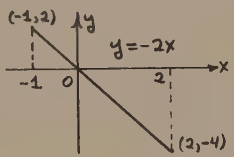
Hence, there is an absolute maximum of 0 at $x = 0$ and an absolute minimum of -4 at $x = -2$.



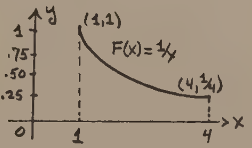
2. Absolute maximum of 11 at 2; absolute minimum of 3 at 0.



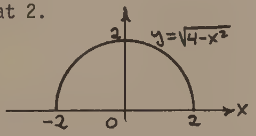
3. Absolute maximum of 2 at -1; absolute minimum of -4 at 2.



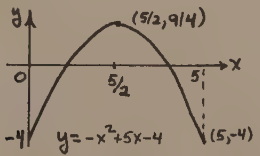
4. Absolute maximum of 1 at 1; absolute minimum of $\frac{1}{4}$ at 4.



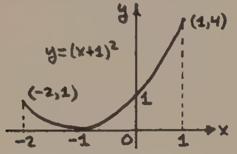
5. Absolute maximum of 2 at 0; absolute minimum of 0 at -2 and at 2.



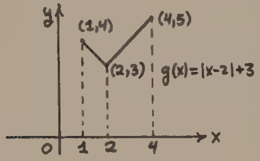
6. Absolute maximum of $\frac{9}{4}$ at $\frac{5}{2}$; absolute minimum of -4 at 0 and at 5.



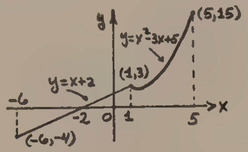
7. Absolute maximum of 4 at 1; absolute minimum of 0 at -1.



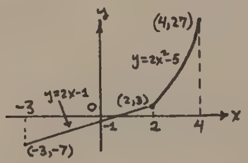
8. Absolute maximum of 5 at 4; absolute minimum of 3 at 2.



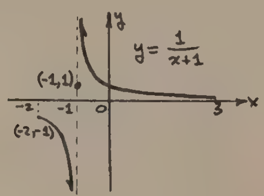
9. Absolute maximum of 15 at 5; absolute minimum of -4 at -6.



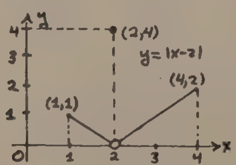
10. Absolute maximum of 27 at 4; absolute minimum of -7 at -3.



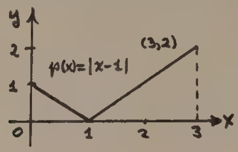
11. No absolute maximum; no absolute minimum.



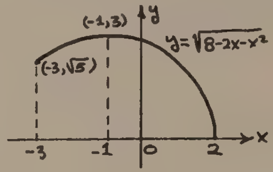
12. Absolute maximum of 4 at 2; no absolute minimum.



13. Absolute maximum of 2 at 3; absolute minimum of 0 at 1.



14. Absolute maximum of 3 at -1; absolute minimum of 0 at 1.



15. $r(x) = \sqrt{1 + x}$ on [-1,8]

$r'(x) = \frac{1}{2}(1 + x)^{-\frac{1}{2}}$; graph is always increasing on [-1,8]; no relative extrema.

$r(-1) = \sqrt{1 - 1} = 0$; $r(8) = \sqrt{9} = 3$.

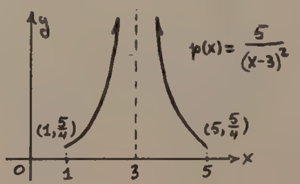
Absolute maximum of 3 at 8; absolute minimum of 0 at -1.

16. $P(x) = 5(x - 3)^{-2}$ on [1,5].

$P'(x) = -10(x - 3)^{-3}$.

As x approaches 3, P(x) get larger and larger. No absolute maximum.

$f(1) = f(5) = \frac{5}{4}$. Absolute minimum of $\frac{5}{4}$ at 1 and 5.



17. $Q(x) = x^3 + 3x^2 - 9x$ on [-5,4].

$Q'(x) = 3x^2 + 6x - 9$.

Critical numbers: -3, 1.

$Q(-5) = -5$; $Q(4) = 76$; $Q(1) = -5$.

Absolute maximum of 76 at 4; absolute minimum of -5 at -5.

18. $R'(x) = 3x^2 + 5 > 0$ on [-4,0]; graph is always increasing on [-4,0].

$R(-4) = -88$; $R(0) = -4$.

Absolute minimum of -88 at -4; absolute maximum of -4 at 0.

19. $F'(x) = \dfrac{-8}{(2x - 3)^2} < 0$ on $[-1,1]$; graph always decreasing on $[-1,1]$. $F(-1) = \frac{1}{5}$; $F(1) = -3$.

Absolute maximum of $\frac{1}{5}$ at -1; absolute minimum of -3 at 1.

20. $G'(x) = \dfrac{-6}{(x - 2)^2} < 0$ on $[-4,1]$; graph is always decreasing on $[-4,1]$.

$G(-4) = 0$; $G(1) = -5$.

Absolute maximum of 0 at -4; absolute minimum of -5 at 1.

21. $H'(x) = \dfrac{-2x}{(x^2 + 1)^2}$.

Critical number = 0.

$H(0) = 1$; $H(-2) = \frac{1}{5}$; $H(1) = \frac{1}{2}$.

Absolute maximum of 1 at 0; absolute minimum of $\frac{1}{5}$ at -2.

22. $f'(x) = \dfrac{-3}{(x - 1)^2} < 0$ on $[0,\frac{2}{3}]$; graph is decreasing $[0,\frac{2}{3}]$. $f(0) = -3$; $f(\frac{2}{3}) = -9$.

Absolute maximum of -3 at 0; absolute minimum of -9 at $\frac{2}{3}$.

23. $g'(x) = \dfrac{2 - x^2}{(x^2 + 2)^2}$.

Critical numbers: $-\sqrt{2}$, $\sqrt{2}$.

$g(-\sqrt{2}) = \frac{-\sqrt{2}}{4}$; $g(\sqrt{2}) = \frac{\sqrt{2}}{4}$; $g(-1) = -\frac{1}{3}$; $g(4) = \frac{2}{9}$.

Absolute maximum of $\frac{\sqrt{2}}{4}$ at $\sqrt{2}$; absolute minimum of $\frac{-\sqrt{2}}{4}$ at $-\sqrt{2}$.

24. $h'(x) = \dfrac{3}{(4x^2 + 1)^{3/2}} > 0$ on $[-1,1]$; graph is decreasing on $[-1,1]$.

Absolute minimum of $-\frac{3}{\sqrt{5}}$ at -1; absolute maximum of $\frac{3}{\sqrt{5}}$ at 1.

25. $f'(x) = \frac{2}{3}(x + 2)^{-1/3}$; relative minimum at $x = -2$.

$f(-2) = 0$; $f(-4) = (-2)^{2/3} = 2^{2/3}$; $f(3) = 5^{2/3}$.

Absolute maximum of $5^{2/3}$ at 3; absolute minimum of 0 at -2.

26. $g'(x) = -\frac{2}{3}(x - 2)^{-1/3}$; relative maximum at $x = 2$.
$g(2) = 1$; $g(-5) = 1 - (-7)^{2/3} = 1 - 7^{2/3}$; $g(5) = 1 - 3^{2/3}$. Absolute maximum of 1 at 2; absolute minimum of $1 - 7^{2/3}$ at -5.

27. $s'(x) = -2 \cos x$ on $[0, \frac{3\pi}{4}]$.

$-2 \cos x = 0$ when $x = \frac{\pi}{2}$; so

critical number: $\frac{\pi}{2}$. $s(\frac{\pi}{2}) = -2$; $s(0) = 0$; $s(\frac{3\pi}{4}) = -2$.

Absolute minimum of -2 at $\frac{\pi}{2}$; absolute maximum of 0 at 0.

28. $S'(x) = \cos x + \sin x$ on $[0,\pi]$.

$\cos x + \sin x = 0$ when $x = \frac{3\pi}{4}$; so

critical number: $\frac{3\pi}{4}$. $S(\frac{3\pi}{4}) = 2$; $S(0) = -1$; $S(\pi) = 1$.

Absolute maximum of 2 at $\frac{3\pi}{4}$; absolute minimum of -1 at 0.

29. $t'(x) = 1 - \sec^2 x$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

$1 - \sec^2 x = 0$ when $x = 0$; so critical number: 0.

$t(0) = 0$; $t(-\frac{\pi}{4}) = -\frac{\pi}{4} + 1$; $t(\frac{\pi}{4}) = \frac{\pi}{4} - 1$.

Absolute maximum of $1 - \frac{\pi}{4}$ at $-\frac{\pi}{4}$; absolute minimum of $\frac{\pi}{4} - 1$ at $\frac{\pi}{4}$.

30. $T'(x) = -6 \sin 2x$ on $[\frac{\pi}{6}, \frac{3\pi}{4}]$.

$-6 \sin 2x = 0$ when $x = \frac{\pi}{2}$; so critical number: $\frac{\pi}{2}$.

$T(\frac{\pi}{2}) = -3$; $T(\frac{\pi}{6}) = \frac{3}{2}$; $T(\frac{3\pi}{4}) = 0$.

Absolute maximum of $\frac{3}{2}$ at $\frac{\pi}{6}$; absolute minimum of -3 at $\frac{\pi}{2}$.

31. $f'(x) = -2x$; relative maximum at 0.

f': + | - (at 0)

Absolute maximum of 4 at 0; no absolute minimum.

32. $g'(x) = 2x - 2 = 2(x - 1)$; relative minimum at 1.

g': - | + (at 1)

Absolute minimum of -9 at 1; no absolute maximum.

33. $f'(x) = 6x - 6 = 6(x - 1)$; relative minimum of -1

at 1. f': $-\ \underset{1}{|}\ +$

From graph, absolute minimum of -1 at 1; no maximum.

34. $g'(x) = 6x^2 - 6x = 6x(x - 1)$. Relative maximum of 3 at 0; relative minimum of 2 at 1; no absolute extrema.

$+\ \underset{0}{|}\ -\ \underset{1}{|}\ +$

35. $h'(x) = 3x^2 - 12 = 3(x^2 - 4)$. Relative maximum of 21 at -2; relative minimum of -11 at 2; no absolute extrema.

$+\ \underset{-2}{|}\ -\ \underset{2}{|}\ +$

36. $F'(x) = -3x^2 - 12x + 15 = -3(x^2 + 4x - 5) = -3(x + 5)(x - 1)$.

F': $-\ \underset{-5}{|}\ +\ \underset{1}{|}\ -$

Relative maximum of 8 at 1; relative minimum of -75 at -5; no absolute extrema.

37. $G'(x) = 4x^3 - 16x = 4x(x^2 - 4)$.

G': $-\ \underset{-2}{|}\ +\ \underset{0}{|}\ -\ \underset{2}{|}\ +$

Relative minimum of -8 at -2; relative maximum of 8 at 0; relative minimum of -8 at 2.

$G(-3) = 17$, for example, so no absolute maximum; G has minimum of -8 at 2 and at -2.

38. $H'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)$.

H': $-\ \underset{-1}{|}\ +\ \underset{0}{|}\ -\ \underset{2}{|}\ +$

Relative minimum of -5 at -1; relative maximum of 0 at 0; relative minimum of -32 at 2; no absolute extrema.

39. $p'(x) = 5x^4 - 20x^3 = 5x^3(x - 4)$.

p': $+\ \underset{0}{|}\ -\ \underset{4}{|}\ +$

Relative maximum of 0 at 0; relative minimum of -256 at 4; no absolute extrema.

40. $q'(x) = 10x^4 - 40x^3 = 10x^3(x - 4)$.

q': $+\ \underset{0}{|}\ -\ \underset{4}{|}\ +$

Relative maximum of 0 at 0; relative minimum of -512 at 4; no absolute extrema.

41. $r'(x) = \dfrac{-x^2 - 6x + 16}{(x^2 + 16)^2} = \dfrac{(-x + 2)(x + 8)}{(x^2 + 16)^2}$.

r': $-\ \underset{-8}{|}\ +\ \underset{2}{|}\ -$

Relative minimum of $-\frac{1}{16}$ at -8; relative maximum of $\frac{1}{4}$ at 2. These are also absolute extrema.

42. $P'(x) = \dfrac{-x^2 + 10x - 16}{(x^2 - 16)^2} = \dfrac{(-x + 8)(x - 2)}{(x^2 - 1)^2}$.

P': $-\ \underset{-4}{|}\ -\ \underset{2}{|}\ +\ \underset{4}{|}\ +\ \underset{8}{|}\ -$

Relative minimum of $\frac{1}{4}$ at 2; relative maximum of $\frac{1}{16}$ at 8; no absolute extrema.

43. $Q'(x) = \dfrac{2x}{(1 + x^2)^2}$.

Relative minimum of 0 at 0; only critical number and Q is decreasing for $x < 0$ and increasing for $x > 0$, so minimum of 0 at 0; there is no maximum.

$-\ \underset{0}{|}\ +$

44. $R'(x) = \dfrac{4x}{(1 - x^2)^2}$.

Relative minimum of 1 at 0; absolute minimum of 1 at 0; no maximum.

$-\ \underset{-1}{|}\ -\ \underset{0}{|}\ +\ \underset{1}{|}\ +$

45. $f'(x) = -\frac{2}{3}(x + 1)^{-1/3}$.

f': $+\ \underset{-1}{|}\ -$

Relative maximum of 0 at -1; absolute maximum of 0 at -1; no minimum.

46. $g'(x) = -\frac{3}{2}(x + 1)^{1/2}$.

g': $+\ \underset{-1}{|}\ -$

Relative maximum of 1 at -1; absolute maximum of 1 at -1; no minimum.

47. Note: Domain of h is $0 \le x < 1$.

$h'(x) = \frac{1}{2}\frac{1}{x^{1/2}(1 - x)^{3/2}}$. h': $0 \quad + \quad 1$

As $x \to 1^-$, h(x) gets larger and larger; absolute minimum of 0 at 0.

48. Note: Domain of F is $x > 1$.

$F'(x) = \frac{3x^5 - 6x^2}{(x^3 - 1)^{3/2}}$.

F': $1 \quad - \quad \sqrt[3]{2} \quad +$

Relative minimum of 2 at $\sqrt[3]{2}$; absolute minimum of 2 at $\sqrt[3]{2}$.

49. $s'(x) = \frac{\sin 2x}{2\sqrt{1 + \sin^2 x}}$.

$\sin 2x = 0$ when $2x = k\pi$ or $x = k(\frac{\pi}{2})$, k an integer.

s': $-\frac{3\pi}{2} \quad - \quad -\pi \quad + \quad -\frac{\pi}{2} \quad - \quad 0 \quad + \quad \frac{\pi}{2} \quad - \quad \pi \quad + \quad \frac{3\pi}{2} \quad - \quad 2\pi$

Absolute minimum of 1 at odd multiples of $\frac{\pi}{2}$; absolute maximum of $\sqrt{2}$ at all multiples of π.

50. $S'(x) = 2 \cos x$.

$2 \cos x = 0$ for x = odd multiples of $\frac{\pi}{2}$.

S': $-\frac{5\pi}{2} \quad + \quad -\frac{3\pi}{2} \quad - \quad -\frac{\pi}{2} \quad + \quad \frac{\pi}{2} \quad - \quad \frac{3\pi}{2} \quad + \quad \frac{5\pi}{2}$

Absolute maximum of 3 for $(4k + 1)\frac{\pi}{2}$, k an integer; absolute minimum of -1 for $(4k - 1)\frac{\pi}{2}$, k an integer.

51. $p'(x) = \frac{(t^2 + 25)(2t + 5) - (t^2 + 5t + 25)(2t)}{(t^2 + 25)^2} =$

$100 \frac{-5t^2 + 125}{(t^2 + 25)^2}$.

$p' = 0$ when $-5t^2 + 125 = 0$ or $t^2 = 25$; thus, $t = \pm 5$.

p': $- \quad -5 \quad + \quad 5 \quad -$

The population p reaches a maximum of 150 animals after 5 years.

52. On $(-\infty,0)$, $f(x) = \frac{1}{1 + x} + \frac{1}{1 + |x - 4|} = \frac{1}{1 - x} +$

$\frac{1}{5 - x}$ and $f'(x) = \frac{1}{(1 - x)^2} + \frac{1}{(5 - x)^2} > 0$.

On $(0,4)$, $f(x) = \frac{1}{1 + x} + \frac{1}{5 - x}$ and

$f'(x) = \frac{-1}{(1 + x)^2} + \frac{1}{(5 - x)^2} = \frac{-12(x - 2)}{(1 + x)^2(5 - x)^2}$;

while on $(4,\infty)$, $f(x) = \frac{1}{1 + x} + \frac{1}{x - 3}$ and

$f'(x) = \frac{-1}{(1 + x)^2} + \frac{-1}{(x - 3)^2} < 0$. It follows that f is increasing on $(-\infty,2]$ and decreasing on $[2,\infty)$. Thus, f has an absolute maximum value of $\frac{2}{3}$ at 2.

53. $\frac{dR}{d\theta} = \frac{v_0^2}{g}(2 \cos 2\theta)$ for $0 \le \theta < \frac{\pi}{2}$. If $\frac{dR}{d\theta} = 0$, then $\cos 2\theta = 0$ or 2θ = odd multiple of $\frac{\pi}{2}$; thus, θ = odd multiple of $\frac{\pi}{4}$.

$\frac{dR}{d\theta}$: $+ \quad \frac{\pi}{4} \quad -$

Relative maximum of $\frac{v_0^2}{g}$ at $\frac{\pi}{4}$.

54. $\frac{dy}{dx} = \frac{p}{3EI}(200 - \frac{3x^2}{8}) = 0$ for $x = \frac{40}{\sqrt{3}}$, $0 \le x \le 40$. When $x = \frac{40}{\sqrt{3}}$, $y = \frac{16000P}{9\sqrt{3}EI}$; when $x = 0$ $y = 0$; when $x = 40$ $y = 0$. Absolute maximum at $40/\sqrt{3}$. The maximum deflection is $\frac{16000P}{9\sqrt{3}\,EI}$ and occurs $\frac{40}{\sqrt{3}}$ feet from the left end of the beam.

55. (a) $\frac{dR}{dx} = 2ABx - 3Ax^2 = Ax(2B - 3x) = 0$ for $x = 0$ or $x = \frac{2B}{3}$. We assume $x \ne 0$ and $x > 0$. Now when $x > \frac{2B}{3}$, $\frac{dR}{dx} < 0$; when $x < \frac{2B}{3}$, $\frac{dR}{dx} > 0$. Hence, the reaction is maximum when $x = \frac{2B}{3}$.

(b) For $x = \frac{2B}{3}$, $R = A(\frac{4B^2}{9})(B - \frac{2B}{3}) = \frac{4AB^3}{27}$.

(c) $\frac{d^2R}{dx^2} = 2AB - 6Ax = 0$, so that $x = \frac{B}{3}$ is the only critical number. When $x > \frac{B}{3}$, $\frac{d^2R}{dx^2} < 0$; when $x < \frac{B}{3}$, $\frac{d^2R}{dx^2} > 0$. Hence, the sensitivity $\frac{dR}{dx}$ is maximum when $x = \frac{B}{3}$.

56. $\frac{dT}{dx} = 4000 \frac{x\frac{x}{\sqrt{324 + x^2}} - \sqrt{324 + x^2}}{x^2} - \frac{972}{x^2} + 3 =$

$\frac{1}{x^2}[3(x^2 - 324) - 4000 \frac{324}{\sqrt{324 + x^2}}]$. Thus, for a critical value of x we must solve $3(x^2 - 324) =$

$4000 \frac{324}{\sqrt{324 + x^2}}$ for x. The solution is $x \approx 76.35$ feet and the corresponding tension in the cable is $T \approx 4351.44$ pounds.

Problem Set 3.5, page 202

1. $\lim\limits_{x\to+\infty} \frac{1 + 6x}{-2 + x} = \lim\limits_{x\to+\infty} \frac{\frac{1}{x} + 6}{-\frac{2}{x} + 1} = 6.$

2. $\lim\limits_{x\to-\infty} \frac{2x^2 + x + 1}{-4x^2 + 5x + 10} = \lim\limits_{x\to-\infty} \frac{2 + \frac{1}{x} + \frac{1}{x^2}}{-4 + \frac{5}{x} + \frac{10}{x^2}} = \frac{2}{-4} = -\frac{1}{2}.$

3. $\lim\limits_{x\to+\infty} \frac{5x^2 - 7x + 3}{8x^2 + 5x + 1} = \lim\limits_{x\to+\infty} \frac{5 - \frac{7}{x} + \frac{3}{x^2}}{8 + \frac{5}{x} + \frac{1}{x^2}} = \frac{5}{8}.$

4. $\lim\limits_{x\to-\infty} \frac{7x^3 + 3x + 1}{x^3 - 2x + 3} = \lim\limits_{x\to-\infty} \frac{7 + \frac{3}{x^2} + \frac{1}{x^3}}{1 - \frac{2}{x} + \frac{3}{x^3}} = \frac{7}{1} = 7.$

5. $\lim\limits_{x\to-\infty} \frac{x^{100} + x^{99}}{x^{101} - x^{100}} = \lim\limits_{x\to-\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x}} = \frac{0}{1} = 0.$

6. $\lim\limits_{x\to+\infty} \frac{x^{99} + x^{98}}{x^{100} - x^{99}} = \lim\limits_{x\to+\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x}} = \frac{0}{1} = 0.$

7. $\lim\limits_{t\to+\infty} \frac{8t}{\sqrt[4]{3t^4 + 5}} = \lim\limits_{t\to+\infty} \frac{8}{\frac{\sqrt[4]{3t^4 + 5}}{t}} = \lim\limits_{t\to+\infty} \frac{8}{\sqrt[4]{\frac{3t^4 + 5}{t^4}}} =$
$\lim\limits_{t\to+\infty} \frac{8}{\sqrt[4]{3 + \frac{5}{t^4}}} = \frac{8}{\sqrt[4]{3}}.$

8. $\lim\limits_{x\to-\infty} \frac{6x^2}{\sqrt[3]{5x^6 - 1}} = \lim\limits_{x\to-\infty} \frac{6}{\frac{\sqrt[3]{5x^6 - 1}}{x^2}} = \lim\limits_{x\to-\infty} \frac{6}{\sqrt[3]{\frac{5x^6 - 1}{x^6}}} =$
$\lim\limits_{x\to-\infty} \frac{6}{\sqrt[3]{5 - \frac{1}{x^6}}} = \frac{6}{\sqrt[3]{5}}.$

9. $\lim\limits_{x\to+\infty} (5x^2 - 3x) = \lim\limits_{x\to+\infty} x(5x - 3) = +\infty$, since each factor $\to +\infty$.

10. $\lim\limits_{x\to-\infty} (\frac{x^3 - 5x^2}{3x}) = \lim\limits_{x\to-\infty} (\frac{x^2 - 5x}{3}) = \lim\limits_{x\to-\infty} \frac{x(x - 5)}{3} =$ $+\infty$, since each factor in the numerator $\to -\infty$.

11. $\lim\limits_{x\to1^+} \frac{2x}{x - 1} = +\infty$, since $\lim\limits_{x\to1^+} 2x = 2$.

12. $\lim\limits_{x\to2^-} \frac{x^2}{x - 2} = -\infty$, since $\lim\limits_{x\to2^-} x^2 = 4$.

13. $\lim\limits_{x\to0^+} \frac{\sqrt{4 + 3x^2}}{5x} = +\infty$, since $\lim\limits_{x\to0^+} 4 + 3x^2 = 2$ and $\lim\limits_{x\to0^+} 5x = 0$.

14. $\lim\limits_{x\to3^+} \frac{x^2 + 5x + 1}{x^2 - 2x - 3} = \lim\limits_{x\to3^+} \frac{x^2 + 5x + 1}{(x - 3)(x + 1)} = +\infty$, since $\lim\limits_{x\to3^+} (x^2 + 5x + 1) = 25$ and $\lim\limits_{x\to3^+} (x + 1) = 4$.

15. $\lim\limits_{x\to4^-} \frac{2x^2 + 3x - 2}{x^2 - 3x - 4} = \frac{2x^2 + 3x - 2}{(x - 4)(x + 1)} = -\infty$, since $\lim\limits_{x\to4^-} (2x^2 + 3x - 2) = 42$ and $\lim\limits_{x\to4^-} (x + 1) = 5$.

16. $\lim\limits_{t\to5^-} \frac{\sqrt{25 - t^2}}{t - 5} = \lim\limits_{t\to5^-} \frac{\sqrt{25 - t^2}\sqrt{25 - t^2}}{(t - 5)\sqrt{25 - t^2}} =$
$\lim\limits_{t\to5^-} \frac{25 - t^2}{(t - 5)\ 25 - t^2} = \lim\limits_{t\to5^-} \frac{-(5 + t)}{25 - t^2} =$
$\lim\limits_{t\to5^-} -(5 + t) = -10$ so $\lim\limits_{t\to5^-} \frac{\sqrt{25 - t^2}}{t - 5} = -\infty.$

17. $\lim\limits_{x\to1^-} \frac{x^2 - 1}{|x^2 - 1|} = \lim\limits_{x\to1^-} \frac{x^2 - 1}{-(x^2 - 1)} = \lim\limits_{x\to1^-} -1 = -1.$

18. $\lim\limits_{x\to2^-} \frac{[\![2 - x]\!]}{2 - x} = 0$, since $2 - x > 0$ but < 1 when x is close to 2 and to the left of 2, and so $[\![2 - x]\!] = 0$.

19. $\lim\limits_{x\to2^-} \frac{x^2 + 1}{x - 2}.$

For values of x slightly smaller than 2, $x^2 + 1 > 0$
$x - 2 < 0$; hence, $\lim\limits_{x\to2^-} \frac{x^2 + 1}{x - 2} = -\infty.$

20. $\lim\limits_{z\to2^+} \frac{z^2 + 1}{z - 2}.$

For values of z slightly larger than 2, $z^2 + 1 > 0$ and $z - 2 > 0$; hence, $\lim\limits_{z\to2^+} \frac{z^2 + 1}{z - 2} = +\infty.$

21. $\lim\limits_{t\to-1^+} (\frac{3}{t + 1} - \frac{5}{t^2 - 1}) = \lim\limits_{t\to-1^+} (\frac{3(t - 1) - 5}{t^2 - 1}) =$
$\lim\limits_{t\to-1^+} (\frac{3t - 8}{t^2 - 1}).$

For values of t slightly larger than -1, $3t - 8 < 0$ and $t^2 - 1 < 0$; hence,

$$\lim_{t \to -1} \left(\frac{3}{t+1} - \frac{5}{t^2 - 1}\right) = +\infty.$$

22. $\displaystyle \lim_{x \to -\infty} \frac{1 + \sqrt[5]{x}}{1 - \sqrt[5]{x}} = \lim_{x \to -\infty} \frac{\frac{1}{\sqrt[5]{x}} + 1}{\frac{1}{\sqrt[5]{x}} - 1} = \frac{1}{-1} = -1.$

23. $\displaystyle \lim_{x \to (\frac{\pi}{2})^+} \sec x = \lim_{x \to (\frac{\pi}{2})^+} \frac{1}{\cos x}.$

For x slightly greater than $\frac{\pi}{2}$, $1 > 0$ and $\cos x < 0$ and close to zero; hence, $\displaystyle \lim_{x \to (\frac{\pi}{2})^+} \sec x = -\infty.$

24. $\displaystyle \frac{\sin\theta}{\theta} = \frac{|\sin\theta|}{|\theta|} \le \frac{1}{|\theta|}$, so $\displaystyle -\frac{1}{|\theta|} \le \frac{\sin\theta}{\theta} \le \frac{1}{|\theta|}.$

But $\displaystyle \lim_{\theta \to +\infty} \left(-\frac{1}{|\theta|}\right) = \lim_{\theta \to +\infty} \frac{1}{|\theta|} = 0.$ Therefore,

$$\lim_{\theta \to +\infty} \frac{\sin\theta}{\theta} = 0.$$

25. (a) $\displaystyle \lim_{x \to +\infty} \frac{4x}{(x-5)^2} = \lim_{x \to +\infty} \frac{\frac{4}{x}}{(1 - \frac{5}{x})^2} = \frac{0}{1} = 0.$ Thus, $y = 0$ or the x axis is a horizontal asymptote.

Similarly, $\displaystyle \lim_{x \to -\infty} \frac{4x}{(x-5)^2} = 0.$

(b) When $y = 0$, $\displaystyle \frac{4x}{(x-5)^2} = 0$ or $x = 0$. Thus, $(0,0)$ is a point on the graph of f.

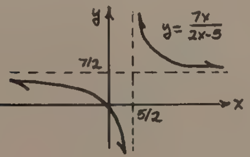
26. $\displaystyle \lim_{x \to +\infty} x \sin\frac{1}{x}.$

Let $t = \frac{1}{x}$, then as $x \to +\infty$, $t \to 0^+$. Thus,

$$\lim_{x \to +\infty} x \sin\frac{1}{x} = \lim_{t \to 0^+} \frac{1}{t}\sin t = \lim_{t \to 0^+} \frac{\sin t}{t} = 1.$$

27. $\displaystyle \lim_{x \to +\infty} \frac{7x}{2x - 5} = \lim_{x \to +\infty} \frac{7}{2 - \frac{5}{x}} = \frac{7}{2} = \lim_{x \to -\infty} \frac{7x}{2x - 5}.$

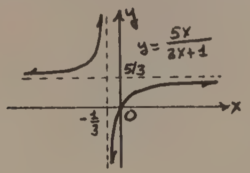
So $y = \frac{7}{2}$ is a horizontal asymptote; $2x - 5 = 0$ or $x = \frac{5}{2}$ is a vertical asymptote.



28. $\displaystyle \lim_{x \to +\infty} \frac{5x}{3x + 1} = \lim_{x \to +\infty} \frac{5}{3 + \frac{1}{x}} = \frac{5}{3} = \lim_{x \to -\infty} \frac{5x}{3x + 1}.$

So $y = \frac{5}{3}$ is a horizontal asymptote.

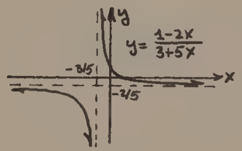
$3x + 1 = 0$ or $x = -\frac{1}{3}$ is a vertical asymptote.



29. $\displaystyle \lim_{x \to +\infty} \frac{1 - 2x}{3 + 5x} = \lim_{x \to +\infty} \frac{\frac{1}{x} - 2}{\frac{3}{x} + 5} = -\frac{2}{5} = \lim_{x \to -\infty} \frac{1 - 2x}{3 + 5x}.$

So $y = -\frac{2}{5}$ is a horizontal asymptote.

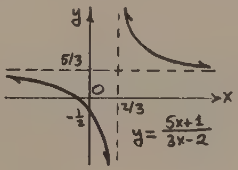
$3 + 5x = 0$ or $x = -\frac{3}{5}$ is a vertical asymptote.



30. $\displaystyle \lim_{x \to +\infty} \frac{5x + 1}{3x - 2} = \lim_{x \to +\infty} \frac{5 + \frac{1}{x}}{3 - \frac{2}{x}} = \frac{5}{3} = \lim_{x \to -\infty} \frac{5x + 1}{3x - 2}.$

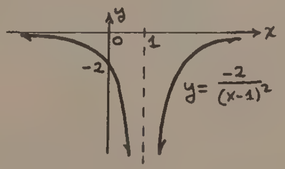
So $y = \frac{5}{3}$ is a horizontal asymptote.

$3x - 2 = 0$ or $x = \frac{2}{3}$ is a vertical asymptote.



31. $\displaystyle \lim_{x \to +\infty} \frac{-2}{(x-1)^2} = 0 = \lim_{x \to -\infty} \frac{-2}{(x-1)^2}.$

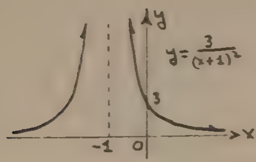
So $y = 0$ a horizontal asymptote; $(x - 1)^2 = 0$ or $x = 1$ is a vertical asymptote.



32. $\displaystyle \lim_{x \to +\infty} \frac{3}{(x+1)^2} = 0 = \lim_{x \to -\infty} \frac{3}{(x+1)^2}.$

So $y = 0$ is a horizontal asymptote.

$(x + 1)^2 = 0$ or $x = -1$ is a vertical asymptote.



33. $\lim\limits_{x\to+\infty} \dfrac{x^2}{x^2+4} = \lim\limits_{x\to+\infty} \dfrac{1}{1+\frac{4}{x^2}} = 1 = \lim\limits_{x\to-\infty} p(x).$

So $y = 1$ is a horizontal asymptote.

No vertical asymptote.

34. $\lim\limits_{x\to+\infty} \dfrac{x^2+1}{x^2+9} = \lim\limits_{x\to+\infty} \dfrac{1+\frac{1}{x^2}}{1+\frac{9}{x^2}} = 1 = \lim\limits_{x\to-\infty} q(x).$

So $y = 1$ is a horizontal asymptote.

No vertical asymptote.

35. $\lim\limits_{x\to+\infty} \dfrac{3x}{\sqrt{2x^2-1}} = \lim\limits_{x\to+\infty} \dfrac{3}{\sqrt{\frac{2x^2-1}{x^2}}} = \lim\limits_{x\to+\infty} \dfrac{3}{\sqrt{2+\frac{1}{x^2}}} = \dfrac{3}{\sqrt{2}}.$

$\lim\limits_{x\to-\infty} \dfrac{3x}{\sqrt{2x^2+1}} = \lim\limits_{x\to-\infty} \dfrac{-3}{\sqrt{\frac{2x^2+1}{x^2}}} = \lim\limits_{x\to-\infty} \dfrac{-3}{\sqrt{2+\frac{1}{x^2}}} = -\dfrac{3}{\sqrt{2}}.$

So $y = \frac{3}{\sqrt{2}}$, $y = -\frac{3}{\sqrt{2}}$ are horizontal asymptotes.

No vertical asymptotes.

36. $\lim\limits_{x\to+\infty} \dfrac{\sqrt[3]{25-x^2}}{5-x} = \lim\limits_{x\to+\infty} \dfrac{\sqrt[3]{\frac{25-x^2}{x^3}}}{\frac{5}{x}-1} = \lim\limits_{x\to+\infty} \dfrac{\sqrt[3]{\frac{25}{x^3}-\frac{1}{x}}}{\frac{5}{x}-1} =$

$\dfrac{0}{0-1} = 0 = \lim\limits_{x\to-\infty} P(x).$

So $y = 0$ is a horizontal asymptote.

$\lim\limits_{x\to5^+} P(x) = \lim\limits_{x\to5^+} \dfrac{\sqrt[3]{25-x^2}\,\sqrt[3]{(25-x^2)^2}}{(5-x)\,\sqrt[3]{(25-x^2)^2}} =$

$\lim\limits_{x\to5^+} \dfrac{25-x^2}{(5-x)\sqrt[3]{(25-x^2)^2}} = \lim\limits_{x\to5^+} \dfrac{5+x}{\sqrt[3]{(25-x^2)^2}} = +\infty.$

Similarly, $\lim\limits_{x\to5^-} P(x) = -\infty.$

Therefore, $x = 5$ is a vertical asymptote.

37. Note: Domain of Q is $x > 2$ or $x \le 0$.

$\lim\limits_{x\to+\infty} \sqrt{\dfrac{x}{x-2}} = \lim\limits_{x\to+\infty} \sqrt{\dfrac{1}{1-\frac{2}{x}}} = \sqrt{1} = 1 = \lim\limits_{x\to-\infty} Q(x).$

So $y = 1$ is a horizontal asymptote.

$\lim\limits_{x\to2^+} Q(x) = +\infty$; $\lim\limits_{x\to2^-} Q(x)$ cannot be calculated since $0 < x \le 2$ is not in domain. $x = 2$ is a vertical asymptote.

38. Since the domain of R is $x < 1$, we cannot calculate $\lim\limits_{x\to+\infty} R(x)$. $\lim\limits_{x\to-\infty} R(x) = \lim\limits_{x\to-\infty} \dfrac{-1-\frac{2}{x}}{\sqrt{\frac{1}{x^2}-\frac{1}{x}}} = -\infty$, since numerator $\to -1$ and denominator $\to 0$. Thus there is no horizontal asymptote.

$\lim\limits_{x\to1^-} \dfrac{x+2}{1-x} = +\infty$; so $x = 1$ is a vertical asymptote.

39. $\lim\limits_{x\to+\infty} u(x) = \lim\limits_{x\to+\infty} (x - \frac{1}{x}) = +\infty$ and $\lim\limits_{x\to-\infty} u(x) = -\infty.$

So no horizontal asymptotes.

$\lim\limits_{x\to0^+} \dfrac{x^2-1}{x} = -\infty$ and $\lim\limits_{x\to0^-} \dfrac{x^2-1}{x} = +\infty.$

So $x = 0$ is a vertical asymptote.

40. $\lim\limits_{x\to+\infty} \dfrac{4x^2+1}{x^3} = \lim\limits_{x\to+\infty} (\frac{4}{x} + \frac{1}{x^3}) = 0 = \lim\limits_{x\to-\infty} V(x).$

So $y = 0$ is a horizontal asymptote.

$\lim\limits_{x\to0^+} \dfrac{4x^2+1}{x^3} = +\infty$ and $\lim\limits_{x\to0^-} \dfrac{4x^2+1}{x^3} = -\infty.$

So $x = 0$ is a vertical asymptote.

41. $\lim\limits_{x\to+\infty} U(x) = \lim\limits_{x\to+\infty} \dfrac{3+\frac{1}{x^2}}{2-\frac{7}{x}} = \dfrac{3}{2} = \lim\limits_{x\to-\infty} U(x).$

So $y = \frac{3}{2}$ is a horizontal asymptote.

$2x^2 - 7x = 0$ or $x(2x-7) = 0$; so $x = 0$ and $x = \frac{7}{2}$ are vertical asymptotes.

42. Note: Domain of V is $x > -1$ and $x < -4$.

$\lim\limits_{x\to+\infty} V(x) = \lim\limits_{x\to+\infty} \dfrac{4x}{\sqrt{1+\frac{5}{x}+\frac{4}{x^2}}} = +\infty$ and

$\lim\limits_{x\to-\infty} V(x) = \lim\limits_{x\to-\infty} \dfrac{-4x}{\sqrt{1+\frac{5}{x}+\frac{4}{x^2}}} = +\infty.$

So no horizontal asymptotes.

$\lim\limits_{x\to-1^+} V(x) = +\infty$ and $\lim\limits_{x\to-4^-} V(x) = +\infty.$

So $x = -1$ and $x = -4$ are vertical asymptotes.

43. $f(x) = \cot x = \dfrac{\cos x}{\sin x}.$

$\lim_{x\to+\infty} \cot x = \lim_{x\to-\infty} \cot x$ does not exist, so there are no horizontal asymptotes. $\sin x = 0$ for $x = k\pi$, k an integer. $\lim_{x\to(k\pi)^+} \cot x = +\infty$ and $\lim_{x\to(k\pi)^-} \cot x = -\infty$, k an integer. So $x = k\pi$, k an integer, are vertical asymptotes.

44. $g(x) = \sec x = \frac{1}{\cos x}$.

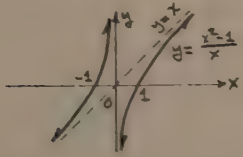
$\lim_{x\to+\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$ do not exist, so there are no horizontal asymptotes. $\cos x = 0$ for $x =$ odd multiples of $\frac{\pi}{2}$. $\lim_{x\to\frac{\pi}{2}^+} \sec x = -\infty$ and $\lim_{x\to\frac{\pi}{2}^-} \sec x = +\infty$. Using periodicity you can show $x =$ odd multiples of $\frac{\pi}{2}$ are vertical asymptotes.

45. $h(x) = \csc x = \frac{1}{\sin x}$.

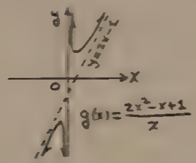
$\lim_{x\to+\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$ do not exist, so there are no horizontal asymptotes. $\sin x = 0$ for $x = k\pi$, k an integer. $\lim_{x\to0^+} h(x) = +\infty$ and $\lim_{x\to0^-} h(x) = -\infty$; $\lim_{x\to\pi^+} h(x) = -\infty$ and $\lim_{x\to\pi^-} h(x) = +\infty$. Using periodicity, you can show $x = k\pi$, k an integer are the vertical asymptotes.

46. $\lim_{x\to+\infty} F(x) = 1$ (see Problem 26) and $\lim_{x\to-\infty} F(x) = 1$. So $y = 1$ is a horizontal asymptote. $|x \sin \frac{1}{x}| \le |x|$ so $-|x| \le x \sin \frac{1}{x} \le |x|$. $\lim_{x\to0} (-|x|) = \lim_{x\to0} |x| = 0$, so $\lim_{x\to0} x \sin \frac{1}{x} = 0$. No vertical asymptotes.

47. $f(x) = \frac{x^2 - 1}{x} = x - \frac{1}{x}$ and $\lim_{x\to+\infty} \frac{1}{x} = 0$. Thus, $y = x$ is an oblique asymptote.



48. $g(x) = \frac{2x^2 - x + 1}{x} = 2x - 1 + \frac{1}{x}$ and $\lim_{x\to+\infty} \frac{1}{x} = 0$. Thus, $y = 2x - 1$ is an oblique asymptote.



49. $\lim_{x\to+\infty} \frac{1}{x^2 + 1} = 0$. So $y = 3x - 2$ is an oblique asymptote.

50. $F(x) = \frac{2x^3 + x^2 + 5x + 1}{x^2 + 2} = 2x + 1 + \frac{x - 1}{x^2 + 2}$ and

$$\lim_{x\to+\infty} \frac{x - 1}{x^2 + 2} = \lim_{x\to+\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1 + \frac{2}{x^2}} = \frac{0}{1} = 0.$$

So $y = 2x + 1$ is an oblique asymptote.

51. $G(x) = \frac{x^2 + 2x}{x + 1} = x + 1 + \frac{1}{x + 1}$ and $\lim_{x\to+\infty} \frac{1}{x + 1} = 0$. So $y = x + 1$ is an oblique asymptote.

52. $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$ (see Problem 24). So $y = 2x + 2$ is an oblique asymptote.

53. $\frac{f(0 + \Delta x) - f(0)}{\Delta x} = \frac{1 + \sqrt[3]{\Delta x} - 1}{\Delta x} = \frac{(\Delta x)^{1/3}}{\Delta x} = \frac{1}{(\Delta x)^{2/3}}$. Therefore, $\lim_{\Delta x\to0} \left|\frac{f(0 + \Delta x) - f(0)}{\Delta x}\right| = \lim_{\Delta x\to0} \left|\frac{1}{(\Delta x)^{2/3}}\right| = +\infty$.

So f has a vertical tangent line at 0.

54. $\frac{g(0 + \Delta x) - g(0)}{\Delta x} = \frac{x + \sqrt[3]{\Delta x} - 0}{\Delta x} = 1 + \frac{1}{(\Delta x)^{2/3}}$. Therefore, $\lim_{\Delta x\to0} \left|\frac{g(0 + \Delta x) - g(0)}{\Delta x}\right| = \lim_{\Delta x\to0} \left|1 + \frac{1}{(\Delta x)^{2/3}}\right| = +\infty$. So g has a vertical tangent line at 0.

55. $\frac{h(1 + \Delta x) - h(1)}{\Delta x} = \frac{1 + (1 + \Delta x - 1)^{2/3} - (1 + 0^{2/3})}{\Delta x} = \frac{1 + (\Delta x)^{2/3} - 1}{\Delta x} = \frac{1}{(\Delta x)^{1/3}}$. Thus,

$$\lim_{\Delta x\to0} \left|\frac{h(1 + \Delta x) - h(1)}{\Delta x}\right| = \lim_{\Delta x\to0} \left|\frac{1}{(\Delta x)^{2/3}}\right| = +\infty.$$

So h has a vertical tangent line at 1.

56. $\dfrac{F(2 + \Delta x) - F(2)}{\Delta x} = \dfrac{-3 - (2 + \Delta x - 2)^{2/3} - (-3 - 0^{2/3})}{\Delta x} =$

$\dfrac{-3 - \Delta x^{2/3} + 3}{\Delta x} = \dfrac{-1}{(\Delta x)^{1/3}}$. Thus,

$\lim\limits_{\Delta x \to 0} \left|\dfrac{F(2 + \Delta x) - F(2)}{\Delta x}\right| = \lim\limits_{\Delta x \to 0} \left|\dfrac{-1}{\Delta x^{1/3}}\right| = +\infty.$

So F has a vertical tangent line at 2.

57. $\dfrac{G(1 + \Delta x) - G(1)}{\Delta x} = \dfrac{-2 - \sqrt[5]{1 + \Delta x - 1} - (2 - \sqrt[5]{0})}{\Delta x} =$

$\dfrac{-2 - \sqrt[5]{\Delta x} + 2}{\Delta x} = -\dfrac{1}{(\Delta x)^{4/5}}$. Therefore,

$\lim\limits_{\Delta x \to 0} \left|\dfrac{G(1 + \Delta x) - G(1)}{\Delta x}\right| = \lim\limits_{\Delta x \to 0} \left|-\dfrac{1}{(\Delta x)^{4/5}}\right| = +\infty.$

So G has a vertical tangent line at 1.

58. $\dfrac{H(-1 + \Delta x) - H(-1)}{\Delta x} =$

$\dfrac{(-1 + \Delta x + 1)^{1/3}(-1 + \Delta x)^{2/3} - 0^{1/3}(-1)^{2/3}}{\Delta x} =$

$\dfrac{(\Delta x)^{1/3}(\Delta x - 1)^{2/3}}{\Delta x} = \dfrac{(\Delta x - 1)^{2/3}}{(\Delta x)^{2/3}} = \left(\dfrac{x - 1}{x}\right)^{2/3} =$

$(1 - \frac{1}{x})^{2/3}$. Therefore, $\lim\limits_{\Delta x \to 0} \left|\dfrac{H(-1 + \Delta x) - H(-1)}{\Delta x}\right| =$

$\lim\limits_{\Delta x \to 0} \left|(1 - \frac{1}{\Delta x})^{2/3}\right| = +\infty$. So H has a vertical tangent line at -1.

59. $\lim\limits_{x \to +\infty} \dfrac{1 + 6x}{-2 + x} = \lim\limits_{x \to +\infty} f(x).$

x	10	100	1000	10,000	100,000
f(x)	7.625	6.1326	6.01307	6.00130	6.00013

$\lim\limits_{x \to +\infty} \dfrac{5x^2 - 7x + 3}{8x^2 + 5x + 1} = \lim\limits_{x \to +\infty} f(x).$

x	10	100	1000	10,000
f(x)	0.508813	0.612452	0.623735	0.624873

60. Given any $\varepsilon > 0$, there is a positive N such that $|f(x) - B| < \varepsilon$ holds whenever $x < -N$.

61. $\lim\limits_{t \to -1^+} \left(\dfrac{3}{t + 1} - \dfrac{5}{t^2 - 1}\right) = \lim\limits_{t \to -1^+} f(t).$

t	-.9	-.99	-.999	-.9999
f(t)	56.316	551.25628	2502.75	25,002.75

$\lim\limits_{x \to (\frac{\pi}{2})^+} \sec x = \lim\limits_{x \to (\frac{\pi}{2})^+} f(x).$

t	$\frac{\pi}{2} + 0.1$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.001$	$\frac{\pi}{2} + 0.00001$
f(x)	-10.0167	-100.00617	-1000.0001	-10,000.0000

62. $f(x) = p(x) + \dfrac{r(x)}{q(x)}$ where the degree of $r(x) <$ degree of $q(x)$. Then $\lim\limits_{x \to +\infty} \dfrac{r(x)}{q(x)} = 0$ by dividing numerator and denominator by the highest power of x. Thus, if $\lim\limits_{x \to +\infty} p(x)$ exists (namely, if p(x) is constant), then $\lim\limits_{x \to +\infty} f(x)$ exists and only has one value.

Thus, there is only one horizontal asymptote.

63. (a) $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - 1}{x} = \lim\limits_{x \to 0} \dfrac{(\sqrt{1 + x} - 1)(\sqrt{1 + x} + 1)}{x(\sqrt{1 + x} + 1)} =$

$\lim\limits_{x \to 0} \dfrac{(1 + x) - 1}{x(\sqrt{1 + x} + 1)} = \lim\limits_{x \to 0} \dfrac{x}{x(\sqrt{1 + x} + 1)} =$

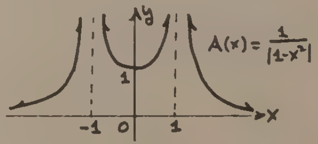
$\lim\limits_{x \to 0} \dfrac{1}{\sqrt{1 + x} + 1} = \dfrac{1}{1 + 1} = \dfrac{1}{2}$. Since $\lim\limits_{x \to 0} f(x) \neq \pm\infty$,

$x = 0$ is not a vertical asymptote of the graph of f.

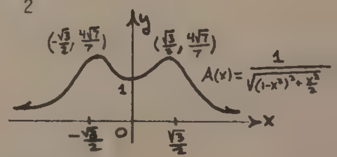
(b) $\lim\limits_{x \to 0^+} \dfrac{1 + \csc x}{1 + x} = \lim\limits_{x \to 0^+} \dfrac{1 + \frac{1}{\sin x}}{1 + x} =$

$\lim\limits_{x \to 0^+} \dfrac{\sin x + 1}{\sin x(1 + x)} = +\infty$. So $x = 0$ is a vertical asymptote of the graph of f.

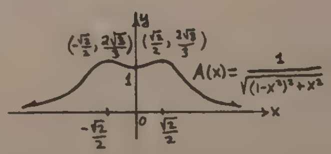
64. (a) $y = A(x)$ for $k = 0$.



$y = A(x)$ for $k = \frac{1}{2}$



$y = A(x)$ for $k = 1$



(b) The only way in which the graph of A can have a vertical asymptote at $x = a$ is for the denominator $\sqrt{(1 - x^2)^2 + kx^2}$ to approach 0 as x approaches a; that is, for $(1 - x^2)^2 + kx^2$ to approach 0 as x approaches a. Since $(1 - x^2)^2 + kx^2 = x^4 + (k - 2)x^2 + 1$ is a polynomial, this would require that a^2 be a root of the quadratic polynomial $x^2 + (k - 2)x + 1$. The latter polynomial will have a root a^2 only if its discriminant $(k - 2)^2 - 4$ is nonnegative. But, for $0 < k < 4$, $(k - 2)^2 - 4 < 0$.

65. (a) $\lim_{t \to +\infty} \frac{a + bt}{t} = \lim_{t \to +\infty} (\frac{a}{t} + b) = b$, so $I = b$ is a horizontal asymptote.

(b) For values of $I < b$, there is no excitation regardless of duration of stimulus.

66. $\lim_{t \to +\infty} w = \lim_{t \to +\infty} (0.012 + \frac{3 + \frac{6}{t}}{4 + \frac{7}{t} + \frac{8}{t^2}}) = 0.012 + \frac{3}{4} =$ 0.762 kilograms.

Problem Set 3.6, page 210

1. $f(x) = (x + 1)^2(x - 2) = x^3 - 3x - 2$; f is neither even nor odd. x intercepts: -1,2; y intercept: -2.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1)$.

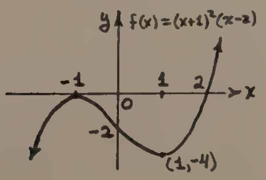
f': + (−1) − (1) +

$f''(x) = 6x$.

Relative maximum at -1; $f(-1) = 0$.

Relative minimum at 1; $f(1) = -4$.

Point of inflection at (0,-2). No asymptotes.



2. $g(x) = x^3 - 6x^2 + 9x - 4$; g is neither even nor odd.

y intercept: -4; x intercepts: 1, 4.

$g'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$.

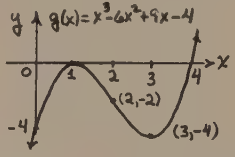
g': + (1) − (3) +

$g''(x) = 6x - 12$.

Relative maximum at 1; $f(1) = 0$.

Relative minimum at 3; $f(3) = -4$.

Point of inflection at (2,-2). No asymptotes.



3. $h(x) = x^3 - 3x + 2$; h is neither even nor odd.

y intercept: 2; x intercept: 1, -2.

$h'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1)$.

Increasing on $(-\infty,-1]$ and on $[1,\infty)$; decreasing on $[-1,1]$.

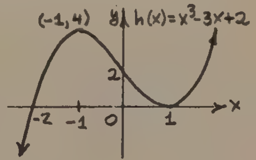
Relative maximum of 4 at -1; relative minimum of 0 at 1.

$h''(x) = 6x$.

Concave downward on $(-\infty,2)$; concave upward on $(2,\infty)$.

Point of inflection at (0,2).

No asymptotes.



4. $F(x) = 10 + 12x - 3x^2 - 2x^3$; F is neither even nor odd.

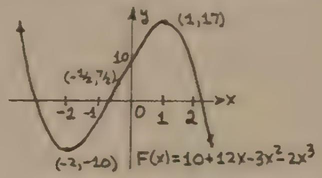
y intercept: 10; x intercepts: approximately 2.22, -0.762, and -2.96.

$F'(x) = 12 - 6x - 6x^2 = -6(x + 2)(x - 1)$.

Decreasing on $[1,\infty)$ and on $(-\infty,-2]$; increasing on $[-2,1]$.

Relative maximum of 17 at 1; relative minimum of -10 at -2.

$F''(x) = -6 - 12x$. Concave upward on $(-\infty, -\frac{1}{2})$; concave downward on $(-\frac{1}{2}, \infty)$. Point of inflection at $(-\frac{1}{2}, \frac{7}{2})$. No absolute extrema. No asymptotes.



5. $G(x) = 4x^2 - x^4$; G is even, so symmetric with respect to y axis. y intercept: 0; x intercepts: 0, ± 2.

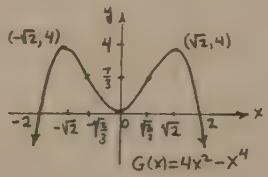
$G'(x) = 8x - 4x^3 = 4x(2 - x^2)$.

Increasing on $(-\infty, -2]$ and on $[0,2]$; decreasing on $[-2,0]$ and on $[2,\infty)$. Relative maximum of 4 at $-\sqrt{2}$ and at $\sqrt{2}$; relative minimum of 0 at 0.

$G''(x) = 8 - 12x^2 = 4(2 - 3x^2)$.

Concave upward on $(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$; concave downward on $(-\infty, -\sqrt{\frac{2}{3}})$ and on $(\sqrt{\frac{2}{3}}, \infty)$.

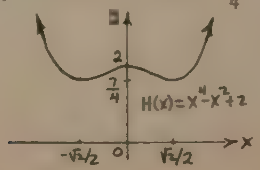
Points of inflection at $(-\sqrt{\frac{2}{3}}, \frac{7}{3})$ and $(\sqrt{\frac{2}{3}}, \frac{7}{3})$.



6. $H(x) = x^4 - x^2 + 2$; H is even, so symmetric with respect to y axis. y intercept: 2; no x intercepts. $H'(x) = 2x(2x^2 - 1)$. Increasing on $[\frac{\sqrt{2}}{2}, \infty)$ and $[-\frac{\sqrt{2}}{2}, 0]$; decreasing on $(-\infty, \frac{\sqrt{2}}{2}]$ and on $[0, \frac{\sqrt{2}}{2}]$. Relative minimum at $-\frac{\sqrt{2}}{2}$ and at $\frac{\sqrt{2}}{2}$; relative maximum at 0.

$H(\frac{\sqrt{2}}{2}) = H(-\frac{\sqrt{2}}{2}) = \frac{7}{4}$; $H(0) = 2$; points of inflection at $(-\frac{\sqrt{6}}{6}, \frac{67}{36})$ and $(\frac{\sqrt{6}}{6}, \frac{67}{36})$.

$H''(x) = 12x^2 - 2$. Concave upward on $(\frac{\sqrt{6}}{6}, \infty)$ and on $(-\infty, \frac{\sqrt{6}}{6})$; concave downward on $(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6})$. No absolute maximum; absolute minimum of $\frac{7}{4}$ at $\pm\frac{\sqrt{2}}{2}$.



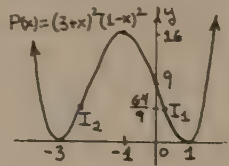
7. $P(x) = (3 + x)^2(1 - x)^2$; P is neither even nor odd. y intercept: 9; x intercepts: -3 and 1.

$P'(x) = 4(x + 3)(x - 1)(x + 1)$.

P is increasing on $[1,\infty)$ and on $[-3,-1]$; decreasing on $[-1,1]$ and on $(-\infty,-3]$. Relative maximum of 16 at -1; relative minima of 0 at 1 and at -3.

$P''(x) = 4(3x^2 + 6x - 1)$. Concave upward on $[\frac{-3 + 2\sqrt{3}}{3}, \infty)$ and on $(-\infty, \frac{-3 - 2\sqrt{3}}{3}]$; concave downward on $[\frac{-3 - 2\sqrt{3}}{3}, \frac{-3 + 2\sqrt{3}}{3}]$. Points of inflection are $I_1 = (\frac{-3 + 2\sqrt{3}}{3}, \frac{64}{9})$ and $I_2 = (\frac{-3 - 2\sqrt{3}}{3}, \frac{64}{9})$.

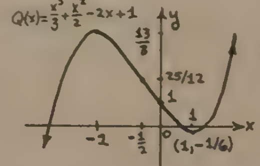
Absolute minimum of 0 at 1 and -3.



8. $Q(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 1$; Q is neither even nor odd. y intercept: 1; x intercepts: approximately 0.653, 1.32, and -3.48.

$Q'(x) = x^2 + x - 2$. Increasing on $(-\infty,-2]$ and $[1,\infty)$; decreasing on $[-2,1]$.

$Q''(x) = 2x + 1$. Concave upward on $(-\frac{1}{2}, \infty)$; concave downward on $(-\infty, -\frac{1}{2})$. Point of inflection at $(-\frac{1}{2}, \frac{25}{12})$.

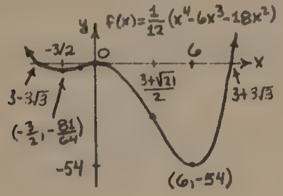


9. $f(x) = \frac{1}{12}(x^4 - 6x^3 - 18x^2)$; f is neither even nor odd. y intercept: 0; x intercepts: 0, $3 \pm 3\sqrt{3}$.

$f'(x) = \frac{1}{12}(4x^3 - 18x^2 - 36x) = \frac{1}{6}(2x^3 - 9x^2 - 18x) = \frac{x}{6}(2x^2 - 9x - 18) = \frac{x}{6}(2x + 3)(x - 6)$. Relative minimum of $-\frac{81}{64}$ at $-\frac{3}{2}$ and of -54 at 6; relative maximum of 0 at 0.

$f''(x) = \frac{1}{6}(6x^2 - 18x - 18) = x^2 - 3x - 3$. Concave upward on $(-\infty, \frac{3 - \sqrt{21}}{2})$ and on $(\frac{3 + \sqrt{21}}{2}, \infty)$; concave downward on $(\frac{3 - \sqrt{21}}{2}, \frac{3 + \sqrt{21}}{2})$. Points of

inflection at $(\frac{3+\sqrt{21}}{2}, -31.6)$ and at $(\frac{3-\sqrt{21}}{2}, -0.659)$.



10. $g(x) = x^5 - 3x^4$; g is neither even nor odd.

y intercept: 0; x intercepts: 0, 3.

$g'(x) = 5x^4 - 12x^3 = x^3(5x - 12)$.

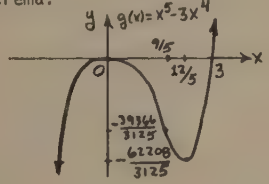
Increasing on $[\frac{12}{5},\infty)$ and on $(-\infty,0]$; decreasing on $[0,\frac{12}{5}]$. Relative maximum of 0 at 0; relative minimum at $\frac{12}{5}$ of $\frac{-62208}{3125} \approx -20$.

$g''(x) = 20x^3 - 36x^2 = 4x^2(5x - 9)$.

Concave upward on $(\frac{9}{5},\infty)$; concave downward on $(-\infty,\frac{9}{5})$.

Point of inflection at $(\frac{9}{5},-\frac{39,366}{3125}) \approx (\frac{9}{5}, 12.6)$.

No absolute extrema.



11. h is odd, so symmetric with respect to the origin.

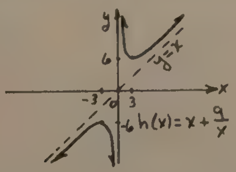
$h(x) = x + \frac{9}{x}$.

$h'(x) = 1 - \frac{9}{x^2} = \frac{x^2 - 9}{x^2}$. Relative maximum of -6 at -3; relative minimum of 6 at 3.

$h''(x) = 18x^{-3}$.

Concave upward on $(0,\infty)$; concave downward on $(-\infty,0)$.

Vertical asymptote: $x = 0$; oblique asymptote: $y = x$.

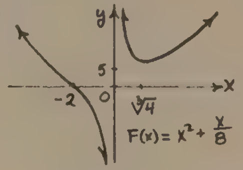


12. $F(x) = x^2 + \frac{8}{x}$; F is neither even nor odd.

x intercept: -2.

$F'(x) = 2x - \frac{8}{x^2}$. Increasing on $[\sqrt[3]{4},\infty)$; decreasing on $(-\infty,0)$ and $(0,\sqrt[3]{4}]$. Relative minimum of ≈ 7.56 at $\sqrt[3]{4}$.

$F''(x) = 2 + \frac{16}{x^3}$. Concave upward on $(-\infty,-2)$ and on $(0,\infty)$; concave downward on $(-2,0)$. Point of inflection at $(-2,0)$. No absolute extrema. Vertical asymptote: $x = 0$.



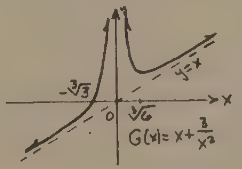
13. $G(x) = x + \frac{3}{x^2}$; G is neither even nor odd.

x intercept: $-\sqrt[3]{3}$; y intercept: none.

$G'(x) = 1 - \frac{6}{x^3} = \frac{x^3 - 6}{x^3}$. Increasing on $(-\infty,0)$ and on $[\sqrt[3]{6},\infty)$; decreasing on $(0,\sqrt[3]{6}]$. Relative minimum of ≈ 2.73 at $\sqrt[3]{6}$.

$G''(x) = \frac{18}{x^4}$.

Concave upward on $(-\infty,0)$ and on $(0,\infty)$. Vertical asymptote: $x = 0$; oblique asymptote: $y = x$.

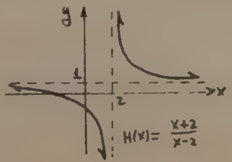


14. $H(x) = \frac{x+2}{x-2}$; H is neither even nor odd.

y intercept: -1; x intercept: -2.

$H'(x) = \frac{-x}{(x-2)^2}$. Decreasing on $(-\infty,2)$ and on $(2,\infty)$. $f''(x) = \frac{1}{(x-2)^2}$. No points of inflection.

Vertical asymptote: $x = 2$; horizontal asymptote: $y = 1$.

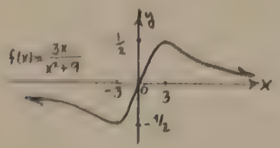


15. $f(x) = \frac{3x}{x^2 + 9}$; f is symmetric about the origin.

y intercept: 0; x intercept: 0.

$f'(x) = \frac{3(3 - x)(3 + x)}{(x^2 + 9)^2}$. Decreasing on $(-\infty,-3]$ and on $[3,\infty)$; increasing on $[-3,3]$. Relative maximum of $\frac{1}{2}$ at 3; relative minimum of $-\frac{1}{2}$ at -3.

$f''(x) = \frac{6x(x^2 - 27)}{(x^2 + 9)^3}$. Concave upward on $(-\sqrt{27},0)$ and on $(\sqrt{27},\infty)$; concave downward on $(-\infty,-\sqrt{27})$ and on $(0,\sqrt{27})$. Points of inflection at $(-\sqrt{27},-\frac{\sqrt{27}}{12})$, $(0,0)$ and $(\sqrt{27},\frac{\sqrt{27}}{12})$. Absolute maximum of $\frac{1}{2}$ at 3; absolute minimum of $-\frac{1}{2}$ at -3.

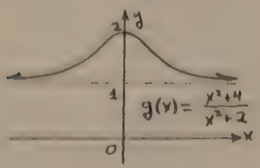


16. $g(x) = \frac{x^2 + 4}{x^2 + 2}$; g is symmetric about the y axis

y intercept: 2.

$g'(x) = \frac{-4x}{(x^2 + 2)^2}$. Increasing on $(-\infty,0]$; decreasing on $[0,\infty)$.

$g''(x) = \frac{-4(-3x^2 + 2)}{(x^2 + 2)^3}$. Concave upward on $(\frac{\sqrt{6}}{3},\infty)$ and $(-\infty,-\frac{\sqrt{6}}{3})$; concave downward on $(-\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{3})$. Points of inflection at $(\frac{\sqrt{6}}{3},\frac{7}{4})$ and $(-\frac{\sqrt{6}}{3},\frac{7}{4})$. Horizontal asymptote: y = 1. Absolute maximum of 2 at 0.



17. $h(x) = \frac{x^2 + 3x}{x + 4}$. h is neither even nor odd.

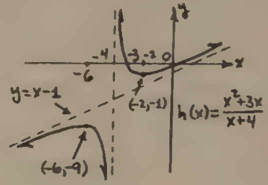
y intercept: 0; x intercept: 0, 3.

$h'(x) = \frac{x^2 + 8x + 12}{(x + 4)^2} = \frac{(x + 2)(x + 6)}{(x + 4)^2}$. Increasing on $(-\infty,-6)$ and on $[-2,\infty)$; decreasing on $(-6,-2)$.

Relative maximum of -9 at -6; relative minimum of -1 at -2.

$f''(x) = \frac{-8}{(x - 4)^2}$. No points of inflection.

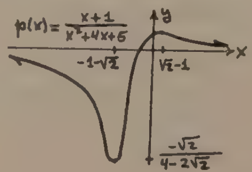
Vertical asymptote: x = -4; oblique asymptote: y = x.



18. $p(x) = \frac{x + 1}{x^2 + 4x + 5}$. p is neither even nor odd.

$p'(x) = \frac{-x^2 - 2x + 1}{(x^2 + 4x + 5)^2}$. Increasing on $[-1 - \sqrt{2}, -1 + \sqrt{2}]$; decreasing on $(-\infty, -1 - \sqrt{2}]$ and on $[-1 + \sqrt{2},\infty)$. Relative maximum at $-1 + \sqrt{2}$; relative minimum at $-1 - \sqrt{2}$.

$p''(x) = \frac{2(x + 3)(x^2 - 3)}{(x^2 + 4x + 5)^3}$. Concave upward on $(-3,-\sqrt{3})$ and on $(\sqrt{3},\infty)$; concave downward on $(-\infty,-3)$ and on $(-\sqrt{3},\sqrt{3})$. Points of inflection are $(-3,-1)$, $(-\sqrt{3}, \frac{1 - \sqrt{3}}{8 - 4\sqrt{3}})$ and $(\sqrt{3}, \frac{1 + \sqrt{3}}{8 + 4\sqrt{3}})$. Absolute minimum of $-\frac{\sqrt{2}}{4 - 2\sqrt{2}}$ at $-1 - \sqrt{2}$; absolute maximum of $\frac{\sqrt{2}}{4 + 2\sqrt{2}}$ at $-1 + \sqrt{2}$. Horizontal asymptote: y = 0.



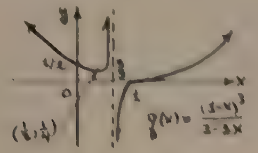
19. $q(x) = \frac{(1 - x)^3}{2 - 3x}$. q is neither even nor odd.

y intercept: $\frac{1}{2}$; x intercept: 1.

$q'(x) = \frac{3(2x - 1)(x - 1)^2}{(3x - 2)^2}$. Decreasing on $(-\infty,\frac{1}{2}]$; increasing on $[\frac{1}{2},\frac{2}{3}]$ and $(\frac{2}{3},\infty)$. Relative minimum of $\frac{1}{4}$ at $\frac{1}{2}$; no relative extremum at 1.

$q''(x) = \frac{6(x - 1)(3x^2 - 3x + 1)}{(3x - 2)^3}$. Concave upward on

$(-\infty,\frac{2}{3})$ and on $(1,\infty)$; concave downward on $(\frac{2}{3},1)$; no absolute extrema. Point of inflection at $(1,0)$. Vertical asymptote: $x = \frac{2}{3}$.



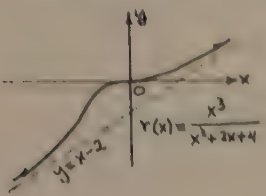
20. $r(x) = \dfrac{x^3}{x^2 + 2x + 4}$; r is neither even nor odd.

y intercept: 0; x intercept: 0.

$r'(x) = \dfrac{x^4 + 4x^3 + 12x^2}{(x^2 + 2x + 4)^2} = \dfrac{x^2(x^2 + 4x + 12)}{(x^2 + 2x + 4)^2} > 0$

except for $x = 0$; r is increasing on $\mathbb{R}$.

$r''(x) = \dfrac{48x(x + 2)}{(x^2 + 2x + 4)^3}$. Concave upward on $(-\infty,-2)$ and $(0,\infty)$; graph is concave downward on $(-2,0)$. Points of inflection at $(-2,-2)$ and $(0,0)$. No vertical asymptotes; oblique asymptote: $y = x - 2$.

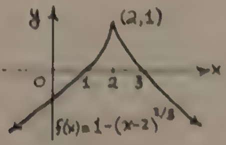


21. $f(x) = 1 - (x - 2)^{2/3}$; f is neither even nor odd.

y intercept: $1 - 2^{2/3}$; x intercepts: 1, 3.

$f'(x) = -\frac{2}{3}(x - 2)^{-1/3}$. Increasing on $(-\infty,2]$; decreasing on $[2,\infty)$. Relative maximum of 1 at 2.

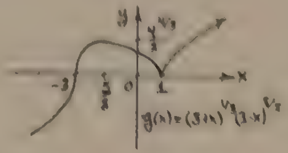
$f''(x) = \frac{2}{9}(x - 2)^{-4/3}$. Concave upward on $(-\infty,2)$ and $(2,\infty)$. Absolute maximum of 1 at 2. Vertical tangent line at $x = 2$.



22. $g(x) = (3 + x)^{1/3}(1 - x)^{2/3}$; g is neither even nor odd. y intercept: $\sqrt[3]{3} \approx 1.44$; x intercepts: -3, 1.

$g'(x) = \dfrac{-(3x + 5)}{3(3 + x)^{2/3}(1 - x)^{1/3}}$. Increasing on $(-\infty,-\frac{5}{3}]$ and on $[1,\infty)$; decreasing on $[-\frac{5}{3},1]$. Relative minimum of 0 at 1; relative maximum of $\frac{4^{4/3}}{3} \approx 2.12$ at $-\frac{5}{3}$.

$g''(x) = \dfrac{-32}{9(3 + x)^{5/3}(1 - x)^{4/3}}$. Concave upward on $(-\infty,-3)$; concave downward on $(-3,1)$ and on $(1,\infty)$. No absolute extrema.

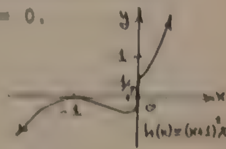


23. $h(x) = (x + 1)^2x^{1/3}$. h is neither even nor odd.

y intercept: 0; x intercept: 0, -1.

$h'(x) = \dfrac{\sqrt[3]{x}(x + 1)(7x + 1)}{3x}$. Increasing on $(-\infty,-1]$ and on $[-\frac{1}{7},\infty)$; decreasing on $[-1,-\frac{1}{7}]$. Relative maximum of 0 at -1; relative minimum of $\dfrac{-36}{49\sqrt[3]{7}} \approx -0.384$ at $-\frac{1}{7}$; no relative extrema at 0.

$h''(x) = \dfrac{2\sqrt[3]{x}(14x^2 + 4x - 1)}{9x^2}$. Concave upward on $(\frac{-2 - 3\sqrt{2}}{14},0)$ and on $(\frac{-2 + 3\sqrt{2}}{14},\infty)$; concave downward on $(-\infty,\frac{-2 - 3\sqrt{2}}{14})$ and on $(0,\frac{-2 + 3\sqrt{2}}{14})$. Points of inflection at $(\frac{-2 - 3\sqrt{2}}{14}, f(\frac{-2 - 3\sqrt{2}}{14})) \approx$ [illegible], $(0,0)$, and $(\frac{-2 + 3\sqrt{2}}{14}, f(\frac{-2 + 3\sqrt{2}}{14})) \approx (0.16,\ 0.73)$. No absolute extrema. Vertical tangent at $x = 0$.



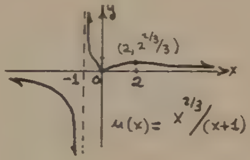
24. $u(x) = \dfrac{x^{2/3}}{x + 1}$. u is neither even nor odd.

y intercept: 0; x intercept: 0.

$u'(x) = \dfrac{-x + 3}{3x^{1/3}(x + 1)^2}$. Decreasing on $(-\infty,-1)$, on

$(-1,0]$ and on $[2,\infty)$; increasing on $[0,2]$. Relative minimum of 0 at 0; realtive maximum of $2^{2/3}/3 \approx 0.529$.

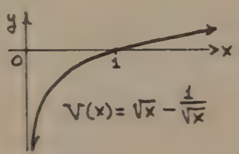
$u''(x) = \dfrac{4x^2 - 16x - 2}{9x^{4/3}(x+1)^3}$. Concave downward on $(\infty,-1)$ and on $(\frac{4-3\sqrt{2}}{2}, \frac{4+3\sqrt{2}}{2})$; concave upward on $(-1, \frac{4-3\sqrt{2}}{2})$ and on $(\frac{4+3\sqrt{2}}{2},\infty)$. Points of inflection at approximately $(\frac{4-3\sqrt{2}}{2}, 0.279)$ and $(\frac{4+3\sqrt{2}}{2}, 0.502)$. Vertical asymptote: $x = -1$. Vertical tangent line: $x = 0$. Horizontal asymptote: $y = 0$.



25. $v(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$. Note: $x > 0$. x is neither even nor odd. y intercept: none; x intercept: 1.

$v'(x) = \dfrac{1}{2x^{1/2}} + \dfrac{1}{2x^{3/2}} = \dfrac{x+1}{2x^{3/2}} > 0$, so increasing on its entire domain.

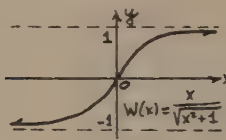
$v''(x) = -\dfrac{x+3}{4x^{5/2}} < 0$, so concave down on its entire domain. Vertical asymptote: $x = 0$.



26. $w(x) = \dfrac{x}{\sqrt{x^2+1}}$; h is odd, so symmetric about the origin. y intercept: 0, x intercept: 0.

$w'(x) = \dfrac{1}{(x^2+1)^{3/2}}$. Increasing on $(-\infty,\infty)$. No relative extrema.

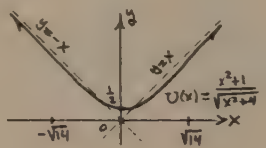
$w''(x) = -\dfrac{-3x}{(x^2+1)^{5/2}}$. Concave downward on $(0,\infty)$; concave upward on $(-\infty,0)$. Point of inflection at $(0,0)$. No absolute extrema. Horizontal asymptote $y = 1$ and $y = -1$.



27. $U(x) = \dfrac{x^2+1}{\sqrt{x^2+4}}$. U is even, so symmetric about the y axis. y intercept: $\frac{1}{2}$; x intercept: none.

$U'(x) = \dfrac{(x(x^2+7)}{(x^2+4)^{3/2}}$. Increasing on $[0,\infty)$; decreasing on $(-\infty,0]$. Relative minimum of $\frac{1}{2}$ at 0.

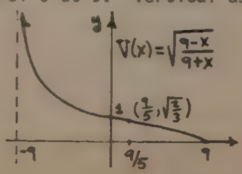
$U''(x) = \dfrac{-2(x^2-14)}{(x^2+4)^{5/2}}$. Concave upward on $(-\sqrt{14},\sqrt{14}$; concave downward on $(-\infty,-\sqrt{14})$ and on $(\sqrt{14},\infty)$. Points of inflection are $(-\sqrt{14},f(-\sqrt{14})) \approx (-3.74,3.54)$ and at $(\sqrt{14},f(\sqrt{14})) \approx (3.74,3.54)$. Absolute minimum of $\frac{1}{2}$ at 0. Oblique asymptotes: $y = x$ and $y = -x$.



28. $V(x) = \sqrt{\dfrac{9-x}{9+x}}$. Note: $-9 < x \le 9$. V is neither even nor odd. y intercept: 1; x intercept: 9.

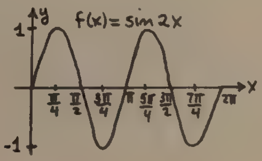
$V'(x) = \dfrac{-9}{(9-x)(9+x)^{3/2}}$. Decreasing on $(-9,9]$. No relative extrema.

$V''(x) = \dfrac{-9(-5x+9)}{2(9-x)^2(9+x)^{5/2}}$. Concave upward on $(-9,\frac{9}{5})$; concave downward on $(\frac{9}{5},9)$. Point of inflection at $(\frac{9}{5},\sqrt{\frac{2}{3}})$, where $\sqrt{\frac{2}{3}} \approx 0.82$. Absolute minimum of 0 at 9. Vertical asymptote: $x = -9$.



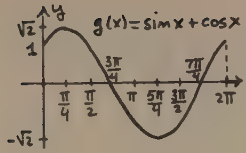
29. $f(x) = \sin 2x$, $0 \le x \le 2\pi$.

$f'(x) = 2 \cos 2x$, so that $f'(x) = 0$ when $\cos 2x = 0$; that is, when $2x = \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \pm 5\frac{\pi}{2}, \pm 7\frac{\pi}{2}, \ldots$. Thus, $f'(x) = 0$ when $x = \pm\frac{\pi}{4}, \pm 3\frac{\pi}{4}, \pm 5\frac{\pi}{4}, \pm 7\frac{\pi}{4}, \ldots$. Since we are requiring $0 \le x \le 2\pi$, then $f'(x) = 0$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. Since $f'(x) = 2 \cos 2x$, it follows that $f''(x) = -4 \sin 2x$; hence, $f''(\frac{\pi}{4}) = -4 \sin \frac{\pi}{2} = -4 < 0$, $f''(\frac{3\pi}{4}) = -4 \sin \frac{3\pi}{2} = 4 > 0$, $f''(\frac{5\pi}{4}) = -4 \sin \frac{5\pi}{2} = -4 < 0$, $f''(\frac{7\pi}{4}) = -4 \sin \frac{7\pi}{2} = 4 > 0$. Therefore, $f(\frac{\pi}{4}) = 1$ is a maximum, and $f(\frac{3\pi}{4}) = -1$ is a minimum, $f(\frac{5\pi}{4}) = 1$ is a maximum, and $f(\frac{7\pi}{4}) = -1$ is a minimum. Here, $f''(x) = -4 \sin 2x = 0$ when $2x = 0, \pi, 2\pi, 3\pi$, and 4π; that is, when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π. Since $f''(x) < 0$ on $(0,\frac{\pi}{2})$, then the graph of f is concave downward on $(0,\frac{\pi}{2})$. Similarly, this graph is concave upward on $(\frac{\pi}{2},\pi)$, downward on $(\pi,\frac{3\pi}{2})$ and upward on $(\frac{3\pi}{2},2\pi)$. Thus, the points of inflection are $(\frac{\pi}{2},0)$, $(\pi,0)$, and $(\frac{3\pi}{2}, 0)$.

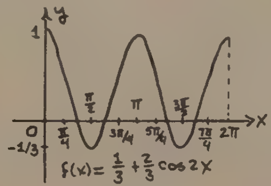


30. $g(x) = \sin x + \cos x$, $0 \le x \le 2\pi$.

$g'(x) = \cos x - \sin x$, so that $g'(x) = 0$ when $\sin x = \cos x$; that is, when $x = \frac{\pi}{4}$ and when $x = \frac{5\pi}{4}$ for $0 \le x \le 2\pi$. Here, $g''(x) = -\sin x - \cos x$, so that $g''(\frac{\pi}{4}) = -\sqrt{2} < 0$ and $g''(\frac{5\pi}{4}) = \sqrt{2} > 0$; hence, $g(\frac{\pi}{4}) = \sqrt{2}$ is a maximum and $g(\frac{5\pi}{4}) = -\sqrt{2}$ is a minimum. Notice that $g''(x) < 0$ on $(0,\frac{3\pi}{4})$ and on $(\frac{7\pi}{4},2\pi)$, so that the graph of g is concave downward on these intervals. Also, $g''(x) > 0$ on $(\frac{3\pi}{4}, \frac{7\pi}{4})$ so that the graph of g is concave upward on $(\frac{3\pi}{4}, \frac{7\pi}{4})$. Evidently, points of inflection occur at $(\frac{3\pi}{4},0)$ and $(\frac{7\pi}{4},0)$.



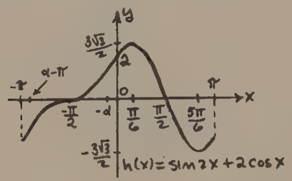
31. $f(x) = \frac{1}{3} + \frac{2}{3} \cos 2x$, $f'(x) = -\frac{4}{3} \sin 2x$, $f''(x) = -\frac{8}{3} \cos 2x$. For $0 \le x \le 2$, $f'(x) = 0$ when $2x = 0, \pi, 2\pi, 3\pi$, and 4π; that is, when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π. Since $f''(0) = f''(\pi) = f''(2\pi) = -\frac{8}{3} < 0$, it follows that $f(x)$ takes on a maximum value of 1 at $x = 0$, at $x = \pi$ and at $x = 2\pi$. Similarly, $f''(\frac{\pi}{2}) = f''(\frac{3\pi}{2}) = \frac{8}{3} > 0$, so that $f(x)$ takes on a minimum value of $-\frac{1}{3}$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Here, $f''(x) = 0$ when $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ and $\frac{7\pi}{2}$; that is, when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. Notice that $f''(x) > 0$ for x in the intervals $(\frac{\pi}{4},\frac{3\pi}{4})$ and $(\frac{5\pi}{4},\frac{7\pi}{4})$; hence, the graph of f is concave upward on these intervals. Similarly, $f''(x) < 0$ for x in the intervals $(0,\frac{\pi}{4})$, $(\frac{3\pi}{4}, \frac{5\pi}{4})$ and $(\frac{7\pi}{4}, 2\pi)$; hence, the graph of f is concave downward on these intervals. The points of inflection are $(\frac{\pi}{4},\frac{1}{3})$, $(\frac{3\pi}{4},\frac{1}{3})$, $(\frac{5\pi}{4},\frac{1}{3})$, and $(\frac{7\pi}{4},\frac{1}{3})$.



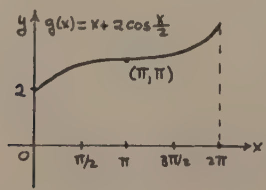
32. $h(x) = \sin 2x + 2 \cos x$, $-\pi \le x \le \pi$.

$h'(x) = 2 \cos 2x - 2 \sin x = 2(1 - 2 \sin^2 x) - 2 \sin x = 2(1 - \sin x - 2 \sin^2 x) = 2(1 - 2 \sin x)(1 + \sin x)$. Hence, $h'(x) = 0$ when $\sin x = \frac{1}{2}$ and also when $\sin x = -1$. Therefore, $h'(x) = 0$ when $x = \frac{\pi}{6}$, when $x = \frac{5\pi}{6}$, and when $x = -\frac{\pi}{2}$ for $-\pi \le x \le \pi$. $h''(x) = 2(-2 \sin 2x - \cos x) = -2(4 \sin x \cos x + \cos x) = -2 \cos x(4 \sin x + 1)$. Thus, $h''(\frac{\pi}{6}) = -3\sqrt{3} < 0$, $h''(\frac{5\pi}{6}) = 3\sqrt{3}$, and $h''(\frac{-\pi}{2}) = 0$,

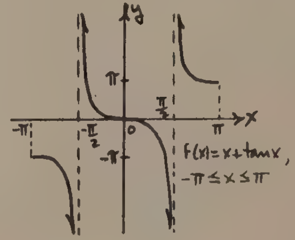
so that $h(x)$ takes on a maximum value of $\frac{3\sqrt{3}}{2}$ at $x = \frac{\pi}{6}$ and a minimum value of $-\frac{3\sqrt{3}}{2}$ at $x = \frac{5\pi}{6}$. Notice that $h''(x) = 0$ when $\cos x = 0$; that is, when $x = \frac{\pi}{2}$ and when $x = \frac{-\pi}{2}$. Also, $h''(x) = 0$ when $4 \sin x + 1 = 0$; that is, when $\sin x = -\frac{1}{4}$. Let α be the angle in the first for which $\sin \alpha = \frac{1}{4}$ (so that $\alpha \approx 0.2527$ radian). Then $\sin x = -\frac{1}{4}$ for $x = \alpha - \pi$ and again for $x = -\alpha$. Thus, $h''(x) = 0$ for $x = \frac{\pi}{2}$, $-\alpha$, $\frac{-\pi}{2}$, and $\alpha - \pi$. Since $h''(x) < 0$ for x in the intervals $(\alpha - \pi, -\frac{\pi}{2})$ and $(-\alpha, \frac{\pi}{2})$, it follows that the graph of h is concave downward in these intervals. Similarly, $h''(x) > 0$ for x in the intervals $(-\pi, \alpha - \pi)$, $(-\frac{\pi}{2}, -\alpha)$ and $(\frac{\pi}{2}, \pi)$; hence, the graph of h is concave upward on these intervals. Consequently, the graph of h has points of inflection at $(\alpha - \pi, \frac{-6\sqrt{15}}{16})$, $(-\frac{\pi}{2}, 0)$, $(-\alpha, \frac{6\sqrt{15}}{16})$, and $(\frac{\pi}{2}, 0)$.



33. $g(x) = x + 2 \cos(\frac{x}{2})$, $0 \leq x \leq 2\pi$. $g'(x) = 1 - \sin(\frac{x}{2}) = 0$ when $\sin(\frac{x}{2}) = 1$; that is, $\frac{x}{2} = \frac{\pi}{2}$ or when $x = \pi$. $g''(x) = -\frac{1}{2} \cos(\frac{x}{2})$. $g''(\pi) = 0$. $g''(x) > 0$ when $-\cos(\frac{x}{2}) > 0$, that is, when $\cos(\frac{x}{2}) < 0$; hence, when $\frac{\pi}{2} \leq \frac{x}{2} \leq \frac{3\pi}{2}$, or when $\pi \leq x \leq 2\pi$. So $g''(x) < 0$ when $0 \leq x \leq \pi$. g is concave upward on $(\pi, 2\pi)$ and concave downward on $(0, \pi)$. Hence, there is a point of inflection at (π, π). There is a minimum point at $(0,2)$ and a maximum at $(2\pi, 2\pi - 2)$.



34. $f(x) = x - \tan x$. f is an odd function, so symmetric about the origin. $f'(x) = 1 - \sec^2 x = -\tan^2 x$ and $f''(x) = -2 \tan x \sec^2 x$. Thus $f'(x) = 0$ when $\sec^2 x = 1$; that is, when $\cos x = \pm 1$. Consequently, for $-\pi \leq x \leq \pi$, $f'(x) = 0$ when $x = -\pi$, 0, and π. Since $f'(x) \leq 0$ for all values of x (other than $\frac{\pi}{2}$ and $\frac{-\pi}{2}$), then f is monotone decreasing on $[-\pi, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, and $(\frac{\pi}{2}, \pi]$. It has no maximum or minimum values on $(-\pi, \pi)$, since the graph has vert cal asymptotes at $x = \frac{\pi}{2}$ and at $x = \frac{-\pi}{2}$. Here $f''(x)$ 0 for $x = 0$, π, and $-\pi$. Also, $f''(x)$ is positive for x in $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \pi)$, so that the graph of f is concave upward in these intervals. Similarly, $f''(x)$ is negative for x in $(-\pi, -\frac{\pi}{2})$ and $(0, \frac{\pi}{2})$, so that the graph of f is concave downward in these intervals. Thus, f has an inflection point at $(0,0)$.



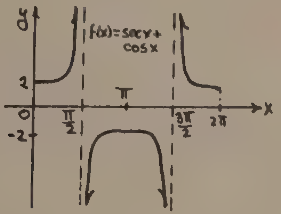
35. $f(x) = \sec x + \cos x = \frac{1}{\cos x} + \cos x$.

$f'(x) = \sec x \tan x - \sin x = \frac{\sin x}{\cos^2 x} - \sin x = \sin x(\frac{1}{\cos^2 x} - 1)$.

$f''(x) = \cos x(\frac{1}{\cos^2 x} - 1) + \sin x(\frac{2 \sin x}{\cos^3 x}) = \frac{1}{\cos^3 x}(\cos^2 x - \cos^4 x + 2 \sin^2 x) = \frac{1}{\cos^3 x}[\cos^2 x - \cos^4 x + 2(1 - \cos^2 x)] = \frac{1}{\cos^3 x}(2 - \cos^2 x - \cos^4 x) = \frac{1}{\cos^3 x}(2+\cos^2 x)(1-\cos^2 x$

$\frac{\sin^2 x}{\cos^2 x}(2 + \cos^2 x)$. Thus, $f'(x) = 0$ when $\sin x = 0$ and when $\cos x = \pm 1$; that is, for $0 \leq x \leq 2\pi$, $f'(x) = 0$ for $x = 0, \pi$, and 2π. For $\frac{\pi}{2} \leq x \leq \pi$, $f'(x) > 0$, while for $\pi < x < \frac{3\pi}{2}$, $f'(x) < 0$; hence,

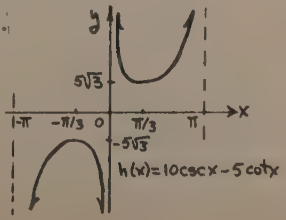
f has a relative maximum value of -2 at π. Here, $f''(x) > 0$ for x in the intervals $(0,\frac{\pi}{2})$ and $(\frac{3\pi}{2},2\pi)$; hence, the graph of f is concave upward on these intervals. Since $f''(x) \le 0$ for x in the interval $(\frac{\pi}{2},\frac{3\pi}{2})$, then the graph of f is concave downward on this interval. There are no points of inflection. The graph has vertical asymptotes at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.



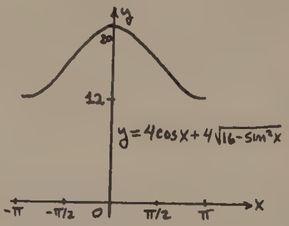
6. $h(x) = 10 \csc x - 5 \cot x = \frac{10}{\sin x} - \frac{5 \cos x}{\sin x} = \frac{5}{\sin x}(2 - \cos x)$. h is odd, so symmetric about the origin.

$h'(x) = -10 \csc x \cot x + 5 \csc^2 x = \frac{5}{\sin x}(-2 \frac{\cos x}{\sin x} + \frac{1}{\sin x}) = \frac{5}{\sin^2 x}(1 - 2 \cos x)$.

$h''(x) = \frac{-10 \cos x}{\sin^3 x}(1 - 2 \cos x) + \frac{10 \sin x}{\sin^2 x} = \frac{10}{\sin^3 x}(\cos^2 x - \cos x + 1)$. Thus, $h'(x) = 0$ when $\cos x = \frac{1}{2}$, that is, for $-\pi < x < \pi$, when $x = \frac{\pi}{3}$ and when $x = \frac{-\pi}{3}$. Since $h''(\frac{\pi}{3}) = \frac{20}{3}\sqrt{3} > 0$ and $h''(\frac{-\pi}{3}) = -\frac{20}{3}\sqrt{3} < 0$, it follows that h has a relative minimum of $5\sqrt{3}$ at $x = \frac{\pi}{3}$ and h has a relative maximum of $-5\sqrt{3}$ at $x = \frac{-\pi}{3}$. Evidently, the graph of h has vertical asymptotes at $x = 0$, $x = \pi$, and $x = -\pi$. Since $h''(x) > 0$ for x in $(0,\pi)$, it follows that the graph of h is concave upward on $(0,\pi)$. Similarly, the graph of h is concave downward on $(-\pi,0)$. There are no points of inflection.



37. For a = 8 and b = 16, $f(x) = y = 4 \cos x + 4\sqrt{16 - \sin^2 x}$, $-\pi \le x \le \pi$. $f(x) = f(-x)$, so f is even. Thus, f is symmetric about the y axis. $f'(x) = -4 \sin x(1 + \frac{\cos x}{\sqrt{16 - \sin^2 x}})$. $f'(x) = 0$ when $\sin x = 0$; that is, when $x = -\pi$, 0, and π. $f''(x) = -4 \cos x - \frac{64 - 128 \sin^2 x + 4 \sin^4 x}{(16 - \sin^2 x)^{3/2}}$. $f''(-\pi) = 3$; $f''(0) = -5$; $f''(\pi) = 3$. Relative minimum of 12 at $-\pi$ and at π; relative maximum of 20 at 0.



38. (a) $F(x) = x^{1/3}(x - 1)^{2/3} = \frac{x^{2/3}x^{1/3}(x - 1)^{2/3}}{x^{2/3}} = \frac{x(x - 1)^{2/3}}{x^{2/3}} = x(\frac{x - 1}{x})^{2/3} = x(1 - \frac{1}{x})^{2/3}$.

(b) Let $t = \frac{1}{x}$. Then $F(x) - x = \frac{1}{\Delta t}(1 - \Delta t)^{2/3} - \frac{1}{\Delta t} = \frac{(1 - \Delta t)^{2/3} - 1}{\Delta t}$. Thus, $\lim_{x\to\pm\infty} [F(x) - x] = \lim_{t\to 0} \frac{(1 - \Delta t)^{2/3} - 1}{\Delta t}$, since $t = \frac{1}{x}$.

(c) $G'(1) = \lim_{h\to 0} \frac{G(1 + h) - G(1)}{h} = \lim_{h\to 0} \frac{(1 + h)^{2/3} - 1}{h} = \frac{2}{3}$. Let $h = -\Delta t$, then $\lim_{\Delta t\to 0} \frac{(1 - \Delta t)^{2/3} - 1}{-\Delta t} = \frac{2}{3}$ or $\lim_{\Delta t\to 0} \frac{(1 - t)^{2/3} - 1}{\Delta t} = -\frac{2}{3}$.

(d) From (b) and (c), $\lim_{x} [F(x) - x] = -\frac{2}{3}$. Therefore, $\lim_{x\to\pm\infty} [F(x) - x + \frac{2}{3}] = 0$ so $y - x + \frac{2}{3} = 0$ or $y = x - \frac{2}{3}$ is an oblique asymptote.

39. $F(x) = x^{1/3}(x - 1)^{2/3} = [x(x - 1)^2]^{1/3}$.

(a) For x = 100, $F(x) - x \approx -0.6678$.

(b) For x = 1000, $F(x) - x \approx -0.6668$.

(c) For x = 1,000,000, $F(x) - x \approx -0.6680$.

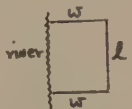
(d) For $x = 10^{10}$, $F(x) - x = -24$. The calculator

has difficulty dealing with the relatively small difference between x and x - 1 for $x = 10^{10}$.

Problem Set 3.7, page 217

1. $p = 2\ell + 2w$, $\ell w = 100$, or $\ell = \frac{100}{w}$, $p = 2(\frac{100}{w}) + 2w = \frac{200}{w} + 2w$. Thus, $\frac{dp}{dw} = 2 - \frac{200}{w^2} = 0$ gives the critical number $w = 10$. Since $\frac{dp}{dw} < 0$ for $0 < w < 10$ and $\frac{dp}{dw} > 0$ for $w > 10$, then $w = 10$ gives a minimum value of p. The required dimensions are $w = 10$ meters, $\ell = \frac{100}{w} = 10$ meters.

2. From the adjacent figure, $10{,}000 = 2w + \ell$; $\ell = 10{,}000 - 2w$; $A = \ell w = 10{,}000w - 2w^2$; $\frac{dA}{dw} = 10{,}000 - 4w$; $\frac{dA}{dw} = 0$ for $w = 2500$. Thus, $w = 2500$ meters and $\ell = 5000$ meters.



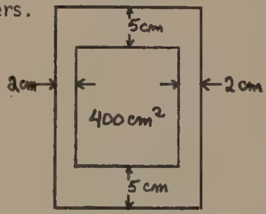
3. (a) Let the numbers be x and y. Then $x + y = 20$, so that $y = 20 - x$. The product p is given by $p = xy$, that is, $p = x(20 - x)$. Now, $\frac{dp}{dx} = x(-1) + (20 - x) = 20 - 2x$; hence, $x = 10$ gives the only critical number. Since $\frac{dp}{dx} > 0$ for $0 < x < 10$ and $\frac{dp}{dx} < 0$ for $10 < x < 20$, then $x = 10$ gives an absolute minimum value of p. When $x = 10$ then $y = 20 - x = 10$; hence, the two numbers are $x = 10$ and $y = 10$.

   (b) Now $p = 0$ if $x = 0$ or $x = 20$. Thus, if one number is 0 and the other is 20, the product is a minimum.

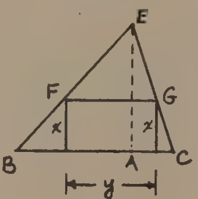
4. From the adjacent figure, $1800 = 3\ell + 2w$; $\ell = 600 - \frac{2}{3}w$; $A = \ell w = 600w - \frac{2}{3}w^2$; $\frac{dA}{dw} = 600 - \frac{4}{3}w$; $\frac{dA}{dw} = 0$ for $w = 450$. Thus, $w = 450$, $\ell = 300$.

5. From the figure below $(\ell - 10)(w - 4) = 400$; $\ell = \frac{360 + 10w}{w - 4}$; $A = \ell w = \frac{360w + 10w^2}{w - 4} = 10w + 400 + \frac{1600}{w - 4}$ thus $\frac{dA}{dw} = 10 - \frac{1600}{(w - 4)^2}$; $\frac{dA}{dw} = 0$ for $w = 4 + 4\sqrt{10}$ (since $4 - 4\sqrt{10} < 0$). $\frac{dA}{dw}$: $-$ | $+$ ($4 + 4\sqrt{10}$)

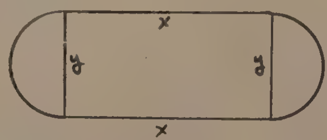
   So A takes on a minimum at $x = 4 + 4\sqrt{10}$. Thus, $w = 4 + 4\sqrt{10}$ centimeters and $\ell = 10 + 10\sqrt{10}$ centimeters.



6. In the figure below, $|\overline{AE}| = 80$ and $|\overline{BC}| = 100$. The area $A = xy$. Now $\Delta BEC \sim \Delta FEG$; therefore, $\frac{80 - x}{80} = \frac{y}{100}$ or $y = 100(1 - \frac{x}{80})$, so $A = 100(x - \frac{x^2}{80})$. $\frac{dA}{dx} = 100(1 - \frac{x}{40}) = 0$ for $x = 40$ and $y = 100(1 - \frac{40}{80}) = 50$. Thus, $A = 40(50) = 2500\ m^2$.



7. It is clear that the field house of maximum area is on the rectangular plot of land enclosed by the 2 parallel lines and the diameter of the two semi-circles. The jogging track $= x + x + \pi(\frac{y}{2}) + \pi(\frac{y}{2}) = 2x + \pi y = 1000$, and the area $A = xy = y(500 - \frac{\pi y}{2}) = 500y - \frac{\pi y^2}{2}$. $\frac{dA}{dy} = 500 - \pi y$. $\frac{dA}{dy} = 0$ when $y = \frac{500}{\pi}$, so $A = 500(\frac{500}{\pi}) = \frac{\pi(500)^2}{2 \cdot \pi^2} = (500)^2(\frac{1}{2\pi}) = \frac{125{,}000}{\pi}$ square meters.



8. If the square has side length x, then it has perimeter 4x, leaving 24 - 4x for the circumference of

the circle. Thus, the radius of the circle is $r = \frac{24 - 4x}{2}$. The total area enclosed by the square and the circle is $A = x^2 + \pi r^2 = x^2 + \pi(\frac{12 - 2x}{\pi})^2$, so $\frac{dA}{dx} = 2x - \frac{4(12 - 2x)}{\pi}$. Solving $\frac{dA}{dx} = 0$, we obtain $x = \frac{24}{\pi + 4} \approx 3.36$ as the only critical number; hence, $4x = \frac{96}{\pi + 4} \approx 13.44$ inches of wire should be used to form the square for minimum total area. Since A takes on a minimum at $x = \frac{24}{\pi + 4}$.

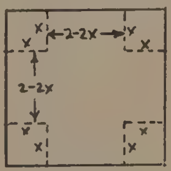
$\frac{dA}{dx}$: $-$ | $+$ (sign change at $\frac{24}{\pi + 4}$)

Now $0 \le 4x \le 24$. When $x = 0$, $A = \frac{144}{\pi} \approx 45.84$, and when $x = 6$, $A = 36$, so the maximum area results when $x = 0$.

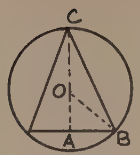
9. From the figure below, we can conclude that the sand box has the following volume: $V = x(2 - 2x)^2$. Thus, $\frac{dV}{dx} = (2 - 2x)(2 - 6x)$. $\frac{dV}{dx} = 0$ for $x = 1$ and $\frac{1}{3}$. But we must reject $x = 1$ since $x < 1$. Thus,

$\frac{dV}{dx}$: $+$ | $-$ | (marks at $1/3$ and 1)

Thus, from the chart above we can see there is a maximum at $\frac{1}{3}$. The size of the squares cut should be $\frac{1}{3}$ meter by $\frac{1}{3}$ meter.

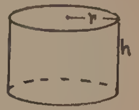


10. $|\overline{OC}| = |\overline{OB}| = a$; let $|\overline{AC}| = h$ $|\overline{AB}| = r$ $|\overline{OA}|^2 + |\overline{AB}|^2 = |\overline{OB}|^2$ by the Pythagorean Theorem, that is, $(h - a)^2 + r^2 = a^2$ or $r^2 = a^2 - (h - a)^2 = 2ha - h^2$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h(2ha - h^2) = \frac{2}{3}\pi a h^2 - \frac{1}{3}\pi h^3$, $0 \le h \le 2a$. $\frac{dV}{dh} = \frac{4}{3}\pi a h - \pi h^2 = 0$ gives $h = 0$ and $h = \frac{4}{3}a$; for $h = 0$, $V = 0$; for $h = \frac{4}{3}a$, $V = \frac{32}{81}\pi a^3$; for $h = 2a$, $V = 0$. Thus $h = \frac{4}{3}a$ yields maximum volume. When $h = \frac{4}{3}a$, $r^2 = 2ha - h^2 = \frac{8}{9}a^2$ so $r = \frac{2}{3}a\sqrt{2}$.



11. $4114\pi = \pi r^2 h$ or $h = \frac{4114}{r^2}$. The lateral surface of a cylinder is $2\pi rh$ and the area of a base is πr^2. The total surface area is given by $S = 2\pi rh + 2\pi r^2 = 2\pi r(\frac{4114}{r^2}) + 2\pi r^2 = \frac{8228}{r}\pi + 2\pi r^2$, $r > 0$. Now $\frac{dS}{dr} = -\frac{8228}{r^2} + 4r = 0$ for $r = \sqrt[3]{2057}$.

$\frac{dS}{dr}$: $-$ | $+$ (sign change at $\sqrt[3]{2057}$)

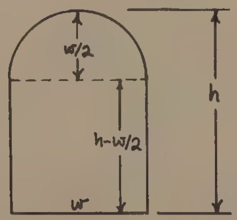


So S takes on an absolute minimum value when $r = \sqrt[3]{2057} \approx 12.72$ inches and $h = \frac{4114}{(2057)^{2/3}} \approx 25.44$ inches.

12. Perimeter $= 7 = w + 2(h - \frac{w}{2}) + \frac{1}{2}\cdot 2\pi(\frac{w}{2})$. Thus, $7 = w + 2h - w + \frac{w\pi}{2} = 2h + \frac{w}{2}$, so $h = \frac{7}{2} - \frac{w\pi}{4}$. Area $A = \frac{1}{2}\pi(\frac{w}{2})^2 + w(h - \frac{w}{2}) = \frac{1}{2}\pi\cdot\frac{w^2}{4} + w(\frac{7}{2} - \frac{w\pi}{4}) - \frac{w^2}{2} = \frac{7w}{2} - \frac{w^2}{2} - \frac{w^2}{8}$. $\frac{dA}{dw} = \frac{7}{2} - w - \frac{w}{4} = 0$ for $w = \frac{14}{4 + \pi}$.

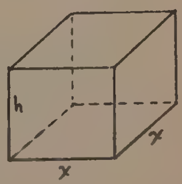
$\frac{dA}{dw}$: $+$ | $-$ (sign change at $14/4 + \pi$)

So A takes on a maximum at $\frac{14}{4 + \pi} = w$. Now $h = \frac{7}{2} - \frac{\pi}{4}(\frac{14}{4 + \pi}) = \frac{7}{2} - \frac{7}{2(4 + \pi)} = \frac{7(4 + \pi) - 7\pi}{2(4 + \pi)} = \frac{14}{4 + \pi}$.



13. $1 = x^2 h$ or $h = \frac{1}{x^2}$.

(a) Lateral surface (open top) $L = x^2 + 4xh = x^2 + 4x(\frac{1}{x^2}) = x^2 + \frac{4}{x}$. $\frac{dL}{dx} = 2x - \frac{1}{x^2} = 0$ for $x = \sqrt[3]{2}$.



$\frac{dL}{dx}$: $-$ on the left, $+$ on the right of $\sqrt[3]{2}$

So L takes on a minimum at $x = \sqrt[3]{2}$ meters. $h = \frac{1}{\sqrt[3]{4}}$ meter.

(b) Lateral surface (closed box)

$L = 2x^2 + 4xh = 2x^2 + \frac{4}{x}$. $\frac{dL}{dx} = 4x - \frac{4}{x^2} = 0$ for $x = 1$.

$\frac{dL}{dx}$: $-$ on the left, $\pm$ on the right of 1

So L takes on a minimum at $x = 1$ meter. $h = 1$ meter.

14. (a) Cost $= ax^2 + 4bxh = ax^2 + \frac{4b}{x}$, where $\frac{a}{b} = c$.

$\frac{dC}{dx} = 2ax - \frac{4b}{x^2} = 0$ for $x = \sqrt[3]{\frac{2b}{a}} = \sqrt[3]{\frac{2}{c}}$. $h = \sqrt[3]{\frac{c^2}{4}}$.

(b) Cost $C = 2ax^2 + 4bxh = 2ax^2 + \frac{4b}{x}$, where $\frac{a}{b} = c$.

$\frac{dC}{dx} = 4ax - \frac{4b}{x^2} = 0$ for $x = \sqrt[3]{\frac{b}{a}} = \sqrt[3]{\frac{1}{c}}$. $h = \frac{1}{\sqrt[3]{(\frac{1}{c})^2}} = \sqrt[3]{c^2}$.

15. Fixed volume $V = \frac{4}{3}\pi r^3 + \pi r^2$ or $\ell = \frac{V - (4/3)\pi r^3}{\pi r^2}$.

Surface area $S = 4\pi r^2 + 2\pi r\ell = 4\pi r^2 + 2\pi r(\frac{V - (4/3)\pi r^3}{\pi r^2}) = 4\pi r^2 + 2(\frac{V - (4/3)\pi r^3}{r}) = 4\pi r^2 + \frac{2V}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{2V}{r}$.

$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{2V}{r^2} = 0$ for $r = \sqrt[3]{\frac{3V}{4\pi}}$, where $r > 0$.

$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{4V}{r^3} > 0$. Therefore, $r = \sqrt[3]{\frac{3V}{4\pi}}$ is a minimum. $\ell = \frac{V - \frac{4}{3}\pi(\frac{3V}{4\pi})}{\pi\sqrt[3]{\frac{9V^2}{16\pi^2}}} = 0$, so the bacterium is a sphere (endpoint extremum).

16. $\Delta ODC \sim \Delta BDA$; hence,

$\frac{|\overline{OC}|}{|\overline{AB}|} = \frac{|\overline{OD}|}{|\overline{BD}|}$

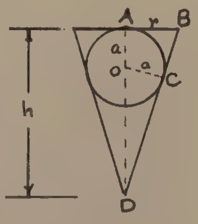
or $\frac{a}{r} = \frac{h - a}{\sqrt{h^2 + r^2}}$

or $a\sqrt{h^2 + r^2} = rh - ra$

or $a^2(h^2 + r^2) = (rh - ra)^2$

or $(a^2 - r^2)h^2 + 2r^2ah = 0$.

Since $h > 0$, we get $(a^2 - r^2)h + 2ar^2 = 0$ or



$h = \frac{2ar^2}{r^2 - a^2}$. Now $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(\frac{2ar^2}{r^2 - a^2}) = \frac{2\pi a}{3}\,\frac{r^4}{r^2 - a^2}$. $\frac{dV}{dr} = \frac{4\,ar^3}{3}\,(\frac{r^2 - 2a^2}{(r^2 - a^2)^2}) = 0$ for $r = 0$ and $a\sqrt{2}$.

$\frac{dV}{dr}$: $-$ on the left, $+$ on the right of $a\sqrt{2}$

So V takes on a minimum at $r = a\sqrt{2}$. $h = \frac{2a(2a^2)}{2a^2 - a^2} = 4a$.

17. Let $x = |\overline{QP}|$ so that $|\overline{AP}| = \sqrt{|\overline{AQ}|^2 + |\overline{QP}|^2} = \sqrt{1 + x^2}$ and $|\overline{PT}| = 3 - x$, $0 \le x \le 3$. The cost C of the cable is given by $C = 1000(15|\overline{AP}| + 9|\overline{PT}|) = 1000[15\sqrt{1 + x^2} + 9(3 - x)]$. Thus, $\frac{dC}{dx} = 1000\left[\frac{15x}{\sqrt{1 + x^2}} - 9\right]$. $\frac{dC}{dx} = 0$ when $\frac{15x}{\sqrt{1 + x^2}} = 9$, that is, $225x^2 = 81(1 + x^2)$ or $144x^2 = 81$. Thus, the only critical number on $(0,3)$ is $x = \frac{9}{12} = \frac{3}{4}$. When $x = 0$, $C = \$42{,}000$; when $x = \frac{3}{4}$, $C = \$39{,}000$; when $x = 3$, $C \approx \$15{,}000\sqrt{10} > \$45{,}000$. Thus, $x = \frac{3}{4}$ gives the desired absolute minimum. The required distance from P to T is $\frac{9}{4}$ km.

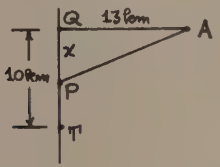
18. Let the cost of laying cable underwater be \$k per meter. Then, proceeding as in Problem 17, we have $C = 1000\left[k\sqrt{1 + x^2} + 9(3 - x)\right]$ and $\frac{dC}{dx} = 1000(\frac{kx}{\sqrt{1 + x^2}} - 9)$. Now $\frac{dC}{dx} = 0$ when $kx = 9\sqrt{1 + x^2}$ or $k^2x^2 = 81 + 81x^2$; that is, $(k^2 - 81)x^2 = 81$. Since $0 \le x \le 3$ and $k \ge 15 > 9$, the only critical number is $x = \frac{9}{\sqrt{k^2 - 81}}$. When $x = 0$, $C = \$1000(k + 27)$; when $x = \frac{9}{\sqrt{k^2 - 81}}$, $C = \$1000\left[k\sqrt{1 + \frac{81}{k^2 - 81}} + 9(3 - \frac{9}{\sqrt{k^2 - 81}})\right] = \$1000(\sqrt{k^2 - 81} + 27)$. When $x = 3$, $C = \$1000(\sqrt{10}k)$. Now for $k \ge 15$, $\$1000(\sqrt{k^2 - 81} + 27) < \$1000(k + 27) < \$1000(\sqrt{10}\,k)$; hence, no matter how large k might be it is (theoretically) not most economical to run the cable straight under water.

19. Let $|\overline{QP}| = x$, then $|\overline{AP}| = 169 + x^2$ and $|\overline{PT}| = 10 - x$.
$C = 90{,}000\sqrt{169 + x^2} + 60{,}000(10 - x)$, $0 \le x \le 10$, and $\frac{dC}{dx} = \frac{90{,}000x}{\sqrt{169 + x^2}} - 60{,}000$. $\frac{dC}{dx} = 0$ when

$\frac{90{,}000x}{\sqrt{169 + x^2}} = 60{,}000$ or $3x = 2\sqrt{169 + x^2}$; that is,

$5x^2 = 676$ so $x = \pm \frac{26}{\sqrt{5}} \approx 11.6$.

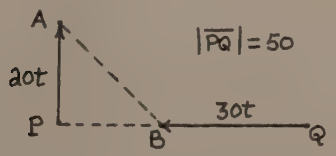
Neither value of x is in the open interval $(0,10)$, so there are no critical points. Hence, there must be an endpoint extremum. When $x = 0$, $C = \$1{,}770{,}000$. When $x = 10$, $C = 90{,}000\sqrt{269} \approx \$1{,}476{,}109.75$. Thus, minimum cost is attained when $x = 10$. So the distance from P to T is 0 kilometers.



20. In the diagram below, P is the original position of ship A and Q is the original position of ship B. After t hours, A will have sailed $20t$ nautical miles due north, while B will have sailed $30t$ nautical miles due west. The distance y between the ships at time t is

$y = \sqrt{(20t)^2 + (50 - 30t)^2} = \sqrt{1300t^2 - 3000t + 2500}$.

To minimize y it will suffice to minimize the quantity $q = 1300t^2 - 3000t + 2500$. Since $\frac{dq}{dt} = 2600t - 3000$, the critical value is $t = \frac{15}{13}$ hours. The corresponding minimum distance is $y = \frac{100}{13} \approx 27.74$ nautical miles.



21. $P = I^2R = \frac{E^2}{(R + r)^2}R$, so $\frac{dP}{dR} = E^2\frac{r - R}{(r + R)^3}$, and it follows that $R = r$ is the only critical value.

Since $\frac{dP}{dR} > 0$ for $R < r$ and $\frac{dP}{dR} < 0$ for $R > r$, it follows that P is maximum when $R = r$.

22. Ignore temporarily that $|\overline{BC}| = 7$. Let $|\overline{DB}| = x$, so the total energy $E = w\sqrt{x^2 + 25} + \ell(13 - x)$. Now $\frac{dE}{dx} = \frac{wx}{\sqrt{x^2 + 25}} - \ell$; so $\frac{dE}{dx} = 0$ when $\frac{wx}{\sqrt{x^2 + 25}} = \ell$, or $\frac{w}{\ell} = \frac{\sqrt{x^2 + 25}}{x}$. We are given that $x + 7 = 13$, so $x = 6$ and $\frac{w}{\ell} = \frac{\sqrt{61}}{6}$.

23. Fixed volume $V = r^2h + \frac{2}{3}r^3$. Let a be the cost of materials for the cylinder. Then $2a$ is cost of the materials for the hemisphere. $C = 2\pi rha + 4\pi r^2a$.

Now $h = \frac{V - (2/3)\pi r^3}{\pi r^2}$, so

$C = 2\pi ra\left(\frac{V - (2/3)\pi r^3}{\pi r^2}\right) + 4\pi r^2a =$

$2a\left(\frac{V}{r} - \frac{2}{3}\pi r^2\right) + 4\pi r^2a$. $\frac{dC}{dr} = 2a\left(-\frac{V}{r^2} - \frac{4}{3}\pi r\right) + 8\pi ra$.

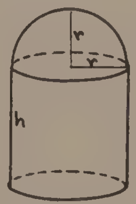
Now $\frac{dC}{dr} = 0$ for $r^3 = \frac{V}{(-4/3)\pi + 4\pi} = \frac{3V}{8\pi}$, so

$r = \sqrt[3]{\frac{3V}{8\pi}}$. $\frac{d^2C}{dr^2} > 0$ for $r = \sqrt[3]{\frac{3V}{8\pi}}$ so this value of r gives a minimum. Thus, $h = \frac{V - (2/3)\pi(3V/8\pi)}{\pi\sqrt[3]{\frac{9V^2}{64\pi^2}}} =$

$\frac{\frac{3}{4}V}{\frac{\pi}{4}\sqrt[3]{\frac{9V^2}{\pi^2}}} = \frac{3V}{\pi\sqrt[3]{\frac{9V^2}{\pi^2}}}$. Now $\frac{r + h}{r} = 1 + \frac{h}{r} = 1 +$

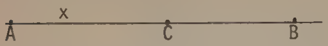
$3V\sqrt[3]{\pi^3 9V^2/\pi^2}/\sqrt[3]{3V/8\pi} = 1 + 3V/\pi\sqrt[3]{27V^3/8\pi^3} =$

$1 + 2 = 3$.



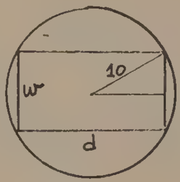
24. Concentration $C = \frac{kr}{x^2} + \frac{ks}{(12 - x)^2}$, where r is rate at which plant A discharges matter and s is rate at which plant B discharges matter. Since $r = 8s$, $C = 8ksx^{-2} + ks(12 - x)^{-2}$. $\frac{dC}{dx} = -2(8ks)x^{-3} + ks(-2)(12 - x)^{-3}(-1)$. Now $\frac{dC}{dx} = 0$ yields $8 = \frac{x^3}{(12 - x)^3}$; that is, $2 = \frac{x}{12 - x}$ or $x = 8$

kilometers.



25. $R = kx(N - x) = k(xN - x^2)$. $\frac{dR}{dx} = k(N - 2x)$. $\frac{dR}{dx} = 0$ for $x = \frac{N}{2}$. Thus, $\frac{x}{N} = \frac{N/2}{N} = \frac{1}{2}$.

26. $(\frac{w}{2})^2 + (\frac{d}{2})^2 = 100$ or $w^2 + d^2 = 400$. $S = kwd^2 = kw(400 - w^2) = k(400w - w^3)$. $\frac{dS}{dw} = k(400 - 3w^2)$. $\frac{dS}{dw} = 0$ for $w = \frac{20}{\sqrt{3}}$. $d^2 = 400 - \frac{400}{3} = 400 \cdot \frac{2}{3}$, and so $d = 20\sqrt{\frac{2}{3}}$ centimeters.



27. The distance from A to B (by the Pythagorean theorem) is $a^2 + x^2$, so the time required to go from A to B at speed c is $\frac{\sqrt{a^2 + x^2}}{c}$. Similarly, the time required to go from B to W at speed v is $\frac{\sqrt{b^2 + (k - x)^2}}{v}$. Thus, the total time required to go from A to W is $T = \frac{\sqrt{a^2 + x^2}}{c} + \frac{\sqrt{b^2 + (k - x)^2}}{v}$.

(b) $\frac{dT}{dx} = \frac{x}{c\sqrt{a^2 + x^2}} - \frac{k - x}{v\sqrt{b^2 + (k - x)^2}}$, so the critical value of x satisfies

$(\frac{x}{\sqrt{a^2 + x^2}}) \quad (\frac{k - x}{\sqrt{b^2 + (k - x)^2}}) = \frac{c}{v}$; that is,

$\frac{\sin \alpha}{\sin \beta} = \frac{c}{v}$.

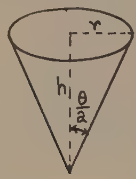
28. $\sin \frac{\theta}{2} = \frac{r}{8}$ or $r = 8 \sin \frac{\theta}{2}$; $\cos \frac{\theta}{2} = \frac{h}{8}$ or $h = 8 \cos \frac{\theta}{2}$.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(64 \sin^2 \frac{\theta}{2})(8 \cos \frac{\theta}{2}) = \frac{512\pi}{3} \sin^2 \frac{\theta}{2} \cos \frac{\theta}{2}$.

$\frac{dV}{d\theta} = \frac{512\pi}{3}[\sin^2 \frac{\theta}{2} (-\frac{1}{2} \sin \frac{\theta}{2}) + \cos \frac{\theta}{2}(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2})(\frac{1}{2})]$. $\frac{dV}{d\theta} = 0$ yields $-\frac{1}{2} \sin^3 \frac{\theta}{2} + \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} = 0$.

$0 = \sin\frac{\theta}{2}[-\frac{1}{2}\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}] = \sin \frac{\theta}{2} [-\frac{1}{2} \sin^2 \frac{\theta}{2} + 1 - \sin^2 \frac{\theta}{2}]$ for

$\sin \frac{\theta}{2} = 0$	$\sin^2 \frac{\theta}{2} = \frac{2}{3}$ (Ignore negative values.)
$\frac{\theta}{2} = 0, \pi, 2\pi, \ldots$	$\sin \frac{\theta}{2} = \frac{2}{3}$
$\theta = 0, 2\pi, 4\pi, \ldots$	$\frac{\theta}{2} = \sin^{-1} \frac{2}{3}$
(Reject - no cone.)	$\theta = 2 \sin^{-1} \frac{2}{3} \approx 109.47°$



29. $\sin \theta = \frac{a}{x}$ or $x = \frac{a}{\sin \theta} = a \csc \theta$;

$\cos \theta = \frac{b}{y}$ or $y = \frac{b}{\cos \theta} = b \sec \theta$.

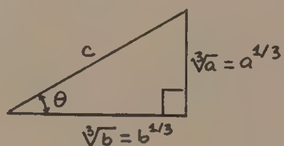
(b) $S = x + y = \frac{a}{\sin \theta} + \frac{b}{\cos \theta} = a \csc \theta + b \sec \theta$.

$\frac{dS}{d\theta} = a(-\csc \theta \cot \theta) + b(\sec \theta \tan \theta) = \frac{-a}{\tan \theta \sin \theta} + \frac{b \tan \theta}{\cos \theta}$.

$\frac{dS}{d\theta} = 0$ yields $a = b \tan^3\theta$ or $\tan \theta = \sqrt[3]{\frac{a}{b}}$.

(c) $c = \sqrt{a^{2/3} + b^{2/3}}$.

$S = x + y = a \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} + b \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}} = a^{2/3}(a^{2/3} + b^{2/3})^{1/2} + b^{2/3}(a^{2/3} + b^{2/3})^{1/2} = (a^{2/3} + b^{2/3})^{1/2}(a^{2/3} + b^{2/3}) = (a^{2/3} + b^{2/3})^{3/2}$.



30. Using Problem 29 with a = 1 and b = 1.5, we have $\ell = [1^{2/3} + (1.5)^{2/3}]^{3/2} = [1 + (1.5)^{2/3}]^{3/2} \approx 3.51174119$ meters.

31. Using Problem 29 with a = 3 and b = 8, we have $\ell = (3^{2/3} + 8^{2/3})^{3/2} \approx 14.99$ meters.

32. Let $h = |\overline{BE}|$, $b = |\overline{BF}|$, and $c = \csc \theta - 1$, noting that $c > 0$. From ΔADO, we have $\frac{|\overline{OD}|}{|\overline{AO}|} = \sin \theta$, so $|\overline{AO}| = |\overline{OD}| \csc \theta = r(c + 1)$. But $|\overline{AO}| = |\overline{AB}| - |\overline{OB}| = |\overline{AB}| - [|\overline{BE}| - |\overline{OE}|] = a - (h - r) =$

$a + r - h$, and so $a - r - h = r(c + 1)$. It follows that $h = a - rc$. From ΔOBF, $|\overline{OB}|^2 + |\overline{BF}|^2 = |\overline{OF}|^2$; that is, $|\overline{OB}|^2 + b^2 = r^2$. Since $|\overline{OB}| = |\overline{BE}| - |\overline{OE}| = h - r$, it follows that $(h - r)^2 + b^2 = r^2$, and therefore $b^2 = 2rh - h^2$. Now, by formula 4e on page 1015, the volume of water displaced by the sphere is given by $V = \frac{1}{6}\pi h(3b^2 + h^2) =$

$\frac{1}{6}\pi h[3(2rh - h^2) + h^2] = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi(3rh^2 - h^3)$.

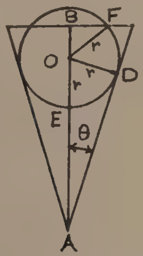
Note that, since $h = a - rc$, $\frac{dh}{dr} = -c < 0$. Thus,

$\frac{dV}{dr} = \frac{1}{3}\pi(3h^2 + 6rh\frac{dh}{dr} - 3h^2\frac{dh}{dr}) =$

$\frac{1}{3}\pi(3h^2 - 6rhc + 3h^2c) = \pi h(h - 2rc + hc) =$

$\pi h[(a - rc) - 2rc + (a - rc)c] =$

$\pi hc(c + 3)[\frac{a(c + 1)}{c(c + 3)} - r]$. Thus, a critical value for V is given by $r = r_c = \frac{a(c + 1)}{c(c + 3)}$. Note that $\frac{dV}{dr} > 0$ for $r < r_c$ and $\frac{dV}{dr} < 0$ for $r > r_c$; hence, V is maximum when $r = r_c$.



$|\overline{AB}| = a$

$|\overline{BE}| = h$

$|\overline{BF}| = b$

33. $\sin\theta = \frac{h}{8}$ or $h = 8\sin\theta$; $\cos\theta = \frac{k}{8}$ of $k = 8\cos\theta$. Thus, the larger base equals $8 + 2k = 8 + 16\cos\theta$. Hence, the area A of the trapezoid equals

$\frac{h}{2}(b + b') = \frac{8\sin\theta}{2}(8 + 8 + 16\cos\theta) =$

$4\sin\theta(16 + 16\cos\theta) = 64\sin\theta(1 + \cos\theta)$.

$\frac{dA}{d\theta} = 64[\sin\theta(-\sin\theta) + (1 + \cos\theta)\cos\theta] =$

$64[-\sin^2\theta + \cos\theta + \cos^2\theta] =$

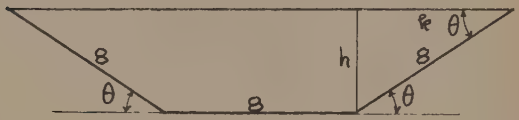
$64[2\cos^2\theta + \cos\theta - 1]$. $\frac{dA}{d\theta} = 0$ when

$(2\cos\theta - 1)(\cos\theta + 1) = 0$; that is, when

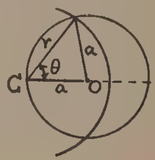
$2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$.

$2\cos\theta = 1$	$\cos\theta = -1$
$\cos\theta = \frac{1}{2}$	$\theta = \pi$ (Reject - no trapezoid.)
$\theta = \frac{\pi}{3}$	

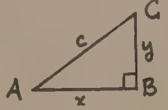
Therefore, $\theta- \frac{\pi}{3}$ produces a trapezoid of maximum area.



34. Using the law of cosines $a^2 = a^2 + r^2 - 2ar\cos\theta$ or $r^2 = 2ar\cos\theta$. Thus, $r = 2a\cos\theta$ since $r \neq 0$. Now $\ell = 2\theta r = 2\theta(2a\cos\theta) = 4a\,\theta\cos\theta$ so $\frac{d\ell}{d\theta} = 4a(-\theta\sin\theta + \cos\theta)$. $\frac{d\ell}{d\theta} = 0$ when $\theta\sin\theta = \cos\theta$ or $\theta = \frac{\cos\theta}{\sin\theta} = \cot\theta$.



35. Let $|\overline{AB}| = x$ and $|\overline{BC}| = y$, so that $A = xy$, where $x^2 + y^2 = c^2$. Maximizing A^2 will also maximize A. $A^2 = x^2y^2 = x^2(c^2 - x^2) = c^2x^2 - x^4$. $\frac{dA^2}{dx} = 2c^2x - 4x^3$. $\frac{dA^2}{dx} = 0$ yields $c^2 = 2x^2$ or $x = \frac{c}{\sqrt{2}}$; $y^2 = c^2 - \frac{c^2}{2} = \frac{c^2}{2}$ so $y = \frac{c}{\sqrt{2}}$. The triangle is isosceles with legs of length $\frac{c}{\sqrt{2}}$.



36. Example 3: $\frac{dc}{dr} = -\frac{2kV}{r^2} + 10k\quad r, r > 0$. Now $\frac{d^2c}{dr^2} = \frac{4kV}{r^3} + 10k > 0$, so $r = \sqrt[3]{\frac{V}{5\pi}}$ gives a minimum value. Since $\frac{d^2c}{dr^2}$ always > 0, the graph is concave upward; thus, this value is an absolute minimum.

Example 4: $\frac{dQ}{dr} = 4\pi^2r^3 - \frac{2(3000)^2}{r^3}$. Now $\frac{d^2Q}{dr^2} = 12\pi^2r^2 + \frac{6(3000)^2}{r^4} > 0$; $r = 8.77$ gives a minimum. Since $\frac{d^2Q}{dr^2}$ is always > 0, the graph is concave upward; thus, this value is an absolute minimum.

37. $I = \frac{c\sin\alpha}{r^2} = \frac{c(x/r)}{r^2} = \frac{cx}{r^3} = \frac{cx}{(x^2 + 900)^{3/2}}$ since $r = \sqrt{x^2 + (30)^2}$. Thus, $\frac{dI}{dx} = \frac{900c - 2cx^2}{(x^2 + 900)^{5/2}}$ so

that $x = 15\sqrt{2}$ gives the desired critical value. Thus, the height of the pole should be $x = 15\sqrt{2} \approx 21.21$ meters.

38. Let c be the velocity with which light travels. Then the total time of transit from $(p,0)$ to (x,y) to $(0,q)$ is given by $T = \frac{1}{c}(\sqrt{x^2 + (y - q)^2} + \sqrt{(x - p)^2 + y^2})$. Thus, $\frac{dT}{dx} = \frac{1}{c}\left[\frac{x + (y - q)\frac{dy}{dx}}{\sqrt{x^2 + (y - q)^2}} + \frac{(x - p) + y\frac{dy}{dx}}{\sqrt{(x - p)^2 + y^2}}\right]$. Now from the law of cosines, $\cos(180 - \beta) = -\cos\beta = \frac{1 + x^2 + (y - q)^2 - q^2}{2\sqrt{x^2 + (y - q)^2}} = \frac{1 - yq}{x^2 + (y - q)^2}$. Similarly, $\cos(180 - \alpha) = -\cos\alpha = \frac{1 - xp}{(x - p)^2 + y^2}$. Now if $\frac{dT}{dx} = 0$, we have $\frac{x + (y - q)(dy/dx)}{\sqrt{x^2 + (y - q)^2}} + \frac{(x - p) + y\frac{dy}{dx}}{\sqrt{(x - p)^2 + y^2}} = 0$. But $\frac{dy}{dx} = -\frac{x}{y}$ since $x^2 + y^2 = 1$. Thus, $2x + 2y\frac{dy}{dx} = 0$. Substituting, we get $\frac{qx}{\sqrt{x^2 + (y - q)^2}} = \frac{py}{\sqrt{(x - p)^2 + y^2}}$. Now $\sin^2\beta = 1 - \cos^2\beta = 1 - \frac{(1 - yq)^2}{x^2 + (y - q)^2} = \frac{q^2x^2}{x^2 + (y - q)^2}$, so $\sin\beta = \frac{qx}{\sqrt{x^2 + (y - q)^2}}$. Similarly, $\sin\alpha = \frac{py}{\sqrt{(x - p)^2 + y^2}}$. Thus, $\sin\beta = \sin\alpha$ or $\beta = \alpha$.

Problem Set 3.8, page 225

1. $A = r^2$; $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. When $r = 5$, $\frac{dr}{dt} = 0.85$. Then $\frac{dA}{dt} = 2\pi(5)(0.85) = 8.5\pi \approx 26.7$ m^2/sec.

2. $A = r^2$; $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. When $r = 2$, $\frac{dA}{dt} = -3$ cm^2/hr. So $-3 = 2\pi(2)\frac{dr}{dt}$ or $\frac{dr}{dt} = \frac{-3}{4\pi} \approx -0.2387$ cm/hr.

3. $A = \pi r^2$; $\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi r(0.02) = (0.04)\pi r$. When $r = 4$, $\frac{dA}{dt} = (0.16)\pi \approx 0.5027$ cm^2/sec.

4. $A = x^2$; $\frac{dA}{dt} = 2x\frac{dx}{dt}$. When $x = 3$, $\frac{dA}{dt} = 2(3)(2) = 12$ m^2/sec. $P = 4x$, so $\frac{dP}{dt} = 4(2) = 8$ m/sec.

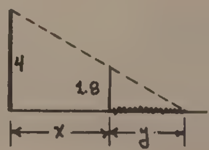
5. (a) $V = x^3$; $\frac{dV}{dt} = 3x^2\frac{dx}{dt} = 3(20)^2(-10) = -12{,}000$ cm^3/min. V is decreasing by 12,000 cm^3/min
(b) $S = 6x^2$; $\frac{dS}{dt} = 12x\frac{dx}{dt} = 12(20)(-10) = -2400$ cm^2/min.; S is decreasing by 2400 cm^2/min.

6. $V = x^3$; $\frac{dV}{dt} = 3x^2\frac{dx}{dt} = 3x^2(0.1) = (0.3)x^2$. Thus, at the instant when $x = 10$, $\frac{dV}{dt} = (0.3)(10)^2 = 30$ in.^{3}/sec. No. The volume at the instant when $x = 10$ is 1000 in.^{3}. One second later, the volume is $(10.1)^3 = 1030.301$ in.^{3}; hence, during this second, the volume has increased by slightly more than 30 in.^{3}. Recall that $\frac{dV}{dt}$ gives the <u>instantaneous</u> rate of change of V.

7. $A = \frac{1}{2}bh$; $\frac{dA}{dt} = \frac{1}{2}(b\frac{dh}{dt} + h\frac{db}{dt})$. Now $\frac{db}{dt} = 6$ and $\frac{dh}{dt} = 2$, so $b = 8$ and $h = 10$. $\frac{dA}{dt} = \frac{1}{2}[(8)(2) + (10)(6)] = 8 + 30 = 38$ m^2/min.

8. $\frac{dA}{dt} = -9$ cm^2/sec. $\frac{d\ell}{dt} = 2\frac{dw}{dt}$ and $A = \ell w$, so $\frac{dA}{dt} = \ell\frac{dw}{dt} + w\frac{d\ell}{dt}$. Thus, at a certain instant, $-9 = 1\frac{dw}{dt} + 1(2\frac{dw}{dt}) = 3\frac{dw}{dt}$ or $\frac{dw}{dt} = -3$ cm/sec. The width is decreasing by 3 cm/sec.

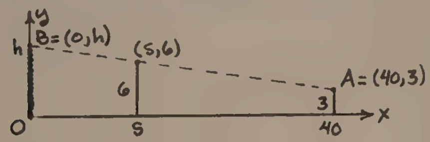
9. In the adjacent figure y denotes the length of the shadow and $x + y$ denotes the distance of one tip of the shadow from the lamppost.



By similar triangles, $\frac{x + y}{4} = \frac{y}{1.8}$. So $y = \frac{9}{11}x$. Since $\frac{dx}{dt} = -2$, then
(a) $\frac{dy}{dt} = \frac{9}{11}\frac{dx}{dt} = \frac{9}{11}(-2) = \frac{-18}{11} \approx -1.64$ m/sec.; shadow is shortening at $\frac{18}{11}$ m/sec.

(b) $\frac{d}{dt}(x + y) = \frac{dx}{dt} + \frac{dy}{dt} = -2 + \frac{9}{11}(-2) = \frac{-40}{11} \approx -3.64$ m/sec. The tip is moving at $\frac{40}{11}$ m/sec.

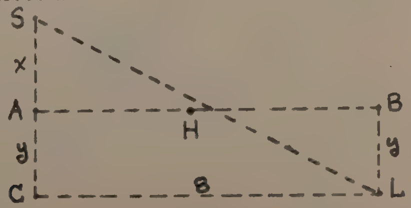
10. For convenience, locate the person at the point $(s,0)$ on the x axis, the wall along the y axis, and the spotlight at the point $(40,3)$ as in the figure below. We use the points $(0,h)$, $(40,3)$ and $(s,6)$ on the straight line through A and B to determine that $(40 - s)h = 240 - 3s$. Thus, $40\frac{dh}{dt} - s\frac{dh}{dt} - h\frac{ds}{dt} = -3\frac{ds}{dt}$. Since $\frac{ds}{dt} = -4$, then $\frac{dh}{dt} = \frac{h - 3}{40 - s}\frac{ds}{dt} = -4\frac{h - 3}{40 - s}$. When $s = 20$, $h = 9$ and $\frac{dh}{dt} = -\frac{6}{5}$ feet per second. The negative sign indicates that the shadow is shortening.



11. In the figure below, H represents the lighthouse, A the position of the ship at 2:00p.m., S the position of the ship t hours later, B the position of the launch at 2:00p.m., L the position of the launch t hours later. We let $x = |\overline{AS}|$, $y = |\overline{BL}|$. Here $|\overline{AH}| = 4$ $|\overline{HB}| = 4$, so $|\overline{AB}| = |\overline{CL}| = 8$. Also $|\overline{SC}| = x + y$. The Pythagorean theorem applied to right triangle SCL gives $|\overline{SL}|^2 = |\overline{SC}|^2 + |\overline{CL}|^2$, $|\overline{SL}| = \sqrt{(x + y)^2 + 64}$,

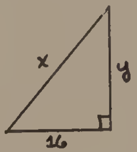
$$\frac{d|\overline{SL}|}{dt} = \frac{2(x + y)}{2\sqrt{(x + y)^2 + 64}}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{x + y}{\sqrt{(x + y)^2 + 64}}(20 + 10) = \frac{30(x + y)}{\sqrt{(x + y)^2 + 64}}.$$

At 3:30 p.m., $x = 30$ and $y = 15$, so $\frac{d(\overline{SL})}{dt} = \frac{(30)(45)}{\sqrt{45^2 + 64}} \approx 29.54$ knots.



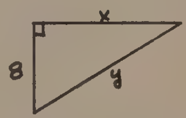
12. Let s denote the distance between the particle and point $(0,b)$ where the line $y = mx + b$ cuts the y axis. Then $|S| = \sqrt{(x - 0)^2 + (y - b)^2} = \sqrt{x^2 + (mx)^2} = \sqrt{1 + m^2}\,|x|$. Recall that $\frac{d}{dt}|u| = \frac{d}{dt}\sqrt{u^2} = \frac{2u}{2\sqrt{u^2}}\frac{du}{dt} = \frac{u}{|u|}\frac{du}{dt}$ for $|u| \neq 0$. Thus, taking the derivative with respect to t on both sides of the last equation, we find that $\frac{s}{|s|}\frac{ds}{dt} = \sqrt{1 + m^2}\frac{x}{|x|}\frac{dx}{dt}$ for $x \neq 0$. Taking absolute values on both sides of the last equation, we obtain speed $= \left|\frac{ds}{dt}\right| = \sqrt{1 + m^2}\left|\frac{dx}{dt}\right|$.

13. $\frac{dy}{dt} = 3$; $16^2 + y^2 = x^2$; $2y\frac{dy}{dt} = 2x\frac{dx}{dt}$. When $y = 14$, $x = \sqrt{452}$, so $\frac{dx}{dt} = \frac{14(3)}{452} = \frac{21}{113} \approx 1.976$ km/sec.

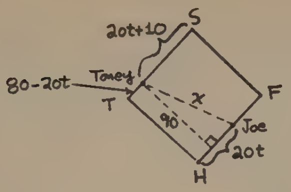


14. $\frac{dy}{dt} = 1$; $x^2 + 9 = y^2$; $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$. When $y = 12$, $x = 135$, so $\frac{dx}{dt} = \frac{12(1)}{\sqrt{135}} \approx 1.0328$ m/sec.

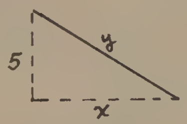
15. $x^2 + 64 = y^2$; $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$; $\frac{dx}{dt} = 16$. When $x = 24$, $y = \sqrt{640}$, so $\frac{dy}{dt} = \frac{24(16)}{8\sqrt{10}} = \frac{48\sqrt{10}}{10} = \frac{24\sqrt{10}}{5} \approx 15.2$ km/hr.



16. Let x denote the distance between Joe and Tony. Consideration of the right triangle in the figure below gives $x = \sqrt{90^2 + (40t - 80)^2}$, $\frac{dx}{dt} = \frac{40(40t - 80)}{\sqrt{90^2 + (40t - 80)^2}}$. When Tony reaches third base, $t = 4$ seconds and $\frac{dx}{dt} = \frac{(40)(80)}{\sqrt{90^2 + 80^2}} \approx 26.57$ ft/sec.

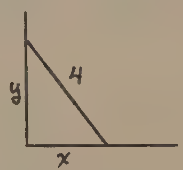


17. In the figure below, y is the length of the rope between the person's hands and the bow of the boat, while x is the distance from the boat to the dock. Here, $x^2 + 25 = y^2$; $2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$; $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} = 72\frac{y}{x}$. When $y = 13$, $x = 12$ and $\frac{dx}{dt} = 72\frac{13}{12} = 78$ ft/min.



18. $V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$; $\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{-0.17}{4\pi r^2}$. When $V = 0.4$, $r = \sqrt[3]{\frac{3(0.4)}{4\pi}}$. $\frac{dr}{dt} = \frac{-0.17}{4\left(\frac{3(0.4)}{4\pi}\right)^{2/3}} \approx$ -0.06 m/min. The radius is decreasing at the rate of 0.06 m/min.

19. In the figure below, $x^2 + y^2 = 16$. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$; $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -(0.7)\frac{x}{y}$. When $y = 2$, $x = \sqrt{12}$ and $\frac{dy}{dt} = -(0.7)\frac{\sqrt{12}}{2} = -(0.7)\sqrt{3} \approx -1.21$ m/sec. It's sliding down at the rate of 1.21 m/second.

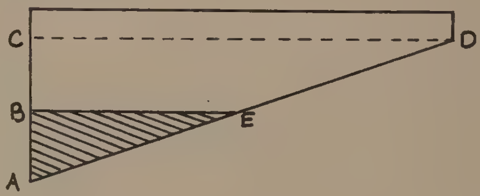


20. (a) $V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. When $r = 0.25$, $\frac{dV}{dt} = 0.75$. Thus, $0.75 = 4(\pi)(0.25)^2\frac{dr}{dt}$, or $\frac{dr}{dt} = \frac{0.75}{4\pi(0.0625)} = 0.955$ m/min.

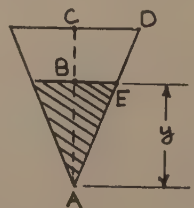
(b) $S = 4\pi r^2$; $\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi(0.25)\left(\frac{0.75}{4\pi(0.0625)}\right) = 6$ m^2/min.

21. $\frac{dV}{dt} = [-\frac{1}{3}\pi h^2 + \frac{2}{3}\pi h(3R - h)]\frac{dh}{dt} = \frac{\pi}{3}[2h(60 - h) - h^2]\frac{dh}{dt}$. Now, $\frac{dV}{dt} = -200(0.134) = -26.8$ ft^3/min. At the instant when $h = 5$, $\frac{dh}{dt} = \frac{-26.8}{\frac{\pi}{3}[(10)(55) - 25]} = \frac{-26.8}{175} \approx -0.04$ ft/min. The level is dropping at approximately 0.0487 ft/min.

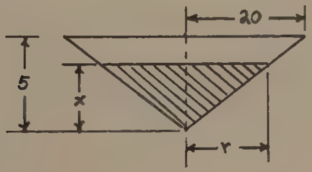
22. In the figure below, which shows a vertical cross section of the pool, triangle ABE is similar to triangle ACD; hence, $\frac{|\overline{BE}|}{|\overline{BA}|} = \frac{|\overline{CD}|}{|\overline{CA}|} = \frac{20}{6} = \frac{10}{3}$. The height h of the water at the deep end is $h = |\overline{BA}|$, so $|\overline{BE}| = \frac{10}{3}h$. The area of triangle ABE is $\frac{1}{2}h\,|\overline{BE}| = \frac{5}{3}h^2$, so the volume of water in the pool is $V = (\frac{5}{3}h^2)(10) = \frac{50h^2}{3}$. Now, $\frac{dV}{dt} = \frac{100\,h}{3}\frac{dh}{dt}$. Since $\frac{dV}{dt} = 1.5$, then, when $h = 6$, $\frac{dh}{dt} = \frac{1.5}{100} = 0.0075$ m/min.



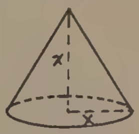
23. In the figure below, which shows a cross section of the trough , let y denote the depth of water in the trough. Here, $|\overline{CD}| = 4$ in., $|\overline{CA}| = 10$ in. and triangle DCA is similar to triangle EBA. Thus, $\frac{|\overline{CD}|}{|\overline{BE}|} = \frac{|\overline{CA}|}{|\overline{BA}|}$, $|\overline{BE}| = 4\frac{y}{10} = \frac{2}{5}y$, and the volume V of water in the trough is $(20)(12)\frac{1}{2}(2|\overline{BE}|)y$ cubic inches. Thus, $V = 96y^2$, so $\frac{dV}{dt} = 192\,y\frac{dy}{dt}$. When $y = 5$ and $\frac{dy}{dt} = \frac{1}{2}$, $\frac{dV}{dt} = 192(5)\frac{1}{2} = 480$ in^3/min.



24. Let x denote the depth of the water in the reservoir. By similar triangles, the radius r of the water surface is given by r = 4x, the area of the surface is $\pi r^2 = 16\pi x^2$, and the rate of evaporation is $\frac{dV}{dt} = -(0.00005)16\pi x^2$ cubic meters per hour. The volume of water in the reservoir is $V = \frac{1}{3}x\ (\pi r^2) = \frac{16\pi x^3}{3}$. $\frac{dV}{dt} = 16\pi x^2\ \frac{dx}{dt}$, so $\frac{dx}{dt} =$ $\frac{1}{16\pi x^2}\ \frac{dV}{dt} = \frac{1}{16\pi x^2}\ (-0.00005)16\pi x^2 = -0.00005$ m/hr. independent of the value of x.

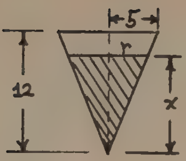


25. $V = \frac{1}{3}(\pi x^2)x = \frac{1}{3}\pi x^3$. $2 = \frac{dV}{dt} = \pi x^2\ \frac{dx}{dt}$. When x = 6, $\frac{dx}{dt} = \frac{2}{\pi x^2} = \frac{2}{36\pi} \approx 0.02$ m/min.



26. The volume of water in the tank is given by $V = \pi(\frac{10}{12})^2 h$ cubic feet, so $\frac{dV}{dt} = \pi(\frac{10}{12})^2\ \frac{dh}{dt}$. Since the water is pouring out of a hole with area $\pi(\frac{1}{12})^2$ square foot with a velocity of $8\sqrt{h}$ feet per second, then it is pouring out at the rate of $[\pi(\frac{1}{12})^2](8\sqrt{h})$ cubic feet per second. Therefore, $-8\pi\sqrt{h}\ \ (\frac{1}{12})^2 = \frac{dV}{dt} = \pi(\frac{10}{12})^2\ \frac{dh}{dt}$, $\frac{dh}{dt} = -\ \frac{8\sqrt{h}}{10^2}$. When h = 5 ft, $\frac{dh}{dt} = -\ \frac{8\sqrt{5}}{100} = -\frac{2}{25}\ \sqrt{5} \approx -0.18$ ft/sec. The water level is dropping at 0.18 ft/sec.

27. Let x denote the depth of the water, r the radius of the water surface. By similar triangles $r = \frac{5}{12}x$. The volume of water in the tank is $V = \frac{1}{3}(\pi r^2)x =$ $\frac{\pi}{3}(\frac{5}{12})^2 x^3$; hence, $\frac{dV}{dt} = \pi(\frac{5}{12})^2 x^2\ \frac{dx}{dt}$. Since, when $x = 6$, $\frac{dV}{dt} = 10\ m^3$/min., then, at this instant, $\frac{dx}{dt} = \frac{10}{\pi(\frac{5}{12})^2 \cdot 6^2} = \frac{8}{5\pi} \approx 0.51$ m/min.



28. Let r be the radius of the surface of the water in the tank. By similar triangles $r = \frac{Rh}{H}$. The volume of water in the tank is $V = \frac{1}{3}\pi r^2 h = \frac{\pi R^2 h^3}{3H^2}$, so the rate of change of this volume is $\frac{dV}{dt} = \pi(\frac{Rh}{H})^2\ \frac{dh}{dt}$. The water is leaking out of the tank at the rate of $k\sqrt{2gh}$ cubic units per second and is being pumped into the tank at the rate of c cubic units per second, so $c - k\sqrt{2gh} = \frac{dV}{dt} = \pi(\frac{Rh}{H})^2\ \frac{dh}{dt}$, and so $\frac{dh}{dt} = \frac{1}{\pi}\ (\frac{H}{Rh})^2[c - k\sqrt{2gh}]$ units/sec.

29. $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = 4\pi r^2\ \frac{dr}{dt}$. $A(10^{-4}) = A(\frac{dr}{dt})$ or $\frac{dr}{dt} = 10^{-4}$. In 2 hours, $r = 10^{-3} + 2(10^{-4}) =$ $10^{-4}(10 + 2) = 1.2\times10^{-3}$ cm.

30. $\frac{dV}{d\theta} = (16.76)(10^{-6})(\theta - 4)$; hence, when $\theta = 10$, $\frac{dV}{d\theta} \approx 0.000101$. Thus, $\frac{dV}{dt} = \frac{dV}{d\theta}\frac{d\theta}{dt} \approx (0.000101)(-1.5) \approx$ $-0.000152\ cm^3$/min.

31. $PV = C$, so $P\ \frac{dV}{dt} + \frac{dP}{dt}\ V = 0$. $\frac{dV}{dt} = -\frac{dP}{dt}\ \frac{V}{P} = 5\ \frac{1000}{150} =$ $\frac{100}{3}$ cubic inches per second per second.

32. $2(x - 800)\ \frac{dx}{dt} + 800\ (y - 50)\ \frac{dy}{dt} = 0$. Now, if $y = 55$, $x = 1100$, and $\frac{dx}{dt} = 40$, then $(1100 - 800)40 +$ $400(5)\ \frac{dy}{dt} = 0$ or $\frac{dy}{dt} = -6$. The population is decreasing by 6 owls per month.

33. $\tan\theta = \frac{y}{x} = \frac{y}{12}$. Thus, $(\sec^2\theta)\ \frac{d\theta}{dt} = \frac{1}{12}\ \frac{dy}{dt}$, so $\sec^2\frac{\pi}{3}\ (-\frac{1}{30}) = \frac{1}{12}\ \frac{dy}{dt}$. Now $\frac{-4}{30} = \frac{1}{12}\ \frac{dy}{dt}$, so $\frac{dy}{dt} =$ $-\frac{8}{5}$ cm/sec.

34. $\csc\theta = \frac{z}{y} = \frac{z}{10\sqrt{2}}$. Thus, $-\csc\theta\ \cot\theta(\frac{d\theta}{dt}) =$ $\frac{1}{10\sqrt{2}}\ \frac{dz}{dt}$, so $-\csc\frac{\pi}{4}\ \cot\frac{\pi}{4}\ (-\frac{1}{30}) = \frac{1}{10\sqrt{2}}\ \frac{dz}{dt}$, and $(-\sqrt{2})1(-\frac{1}{30}) = \frac{1}{10\sqrt{2}}\ \frac{dz}{dt}$, so $\frac{dz}{dt} = \frac{2}{3}$ cm/sec.

35. $\cos\theta = \frac{x}{z} = \frac{1}{40}x$. Thus, $-\sin\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{40}\frac{dx}{dt}$, so $-\sin\theta\left(-\frac{1}{30}\right) = \frac{1}{40}\frac{dx}{dt}$. Now $\sin\theta = \frac{y}{z} = \frac{20}{40} = \frac{1}{2}$, so $-\frac{1}{2}\left(-\frac{1}{30}\right)40 = \frac{dx}{dt}$, and $\frac{dx}{dt} = \frac{2}{3}$ cm/sec.

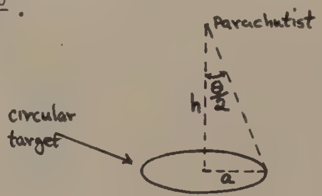
36. $\sec\theta = \frac{z}{x}$. Thus, $\sec\theta\tan\theta\frac{d\theta}{dt} = \frac{1}{x}\frac{dz}{dt}$, so $(\sec\theta)(\tan\theta)\left(-\frac{1}{30}\right) = \frac{1}{1.6}\frac{dz}{dt}$. Since $x^2 + y^2 = z^2$, $2x^2 = z^2$; thus, $2(1.6)^2 = z^2$ or $\sqrt{2}(1.6) = z$. $\sec\theta = \frac{z}{x} = \frac{(1.6)\sqrt{2}}{1.6} = \sqrt{2}$; $\tan\theta = 1$. Hence, $\sqrt{2}\cdot 1\left(-\frac{1}{30}\right)(1.6) = \frac{dz}{dt}$ or $\frac{dz}{dt} = \frac{-16\sqrt{2}}{300}$ or $\frac{dz}{dt} = \frac{-4\sqrt{2}}{75}$ km/sec.

37. $\sin\theta = \frac{y}{z}$, so $\cos\theta\frac{d\theta}{dt} = \frac{z\frac{dy}{dt} - y\frac{dz}{dt}}{z^2}$. When $x = 1$ and $z = 2$, $y = \sqrt{3}$; so $\frac{1}{2}(-30) = \frac{2\left(\frac{2}{15}\right) - \sqrt{3}\frac{dz}{dt}}{4}$ or $-\frac{1}{15} = \frac{4}{15} - \sqrt{3}\frac{dz}{dt}$ or $\frac{dz}{dt} = \frac{\sqrt{3}}{9} \approx 0.19$ m/sec.

38. We use distance = rate $\times$ time. $\tan\frac{\theta}{2} = \frac{a}{h}$, so $h = a\cot\frac{\theta}{2}$. $\frac{dh}{dt} = -\frac{a}{2}\csc^2\frac{\theta}{2}\frac{d\theta}{dt}$. Now $h = -\frac{dh}{dt}\cdot T$, so $T = h/(-dh/dt)$. Hence,

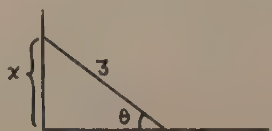
$$T = \frac{a\cot\frac{\theta}{2}}{\frac{a}{2}\csc^2\frac{\theta}{2}\frac{d\theta}{dt}} = \frac{\frac{2\cos(\theta/2)}{\sin(\theta/2)}}{\frac{1}{\sin^2\frac{\theta}{2}}\frac{d\theta}{dt}} = \frac{2\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}}{\frac{d\theta}{dt}};$$

so $T = \frac{\sin\theta}{\frac{d\theta}{dt}}$.

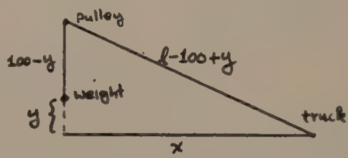


39. $\frac{dx}{dt} = -1.5 = -\frac{3}{2}$ m/sec. $\sin\theta = \frac{x}{3}$, so $\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{3}\frac{dx}{dt}$. When $\theta = \frac{\pi}{6}$, $\cos\theta = \frac{\sqrt{3}}{2}$. Hence, $\frac{\sqrt{3}}{2}\frac{d\theta}{dt} = \frac{1}{3}\left(-\frac{3}{2}\right)$. So $\sqrt{3}\frac{d\theta}{dt} = -1$. Hence, $\frac{d\theta}{dt} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

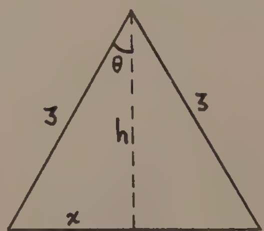
The ladder is turning at the rate of $\frac{\sqrt{3}}{3} \approx 0.58$ rad/sec.



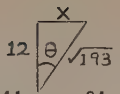
40. Let ℓ be the length of the rope. We want to find $\frac{dy}{dt}$ when $\frac{dx}{dt} = 5$ and $\theta = \frac{\pi}{4}$. $x = 100$ when $\theta = \frac{\pi}{4}$. Now $\cos\theta = \frac{x}{\ell - 100 + y}$, so $-\sin\theta\frac{d\theta}{dt} = \frac{(\ell - 100 + y)\frac{dx}{dy} - x\left(\frac{dy}{dx}\right)}{(\ell - 100 + y)^2}$. When $\theta = \frac{\pi}{4}$, $\sin\theta = \frac{\sqrt{2}}{2}$, and since $\sin\theta = \frac{100}{\ell - 100 + y}$, then $\ell - 100 + y = \frac{200}{\sqrt{2}} = 100\sqrt{2}$. Also since $\tan\theta = \frac{100}{x}$, then $\sec^2\theta\frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}$, and for $\theta = \frac{\pi}{4}$, $\frac{d\theta}{dt} = \left[\frac{-100}{(100)^2}\right]\cdot(5)\left(\frac{1}{2}\right) = -\frac{1}{40}$. Hence, $-\sin\theta\frac{d\theta}{dt} = \frac{(\ell - 100 + y)\frac{dx}{dt} - x\frac{dy}{dt}}{(\ell - 100 + y)^2}$ becomes, for $\theta = \frac{\pi}{4}$, $\frac{-\sqrt{2}}{2}\left(-\frac{1}{40}\right) = \frac{100\sqrt{2}(5) - 100\frac{dy}{dt}}{(100\sqrt{2})^2}$. So $\frac{\sqrt{2}}{80}(2)(100)^2 -$ $500\sqrt{2} = -100\frac{dy}{dt}$ and $\frac{dy}{dt} = 5\sqrt{2} - \frac{5\sqrt{2}}{2}$. Therefore, $\frac{dy}{dt} = \frac{5\sqrt{2}}{2} \approx 3.54$ ft/sec.



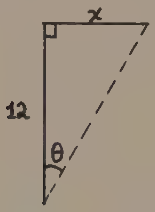
41. $\frac{d\theta}{dt} = \frac{\pi}{100}$ radians per second. We want to find $\frac{dA}{dt}$ when $2\theta = \frac{\pi}{3}$. $h = \sqrt{9 - x^2}$ and $A = \frac{1}{2}(2x)h = x\sqrt{9 - x^2}$. Now $\sin\theta = \frac{x}{3}$, so that $x = 3\sin\theta$. Hence, $A = 3\sin\theta\sqrt{9 - 9\sin^2\theta} = 9\sin\theta\cos\theta = \frac{9}{2}\sin 2\theta$. $\frac{dA}{dt} = \frac{9}{2}\cos 2\theta\left(2\frac{d\theta}{dt}\right) = 9\cos 2\theta\frac{d\theta}{dt} = 9\left(\frac{1}{2}\right)\left(\frac{\pi}{180}\right) = \frac{\pi}{40} \approx 0.079$ cm^2/sec.



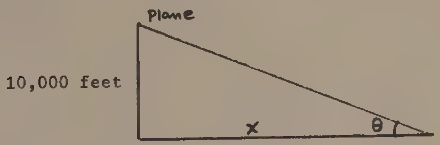
42. We want $\frac{d\theta}{dt}$ when $x = 7$ and $\frac{dx}{dt} = 2$. $\tan\theta = \frac{x}{12}$, so $\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{12}\frac{dx}{dt}$. When $x = 7$, $\sec\theta = \frac{\sqrt{193}}{12}$

according to the triangle 

. Hence, $\frac{193}{144}\left(\frac{d\theta}{dt}\right) = \frac{1}{12}(2)$; so $\frac{d\theta}{dt} = \frac{144}{6(193)} = \frac{24}{193}$ rad/sec.

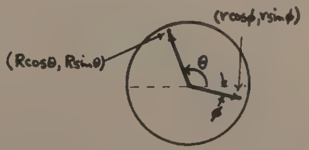


43. We want $\frac{d\theta}{dt}$ when $\theta = \frac{\pi}{4}$ and $\frac{dx}{dt} = -(400)(5280)$ ft/hr. $\cot\theta = \frac{x}{10{,}000}$, so $-\csc\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{10{,}000}\frac{dx}{dt}$. When $\theta = \frac{\pi}{4}$, we have $-2\frac{d\theta}{dt} = \frac{1}{10{,}000}[-400(5280)]$; and $\frac{d\theta}{dt} = \frac{528}{5} = 105.6$ rad/hr.

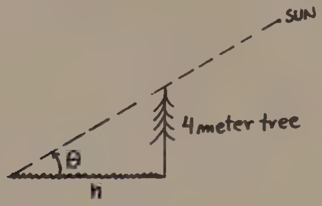


44. We want to find $\frac{ds}{dt}$. $\frac{d\theta}{dt} = -\frac{2\pi}{60}$ rad/min $= \frac{-\pi}{30}$. $\frac{d\phi}{dt} = \frac{-2\pi}{720}$ rad/min $= \frac{-\pi}{360}$. Now $s = \sqrt{(R\cos\theta - r\cos\phi)^2 + (R\sin\theta - r\sin\phi)^2}$; $s = \sqrt{R^2 + r^2 - 2Rr\sin\theta\sin\phi - 2Rr\cos\theta\cos\phi}$.

$\frac{ds}{dt} = \frac{1}{2}(R^2 + r^2 - 2Rr\sin\theta\sin\phi - 2Rr\cos\theta\cos\phi)^{-\frac{1}{2}}\cdot$ $(-2Rr\cos\theta\sin\phi\frac{d\theta}{dt} - 2Rr\sin\theta\cos\phi\frac{d\phi}{dt} +$ $2Rr\sin\theta\cos\phi\frac{d\theta}{dt} + 2Rr\cos\theta\sin\phi\frac{d\phi}{dt})$. When $R = 6$, $r = 4.5$, $\theta = \frac{\pi}{2}$, and $\phi = \frac{-2}{12} = -\frac{\pi}{6}$, $\frac{ds}{dt} =$ $\frac{1}{2}[36 + 20.25 - 54(1)(-\frac{1}{2}) - 54(0)]^{-1/2}\cdot$ $[-54(0) - 54(1)\frac{\sqrt{3}}{2}(-\frac{\pi}{360}) + 54(1)\frac{\sqrt{3}}{2}(-\frac{\pi}{30}) + 54(0)]$. $\frac{ds}{dt} = \frac{1}{2}(83.25)^{-1/2}(\frac{-33\sqrt{3}\pi}{40}) \approx -0.246$. The tips of the hands are approaching each other at the rate of 0.246 ft/min.



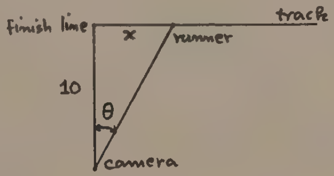
45. $\tan\theta = \frac{4}{h}$, so $\sec^2\theta\frac{d\theta}{dt} = \frac{-4}{h^2}\frac{dh}{dt}$. When $\theta = \frac{\pi}{6}$, $h = 4\sqrt{3}$; thus, $(\sec^2\frac{\pi}{6})(-14°\cdot\frac{\pi}{180}) = \frac{-4}{(4\sqrt{3})^2}\frac{dh}{dt} = \frac{dh}{dt} = \frac{16\cdot 14\cdot\pi}{180} = 3.91$ m/hr.



46. (a) Using the distance formula, we have $b^2 = (x - a\cos\theta)^2 + (0 - a\sin\theta)^2 = x^2 - 2ax\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta = x^2 - 2ax\cos\theta + a^2$. So $x^2 - 2ax\cos\theta + a^2 - b^2 = 0$. Thus, $x = \frac{2a\cos\theta \pm \sqrt{4a^2\cos^2\theta - 4(a^2 - b^2)}}{2} =$ $a\cos\theta \pm \sqrt{a^2\cos^2\theta - a^2 + b^2} =$ $a\cos\theta \pm \sqrt{b^2 - a^2\sin^2\theta}$. We reject $a\cos\theta - \sqrt{b^2 - a^2\sin^2\theta}$ since x must be positive; so $x = a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta}$.

(b) $\frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt} + \frac{1}{2}(b^2 - a^2\sin^2\theta)^{-1/2}\cdot$ $(-2a^2\sin\theta\cos\theta)\frac{d\theta}{dt}$, so $\frac{dx}{dt} = -\sin\theta(2\pi N) +$ $(b^2 - a^2\sin^2\theta)^{-1/2}(-a^2\sin\theta\cos\theta)2\pi N =$ $-a2\pi N\sin\theta - \frac{2\pi Na^2\sin\theta\cos\theta}{(b^2 - a^2\sin^2\theta)^{1/2}}$.

47. $\tan\theta = \frac{x}{10}$, so $\sec^2\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$. When $x = 10$, $\theta = 45°$; so $\sec^2\theta = 2$. It follows that $2(0.5) = \frac{1}{10}\frac{dx}{dt}$ or $\frac{dx}{dt} = 10$ m/sec.

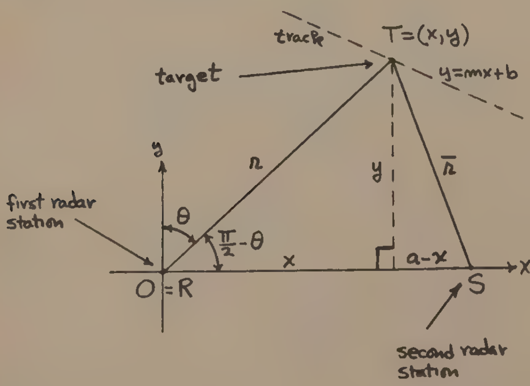


48. Suppose that the second radar is located on the positive x axis, a units from 0 = R, and that the target is at the point T = (x,y) and moving with speed $\left|\frac{ds}{dt}\right|$ along the track $y = mx + b$. We have

$x^2 + y^2 = r^2$ and $(a - x)^2 + y^2 = \bar{r}^2$. By implicit differentiation, $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt}$ and $-2(a - x) \frac{dx}{dt} + 2y \frac{dy}{dt} = 2\bar{r} \frac{d\bar{r}}{dt}$, and so we have the simultaneous equations:

$$\begin{cases} x \dfrac{dx}{dt} + y \dfrac{dy}{dt} = r \dfrac{dr}{dt} \\ -(a - x) \dfrac{dx}{dt} + y \dfrac{dy}{dt} = \bar{r} \dfrac{d\bar{r}}{dt}. \end{cases}$$

Subtracting the second equation from the first, we find that $[x + (a - x)] \frac{dx}{dt} = (r - \bar{r}) \frac{d\bar{r}}{dt}$, or $\frac{dx}{dt} = \frac{r - \bar{r}}{a} \frac{d\bar{r}}{dt}$. Since a, r, $\bar{r}$, and $\frac{d\bar{r}}{dt}$ are known, this determines $\frac{dx}{dt}$. Now $x = r \cos (\frac{\pi}{2} - \theta) = r \sin \theta$ and $y = r \sin (\frac{\pi}{2} - \theta) = r \cos \theta$. Since r and θ are known, these equations determine x and y. Substituting $\frac{dx}{dt} = \frac{r - \bar{r}}{a} \frac{d\bar{r}}{dt}$ into $x \frac{dx}{dt} + y \frac{dy}{dt} = r \frac{dr}{dt}$ and solving for $\frac{dy}{dt}$, we find that $\frac{dy}{dt} = \frac{r}{y} \frac{dr}{dt} - \frac{x}{y} (\frac{r - \bar{r}}{a} \frac{d\bar{r}}{dt}) = \sec \theta \frac{dr}{dt} - \tan \theta (\frac{r - \bar{r}}{a} \frac{d\bar{r}}{dt})$, an equation which determines $\frac{dy}{dt}$. Now $m = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, and so m is determined. Since $y = mx + b$, the y intercept is determined by $b = y - mx = r(\cos \theta - m \sin \theta)$. Finally, using the result of Problem 12, we see that the speed is determined by $\left|\frac{ds}{dt}\right| = 1 + m^2 \left|\frac{dx}{dt}\right|$.



Problem Set 3.9, page 233

1. (a) $C(0) = 10,000$ fixed cost
   (b) $C(x) - C(0) = -x^2 + 2500x$ variable cost
   (c) $C'(x) = -2x + 2500$ marginal cost
   (d) $C(901) = -901^2 + 2500(901) + 10,000 = 1,450,69$
   $C(900) = 1,450,000$. Thus \$699 is the exact cost of producing the 901st unit.
   (e) $C'(900) = -2(900) + 2500 = 700$. Thus, \$700 is the approximate cost of producing the 901st unit.
   (f) $C''(x) = -2$. There are no critical points in th interval, so test the end points: $C'(0) = 2500$ and $C'(1000) = 500$, so there is an absolute minimum when $x = 1000$. Thus, the most efficie production level is that at the 1000th unit.
2. (a) $C(0) = 150$
   (b) $C(x) - C(0) = 8\sqrt{x}$
   (c) $C'(x) = 4/\sqrt{x}$
   (d) $C(901) - C(900) = 8(\sqrt{901} - \sqrt{900}) \approx \$0.13330$
   (e) $C'(900) = \frac{4}{\sqrt{900}} = \frac{4}{30} = \frac{2}{15} \approx \$0.13333$
   (f) $C''(x) = -2x^{-3/2}$. There are no critical points in interval; so test the end points. $C'(0)$ is not defined and $C'(1500) = 4/\sqrt{1500} \approx \$0.10$, so there is an absolute minimum when $x = 1500$.
3. (a) $C(0) = 5000$
   (b) $C(x) - C(0) = (0.00003)x^3 - (0.18)x^2 + 500x$
   (c) $C'(x) = (0.00009)x^2 - (0.36)x + 500$
   (d) $C(901) - C(900) = 326,318.801 - 326,070.00 =$ \$248.801
   (e) $C'(900) = \$248.90$
   (f) $C''(x) = 0.00018x - 0.36$. There is a critical point at $x = 2000$. Since $C'''(x) = 0.00018 > 0$ $x = 2000$ gives most efficient production level
4. (a) $C(0) = 150$
   (b) $C(x) - C(0) = \frac{25,500x}{3x + 1000}$
   (c) $C'(x) = \frac{(3x + 1000)(26,000) - (26,000x + 150,000)}{(3x + 1000)^2}$

$\frac{25{,}550{,}000}{(3x + 1000)^2}$.

(d) $C(901) - C(900) = 6366.7297 - 6364.8649 =$ \$1.86481

(e) $C'(900) = \frac{25{,}550{,}000}{(2703 + 1000)^2} \approx \$1.86633$

(f) $C''(x) = -153{,}300{,}000\ (3x + 1000)^{-3} < 0$. Test the end points. $C'(0) = 25.55$ and $C'(1500) = 0.8446$, so there is a minimum when $x = 1500$.

5. Problem 1: $\bar{C}(900) = \frac{C(900)}{900} = \frac{1{,}450{,}000}{900} = \$1611.11$ per unit.

Problem 3: $\bar{C}(900) = \frac{C(900)}{900} = \frac{331{,}070}{900} = \$367.86$ per unit.

6. Production cost per unit is given by $\bar{C}(x) = \frac{C(x)}{x}$. For minimum production cost per unit, $\bar{C}'(x) = 0$; that is, $\frac{xC'(x) - C(x)}{x^2} = 0$, or $xC'(x) = C(x)$, or $C'(x) = \frac{C(x)}{x} = \bar{C}(x)$.

7. Often, the point of inflection has abscissa x with $C''(x) = 0$, so x is a critical number for C'. If this critical number corresponds to a minimum value of C', then it is the most efficient production level.

8. At the most efficient production level, the cost of producing one more unit is minimized, but higher cost per unit for units already produced will ordinarily offset this advantage. To further reduce overall production costs per unit, even more units must then be produced.

9. (a) $C(x) = \frac{x^3}{900} - 3x^2 + 4000x + 100{,}000,\ 0 \le x \le 1000$.

(b) $C'(x) = \frac{x^2}{300} - 6x + 4000,\ 0 < x < 1000$.

(c) $C(500) - C(499) = 1{,}488{,}888.889 - 1{,}487{,}054.221 =$ \$1834.668.

(d) $C'(499) \approx \$1836.00$.

(e) $C''(x) = \frac{x}{150} - 6$. A critical point is $x = 900$. $C'''(x) = 1 > 0$, so $x = 900$ tons gives a minimum value.

10. $C(x) = (2 \times 10^{-7})x^3 - (3 \times 10^{-3})x^2 + 20x + 15{,}000,\ 0 \le x \le 10{,}000$.

(b) $C'(x) = (6 \times 10^{-7})x^2 - (6 \times 10^{-3})x + 20,\ 0 < x < 10{,}000$.

(c) $C(1001) - C(1000) = 32{,}214.5976 - 32{,}200 =$ \$14.598.

(d) $C'(1000) = \$14.000$.

(e) $C''(x) = (12 \times 10^{-7})x - (6 \times 10^{-3})$. A critical point is $x = 5000$. $C'''(x) = 12 \times 10^{-7} > 0$, so $x = 5000$ gives minimum value.

11. (a) $C(x) = (5 \times 10^{-6})x^3 - (1.5 \times 10^{-2})x^2 + 40x + 4000,\ 0 \le x \le 3000$.

(b) $C'(x) = (1.5 \times 10^{-5})x^2 - (3.0 \times 10^{-2})x + 40,\ 0 < x < 3000$.

(c) $C(201) - C(200) \approx 11{,}474.58801 - 11{,}440 =$ \$34.58801.

(d) $C'(200) = \$34.60$.

(e) $C''(x) = (30 \times 10^{-6})x - 3.0 \times 10^{-2}$. A critical point is $x \doteq 1000$. $C'''(x) = 30 \times 10^{-6} > 0$, so $x = 1000$ gives minimum value.

12. (a) $C(x) = 30\sqrt{x} + 500,\ 0 \le x \le 10{,}000$.

(b) $C'(x) = 15/\sqrt{x},\ 0 < x < 10{,}000$.

(c) $C(5001) - C(5000) = 30(\sqrt{5001} - \sqrt{5000}) \approx$ \$0.2121213.

(d) $C'(5000) \approx \$0.212132$.

(e) $C''(x) = -\frac{15}{2}x^{-3/2} < 0$, so the graph of C' is always decreasing. Test the end points. $C'(0)$ is undefined and $C'(10{,}000) = 15/100 = 0.15$, so there is a minimum at $x = 10{,}000$.

13. (a) Let n be the number of \$500 increases in the price per ton. Then the price p per ton is $p = 4000 + 500n$ and the daily demand is $x = 600 - 100n$, so $n = (600 - x)/100$. Thus, $p = 4000 + 500(\frac{600 - x}{100}) = 7000 - 5x$.

(b) $R(x) = px = 7000x - 5x^2$.

(c) $R'(x) = 7000 - 10x$.

(d) $R'(500) = 7000 - 10(500) = \$2000$.

(e) $R'(x) = 0$ yields $x = 700$ tons. $R''(x) = -10 < 0$ for all x, so $x = 700$ will give a maximum value for R.

14. (a) Let n be the number of \$1 increases in the price per tire. Then the price p per tire is $p = 24 + (1)n = 24 + n$ and the daily demand is $x = 3000 - 500n$. Now, $n = (3000 - x)/500$; so $p = 24 + \frac{3000 - x}{500} = \frac{15,000 - x}{500}$.

(b) $R(x) = px = \frac{15,000x - x^2}{500}$.

(c) $R'(x) = \frac{15,000 - 2x}{500} = \frac{7500 - x}{250}$.

(d) $R'(1000) = \frac{5600}{250} = \$26$.

(e) $R'(x) = 0$ gives $x = 7500$. $R''(x) = -\frac{1}{250} < 0$ for all x, so $x = 7500$ gives a maximum value for R.

15. (a) Let n be the number of \$5 increases per player. Then the price p per player is $p = 60 + 5n$ and the daily demand is $x = 2000 - 250n$ so $n = (2000 - x)/250$. Thus, $p = 60 + 5(\frac{2000 - x}{250}) = \frac{5000 - x}{50} = 100 - \frac{x}{50}$.

(b) $R(x) = px = \frac{5000x - x^2}{50} = 100x - \frac{x^2}{50}$.

(c) $R'(x) = \frac{5000 - 2x}{50} = \frac{2500 - x}{25} = 100 - \frac{x}{25}$.

(d) $R'(200) = \frac{2300}{25} = \$92$.

(e) $R'(x) = 0$ yields $x = 2500$. $R''(x) = -\frac{1}{25} < 0$ for all x, so $x = 2500$ yields a maximum for R.

16. (a) Let n be the number of 25¢ increases in the price per toothbrush. Then the price per toothbrush is $p = 0.90 + 0.25n$ and the daily demand $x = 6000 - 1000n$, so $n = (6000 - x)/1000$. Thus, $p = 0.90 + 0.25(\frac{6000 - x}{1000}) = \frac{9600 - x}{4000}$.

(b) $R(x) = px = \frac{9600 - x^2}{4000}$.

(c) $R'(x) = \frac{9600 - 2x}{4000} = \frac{4800 - x}{2000}$.

(d) $R'(5000) = \frac{-200}{2000} = -\frac{1}{10}$.

(e) $R''(x) = 0$ yields $x = 4800$. $R''(x) = -\frac{1}{2000} < 0$ for all x, so $x = 4800$ gives maximum value.

17. (a) $P(x) = R(x) - C(x) = 7000x - 5x^2 - (\frac{x^3}{900} - 3x^2 + 4000x + 100,000) = -\frac{x^3}{900} - 2x^2 + 3000x - 100,000$, $0 \le x \le 1000$.

(b) $P'(x) = -\frac{x^2}{300} - 4x + 3000$, $0 < x < 1000$.

(c) $P'(x) = 0$ yields $x = -1722.497216$ (reject since not in $0 < x < 1000$) and $x = 522.497216 \approx 522.5$. $P(0) = -100,000$, $P(1000) = -11,111.\bar{1}$ and $P(522.5) = \$76,299.00$. So the maximum profit occurs when $x = 522.5$ tons.

(d) $R(x) = px$, so we have $p = \frac{R(x)}{x} = \frac{7000x - 5x^2}{x} = 7000 - 5x$. When $x = 522.497216$, the price per unit is $7000 - 5(522.497216) = \$4387.51$ per ton

(e) $P(x) = 0$ implies $-\frac{x^3}{900} - 2x^2 + 3000x - 100,000 =$ Using Newton's method, we find that the approxi zeros of the profit function P on $[0,1000]$ are 34.12 and 947.99. Thus, 34.12 tons and 947.99 tons allow the producer to break even.

18. (a) $P(x) = R(x) - C(x) = \frac{15,000x - x^2}{500} - (2 \times 10^{-7}x^3 - 3 \times 10^{-3}x^2 + 20x + 15,000) = -2 \times 10^{-7}x^3 + 10^{-3}x^2 + 10x - 15,000$, $0 \le x \le 10,000$.

(b) $P'(x) = -6 \times 10^{-7}x^2 + 2 \times 10^{-3}x + 10$, $0 < x < 10,000$.

(c) $P'(x) = 0$ yields $x = -2742.9$ (reject since not in $0 < x < 10,000$) and $x = 6076.3$. $P(0) = -15,000$, $P(10,000) = -15,000$, and $P(6076.3) \approx \$37,815.29$. So the profit is maximum for $x = 6076.3$ tires.

(d) Since $R(x) = px$, we have $p = \frac{R(x)}{x} = \frac{15,000 - x}{500}$ $30 - \frac{x}{500}$. When $x = 6076.3$, the price per unit is $30 - \frac{6076.3}{500} = \$17.85$.

(e) $P(x) = 0$ implies $-2 \times 10^{-7}x^3 + 10^{-3}x^2 + 10x - 15,000 = 0$. Using Newton's method, we find

that the approximate zeros of the profit function P in [0,10,000] are 1364 or 9450. Production levels between 1364 and 9450 cause the producer to break even.

19. (a) $P(x) = R(x) - C(x) = \frac{5000x - x^2}{50} - (5 \times 10^{-6}x^3 - 1.5 \times 10^{-2}x^2 + 40x + 4000) = -(5 \times 10^{-6})x^3 - (5 \times 10^{-3})x^2 + 60x - 4000$, $0 \le x \le 3000$.

(b) $P'(x) = -(1.5 \times 10^{-5})x^2 - 10^{-2}x + 60$, $0 < x < 3000$.

(c) $P'(x) = 0$ yields $x = -2360.9$ (reject since not in $0 < x < 3000$) and $x = 1694.25 \approx 1694$. $P''(x) < 0$, so x value gives maximum.

(d) $P = \frac{R(x)}{x} = \frac{5000 - x}{50} = 100 - \frac{x}{50}$. When $x = 1694.25$, the price per unit is $100 - \frac{1694.25}{50} =$ \$66.12.

(e) $P(x) = 0$ implies $(-5 \times 10^{-6})x^3 - 0.5 \times 10^{-2} + 60x - 4000 = 0$. Using Newton's method, we find that the approximate zeros of the profit function on [0,3000] are 67 and 2961. Production levels between 67 and 2961 cause the producer to break even.

20. (a) $P(x) = R(x) - C(x) = \frac{9600x - x^2}{4000} - (30\sqrt{x} + 500) = \frac{12}{5}x - \frac{x^2}{4000} - 30\sqrt{x} - 500$, $0 \le x \le 10{,}000$.

(b) $P'(x) = \frac{12}{5} - \frac{x}{2000} - \frac{15}{\sqrt{x}}$, $0 < x < 10{,}000$.

(c) $P'(x) = 0$ yields $x = 39.72$ and 4344.87 using Newton's method. $P(0) = -500$, $P(10{,}000) = -4500$, $P(39.72) = -594.1$, and $P(4344.87) = 3230.75$.

(d) $P = \frac{R(x)}{x} = \frac{9600 - x}{4000} = \frac{12}{5} - \frac{x}{4000}$. When $x = 4344.87$, the price per unit is \$1.31.

(e) $P(x) = 0$ implies $\frac{12}{5}x - \frac{x^2}{400} - 30\sqrt{x} - 500 = 0$. Using Newton's method, we find that the approximate zeros of the profit function on [0, 10,000] are 522.5 and 8009. Production levels of less than 522.5 or greater than 8009 cause the producer to lose money.

21. $C(x) = 1700 + \frac{23x^3}{10{,}000} - \frac{69x^2}{100} + 159x$, $0 \le x \le 125$.

If n is the number of \$10 increases, then $p = 150 + 10n$. $x = 75 - 15n$, so $n = \frac{75 - x}{15}$. Thus, $p = 150 + 10(\frac{75 - x}{15}) = 200 - \frac{2}{3}x$. Thus, $R(x) = px = 200x - \frac{2}{3}x^2$, $0 \le x \le 125$.

(a) $P(x) = R(x) - C(x) = 200x - \frac{2}{3}x^2 - (1700 + \frac{23x^3}{10{,}000} - \frac{69x^2}{100} + 159x) = 41x + \frac{7}{300}x^2 - \frac{23x^3}{10{,}000} - 1700$, $0 \le x \le 125$.

(b) $P'(x) = 41 + \frac{7}{150}x - \frac{69x^2}{10{,}000}$, $0 < x < 125$.

(c) The roots of $P'(x) = 0$ are $x = 81$ and -74. Rejecting -74 because it does not lie in the interval (0,125), we find that 81 is the only critical number for P in (0,125). Checking the values of P(x) at this critical number and at the endpoints of [0,125], we find a maximum profit of \$551.78 per day is made by producing 81 bikes.

(d) $R(x) = px$, so $p = 200 - \frac{2}{3}x$. When $x = 81$, the price per bike is $200 - \frac{2}{3}(81) = 200 - 54 =$ \$146.

(e) Using Newton's method, we find that the approximate zeros of the profit function is [0,125] are 45.6 and 110.8. Production levels between 46 and 111 cause the producer to break even.

22. $C(x) = 800 + (6.5 \times 10^{-7})x^3 - (3.9 \times 10^{-3})x^2 + 9.8x$.

If n is the number of \$1 increases, then $p = 10 + n$. $x = 500 - 50n$, so $n = \frac{500 - x}{50}$. $p = 10 + \frac{500 - x}{50} = 20 - \frac{x}{50}$. $R(x) = px = 20x - \frac{x^2}{50}$, $0 \le x \le 1000$.

(a) $P(x) = R(x) - C(x) = 20x - \frac{x^2}{50} - [800 + (6.5 \times 10^{-7})x^3 - (3.9 \times 10^{-3})x^2 + 9.8x] = 10.2x - (16.1)(10^{-3})x^2 - (6.5 \times 10^{-7})x^3 - 800$.

(b) $P'(x) = 10.2 - (32.2)(10^{-3})x - (19.5 \times 10^{-7})x^2$.

(c) The roots of $P'(x) = 0$ are 311 and -16,824. Rejecting -16,824 because it does not lie in the interval (0,1000), we find 311 is the only critical point for P in (0,1000). Checking the values of P(x) at the critical point and at the endpoints of [0,1000], we find a maximum profit of \$795 per day by making 311 albums.

(d) $P = 20 - \frac{x}{50}$. When x = 311, $P = 20 - \frac{311}{50} =$ \$13.75.

(e) Using Newton's method, we find the approximate zero of the profit function in [0,1000] to be 91.77. A production level of 92 albums per day causes the producer to break even.

23\. Let n be the number of 10¢ increases above the price of \$10. Then the price per subscriber is $p = 10 + 0.10n$ and the estimated number of subscribers is $600 - 4n$.

(a) Revenue $R = (10 + 0.10n)(600 - 4n)$.
$\frac{dR}{dn} = (10 + 0.10n)(-4) + (600 - 4n) = 0$ gives n = 25, so the price per month per subscription that gives the greatest revenue is 10 + 0.10(25) = \$12.50.

(b) Profit P =
$(10 + 0.10n)(600 - 4n) - 2000 - 3(600 - 4n)$.
$\frac{dP}{dn} = (10 + 0.10n)(-4) + (600 - 4n)(0.10) - 3(-4) =$ 0 gives n = 40. So the price per month per subscription that will bring the greatest profit is 10 + 0.10(40) = \$14.00.

24\. Total receipts to the rental agency are given by

$$R(x) = \begin{cases} ax & 0 \le x \le 12 \\ ax - [0.02(x - 12)]ax & x > 12, \end{cases}$$

where x is the number of cars rented to the members and \$a is the undiscounted rental fee per car. We have

$$R'(x) = \begin{cases} a & 0 < x < 12 \\ (1.24 - 0.04x)a & 12 < x \end{cases}$$

Therefore, R(x) has critical numbers x = 12 and $x = \frac{1.24}{0.04} = 31$. Since $R'(x) > 0$ for $0 < x < 12$ and also for $12 < x < 31$, R(x) is increasing on [0,31]. Since $R'(x) < 0$ for $12 < x$, R(x) is decreasing on $[31,\infty)$. Hence x = 31 gives the maximum value of R(x).

25\. (a) Let m be the number of k dollar increases in price, so that $p = p_1 + nk$ and $q = q_1 - nr$. Solving the first equation for n and substituting into the second equation, we find that $q = q_1 - (\frac{p - p_1}{k})r$ or $\alpha q + p = \beta$, where $\alpha = k/r$ and $\beta = \alpha q_1 + p_1$.

(b) In order for q to be positive, we require $\frac{\beta - p}{\alpha} > 0$; that is, $\beta - p > 0$, or $p < \beta$.

(c) $q = \frac{\beta}{\alpha} - \frac{p}{\alpha}$ and $0 \le \frac{p}{\alpha}$; hence $0 \le q \le \beta/\alpha$.

(d) $R(x) = xp = x(\beta - \alpha x) = \beta x - \alpha x^2$.

(e) $R'(x) = \beta - 2\alpha x$, so R(x) is maximum when $x = \beta/2\alpha$.

26\. (a) Under normal economic conditions, when the price p increases, the demand q decreases; hence, $\frac{dq}{dp} \le 0$, and it follows that $E = -\frac{p}{q}\frac{dq}{dp} \ge 0$.

(b) Given q = x, we have $E = -\frac{p}{x}\frac{dx}{dp}$ and $R(x) = px$. Thus, $\frac{dR}{dp} = x + p\frac{dx}{dp} = x + p(-\frac{xE}{p}) = x - xE = x(1 - E)$.

(c) If E > 1, then $\frac{dR}{dp} = x(1 - E) < 0$, so a slight increase in price will result in a loss of revenue. (Note that a sufficiently large increase in price could cause E to change enough so that E > 1 no longer holds.)

27\. (a) $P(x) = R(x) - C(x) = px - C_0 - Ax = (\beta - \alpha x)x - C_0 - Ax = -\alpha x^2 + (\beta - A)x - C_0$ for $0 \le x \le M$ and $q = x \le \beta/\alpha$ (see part (c) of Problem 25).

(b) $P'(x) = -2\alpha x + (\beta - A)$, so the critical value of x is $x = \frac{\beta - A}{2\alpha} = \frac{1}{2}(\frac{\beta}{\alpha}) - \frac{A}{2\alpha} < \frac{\beta}{\alpha}$, provided that $\frac{\beta - A}{2} \le M$; that is, $\beta - A \le 2\alpha M$. Since the critical value must be positive, $0 < \beta - A$ is also necessary.

28. Suppose units are changed, and let P and Q denote the price per unit and the demand expressed in terms of the new units. Then $P = Kp$ and $Q = kq$, where K and k are constants. Then $dP = Kdp$, $dQ = kdq$ and $-\frac{P}{Q}\frac{dQ}{dP} = -\frac{Kp}{kq}\frac{kdq}{Kdp} = -\frac{p}{q}\frac{dq}{dp}$.

29. (a) $C'(x) = b - 2ax$, so $a > 0$ corresponds to the condition that the marginal cost (approximate cost of producing one more unit) decreases as the production level x increases.

(b) The condition $0 < b - 2ax$ for $0 < x < M$ is equivalent to the condition $0 < x < b/2a$ for $0 < x < M$; hence, it holds if and only if $b/2a \ge M$ and $0 < b$; that is, $a \le b/(2M)$.

(c) $\bar{C}(x) = C(x)/x = C_0/x + b - ax$, so $\bar{C}(M) = C_0/M + b - aM$.

(d) The most efficient production level occurs when $C'(x) = 0$; that is, when $x = b/(2a)$.

(e) Suppose that $0 < a < b/2M$ and $0 \le x \le M$. Then, by part (b), $C'(x) > 0$ for $0 < x < M$; hence, $C(x)$ is an increasing function for $0 \le x \le M$. In particular, for $0 \le x \le M$ we have $C(x) \ge C(0)$; that is $C(x) \ge C_0$. Since $C_0 \ge 0$, it follows that $C(x) \ge 0$ for $0 \le x \le M$.

30. (a) $P(x) = R(x) - C(x) = px - C_0 - bx + x^2 = x - x^2 - C_0 - bx + ax^2 = (a - \alpha)x^2 + (\beta - b)x - C_0$.

(b) $P'(x) = 2(a - \alpha)x + (\beta - b)$. Hence, if $a - \alpha < 0$, $P(x)$ is maximum when $x = \frac{\beta - b}{2(\alpha - a)}$.

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1. $f(x) = x^2 + 6x - 7$, $[a,b] = [-7,1]$. f is continuous on $[-7,1]$ and differentiable on $(-7,1)$. $f(-7) = f(1) = 0$. So there is a c in $(-7,1)$ such that $f''(c) = 2c + 6 = 0$. Thus, $c = -3$.

2. $f(x) = x^3 - x$, $[a,b] = [0,1]$. f is continuous on $[0,1]$ and differentiable on $(0,1)$. $f(0) = f(1) = 0$. So there is a c in $(0,1)$ such that $f'(c) = 3c^2 - 1 = 0$. $c^2 = \frac{1}{3}$, so $c = \pm\sqrt{\frac{1}{3}}$. Reject $c = -\sqrt{\frac{1}{3}}$ since it is not in the interval $(0,1)$, so $c = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$.

3. $f(x) = 4x^3 - 21x^2 + 25$, $[a,b] = [-1,5]$. f is continuous on $[-1,5]$ and differentiable on $(-1,5)$. $f(-1) = f(5) = 0$. So there is a c in $(-1,5)$ such that $f'(c) = 12c^2 - 42c = 6c(2c - 7) = 0$. Thus, $c = 0$ and $c = \frac{7}{2}$, both in the interval $(-1,5)$.

4. $f(x) = 2x^3 - 27x^2 + 25x$, $[a,b] = [0,1]$. f is continuous on $[0,1]$ and differentiable on $(0,1)$. $f(0) = f(1) = 0$. So there is a c in $(0,1)$ such that $f'(c) = 6c^2 - 54c + 25 = 0$. Thus,
$$c = \frac{54 \pm \sqrt{54^2 - 4(6)(25)}}{12} = \frac{54 \pm \sqrt{2316}}{12} = \frac{27 \pm \sqrt{579}}{6}.$$
Reject $\frac{27 + \sqrt{579}}{6}$ since it is not in the interval $(0,1)$, so $c = \frac{27 - \sqrt{579}}{6} \approx 0.49$.

5. $f(x) = \cos x$, $[a,b] = [-\frac{\pi}{2}, \frac{3\pi}{2}]$. f is continuous on $[a,b]$ and differentiable on (a,b). $f(-\frac{\pi}{2}) = f(\frac{3\pi}{2}) = 0$. So there is a c in (a,b) such that $f'(c) = -\sin c = 0$. Thus, $c = 0$ and π.

6. $f(x) = \sin^2 x$, $[a,b] = [0,2\ ]$. f is continuous on $[0,2\pi]$ and differentiable on $(0,2\pi)$. $f(0) = f(2\pi) = 0$. So there is a c in $(0,2\pi)$ such that $f'(x) = 2\sin c\cos c = \sin 2c = 0$. Thus, $c = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

7. (a) f satisfies all hypotheses.

(b) f is not differentiable on $(-1,1)$ at $x = 0$.

(c) f is not differentiable on $(1,3)$ at $x = 2$.

(d) $\bar{f}$ satisfies all hypotheses.

8. Suppose there are two values at which f is 0; that is, $f(x_1) = f(x_2) = 0$, $x_1 \neq x_2$ and x_1 and x_2 are in (a,b). Since f is differentiable on (a,b), f is continuous on (a,b), and so f is differentiable on (x_1,x_2) and continuous on $[x_1,x_2]$. By Rolle's theorem, there exists c in (x_1,x_2) such that $f'(c) = 0$. But this contradicts the fact that $f'(x) > 0$, $a < x < b$. Hence, there is at most one value of x, $a < x < b$, such that $f(x) = 0$.

9. (a) All hypotheses hold for f.
   (b) f is not continuous on $[a,b]$.
   (c) f is not differentiable at 0.
   (d) All hypotheses hold for f.
   (e) All hypotheses hold for f.
   (f) f is not differentiable at 0.
   (g) f is not continuous on $[a,b]$.
   (h) f is not differentiable on $[a,b]$.

10. Applying the mean value theorem to f on $[-1,1]$, we have $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$. Now, $f'(x) = 12x^3 + 6x^2 - 2x + 1$, so $12c^3 + 6c^2 - 2c + 1 = \frac{4 - (-2)}{2} = 3$. Thus, $12c^3 + 6c^2 - 2c - 2 = 0$ or $6c^3 + 3c^2 - c - 1 = 0$. Using synthetic division, we obtain $c = \frac{1}{2}$. Now $f'(\frac{1}{2}) = 12(\frac{1}{8}) + 6(\frac{1}{4}) - 2(\frac{1}{2}) + 1 = 3$ and $f(\frac{1}{2}) = -\frac{5}{16}$. Hence, the equation of the tangent line at $(c,f(c))$ is $y + \frac{5}{16} = 3(x - \frac{1}{2})$.

11. $f'(x) = \frac{1}{2\sqrt{x}}$. $f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}$. Thus, $\sqrt{c} = \frac{3}{2}$, so $c = \frac{9}{4}$.

12. $f'(x) = \frac{(x + 4)(1) - (x - 4)(1)}{(x + 4)^2} = \frac{8}{(x + 4)^2}$.
$f'(c) = \frac{8}{(c + 4)^2} = \frac{f(4) - f(0)}{4 - 0} = \frac{0 - (-1)}{4} = \frac{1}{4}$.
Thus, $(c + 4)^2 = 32$ or $c + 4 = \pm\sqrt{32} = \pm 4\sqrt{2}$, so $c = -4 \pm 4\sqrt{2}$. Reject $-4 - 4\sqrt{2}$ since it is not in $(0,4)$, so $c = -4 + 4\sqrt{2}$.

13. $f'(x) = 3x^2 - 4x + 3$. $f'(c) = 3c^2 - 4c + 3 = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - (-2)}{2} = 3$. So $3c^2 - 4c = 0$ or $c(3c - 4) = 0$; thus, $c = 0$ and $\frac{4}{3}$. Reject 0 since it is not in $(0,2)$, so $c = \frac{4}{3}$.

14. $$f'(x) = \begin{cases} -x & x \leq 1 \\ -\frac{1}{x^2} & x > 1 \end{cases}.$$
So $f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{(1/2) - (3/2)}{2} = -\frac{1}{2}$.
Now $-c = -\frac{1}{2}$, or $-\frac{1}{c^2} = -\frac{1}{2}$; that is, $c = \frac{1}{2}$ or $c = \pm\sqrt{2}$. Reject $-\sqrt{2}$ since it is not in $(0,2)$, so $c = \frac{1}{2}$ and $\sqrt{2}$.

15. $f'(x) = 1 + 2 \sin x \cos x$.
$f'(c) = 1 + 2 \sin c \cos c = \frac{f(2\pi) - f(0)}{2\pi - 0} = \frac{2\pi - 0}{2\pi} = 1$. Thus, $2 \sin c \cos c = 0$ or $\sin 2c = 0$, so $2c = 0, \pi, 2\pi, 3\pi, 4\pi$. Thus, $c = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π. Reject 0 and 2π since they are not in $(0,2\pi)$, so $c = \frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

16. $f'(x) = 1 + \frac{1}{2}(1 - \cos x)^{-1/2}(\sin x)$.
$f'(c) = 1 + \frac{\sin c}{2\sqrt{1 - \cos c}} = \frac{f(\pi/2) - f(-\pi/2)}{(\pi/2) - (-\pi/2)} = \frac{(\frac{\pi}{2} + 1) - (-\frac{\pi}{2} + 1)}{\pi} = 1$. Thus, $\frac{\sin c}{2\sqrt{1 - \cos c}} = 0$, or $\sin c = 0$, so $c = 0$.

17. $f'(x) = -\sin x$. $f'(c) = -\sin c = \frac{f(\pi/3) - f(0)}{(\pi/3) - 0} = \frac{(1/2) - 1}{\pi/3} = -\frac{3}{2\pi}$. We want to solve $\sin c = \frac{3}{2\pi} \approx 0.48$ by Newton's method. Let $x_1 = \frac{\pi}{6}$. Then $x_2 = \frac{\pi}{6} - \frac{(\sin(\pi/6) - (3/2\pi))}{\cos(\pi/6)} \approx 0.497577402$,
$x_3 \approx 0.497767134$, $x_4 \approx 0.497767144$, and $x_5 \approx 0.497767144$. $x_4 = x_5$, so $c \approx 0.497767144$.

18. $f'(x) = 4x^3 + 3x^2 + 4x - 1$. $f'(c) = 4c^3 + 3c^2 + 4c - 1 = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{6 - 6}{2} = 0$ or $4c^3 + 3c^2 + 4c - 1 = 0$. Notice that the equation has a root between 0 and 0.5, so let $x_1 = 0.25$ as a first approximation. Thus, $x_2 = 0.25 -$

$$\frac{4(0.25)^3 + 3(0.25)^2 + 4(0.25) - 1}{12(0.25)^2 + 6(0.25) + 4} \approx 0.210000000,$$

$x_3 \approx 0.208385960$, $x_4 \approx 0.208383471$, $x_5 \approx 0.208383471$.

Since $x_4 = x_5$, $c = 0.208383471$.

19. Let c and d be 2 critical points of f on (a,b). Then, $f'(c) = f'(d) = 0$. Hence, f' is differentiable on (c,d) and $f'(c) = f'(d)$; so by Rolle's theorem, there is an α in (a,b) such that $f''(\alpha) = 0$, which contradicts the assumption that f" maintains a constant algebraic sign on (a,b).

20. Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$, $0 < a < 1$.

Consider [a,1]. Then, $f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(1) - f(a)}{1 - a} = \frac{1 - \sqrt{a}}{1 - a}$, $a < c < 1$. Now $0 < c < 1$, so $0 < \sqrt{c} < 1$ or $1 < \frac{1}{\sqrt{c}}$; thus, $\frac{1}{2} < \frac{1}{2\sqrt{c}} = \frac{1 - \sqrt{a}}{1 - a}$ or $1 - a < 2 - 2\sqrt{a}$ $(1 - a > 0)$ or $2\sqrt{a} < a + 1$ or $\sqrt{a} < \frac{a + 1}{2}$.

21. (a) f is increasing on $(-\infty,a]$ and on $[b,\infty)$;
f is decreasing on [a,b].
(b) f is decreasing on $(-\infty,0]$ and on $[b,\infty)$;
f is increasing on [0,b].
(c) f is increasing on $(-\infty,a]$, on [b,0), on $[0,\infty)$;
f is decreasing on [a,b].
(d) f is increasing on $(-\infty,a)$ and on [0,b];
f is decreasing on [a,0] and on (b,∞).

22. (a) Yes. If $x < y$, then $f(x) < f(y)$ so $3f(x) < 3f(y)$. 3f is increasing on I.
(b) No. If $x < y$, then $f(x) < f(y)$ so $-3f(x) > -3f(y)$. -3f is decreasing on I.
(c) Yes. If $x < y$, then $f(x) < f(y)$ and $g(x) < g(y)$. Thus, $f(x) + g(x) < f(y) + g(y)$ or $(f + g)(x) < (f + g)(y)$ or f + g is increasing on I.
(d) No. If $x < y$, then $f(x) < f(y)$ and $g(x) < g(y)$. But there is no guarantee that $f(x)g(x) < f(y)g(y)$. For example, let $f(x) = x$ and $g(x) = 2x$. Both f and g are increasing on [-1,0], but $f \cdot g$ is decreasing on [-1,0].

23. $f'(x) = 3x^2 + 6x = 3x(x + 2)$.

f': + (−2) − (0) +

f is increasing on $[0,\infty)$ and on $(-\infty,-2]$; f is decreasing on [-2,0].

24. $g'(x) = 3x^2 + 12x + 9 = (3x + 9)(x + 1)$.

g': + (−3) − (−1) +

g is increasing on $[-1,\infty)$ and on $(-\infty,-3]$; g is decreasing on [-3,-1].

25. $g'(x) = \sqrt[3]{x} \cdot 2(x - 4) + \frac{1}{3}x^{-2/3}(x - 4)^2 = (x - 4)x^{-2/3}[2x + \frac{x - 4}{3}] = \frac{(x - 4)(7x - 4)}{3x^{2/3}}$.

g': + (4/7) − (4) +

g is increasing on $[4,\infty)$ and on $(-\infty,4/7]$; g is decreasing on [4/7,4].

26. $h'(x) = 3x^2 - \frac{4}{x^2} = \frac{3x^4 - 4}{x^2}$.

h': + $\left(-\sqrt[4]{\frac{4}{3}}\right)$ − (0) − $\left(\sqrt[4]{\frac{4}{3}}\right)$ +

h is increasing on $[\sqrt[4]{\frac{4}{3}},\infty)$ and on $(-\infty,-\sqrt[4]{\frac{4}{3}}]$; h is decreasing on $[-\sqrt[4]{\frac{4}{3}},0)$ and on $(0,\sqrt[4]{\frac{4}{3}}]$.

27. $f'(x) = 1$, $x \le 0$; $f'(x) = 2x$, $x > 0$.

f': + (0) +

f is increasing on $(-\infty,0]$ and on $(0,+\infty)$.

28. $$g'(x) = \begin{cases} -1/x^2 & x < 0 \\ 2(x - 1) & 0 < x < 2 \\ -3/x^4 & x > 2 \end{cases}$$

g': − (0) − (1) + (2) −

g is decreasing on [0,1], on $(-\infty,0)$, and on $(2,\infty)$; g is increasing on (1,2).

29. $F'(x) = 2\cos x(-\sin x) = -\sin 2x$, $-2\pi < x < 2\pi$.

$-\sin 2x = 0$ when $2x = -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi$; that is, when $x = -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

F': $-$ $+$ $-$ $+$ $-$ $+$ $-$ $+$

$-2\pi \quad -\frac{3\pi}{2} \quad -\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

F is increasing on $[-\frac{3\pi}{2},-\pi][-\frac{\pi}{2},0]$, $[\frac{\pi}{2},\pi]$, and $[\frac{3\pi}{2},2\pi]$; F is decreasing on $[-2\pi,-\frac{3\pi}{2}]$, $[-\pi,-\frac{\pi}{2}]$, $[0,\frac{\pi}{2}]$, and $[\pi,\frac{3\pi}{2}]$.

30. $G'(x) = \frac{1}{2} - \cos x$, $0 < x < 4\pi$. $\frac{1}{2} - \cos x = 0$ when $\cos x = \frac{1}{2}$; that is, when $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$.

G': $-$ $+$ $-$ $+$ $-$

$0 \quad \frac{\pi}{3} \quad \frac{5\pi}{3} \quad \frac{7\pi}{3} \quad \frac{11\pi}{3} \quad 4\pi$

G is increasing on $[\frac{\pi}{3},\frac{5\pi}{3}]$ and $[\frac{7\pi}{3},\frac{11}{3}]$; G is decreasing on $[0,\frac{\pi}{3}]$, $[\frac{5\pi}{3},\frac{7\pi}{3}]$, and $[\frac{11\pi}{3},4\pi]$.

31. $f'(x) = 3x^2 - 2x = x(3x - 2)$. Critical numbers: $0, \frac{2}{3}$. f': $+$ $-$ $+$ (0, $\frac{2}{3}$)

Relative maximum at $x = 0$; relative minimum at $x = \frac{2}{3}$.

32. $g'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1)$. Critical numbers: $-2, 1$. $g''(x) = 12x + 6$. $g''(-2) < 0$; $g''(1) > 0$. Relative maximum at $x = -2$, relative minimum at $x = 1$.

33. $h'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 1)$. Critical numbers: $2, 1$. $h''(x) = 12x - 18$. $h''(2) = 6 > 0$; $h''(1) = -6 < 0$. Relative minimum at $x = 2$; relative maximum at $x = 1$.

34. $F'(x) = x^2 \pm x - 2 = (x + 2)(x - 1)$. Critical numbers $-2,1$. F': $+$ $-$ $+$ (-2, 1)

Relative maximum at $x = -2$; relative minimum at $x = 1$.

35. $G'(x) = x^2 + 2x - 8 = (x + 4)(x - 2)$. Critical numbers: $-4, 2$. G': $+$ $-$ $+$ (-4, 2)

Relative maximum at $x = -4$; relative minimum at $x = 2$.

36. $H'(x) = 4x^3 - 12x + 8 = 4(x^3 - 3x + 2) = 4(x - 1)(x^2 + x - 2) = 4(x - 1)(x + 2)(x - 1)$. Critical numbers: $1, -2$. $H''(x) = 12x^2 - 12$. $H''(1) = 0$; $H''(-2) > 0$. Relative minimum at -2. First derivative shows H': $+$ $+$ (1) so there is no extremum at $x = 1$.

37. $f'(x) = 2(x^2 - 9)(2x) = 4x(x + 3)(x - 3)$. Critical numbers: $0, 3, -3$.

f': $-$ $+$ $-$ $+$ (-3, 0, 3)

Relative maximum at $x = 0$; relative minimum at $x = -3,3$.

38. $g'(x) = (x - 3)^2 2(x + 1) + (x + 1)^2[3(x - 3)] = (x - 3)(x + 1)(5x - 3)$. Critical numbers: $3, -1, \frac{3}{5}$. g': $-$ $+$ $-$ $+$ (-1, $\frac{3}{5}$, 3)

Relative maximum at $x = \frac{3}{5}$; relative minimum at $x = -1,3$.

39. $h'(x) = 2x + 2x^{-3} = 2x + \frac{2}{x^3} = \frac{2x^4 + 2}{x^3}$. Since $h(x)$ is not defined at $x = 0$, there are no critical numbers and no extrema. h': $-$ $+$ (0)

40. F': $+$ $+$ (-1)

$F'(x) = \frac{x + 1 - x}{(x + 1)^2} = \frac{1}{(x + 1)^2}$. No extrema.

41. $G'(x) = 2x - 2x^{-2} = 2x - \frac{2}{x^2} = \frac{2x^3 - 2}{x^2} = \frac{2(x^3 - 1)}{x^2}$.

G': $-$ $-$ $+$ (0, 1)

Relative minimum at $x = 1$.

42. $H'(x) = \dfrac{-2x}{(x^2 - 16)^2}$.

Critical number: 0.

$H''(x) = \dfrac{6x^2 + 32}{(x^2 - 16)^3}$. $H''(0) < 0$.

Relative maximum at $x = 0$.

43. $p'(x) = \dfrac{-32x}{(x^2 + 4)^2}$. Critical number: 0.

$p''(x) = \dfrac{32(3x^2 - 4)}{(x^2 + 4)^3}$. Relative maximum at $x = 0$, since $p''(x) < 0$.

44. $q'(x) = x \cdot \frac{1}{2}(x - 1)^{-1/2} + \sqrt{x - 1} =$

$(x - 1) - \frac{1}{2}[\frac{x}{2} + x - 1] = \dfrac{3x - 2}{2\sqrt{x - 1}}$.

Critical numbers: $\frac{2}{3}$, 1. However, we must have $x - 1 \geq 0$; that is, $x \geq 1$.

q': +, with marks at $\frac{2}{3}$ and 1 (+ to the right of 1)

So, there is no relative extrema, since q is always increasing.

45. $r'(x) = \dfrac{2x}{\sqrt{2x^2 + 9}}$.

Critical number: 0.

$r''(x) = \dfrac{18}{(2x^2 + 9)^{3/2}}$. $r''(0) > 0$.

Relative minimum at 0.

46. $p'(x) = \dfrac{-5}{(x + 2)^{3/2}} < 0$. There are no critical numbers and no relative extrema.

47. $Q'(x) = 1 + \cos x$, $-2\pi < x < 2\pi$. $1 + \cos x = 0$ when $x = -\pi, \pi$.

$Q''(x) = -\sin x$. $Q''(-\pi) = Q''(\pi) = 0$. $Q'(\frac{5\pi}{6}) < 0$ and $Q'(\frac{7\pi}{6}) < 0$; so there is no extremum at π. Similarly, there is no extremum at $-\pi$.

48. $R'(x) = 1 - 2\sin 2x$, $0 < x < 2\pi$. $1 - 2\sin 2x = 0$ when $2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$; that is, when $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

$R''(x) = -4\cos 2x$.

$R''(\frac{\pi}{12}) = -4\cos\frac{\pi}{6} < 0$; relative maximum at $\frac{\pi}{12}$.

$R''(\frac{5\pi}{12}) = -4\cos\frac{5\pi}{6} > 0$; relative minimum at $\frac{5\pi}{12}$.

$R'(\frac{13\pi}{12}) = -4\cos\frac{13\pi}{6} > 0$; relative minimum at $\frac{13\pi}{12}$.

$R''(\frac{17\pi}{12}) = -4\cos\frac{17\pi}{6} < 0$; relative maximum at $\frac{17\pi}{12}$.

49. $f'(x) = 2\cos x(-\sin x) - 2\sin x = 2\sin x(-\cos x - 1)$, $-2\pi < x < 2\pi$. Thus, $f'(x) = 0$ when $2\sin x = 0$ or $-\cos x - 1 = 0$, so $x = -\pi, 0, \pi$ or $x = \pi, -\pi$.

Critical numbers: $-\pi, 0, \pi$. $f'(x) = -\sin 2x - 2\sin x$.

f': -2π − $-\pi$ + 0 − π + 2π

Relative minimum at $x = -\pi, \pi$; relative maximum at $x = 0$.

50. $g'(x) = 2\cos 2x - 2\sin x = 2(1 - \sin^2 x) - 2\sin x = 2(1 - \sin x - \sin^2 x) = 2(1 - 2\sin x)(1 + \sin x)$.

Thus, $g'(x) = 0$ when $\sin x = \frac{1}{2}$ or $\sin x = -1$, so $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

$g''(x) = 2(-2\sin 2x - \cos x)$.

$g''(\frac{\pi}{6}) = -3\sqrt{3} < 0$; $g''(\frac{5\pi}{6}) = 3\sqrt{3} > 0$;

$g''(\frac{3\pi}{2}) = 0$. Relative maximum at $x = \frac{\pi}{6}$; relative minimum at $x = \frac{5\pi}{6}$.

g': + $\frac{3\pi}{2}$ +

No extremum at $\frac{3\pi}{2}$.

51. $$h'(x) = \begin{cases} 2 & x > 1 \\ -2x & x < 1 \end{cases}.$$

Critical numbers: 1, 0.

h': + 0 − 1 +

Relative maximum at $x = 0$; relative minimum at $x = 1$.

52. $$F'(x) = \begin{cases} \dfrac{1}{2\sqrt{x + 1}} & -1 < x < 0 \\ \dfrac{1}{2\sqrt{x}} & x > 0 \end{cases}$$

Critical number: 0.

F': $+ \quad \underset{0}{|} \quad +$

No relative extrema.

53. $f'(x) = 3x^2 - 8$.

$f''(x) = 6x$. $\quad f''$: $- \quad \underset{0}{|} \quad +$

f is concave upward on $(0,\infty)$; f is concave downward on $(-\infty,0)$. $f(0) = 0$; $(0,0)$ is a point of inflection.

54. $g'(x) = 3x^2 - 12x + 9$.

$g''(x) = 6x - 12 = 6(x - 2)$.

g'': $- \quad \underset{2}{|} \quad +$

g is concave upward on $(2,\infty)$; g is concave downward on $(-\infty,2)$. $g(2) = -3$; $(2,-3)$ is a point of inflection.

55. $h'(x) = -6x^2 + 8x$.

$h''(x) = -12x + 8 = 4(-3x + 2)$.

h'': $+ \quad \underset{2/3}{|} \quad -$

h is concave upward on $(-\infty,\frac{2}{3})$; h is concave downward on $(\frac{2}{3},\infty)$. $h(\frac{2}{3}) = \frac{167}{27}$; $(\frac{2}{3}, \frac{167}{27})$ is a point of inflection.

56. $F(x) = x^2(x^2 - 6) = x^4 - 6x^2$.

$F'(x) = 4x^3 + 12x$.

$F''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x+1)(x-1)$.

F'': $+ \quad \underset{-1}{|} \quad - \quad \underset{1}{|} \quad +$

F is concave upward on $(-\infty,-1)$ and $(1,\infty)$; F is concave downward on $(-1,1)$. $F(-1) = -5$, $F(1) = -5$; $(-1,-5)$ and $(1,-5)$ are points of inflection.

57. $G'(x) = 6x^2 + 8x + 2$.

$G''(x) = 12x + 8 = 4(3x + 2)$.

G'': $- \quad \underset{-2/3}{|} \quad +$

G is concave upward on $(-\frac{2}{3},\infty)$; G is concave downward on $(-\infty,-\frac{2}{3})$. $G(-\frac{2}{3}) = \frac{23}{27}$; $(-\frac{2}{3},\frac{23}{27})$ is a point of inflection.

58. $H'(x) = 4x^3 - 24x^2 + 64$.

$H''(x) = 12x^2 - 48x = 12x(x - 4)$.

H'': $+ \quad \underset{0}{|} \quad - \quad \underset{4}{|} \quad +$

H is concave upward on $(-\infty,0)$ and $(4,\infty)$; H is concave downward on $(0,4)$. $H(0) = 8$, $H(4) = 0$; $(0,8)$ and $(4,8)$ are points of inflection.

59. $p'(x) = 8x^3 + 12x^2 - 48x + 1$.

$p''(x) = 24x^2 + 24x - 48 = 24(x^2 + x - 2) = 24(x + 2)(x - 1)$. $\quad p''$: $+ \quad \underset{-2}{|} \quad - \quad \underset{1}{|} \quad +$

p is concave upward on $(-\infty,-2)$ and $(1,\infty)$; p is concave downward on $(-2,1)$. $p(-2) = -101$, $p(1) = -20$; $(-2,-101)$ and $(1,-20)$ are points of inflection

60. $q'(x) = \dfrac{6 - 2x}{(x + 3)^3}$.

$q''(x) = \dfrac{4x - 24}{(x + 3)^4}$. $\quad q''$: $- \quad \underset{-3}{|} \quad - \quad \underset{6}{|} \quad +$

q is undefined at -3; q is concave downward on $(-\infty,-3)$ and $(-3,6)$; q is concave upward on $(6,\infty)$. $q(6) = \frac{4}{27}$; $(6,\frac{4}{27})$ is a point of inflection.

61. $r'(x) = \dfrac{-3x^2 - 27}{(x^2 - 9)^2}$.

$r''(x) = \dfrac{6x^3 + 162x}{(x^2 - 9)^3} = \dfrac{6x(x^2 + 27)}{(x^2 - 9)^3}$.

r'': $- \quad \underset{-3}{|} \quad + \quad \underset{0}{|} \quad - \quad \underset{3}{|} \quad +$

r is not defined at 3 and -3. r is concave upward on $(3,+\infty)$ and $(-3,0)$; r is concave downward on $(-\infty,-3)$ and $(0,3)$. $r(0) = 0$; $(0,0)$ is a point of inflection.

62. $s'(x) = \dfrac{-x^2 - 2x + 1}{(x^2 + 1)^2}$.

$s''(x) = \dfrac{2(x - 1)(x^2 + 4x + 1)}{(x^2 + 1)^3}$. $s''(x) = 0$ when $x^2 + 4x + 1 = 0$ or $x - 1 = 0$; that is, when $x = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$ or when $x = 1$.

s": − $-2-\sqrt{3}$ + $-2+\sqrt{3}$ − 1 +

s is concave upward on $(-2 - \sqrt{3}, -2 + \sqrt{3})$ and $(1,+\infty)$; s is concave downward on $(-\infty,-2 - \sqrt{3})$ and $(-2 + \sqrt{3},1)$. $s(-2 - \sqrt{3}) = \frac{-1 - \sqrt{3}}{8 + 4\sqrt{3}} = \frac{1 - \sqrt{3}}{4}$; $s(-2 + \sqrt{3}) = \frac{-1 + \sqrt{3}}{8 - 4\sqrt{3}} = \frac{1 + \sqrt{3}}{4}$; $s(1) = 1$.

$(-2 - \sqrt{3}, \frac{1 - \sqrt{3}}{4})$, $(-2 + \sqrt{3}, \frac{1 + \sqrt{3}}{4})$, and $(1,1)$ are points of inflection.

63. $P'(x) = -4 \sin 2x$, $0 < x < 2\pi$.

$P''(x) = -8 \cos 2x$, $0 < x < 2\pi$. $P''(x) = 0$ when $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$; that is, when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

P": 0 − $\frac{\pi}{4}$ + $\frac{3\pi}{4}$ − $\frac{5\pi}{4}$ + $\frac{7\pi}{4}$ − 2π

P is concave upward on $(\frac{\pi}{4}, \frac{3\pi}{4})$, $(\frac{5\pi}{4}, \frac{7\pi}{4})$; P is concave downward on $(0,\frac{\pi}{4})$, $(\frac{3\pi}{4}, \frac{5\pi}{4})$, $(\frac{7\pi}{4}, 2\pi)$. $P(\frac{\pi}{4}) = 0$; $P(\frac{3\pi}{4}) = 0$; $P(\frac{5\pi}{4}) = 0$; $P(\frac{7\pi}{4}) = 0$.

$(\frac{\pi}{4},0)$, $(\frac{3\pi}{4},0)$, $(\frac{5\pi}{4},0)$ and $(\frac{7\pi}{4},0)$ are points of inflection.

64. $Q'(x) = 8 \cos x + 2 \cos 2x$, $0 < x < 2\pi$.

$Q''(x) = -8 \sin x - 4 \sin 2x = -8 \sin x - 4(2 \sin x \cos x) = -8 \sin x(1 + \cos x)$, $0 < x < 2\pi$.

$Q''(x) = 0$ when $-8 \sin x = 0$ or $\cos x + 1 = 0$; that is, when $x = \pi$ or $x = \pi$.

Q": 0 − π + 2π

Q is concave upward on $(\pi,2\pi)$; Q is concave downward on $(0,\pi)$. $Q(\pi) = 0$; $(\pi,0)$ is a point of inflection.

65. $R'(x) = 1 + \sec^2 x$.

$R''(x) = 2 \sec x \sec x \tan x = 2 \sec^2 x \tan x = 2 \frac{1}{\cos^2 x} \frac{\sin x}{\cos x} = \frac{2 \sin x}{\cos^3 x}$. $R''(x) = 0$ when $2 \sin x = 0$, so $x = k\pi$, k an integer.

R": -2π + $-\frac{3\pi}{2}$ − $-\pi$ + $-\frac{\pi}{2}$ − 0 + $\frac{\pi}{2}$ − π + $\frac{3\pi}{2}$ − 2π

R is concave upward on $(k\pi, \frac{\pi}{2} + k\pi)$, k an integer; R is concave downward on $(\frac{\pi}{2} + k\pi, k\pi + \pi)$, k an integer. $R(k\pi) = k\pi + \tan k\pi = k\pi$; $(k\pi,k\pi)$ is a point of inflection, k an integer.

66.
$$f'(x) = \begin{cases} 2 \cos 2x & -\pi < x < 0 \\ -2 \sin 2x & 0 < x < \pi \end{cases}.$$
$$f''(x) = \begin{cases} -4 \sin 2x & -\pi < x < 0 \\ -4 \cos 2x & 0 < x < \pi \end{cases}.$$

f is concave upward on $(-\frac{\pi}{2},0)$, $(\frac{\pi}{4},\frac{3\pi}{4})$; f is concave downward on $(-\pi,-\frac{\pi}{2})$, $(0,\frac{\pi}{4})$, $(\frac{3\pi}{4},\pi)$. $f(-\frac{\pi}{2}) = 1$, $f(\frac{\pi}{4}) = 0$, $f(\frac{3\pi}{4}) = 0$, $f(0) = 1$; $(0,1)$, $(-\frac{\pi}{2},1)$, $(\frac{\pi}{4},0)$, and $(\frac{3\pi}{4},0)$ are points of inflection.

f": $-\pi$ − $-\frac{\pi}{2}$ + 0 − $\frac{\pi}{4}$ + $\frac{3\pi}{4}$ − π

67. $y = \frac{3x^2}{25} - \frac{x^3}{150}$, $0 \le x \le 16$.

(a) $y' = \frac{6x}{25} - \frac{x^2}{50}$.

$y'' = \frac{6}{25} - \frac{x}{25}$. $y'' = 0$ for $x = 6$.

y": + 6 −

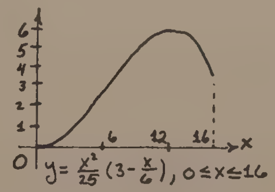
At $x = 6$, the point of diminishing returns occurs. Note: When $x = 6$, $y = 2.88$.

(b) $y' = 0$ or $\frac{12x - x^2}{50} = \frac{x(12 - x)}{50} = 0$.

y': 0 + 12 − 16

$x = 12$ is a relative maximum. When $x = 0$, $y = 0$; when $x = 12$, $y = 5.76$; when $x = 16$, $y \approx 3.41$. So $x = 12$ gives an absolute maximum.

(c)

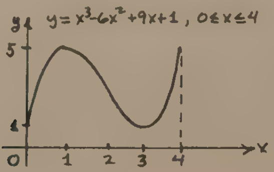


68. Since f and g are continuous and since $f(x_1) > 0$ and $g(x_1) > 0$, there will exist numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $f(x) > 0$ for $x_1 - \varepsilon_1 < x < x_1 + \varepsilon_1$ and $g(x) > 0$ for $x_1 - \varepsilon_2 < x < x_1 + \varepsilon_2$. Since f and g have relative minima at x_1, there will exist numbers $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ such that $f(x) \geq f(x_1)$ for $x_1 - \varepsilon_3 < x < x_1 + \varepsilon_3$ and $g(x) \geq g(x_1)$ for $x_1 - \varepsilon_4 < x < x_1 + \varepsilon_4$. Let ε be the smallest of the four numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$. Then, for $x_1 - \varepsilon < x < x_1 + \varepsilon$, $f(x) > 0$, $g(x) > 0$, $f(x_1) > 0$, $g(x_1) > 0$, $f(x) \geq f(x_1)$, and $g(x) \geq g(x_1)$; hence, $f(x)g(x) \geq f(x_1)g(x_1)$. Therefore, fg has a relative minimum at x_1.

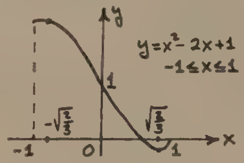
69. $f'(x) = \frac{ad - bc}{(cx + d)^2}$, $f''(x) = \frac{2c(bc - ad)}{(cx + d)^3}$ for $x \neq -\frac{d}{c}$. At $x = -\frac{d}{c}$, the graph of f has a vertical asymptote. Although the direction of concavity changes as one passes through $x = -\frac{d}{c}$, the function is not defined there; hence, there is no inflection point.

70. $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b = 0$ when $x = -\frac{b}{3a}$. When $x > -\frac{b}{3a}$, f is concave upward; when $x < -\frac{b}{3a}$, f is concave downward, so there is one point of inflection at $x = -\frac{b}{3a}$. Since f" is positive for $x > -\frac{b}{3a}$, we can conclude that f' is increasing on $[-\frac{b}{3a}, \infty)$.

71. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 3)(x - 1)$. Critical numbers: 1, 3. $f(3) = 1$, $f(1) = 5$, $f(0) = 1$, $f(4) = 5$. The absolute maximum is 5 and occurs at 1 and 4; the absolute minimum is 1 and occurs at 3 and 0.



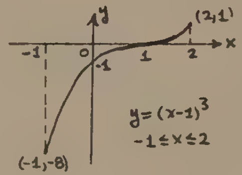
72. $g'(x) = 3x^2 - 2$. Critical numbers: $\pm\sqrt{\frac{2}{3}}$. $g(+\sqrt{\frac{2}{3}}) \approx -0.09$, $g(-\sqrt{\frac{2}{3}}) \approx 2.09$, $g(-1) = 2$, $g(1) = 0$. The absolute maximum is $g(-\sqrt{2/3}) \approx 2.09$ and occurs at $-\sqrt{2/3}$; the absolute minimum is $g(\sqrt{2/3}) \approx -0.09$ and occurs at $+\sqrt{2/3}$.



73. $h'(x) = 3(x - 1)^2$. Critical number: 1. $h(1) = 0$, $h(-1) = -8$, $h(2) = 1$. Absolute minimum is -8 and occurs at -1; absolute maximum is 1 and occurs is 2. Graph is always increasing.

h': + +

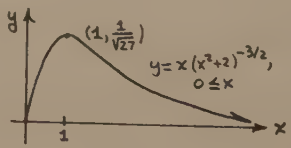
1



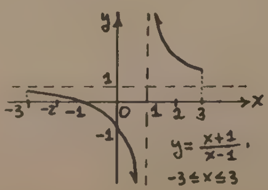
74. $F'(x) = \frac{2 - 2x^2}{(x^2 + 2)^{5/2}}$. Critical number: 1.

$F(0) = 0$, $F(1) = 3^{-3/2} = \frac{1}{3^{3/2}} = \frac{1}{\sqrt{27}} \approx 0.19$.

Absolute maximum is $\frac{1}{\sqrt{27}}$ and occurs at 1; absolute minimum is 0 and occurs at 0.

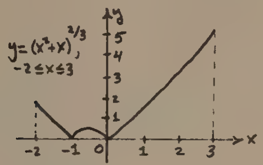


75. $G'(x) = \frac{-2}{(x - 1)^2}$. No maximum and no minimum.



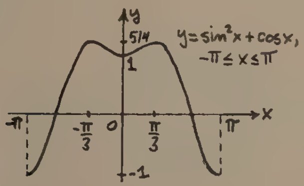
76. $H'(x) = \frac{2}{3}(x^2 + x)^{-1/3}(2x + 1) = \frac{2(2x + 1)}{3(x^2 + x)^{1/3}}$.

$H(-2) = 2^{2/3} \approx 1.59$; $H(3) = 12^{2/3} \approx 5.24$; $H(-\frac{1}{2}) = (\frac{1}{2})^{2/3} \approx 0.63$; $H(-1) = H(0) = 0$. Absolute minimum of 0 at -1 and at 0; absolute maximum of 5.24 at 3.

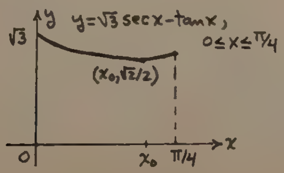


77. Note: $f'(x) = f(-x)$ for all x between $-\pi$ and π.

$f'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1) = \sin 2x - \sin x$. $f'(x) = 0$ when $\sin x = 0$ and $2 \cos x - 1 = 0$; that is, when $x = 0$ and $x = -\frac{\pi}{3}$ and $\frac{\pi}{3}$. $f(-\pi) = f(\pi) = -1$, $f(-\frac{\pi}{3}) = f(\frac{\pi}{3}) = \frac{5}{4}$, $f(0) = 1$. Absolute maximum is $\frac{5}{4}$ and occurs at $-\frac{\pi}{3}$ and $\frac{\pi}{3}$; absolute minimum is -1 and occurs at $-\pi$ and π.



78. $g'(x) = \sqrt{3} \sec x \tan x - \sec^2 x = \sec x (\sqrt{3} \tan x - \sec x)$. $g'(x) = 0$ when $\sec x = 0$ and $\sqrt{3} \tan x - \sec x = 0$. $\sec x = 0$ has no solution. $\sqrt{3} \tan x - \sec x = 0$, or $\sqrt{3} \sin x = 1$, or $\sin x = \frac{1}{\sqrt{3}}$, so $x \approx 0.6155$. Call this solution x_0. $g(0) = \sqrt{3} = 1.73$, $g(\frac{\pi}{4}) = \sqrt{6} - 1 \approx 1.45$, $g(x_0) = \sqrt{2}/2$. Absolute maximum is $\sqrt{3}$ and occurs at 0; absolute minimum is $\sqrt{2}/2$ and occurs at x_0.



79. $f'(x) = 1 - 2x = 0$; $x = \frac{1}{2}$. f': + | - (at 1/2)

Maximum of $\frac{5}{4}$ at $x = \frac{1}{2}$; no minimum.

80. $g'(x) = 3x^2 + 1 = 0$; no solution. g is always increasing; no extrema.

81. $h'(x) = 4x^3 - 6x = 2x^2(2x - 3) = 0$; $x = 0, \frac{3}{2}$.

h': - | - | + (at 0, $\frac{3}{2}$)

Minimum of $-\frac{11}{16}$ at $x = \frac{3}{2}$; no maximum.

82. $F'(x) = 4x^3 + 4 = 0$; $x = -1$. F': - | + (at -1)

Minimum of -6 at $x = -1$.

83. $G'(x) = \frac{-x^2 - 2x + 1}{(x^2 + 1)^2} = 0$; $-x^2 - 2x + 1 = 0$ so $x = \frac{2 \pm \sqrt{8}}{-2} = -1 \mp \sqrt{2}$.

G': - | + | - (at $-1 - \sqrt{2}$, $-1 + \sqrt{2}$)

Minimum of $G(-1 - \sqrt{2}) = \frac{-\sqrt{2}}{4 + 2\sqrt{2}} = \frac{1 - \sqrt{2}}{2}$ at $x = -1 - \sqrt{2}$; maximum of $G(-1 + \sqrt{2}) = \frac{\sqrt{2}}{4 - 2\sqrt{2}} = \frac{1 + \sqrt{2}}{2}$ at $x = -1 + \sqrt{2}$.

84. $H'(x) = \frac{8x}{(x^2 + 4)^2} = 0$; $x = 0$.

H': - | + (at 0)

Minimum of 0 at $x = 0$.

85. Note: For p(x), $x > 0$. $p'(x) = 1 - \frac{1}{2}x^{-3/2} = 1 - \frac{1}{2x^{3/2}} = \frac{2x^{3/2} - 1}{2x^{3/2}} = 0$; $2x^{3/2} = 1$ so $x = (1/2)^{2/3} = 1/\sqrt[3]{4}$. p': | - | + (at 0, $1/\sqrt[3]{4}$)

Minimum of $p(\frac{1}{\sqrt[3]{4}}) = \frac{1}{\sqrt[3]{4}} + \sqrt[3]{2}$ at $x = \frac{1}{\sqrt[3]{4}}$.

86. Note: For q(x), $x \ge -1$. $q'(x) = \frac{x}{2}(x + 1)^{-1/2} + \sqrt{x + 1} = \frac{3x + 2}{2\sqrt{x + 1}} = 0$. Critical numbers: $-1, -\frac{2}{3}$.

q': | - | + (at -1, $-\frac{2}{3}$)

Minimum of $\frac{-2\sqrt{3}}{9}$ at $x = -\frac{2}{3}$.

87. $r(x) = 1 - 2\sin^2 x = \cos 2x$.

$r'(x) = -2\sin 2x = 0$; $2x = k\pi$, k an integer, so $x = \frac{k}{2}\pi$. Maximum of 1 at $x = k\pi$, k an integer; minimum of -1 at $x = (2k+1)\frac{\pi}{2}$, k an integer.

88. $R'(x) = \frac{1}{2}(\sec x)^{-1/2}\sec x \tan x = \frac{1}{2}(\sec x)^{1/2}\tan x = 0$; $\tan x = 0$ for $x = k\pi$, k an integer. Minimum occurs at $x = 2k\pi$, k an integer; minimum value is 1, since $\sqrt{\sec x} = \sqrt{1/\cos x}$ is greater than or equal to 1.

89. $\lim\limits_{x\to+\infty} \dfrac{8x^2 + x - 3}{4x^2 + 71} = \lim\limits_{x\to+\infty} \dfrac{8 + \frac{1}{x} - \frac{3}{x^2}}{4 + \frac{71}{x^2}} = \dfrac{8 + 0 - 0}{4 + 0} = \dfrac{8}{4} = 2.$

90. $\lim\limits_{x\to-\infty} \dfrac{x^2 + 1}{5x + 3} = \lim\limits_{x\to-\infty} \dfrac{1 + (1/x^2)}{(5/x) + (3/x^2)} = -\infty.$

91. $\lim\limits_{t\to-\infty} \dfrac{5t}{t^2 + 1} = \lim\limits_{t\to-\infty} \dfrac{5/t}{1 + (1/t^2)} = 0.$

92. $\lim\limits_{t\to+\infty} \dfrac{3t^{-2} + 7t^{-3}}{7t^{-2} + 5t^{-3}} = \lim\limits_{t\to+\infty} \dfrac{3 + 7t^{-1}}{7 + 5t^{-1}} = \dfrac{3}{7}.$

93. $\lim\limits_{h\to+\infty} \dfrac{h^2 - 3h}{\sqrt{5h^4 + 7h^2 + 3}} = \lim\limits_{h\to+\infty} \dfrac{1 - (3/h)}{\sqrt{5 + \frac{7}{h^2} + \frac{3}{h^4}}} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}.$

94. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{7x^6 + 5x^4 + 7}}{x^4 + 2} = \lim\limits_{x\to-\infty} \dfrac{\sqrt{\frac{7}{x^2} + \frac{5}{x^4} + \frac{7}{x^5}}}{1 + \frac{2}{x^4}} = 0.$

95. $\lim\limits_{y\to-\infty} (4y^2 - 7y) = \lim\limits_{y\to-\infty} y(4y - 7) = +\infty.$

96. $\lim\limits_{\theta\to+\infty} (\theta - \theta\cos\frac{1}{\theta}) = \lim\limits_{\theta\to+\infty} \theta(1 - \cos\frac{1}{\theta}) = \lim\limits_{\theta\to+\infty} \dfrac{1 - \cos\frac{1}{\theta}}{\frac{1}{\theta}} = \lim\limits_{t\to0} \dfrac{1 - \cos t}{t} = 0.$

97. $\lim\limits_{t\to3^-} \dfrac{t}{t^2 - 9} = -\infty.$

98. $\lim\limits_{y\to2^+} \dfrac{\sqrt{y} - 2}{y^2 - 4}\,\dfrac{\sqrt{y - 2}}{\sqrt{y - 2}} = \lim\limits_{y\to2^+} \dfrac{y - 2}{(y + 2)(y - 2)\sqrt{y - 2}} = \lim\limits_{y\to2^+} \dfrac{1}{(y + 2)\sqrt{y - 2}} = +\infty.$

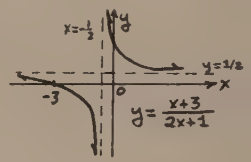
99. $\lim\limits_{x\to0^+} \dfrac{x^2 - 3}{x^2 - x} = \lim\limits_{x\to0^+} \dfrac{x^2 - 3}{x(x - 1)} = +\infty.$

100. $\lim\limits_{x\to0^-} \csc x = \lim\limits_{x\to0^-} \dfrac{1}{\sin x} = -\infty.$

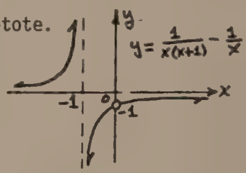
101. $\lim\limits_{x\to\pi^-} \cot x = \lim\limits_{x\to\pi^-} \dfrac{\cos x}{\sin x} = -\infty.$

102. $\lim\limits_{x\to2^-} (3 + [\![2x - 4]\!]) = 3 + (-1) = 2.$

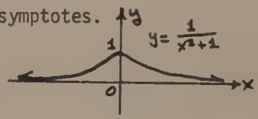
103. $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{x + 3}{2x + 1} = \dfrac{1}{2} = \lim\limits_{x\to-\infty} f(x)$; so $y = \frac{1}{2}$ is a horizontal asymptote. $x = -\frac{1}{2}$ is a vertical asymptote.



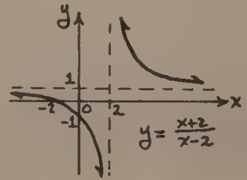
104. $g(x) = \dfrac{1}{x(x + 1)} - \dfrac{1}{x} = \dfrac{-x}{x(x + 1)} = -\dfrac{1}{x + 1}$. Note: Domain is all x except 0 and -1. $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to-\infty} g(x) = 0$; so $y = 0$ is a horizontal asymptote. $x = -1$ is a vertical asymptote.



105. $\lim\limits_{x\to+\infty} h(x) = \lim\limits_{x\to-\infty} h(x) = 0$; so $y = 0$ is a horizontal asymptote. No vertical asymptotes.



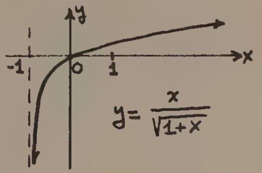
106. $\lim\limits_{x\to+\infty} \dfrac{x + 2}{x - 2} = \lim\limits_{x\to+\infty} \dfrac{1 + \frac{2}{x}}{1 - \frac{2}{x}} = 1 = \lim\limits_{x\to-\infty} F(x)$. Hence, $y = 1$ is a horizontal asymptote. $x = 2$ is a vertical asymptote.



107. Note: Domain is all $x > -1$. $\lim\limits_{x\to+\infty} \dfrac{x}{\sqrt{1 + x}} = \lim\limits_{x\to+\infty} \dfrac{1}{\sqrt{\frac{1}{x^2} + \frac{1}{x}}} = +\infty$; so no horizontal asymptotes.

$\lim_{x\to -1^+} G(x) = -\infty$; so $x = -1$ is a vertical asymptote. $G'(x) = \dfrac{2 + x}{2(1 + x)^{3/2}}$. G': $\underset{-2}{\overset{+}{\text{———|———}}}$

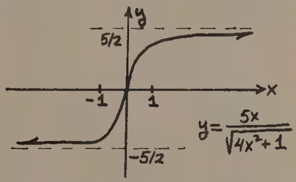
G is always increasing.



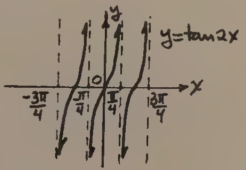
108. Note: $H(-x) = H(x)$.

$\lim_{x\to+\infty} \dfrac{5x}{\sqrt{4x^2 + 1}} = \lim_{x\to\infty} \dfrac{5}{\sqrt{4 + (1/x^2)}} = \dfrac{5}{2}$; so $y = \dfrac{5}{2}$ is a horizontal asymptote. By symmetry, $y = -\dfrac{5}{2}$ is also a horizontal asymptote. No vertical asymptotes.

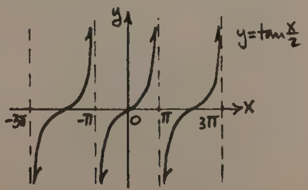
$H'(x) = \dfrac{5}{(4x^2 + 1)^{3/2}} > 0$; so H is always increasing.



109. $\lim_{x\to+\infty} \tan 2x$ and $\lim_{x\to-\infty} \tan 2x$ are not defined; so no horizontal asymptotes. Vertical asymptotes occur at those values of x that satisfy $\cos 2x = 0$; that is, when $2x$ = odd multiples of $\frac{\pi}{2}$, or x = odd multiples of $\frac{\pi}{4}$. Hence, the vertical asymptotes are the lines $x = (2k - 1)\frac{\pi}{4}$, k an integer.



110. $\lim_{x\to+\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$ are not defined. Vertical asymptotes occur when $\cos \frac{x}{2} = 0$; that is, when $\frac{x}{2} = (2k + 1)\frac{\pi}{2}$, k an integer, or when $x = (2k + 1)$, k an integer.



111. $f(x) = 2x + 3 + \frac{1}{x}$. $\lim_{x\to+\infty} \frac{1}{x} = 0$. Therefore, $y = 2x + 3$ is an oblique asymptote.

112. $g(x) = 3x - 2 + \dfrac{2x}{x^2 + 1}$. $\lim_{x\to+\infty} \dfrac{2x}{x^2 + 1} = \lim_{x\to+\infty} \dfrac{2/x}{1 + (1/x^2)} = \dfrac{0}{1 + 0} = 0$. Hence, $y = 3x - 2$ is an oblique asymptote.

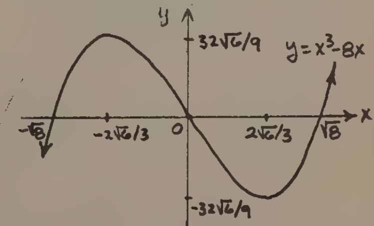
113. $\lim_{x\to+\infty} \dfrac{x - 2}{x^2 - 1} = \lim_{x\to+\infty} \dfrac{1/x - (2/x^2)}{1 - \frac{1}{x^2}} = \dfrac{0 - 0}{1 - 0} = 0$. Hence, $y = 1 - x$ is an oblique asymptote.

114. $\lim_{x\to+\infty} x \cos \frac{1}{x} = \lim_{x\to\infty} \dfrac{\cos(1/x)}{1/x} = \lim_{t\to 0} \dfrac{\cos t}{t} = 0$. Thus, $y = 2x$ is an oblique asymptote.

115. f is continuous at $x = -1$. $\dfrac{f(-1 + \Delta x) - f(-1)}{\Delta x} = \dfrac{\sqrt[3]{\Delta x} - 0}{\Delta x} = \dfrac{1}{\Delta x^{2/3}}$. Now, $\lim_{\Delta x\to 0} \left|\dfrac{1}{\Delta x^{2/3}}\right| = +\infty$; so f has a vertical tangent $x = -1$ at $(-1,0)$.

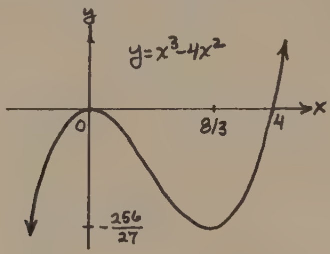
116. g is continuous at $x = 0$. $\dfrac{g(0 + \Delta x) - g(0)}{\Delta x} = \dfrac{(\Delta x)^{1/5} - 1 - (-1)}{\Delta x} = \dfrac{1}{\Delta x^{4/5}}$. Now, $\lim_{\Delta x\to 0} \left|\dfrac{1}{\Delta x^{4/5}}\right| = +\infty$; so g has a vertical tangent $x = 0$ at $(0,-1)$.

117. $f(x) = x^3 - 8x$. f is odd, so symmetric about the origin. y intercept: 0; x intercept: 0, $\pm\sqrt{8} \approx \pm 2.83$. $f'(x) = 3x^2 - 8$. Increasing on $(-\infty, -\sqrt{\frac{8}{3}}]$ and on $[\sqrt{\frac{8}{3}}, \infty)$; decreasing on $[-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}]$. Relative maximum at $-\sqrt{\frac{8}{3}} \approx 1.63$ of $\frac{32}{9}\sqrt{6} \approx 8.71$; relative minimum at $\sqrt{\frac{8}{3}} \approx 1.63$ of $-\frac{32}{9}\sqrt{6} \approx -8.71$. $f''(x) = 6x$. Concave downward on $(-\infty,0)$; concave upward on $(0,\infty)$. Inflection point at $(0,0)$.

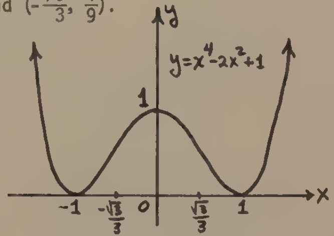


118. $g(x) = x^3 - 4x^2$. g is neither even nor odd. y intercept: 0; x intercept: 0, 4. $g'(x) = 3x^2 - 8x = x(3x - 8)$. Increasing on $(-\infty,0]$ and on $[\frac{8}{3},\infty)$;

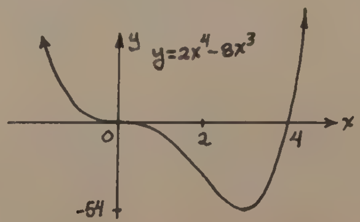
decreasing on $[0,\frac{8}{3}]$. Relative maximum of 0 at 0; relative minimum of $-\frac{256}{27} \approx -9.5$ at $\frac{8}{3}$. $g''(x) = 6x - 8$. Concave downward on $(-\infty,\frac{4}{3})$; concave upward on $(\frac{4}{3},\infty)$. Inflection point at $(\frac{4}{3}, \frac{128}{27})$.



119. $h(x) = x^4 - 2x^2 + 1$. h is even, so symmetric about the y axis. x intercepts: ±1; y intercept: 1. $h'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1)$. Decreasing on $(-\infty,-1]$ and on $[0,1]$; increasing on $[-1,0]$ and on $[1,\infty)$. Relative minimum of 0 at -1 and at 1; relative maximum of 1 at 0. $h''(x) = 12x^2 - 4$. Concave upward on $(-\infty,-\frac{\sqrt{3}}{3})$ and on $(\frac{\sqrt{3}}{3},\infty)$; concave downward on $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$. Points of inflections at $(\frac{\sqrt{3}}{3}, \frac{4}{9})$ and $(-\frac{\sqrt{3}}{3}, \frac{4}{9})$.

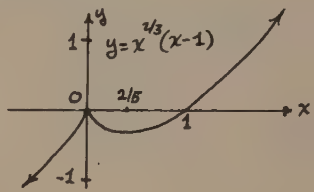


120. $F(x) = 2x^4 - 8x^3$. F is neither even nor odd. y intercept: 0; x intercept = 0,4. $F'(x) = 8x^3 - 24x^2 = 8x^2(x - 3)$. Decreasing on $(-\infty,3]$; increasing on $[3,\infty)$. Relative minimum of -54 at 3. $F''(x) = 24x^2 - 48x = 24x(x - 2)$. Concave upward on $(\infty,0)$ and on $(2,\infty)$; concave downward on $(0,2)$. Points of inflection at $(2,-32)$ and $(0,0)$.

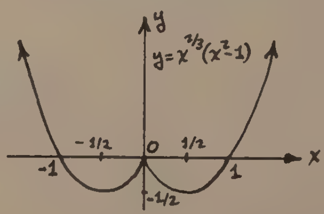


121. $G(x) = x^{2/3}(x - 1)$. G is neither even nor odd. y intercept: 0; x intercepts: 0, 1. $G'(x) = x^{2/3} + (x - 1)^{2/3}x^{-1/3} = \frac{5x - 2}{3x^{1/3}}$. Increasing on $(-\infty,0]$ and on $[\frac{2}{5},\infty)$; decreasing on $[0,\frac{2}{5}]$. Relative maximum of 0 at 0; relative minimum of $-\frac{3}{5}(\frac{2}{5})^{2/3} \approx -0.33$ at $\frac{2}{5}$. $G''(x) =$

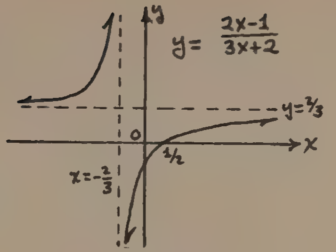
$$\frac{3x^{1/3}(5) - (5x - 2)[3(\frac{1}{3}x^{-2/3})]}{9x^{2/3}} = \frac{10x + 2}{9x^{2/3}},$$ so G is concave upward on all $\mathbb{R}$ except 0.



122. $H(x) = x^{2/3}(x^2 - 1)$. H is even, so symmetric about y axis. y intercept: 0; x intercept: 0, ±1. $H'(x) = x^{2/3}(2x) + (x^2 - 1)\frac{2}{3}x^{-1/3} = \frac{8x^2 - 2}{3x^{1/3}}$. Increasing on $[-\frac{1}{2},0]$ and on $[\frac{1}{2},\infty)$; decreasing on $(-\infty,-\frac{1}{2}]$ and on $[0,\frac{1}{2}]$. Relative minima of $-\frac{3}{4}(\frac{1}{2})^{2/3} \approx -0.47$ at $-\frac{1}{2}$ and at $\frac{1}{2}$; relative maximum of 0 at 0. $H''(x) = \frac{40x^2 + 2}{9x^{4/3}} > 0$ for all x, so H is concave upward. No points of inflection.



123. $f(x) = \frac{2x - 1}{3x + 2}$. f is neither even nor odd. x intercept: $\frac{1}{2}$; y intercept: $-\frac{1}{2}$. Horizontal asymptote: $y = \frac{2}{3}$; vertical asymptote: $x = -\frac{2}{3}$. $f'(x) = \frac{7}{(3x + 2)^2}$, so f is increasing on all $\mathbb{R}$ except at $-\frac{2}{3}$.



124. $g(x) = \dfrac{x^3}{x^2 + x - 6}$. g is neither even nor odd.

y intercept: 0; x intercept: 0. $g'(x) = \dfrac{x^2(x^2 + 2x - 18)}{(x^2 + x - 6)^2}$. $g'(x) = 0$ if $x^2 + 2x - 18 = 0$,

so $x = -1 \pm \sqrt{19}$.

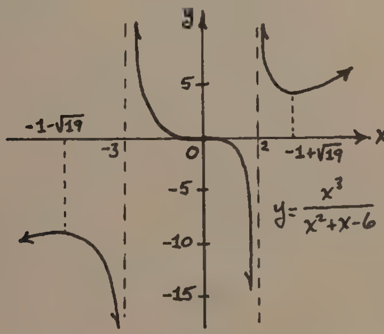
g': + | − | − | − | − | +

$-1-\sqrt{19} \approx -5.36$ -3 0 2 $-1+\sqrt{19} \approx 3.36$

Relative maximum of −8.87 at $-1 - \sqrt{19} \approx -5.36$,

relative minimum of 4.39 at $-1 + \sqrt{19} \approx 3.36$.

Vertical asymptotes: x = −3 and x = 2.



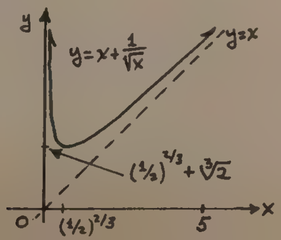
125. $h(x) = x + \dfrac{1}{\sqrt{x}}$. Note: $x > 0$. h is neither even nor odd. y intercept: none. x intercept: none.

$h'(x) = 1 - \frac{1}{2}x^{-3/2} = 1 - \dfrac{1}{2x^{3/2}} = \dfrac{2x^{3/2} - 1}{2x^{3/2}}$.

h': | − | +

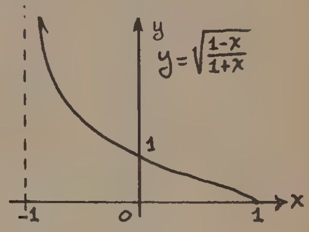
0 $(\frac{1}{2})^{2/3} \approx 0.63$

Relative minimum at $(\frac{1}{2})^{2/3} \approx 0.63$ of $(\frac{1}{2})^{2/3} + \sqrt[3]{2} \approx 1.89$. Vertical asymptote: y = 0. Oblique asymptote: y = x.



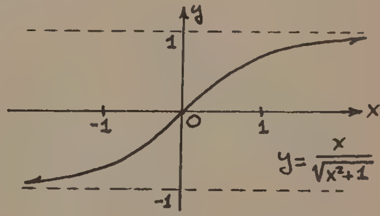
126. $F(x) = \sqrt{\dfrac{1 - x}{1 + x}}$. Note: $-1 < x \le 1$. F is neither even nor odd. y intercept: 1; x intercept: 1.

$F'(x) = \dfrac{-1}{(1 - x)^{1/2}(1 + x)^{3/2}} < 0$ for all x, so F is always decreasing on (−1,1). Vertical asymptote: x = −1.



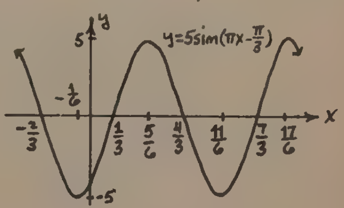
127. $G(x) = \dfrac{x}{\sqrt{x^2 + 1}}$. G is odd, so symmetric about origin. y intercept: 0; x intercept: 0.

$G'(x) = \dfrac{1}{(x^2 + 1)^{3/2}} > 0$ for all x, so G is always increasing.



128. $H(x) = 5 \sin (\pi x - \frac{\pi}{3})$. y intercept: $\frac{-5\sqrt{3}}{2} \approx -4.33$; x intercepts: $\frac{2}{6} + 2k$, k an integer.

$H'(x) = 5\pi \cos(\pi x - \frac{\pi}{3})$. Absolute maximum of 5 occurs at $x = \frac{5}{6} + 2k$, k an integer; absolute minimum of −5 occurs at $x = -\frac{1}{6} + 2k$, k an integer.

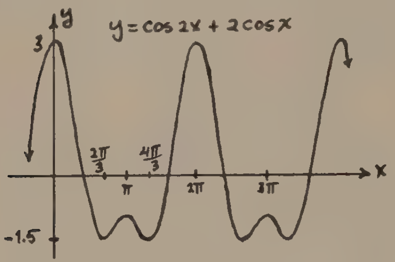


129. $f(x) = \cos 2x + 2 \cos x$. f is even, so symmetric about the y axis. y intercept: 3.

$f'(x) = -2 \sin 2x - 2 \sin x = -4 \sin x \cos x - 2 \sin x = -2 \sin x(2 \cos x + 1)$. $f'(x) = 0$ when $\sin x = 0$ or $2 \cos x + 1 = 0$; that is, when $x = k\pi$, k an integer, or $x = \frac{2}{3}\pi + 2k\pi$ or $\frac{4}{3}\pi + 2k\pi$, k an integer.

f': $-$ on $(0, \frac{2}{3}\pi)$; $+$ on $(\frac{2}{3}\pi, \pi)$; $-$ on $(\pi, \frac{4}{3}\pi)$; $+$ on $(\frac{4}{3}\pi, 2\pi)$

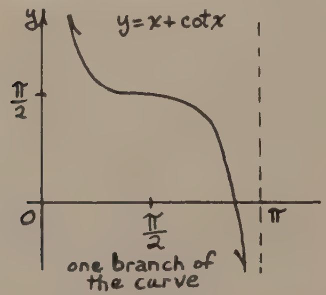
Relative minima of -1.5 at $x = \frac{2\pi}{3} + 2k\pi$ or $\frac{4\pi}{3} + 2k\pi$, k an integer; relative maxima of -1 at $x =$ odd multiples of π; relative maxima of 3 at $x = 2k\pi$, k an integer.



130. $g(x) = x + \cot x$. g is odd, so symmetric about origin. $g'(x) = 1 - \csc^2 x$. $g'(x) = 0$ if $\csc^2 x = 1$ or $x = (\frac{4k+1}{2})\pi$, k an integer, or $x = (\frac{4k+3}{2})\pi$, k an integer. $g''(x) = 2\csc^2 x \cot x$. $g''(x) = 0$ if $x =$ odd multiples of $\frac{\pi}{2}$.

$g''(x)$: $+$ before $\frac{\pi}{2}$, $-$ after $\frac{\pi}{2}$

Vertical asymptotes: $x = k\pi$, k an integer.



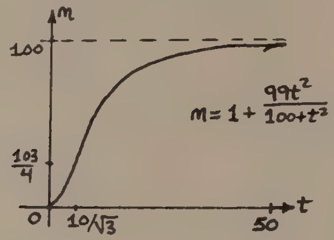
131. $\lim\limits_{t\to+\infty} n = 1 + \lim\limits_{t\to+\infty} \frac{99t^2}{100 + t^2} = 1 + 99 = 100$, so $n = 100$ is a horizontal asymptote. Let $n = f(t)$, then $f'(t) = \frac{19{,}800t}{(100 + t^2)^2} > 0$; $t > 0$; so f is always increasing.

$f''(t) = \frac{19{,}800(100 - 3t^2)}{(100 + t^2)^3}$. f'': $+$ before $10/\sqrt{3}$, $-$ after $10/\sqrt{3}$

f is concave upward on $(0, \frac{10}{\sqrt{3}})$; f is concave downward on $(\frac{10}{\sqrt{3}}, +\infty)$.

$(\frac{10}{\sqrt{3}}, \frac{103}{4})$ is a point of inflection.

(a)



(b) $t = \frac{10}{\sqrt{3}} \approx 5.77$ hours.

(c) No matter how well trained a worker is, he or she can't assemble more than 100 calculators per hour.

132.

Forewing	Hindwing
$0 \le 107t - 2.10 \le 2\pi$	$0 \le 107t - 1.57 \le 2\pi$
$2.10 \le 107t \le 2.10 + 2\pi$	$1.57 \le 107t \le 1.57 + 2\pi$
$\frac{2.10}{107} \le t \le \frac{2.10 + 2\pi}{107}$	$\frac{1.57}{107} \le t \le \frac{1.57 + 2\pi}{107}$
$0.019 \le t \le 0.078$	$0.015 \le t \le 0.073$
amplitude = 0.6	amplitude = 1

133. (a) $|\overline{AP}| = \sqrt{4 + x^2}$ and $|\overline{PB}| = \sqrt{9 + (5 - x)^2}$.

$f(x) = |\overline{AP}| + |\overline{PB}| = \sqrt{4 + x^2} + \sqrt{9 + (5 - x)^2}$.

$f'(x) = \frac{x}{\sqrt{4 + x^2}} - \frac{(5 - x)}{\sqrt{9 + (5 - x)^2}}$. If $f'(x) = 0$, then $5x^2 + 40x - 100 = 0$, so $x = 2$ or $x = -10$. Since $0 \le x \le 5$, $x = 2$.

(b) $f(x) = (x^2 + 4) + 9 + 25 - 10x + x^2$.

$f'(x) = 4x - 10 = 0$; $x = \frac{5}{2}$.

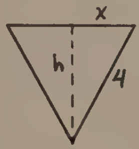
(c) $f(x) = x^2 + 4 - 9 - 25 + 10x - x^2$.

$f'(x) = 10$, so f is increasing. $x = 0$ minimizes the expression.

(d) $A(x) = x + \frac{15}{2} - \frac{3}{2}x = \frac{15}{2} - \frac{1}{2}x$. $A'(x) = -\frac{1}{2}$, so A is decreasing. $x = 5$ minimizes the expression.

134. $A = \frac{1}{2}(2x)h$, $A = xh$, $A = \sqrt{16 - h^2} \cdot h$.

$A'(h) = \sqrt{16 - h^2} - \frac{h^2}{\sqrt{16 - h^2}} = 0$ for $h = \sqrt{18} =$ $2\sqrt{2}$ inches.



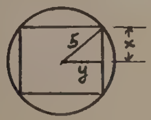
135. (a) $x^2 + y^2 = 25$. $A = 2x \cdot 2y = 4xy$.

$A(x) = 4x\sqrt{25 - x^2}$.

$A'(x) = \frac{-4x^2}{\sqrt{25 - x^2}} + 4\sqrt{25 - x^2} = 0$; $x^2 = \frac{50}{4}$,

so $x = \frac{5\sqrt{2}}{2}$. If $x = \frac{5\sqrt{2}}{2}$, then $y = \frac{\sqrt{50}}{4} = \frac{5\sqrt{2}}{2}$.

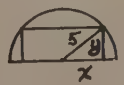
It is a square, $5\sqrt{2}$ meters by $5\sqrt{2}$ meters.



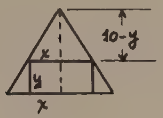
(b) $y = \sqrt{25 - x^2}$. $A = 2xy$. $A(x) = 2x\sqrt{25 - x^2}$.

$A'(x) = \frac{-2x^2}{\sqrt{25 - x^2}} + 2\sqrt{25 - x^2} = 0$; $x^2 = \frac{25}{2}$, so

$x = \frac{5}{\sqrt{2}}$. If $x = \frac{5}{\sqrt{2}}$, then $y = \frac{5}{\sqrt{2}}$. Base $5\sqrt{2}$ meters, height $\frac{5}{\sqrt{2}}$ meters.



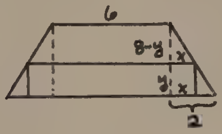
(c) $A = 2xy$. $\frac{x}{5} = \frac{10 - y}{10}$; $x = 5 - \frac{y}{2}$. $A(y) =$ $10y - y^2$. $A'(y) = 10 - 2y = 0$; $y = 5$. If $y = 5$, then $x = 5 - \frac{5}{2}$ or $x = \frac{5}{2}$. Dimensions are 5 meters by 5 meters.



(d) $A = (6 + 2x)y$, $0 < y \leq 8$. $\frac{8 - y}{8} = \frac{x}{2}$;

$x = 2 - \frac{y}{4}$. $A(y) = y(6 + 4 - \frac{y}{2}) = 10y - \frac{y^2}{2}$.

$A'(y) = 10 - y = 0$; $y = 10$. But $0 < y \leq 8$, so the rectangle of maximum area is 6 meters by 8 meters.



136. Minimize area $A = \frac{1}{2}xy$. $a^2 + 64 = y^2$.

$a + \sqrt{x^2 - (8 - x)^2} = y$, $a + \sqrt{16x - 64} = y$.

$\sqrt{16x - 64} = y - \sqrt{y^2 - 64}$. $x = \frac{y^2 - y\sqrt{y^2 - 64}}{8}$.

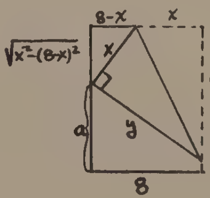
$A = \frac{1}{2}xy = \frac{1}{16}(y^3 - y^2\sqrt{y^2 - 64})$. $A'(y) =$

$\frac{y}{16\sqrt{y^2 - 64}}(3y\sqrt{y^2 - 64} - 3y^2 + 128) = 3y\sqrt{y^2 - 64} -$

$3y^2 + 128 = 0$ for $y = \sqrt{\frac{256}{3}} = \frac{16\sqrt{3}}{3}$. So $x =$

$$\frac{\frac{256}{3} - \sqrt{\frac{256}{3}}\sqrt{\frac{256}{3} - \frac{192}{3}}}{8} = \frac{\frac{128}{3}}{8} = \frac{128}{24} = \frac{32}{6} =$$

$\frac{16}{3}$ inches.



137. $R' = \frac{2v_0^2}{g}\cos 2\theta = 0$ when $\cos 2\theta = 0$; that is, when

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. Now $R'' = -\frac{4v_0^2}{g}\sin 2\theta$.

When $\theta = \frac{\pi}{4}$, $R'' = \frac{-4v_0^2}{g} < 0$. When $\theta = \frac{3\pi}{4}$,

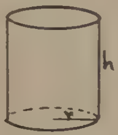
$R'' = -\frac{4v_0^2}{g}(-1) > 0$. R has a maximum when $\theta = \frac{\pi}{4}$.

138. $S = 2\pi rh + \pi r^2 = k$, $h = \dfrac{k - \pi r^2}{2\pi r}$.

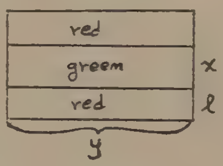
$V = \pi r^2 h = r^2\left(\dfrac{k - \pi r^2}{2\pi r}\right) = \dfrac{r}{2}(k - \pi r^2) = \dfrac{1}{2}(kr - \pi r^3)$.

$V'(r) = \dfrac{1}{2}(k - 3\pi r^2) = 0$ for $r = \sqrt{\dfrac{k}{3\pi}}$.

$h = \dfrac{k - (k/3)}{2\pi\sqrt{k/3\pi}} = \dfrac{2k/3}{2\pi\sqrt{k/3\pi}} = \dfrac{k/3\pi}{\sqrt{k/3\pi}} = \sqrt{\dfrac{k}{3\pi}}$. The volume will be a maximum when the height and base radius are the same.



139. Maximize area $A = xy$. Since $2\ell y = 2$, $\ell = \dfrac{1}{y}$.
Also, $2(2\ell + x) + 2y = 8$; $(2\ell + x) + y = 4$,
$\ell = \dfrac{4 - y - x}{2}$. The two expressions for ℓ yield
$\dfrac{1}{y} = \dfrac{4 - y - x}{2}$, $2 = 4y - y^2 - xy$, $x = \dfrac{-y^2 + 4y - 2}{y}$.
$A(y) = -y^2 + 4y - 2$. $A'(y) = -2y + 4 = 0$ for $y = 2$ and $x = 1$. Dimensions are 1 meter by 2 meters.



140. $L'(p) = \dfrac{n!}{k!(n-k)!}[kp^{k-1}(1-p)^{n-k} - p^k(n-k)(1-p)^{n-k-1}]$.

$L'(p) = \dfrac{n!}{k!(n-k)!}[p^{k-1}(1-p)^{n-k-1}\{k(1-p) - p(n-k)\}] = 0$ when $p^{k-1}(1-p)^{n-k-1}(k - np) = 0$.
So either $p = 0$, $p = 1$, or $p = \dfrac{k}{n}$. $p = \dfrac{k}{n}$ maximizes L.

141. $I = \dfrac{xE}{R + (x^2r/n)}$, so that $\dfrac{dI}{dx} = \dfrac{RE - (rEx^2/n)}{[R + (x^2r/n)]^2}$, and the critical value is found by setting $RE - (rEx^2/n)$ equal to 0. Thus, $x = \sqrt{\dfrac{nR}{r}}$.

142. (a) $\dfrac{dy}{dx} = m - \dfrac{2g(1 + m^2)}{v^2}x$; $\dfrac{d^2y}{dx^2} = -\dfrac{2g(1 + m^2)}{v^2} < 0$.

Thus, the critical value $x = \dfrac{mv^2}{2g(1 + m^2)}$ gives a maximum value of y of $\dfrac{m^2v^2}{4g(1 + m^2)}$.

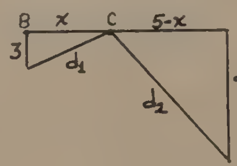
(b) If $y = 0$, then $mx - \dfrac{g}{2}(1 + m^2)\dfrac{x^2}{v^2} = 0$, or $x\left[m - \dfrac{g}{2}(1 + m^2)\dfrac{x}{v^2}\right] = 0$; so $x = 0$ or $x = \dfrac{2mv^2}{g(1 + m^2)}$. Reject $x = 0$.

(c) From (b), $x = \dfrac{2mv^2}{g(1 + m^2)}$. $D_m\left[\dfrac{2mv^2}{g(1 + m^2)}\right] = \dfrac{2v^2(1 - m^2)}{g(1 + m^2)}$; so $m = \pm 1$ gives critical values.
Reject $m = -1$. Hence, hold nozzle at a 45° angle.

(d) $y = mx - \dfrac{g}{2}(1 + m^2)\left(\dfrac{x}{v}\right)^2$, where x and v are constants. $\dfrac{dy}{dm} = x - \dfrac{gx^2}{v^2}m$; $\dfrac{d^2y}{dm^2} = -\dfrac{gx^2}{v^2} < 0$.

Hence, the critical value of $m = \dfrac{v^2}{xg}$ gives maximum value for g.

143. Find C. $d_1 = \sqrt{x^2 + 9}$; $d_2 = \sqrt{81 + (5 - x^2)}$.
$f(x) = d_1 + d_2$. $f'(x) = \dfrac{x}{\sqrt{x^2 + 9}} + \dfrac{x - 5}{\sqrt{x^2 - 10x + 106}} = 0$, or $8x^2 + 10x - 25 = 0$, or $(4x - 5)(2x + 5) = 0$.
$x = \dfrac{5}{4}$. Shortest course when boat lands at a point C $\dfrac{5}{4}$ miles from the point B.

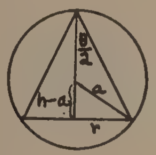


144. The turning moments about the fulcrum must cancel, so that $xF = \dfrac{x}{2}(10x) + 2(1000)$, or $F = 5x + \dfrac{2000}{x}$.
Thus, $\dfrac{dF}{dx} = 5 - \dfrac{2000}{x^2}$, and $\dfrac{dF}{dx} = 0$ when $x = 20$ feet.

145. Let r be the radius of the base of the cone and let h be its height. The slant height of the cone will be a units; hence, $a^2 = h^2 + r^2$, or $h = \sqrt{a^2 -}$
Since $V = \dfrac{1}{3}h(\pi r^2) = \dfrac{\pi}{3}r^2\sqrt{a^2 - r^2}$, then $\dfrac{dV}{dr} = \dfrac{2\pi r}{3}\sqrt{a^2 - r^2} - \dfrac{\pi r^3}{3\sqrt{a^2 - r^2}}$. Setting $\dfrac{dV}{dr} = 0$, we obtain $2r(a^2 - r^2) = r^3$, or $r^3 = \dfrac{2a^2}{3}r$. Rejecting the root $r = 0$, we obtain $r^2 = \dfrac{2}{3}a^2$; hence, the

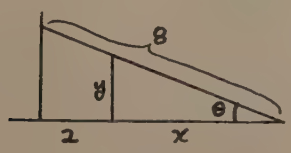
maximum volume is given by $V = \frac{\pi}{3}(\frac{2}{3}a^2)\sqrt{a^2 - \frac{2}{3}a^2} = \frac{2\pi a^3}{9\sqrt{3}}$ cubic units.

146. $(h - a)^2 + r^2 = a^2$, or $r^2 = 2ha - h^2$. Lateral area $A = \pi r\ell = \pi r\sqrt{r^2 + h^2} = \pi\sqrt{2ha - h^2}\,\sqrt{2ha} = \pi\sqrt{4h^2a^2 - 2h^3a}$. $\frac{dA}{dh} = \frac{\pi}{2}(4h^2a^2 - 2h^3a)^{-1/2}(8ha^2 - 6h^2a) = 0$, or $8ha^2 - 6h^2a = 0$, or $4a - 3h = 0$. $h = \frac{4}{3}a$. It follows that $r = \sqrt{2ha - h^2} = \frac{2\sqrt{2}}{3}a$. Thus, θ is a solution of the equation $\tan\frac{\theta}{2} = \frac{r}{h} = \frac{\sqrt{2}}{2}$. By Newton's method (or a calculator), $\theta \approx 1.23$ radians or $\theta \approx 70.53°$.

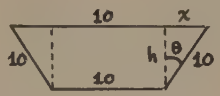


147. $\frac{dx}{d\theta} = \frac{v_0^2}{g}2[\cos 2\theta - \frac{1}{\sqrt{3}}(-\sin 2\theta)] = 0$ when $\sqrt{3}\cos 2\theta + \sin 2\theta = 0$. Thus, we get 0 when $\tan 2\theta = -\sqrt{3}$, or when $2\theta = \frac{2\pi}{3}$; that is, when $\theta = \frac{\pi}{3}$.

148. We want x so that y is maximum. $\tan\theta = \frac{y}{x}$, so $y = x\tan\theta$. Now $\cos\theta = \frac{2 + x}{8}$, so $x = 8\cos\theta - 2$. Hence, $y = (8\cos\theta - 2)\tan\theta = 8\sin\theta - 2\tan\theta$. $y' = 8\cos\theta = 2\sec^2\theta = 0$; so we want $8\cos^3\theta - 2 = 0$. So y is maximum for $\cos\theta = \sqrt[3]{\frac{1}{4}}$, that is, for $x = 8\sqrt[3]{\frac{1}{4}} - 2$. Therefore, the ladder should be placed $8\sqrt[3]{\frac{1}{4}}$ meters from the building.



149. $A = \frac{1}{2}(10 + 10 + 2x)h = (10 + x)h$. Now, $\sin\theta = \frac{x}{10}$ and $\cos\theta = \frac{h}{10}$. Hence, $A(\theta) = (10 + 10\sin\theta)\cdot(10\cos\theta) = 100(1 + \sin\theta)\cos\theta$.
$A'(\theta) = 100[\cos\theta(\cos\theta) + (1 + \sin\theta)(-\sin\theta)] = 100(\cos^2\theta - \sin^2\theta - \sin\theta) = 100(-2\sin^2\theta + 1 - \sin\theta) = 0$. So $2\sin^2\theta + \sin\theta - 1 = 0$ and $(2\sin\theta - 1)(\sin\theta + 1) = 0$. So $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$. We must have $\sin\theta = \frac{1}{2}$, and so $\theta = \frac{\pi}{6}$.



150. $f'(x) = 2(x - x_1) + 2(x - x_2) + \ldots + 2(x - x_n)$.
$f''(x) = 2 + 2 + \ldots + 2 = 2n > 0$.
$2(x - x_1) + 2(x - x_2) + \ldots + 2(x - x_n) = 0$, $2(x - x_1 + x - x_2 + \ldots + x_n) = 0$, $2[nx - (x_1 + \ldots + x_n)] = 0$, $nx = x_1 + \ldots + x_n$, or $x = \frac{x_1 + \ldots + x_n}{n} = \bar{x}$. Since $f''(x) > 0$, this value of x minimizes f(x).

151. $F(\theta) = 400(0.4\sin\theta + \cos\theta)^{-1}$.
$F'(\theta) = 400(-1)(0.4\sin\theta + \cos\theta)^{-2}(0.4\cos\theta - \sin\theta) = \frac{-400}{(0.4\sin\theta + \cos\theta)^2}(0.4\cos\theta - \sin\theta) = 0$ when $0.4\cos\theta - \sin\theta = 0$; that is, when $\tan\theta = 0.4$. By Newton's method, $\theta \approx 0.38$ radian.

152. Let x be the number of \$10 increases. So x is the number of vacant apartments and 80 - x is the number of occupied apartments. Cost C = $15x + 65(80 - x)$; rent $R = 250 + 10x$; profit $P = R - C$. $P = (80 - x)(250 + 10x) - [15x + 65(80 - x)] = (80 - x)(250 + 10x) + 50x - 5200$. $\frac{dP}{dx} = (80 - x)(10) + (250 + 10x)(-1) + 50 = 0$; $20x = 600$ or $x = 30$. $\frac{d^2P}{dx^2} = -20 < 0$. For maximum profit, the rent charged should be 250 + 300 = \$550.

153. Let x be the number of \$1 increases. So 10x is the number of bikes not rented; 100 - 10x is the number rented; 10 + x is the fee per bike. revenue $R = (100 - 10x)(10 + x)$. $\frac{dR}{dx} = (100 - 10x) + (10 + x)(-10) = 0$; $x = 0$. The concessionaire must charge \$10 per bike per day to maximize revenue.

154. Profit $P = 30x - \frac{2}{5}x^2 - (60 + 6x)$. $\frac{dP}{dx} = 30 - \frac{4}{5}x - 6 = 0$ when $x = 30$. $\frac{d^2P}{dx^2} = -\frac{4}{5} < 0$. The manufacturer must produce 30 trophies a day to maximize profits.

155. $R(x) = x\sqrt{5{,}000{,}000 - 2x^2}$.
$R'(x) = x \cdot \frac{1}{2}(5{,}000{,}000 - 2x^2)^{-1/2}(-4x) + \sqrt{5{,}000{,}000 - 2x^2} = 0$, so that $-2x^2 + 5{,}000{,}000 - 2x^2 = 0$, $4x^2 = 5{,}000{,}000$, $x^2 = 1{,}250{,}000$, $x \approx 1118.03$. So 1118 microcomputers should be sold to bring in the largest total revenue.

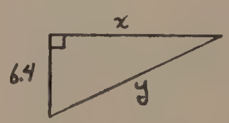
156. Profit $= P(x) = x(\frac{1}{3})(375 - 5x) - (500 + 15x + \frac{x^2}{4}) = 125x - \frac{5}{3}x^2 - 500 - 15x - \frac{x^2}{5} = 110x - \frac{28}{15}x^2 - 500$. $P'(x) = 110 - \frac{56}{15}x = 0$; $56x = 1650$; $x = \frac{825}{28}$, so $x \approx 29.46$. $P(30) = \frac{1}{3}(375 - 150) = \frac{1}{3}(225) = 75$. She must sell 30 tires at $75.00 per tire.

157. $y = \frac{1}{2}\sqrt{x} = \frac{1}{2}x^{1/2}$.
$\frac{dy}{dt} = \frac{1}{4}x^{-1/2}\frac{dx}{dt} = \frac{1}{4}(40{,}000)^{-1/2}(10{,}000) = \frac{2500}{(40{,}000)^{1/2}} = \frac{2500}{200} = \frac{25}{2} = 12.5$ persons per yr.

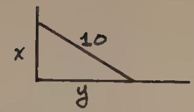
158. $\frac{dq}{dt} = -\frac{1000}{p^2}\frac{dq}{dt} = \frac{-1000}{(0.83)^2}(0.01) = -14.52$ thousand boxes per week per month.

159. $\frac{dp}{dt} = -60(30 + x)^{-2}\frac{dx}{dt}$. Now $\frac{dx}{dt} = -\frac{200}{1000} = -0.2$. $\frac{dp}{dt} = -60(30 + 10)^{-2}(-0.2) = \frac{12}{1600} = \frac{3}{400} \approx \$0.01$ per bushel per day.

160. $6.4^2 + x^2 = y^2$, so $2x\frac{dy}{dt} = 2y\frac{dy}{dt}$, or $x\frac{dx}{dt} = y\frac{dy}{dt}$. Now, $\frac{dx}{dt} = 19.2$; $y = 8$, so $x = \sqrt{64 - 6.4^2} = \sqrt{23.04} = 4.8$. Thus, $4.8(19.2) = 8\frac{dy}{dt}$, or $\frac{dy}{dt} = \frac{92.16}{8} = 11.52$ km/hr.



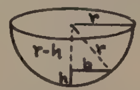
161.



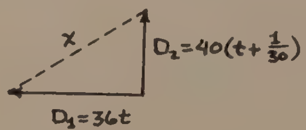
(a) $100 = y^2 + x^2 = 9 + x^2$ when $y = 3$; so $x^2 = 91$, or $x = \sqrt{91}$. Now $2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 0$, or $y\frac{dy}{dt} + x\frac{dx}{dt} = 0$. Thus, $3(2) + \sqrt{91}\frac{dx}{dt}$, or $\frac{dx}{dt} = -\frac{6}{\sqrt{91}}$ meters/sec.

(b) When $x = 9$, $y^2 = 100 - 64 = 36$, or $y = 6$. Thus, $6(2) + 8\frac{dy}{dt} = 0$, or $\frac{dx}{dt} = -\frac{12}{8} = -\frac{3}{2}$ meters/sec.

162. $V(\text{segment}) = \frac{\pi h}{6}(h^2 + 3b^2)$. But $b^2 + (r - h^2) = r^2$ and $b^2 = 2rh - h^2$. So $V = \frac{\pi h^2}{3}(3r - h)$. $r = 10$, so $V = \frac{\pi h^2}{3}(30 - h)$, or $V = 10\pi h^2 - \frac{\pi h^3}{3}$. $\frac{dV}{dt} = (20\pi h - \pi h^2)\frac{dh}{dt}$. $-5 = (20\pi \cdot 6 - \pi \cdot 36)\frac{dh}{dt}$, or $-5 = (120\pi - 36\pi)\frac{dh}{dt}$. $\frac{-5}{84\pi} = \frac{dh}{dt}$. h is decreasing at the rate of $\frac{5}{84\pi} \approx 0.018947$ cm per minute.



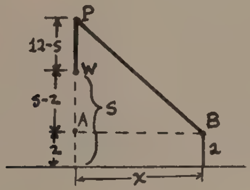
163. $x^2 = D_1^2 + D_2^2$; $2x\frac{dx}{dt} = 2D_1\frac{dD_1}{dt} + 2D_2\frac{dD_2}{dt}$. When $t = \frac{1}{6}$ hours, $D_1 = 6$, $D_2 = 8$, and $x = 10$; so $20\frac{dx}{dt} = (12)(36) + (16)(40)$, or $\frac{dx}{dt} = 53.6$ mi/hr.



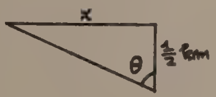
164. $V = \pi r^2 h$. $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$. $\frac{dV}{dt} = 2\pi \cdot 4 \cdot 10(-2) + \pi(16)3 = -160\pi + 48\pi = -112\pi$. V is decreasing at the rate of 112π cubic centimeters per minute.

165. The total length of rope between the weight W and the boy's hands B is 12 + 10 = 22 m. In the

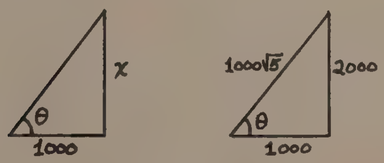
figure below, $|\overline{AP}| = 10$ m, $|\overline{AB}| = x$ m, and $|\overline{PB}| = 22 - (12 - s) = 10 + s$ m. From the right triangle PAB, $x^2 + 10^2 = (10 + s)^2$. Differentiate both sides of the latter equation with respect to time t to obtain $2x\,\frac{dx}{dt} = 2(10 + s)\,\frac{ds}{dt}$. Since $\frac{dx}{dt} = 1$ foot per second, $\frac{ds}{dt} = \frac{x}{10 + s}$. When t = 2 seconds, $x = 2(1) = 2$ m; and $10 + s = \sqrt{2^3 + 10^2} = \sqrt{104}$ and $\frac{ds}{dt} = \frac{2}{\sqrt{104}} = \frac{1}{\sqrt{26}}$ meter/sec.



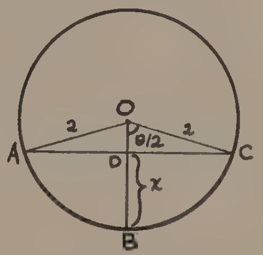
166. Want $\frac{d\theta}{dt}$ when x = 0. $\tan\theta = x/(1/2) = 2x$. $\sec^2\theta\,\frac{d\theta}{dt} = 2\,\frac{dx}{dt}$. $\frac{dx}{dt} = 80$. When x = 0, $\sec^2\theta = 1$, so $\frac{d\theta}{dt} = 2(80) = 160$ radians/hour.



167. Want $\frac{d\theta}{dt}$ when x = 2000. $\frac{dx}{dt} = 20$. $\tan\theta = \frac{x}{1000}$. $\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{1000}\,\frac{dx}{dt}$. When x = 2000, $\cos\theta = \frac{1000}{1000\sqrt{5}} = \frac{1}{\sqrt{5}}$, so $\sec\theta = \sqrt{5}$. Thus, $5\,\frac{d\theta}{dt} = \frac{1}{1000}(20)$, or $\frac{d\theta}{dt} = \frac{1}{150}$ radian/sec.



168. In the figure below, which shows a cross section of the tank, $|\overline{OA}| = |\overline{OB}| = |\overline{OC}| = 2$ meters and $|\overline{OD}| = 2 - x$. When x = 2, $\frac{dx}{dt} = -0.02$ meter/hr. The area of the circular sector OABC is given by $(\frac{\theta}{2\pi})(\pi \cdot 2^2) = 2\theta$ square meters and the area of triangle AOC is given by $\frac{1}{2}|\overline{OD}| \cdot |\overline{AC}| = |\overline{OD}| \cdot |\overline{DC}| = (2\cos\frac{\theta}{2})(2\sin\frac{\theta}{2}) = 2\sin\theta$. Hence, the area of the segment ABCD is given by $2\theta - 2\sin\theta = 2(\theta - \sin\theta)$ square meters. Since the tank is 10 meters long, the volume of liquid nitrogen in the tank is given by $V = 2(\theta - \sin\theta)(10) = 20(\theta - \sin\theta)$ cubic meters. Thus, $\frac{dV}{dt} = 20(1 - \cos\theta)\,\frac{d\theta}{dt}$. From right triangle ODC, $\cos\frac{\theta}{2} = \frac{|\overline{OD}|}{|\overline{OC}|} = \frac{2 - x}{2} = 1 - \frac{x}{2}$. Differentiating the latter equation with respect to time, we obtain $(-\frac{1}{2}\sin\frac{\theta}{2})\,\frac{d\theta}{dt} = -\frac{1}{2}\,\frac{dx}{dt}$, so that $\frac{d\theta}{dt} = \frac{1}{\sin(\theta/2)}\,\frac{dx}{dt}$. When x = 2, $\theta = \pi$, and $\frac{dx}{dt} = -0.02$, we obtain $\frac{dV}{dt} = 20(1 - \cos\theta)\,\frac{d\theta}{dt} = 20[1 - (-1)]\,\frac{1}{1}\,(-0.02) = -0.8$ cubic meter per hour.



169. Let n be the number of \$1 increases above the price of \$40 per barrel, and let p be price of each barrel. Then, p = 40 + n. Daily demand x for barrels is given by: x = 700 - 100n, or $n = \frac{700 - x}{100}$. Thus, $p = 40 + \frac{700 - x}{100} = 47 - \frac{x}{100}$. $R(x) = px = 47x - \frac{x^2}{100}$. $C(x) = (3.5 \times 10^{-6})x^3 - (1.05) \times 10^{-2}x^2 + 40.5x + 1500$. $P(x) = R(x) - C(x) = 47x - \frac{x^2}{100} - [(3.5 \times 10^{-6})x^3 - (1.05) \times 10^{-2}x^2 + 40.5x + 1500] = 6.5x + (5 \times 10^{-4})x^2 - (3.5 \times 10^{-6})x^3 - 1500$. $P'(x) = 6.5 + 10^{-3}x - (1.05 \times 10^{-5})x^2$. Finding the roots of $P'(x) = 0$ and rejecting the negative root, we obtain a maximum for x = 835.85 bbls/day. The maximum profit is \$2238.48. Now $R(x) = px$ or $p = \frac{R(x)}{x} = 47 - \frac{x}{100}$. $R'(x) = 47 - \frac{x}{50} = 0$ when

$x = 2350$, but 2350 is not in the interval $[0,1400]$; so revenue is maximum when $x = 1400$ bbls. Using Newton's method to find the zeros of the profit function $P(x)$ on $[0,1400]$, we find that production levels of $x = 233.42$ and $x = 1310.46$ bbls/day cause the producer to break even.

$C'(x) = (1.05 \times 10^{-5})x^2 - (2.1 \times 10^{-2})x + 40.5.$ The most efficient production level occurs when C' has a maximum. Now $C''(x) = (2.1) \times 10^{-5}x - (2.1) \times 10^{-2} = 0$ when $x = \dfrac{10^{-2}}{10^{-5}} = 10^{-2} \times 10^5 = 1000$. Thus, the most efficient production level occurs when $x = 1000$ barrels per day.

170. If n = the number of \$1 increases in prices per barrel, then $p = 40 + n$ dollars per barrel and $q = 700 - 100n$ barrels per day. From the last equation, $n = 7 - (q/100)$, so $p = 40 + 7 - (q/100) = 47 - (q/100)$. Thus, $q = 4700 - 100p$; $dq/dp = -100$. When $p = \$40$ per barrel, $q = 700$ barrels and $E = -(p/q)(dq/dp) = -(40/700)(-100) = 40/7$.

4 ANTIDIFFERENTIATION AND DIFFERENTIAL EQUATIONS

Problem Set 4.1, page 250

1. $dy = 6x\,dx.$

2. $dy = -dx - 4x^3 dx = (-1 - 4x^3)dx.$

3. $dy = 3x^2 dx - 4dx = (3x^2 - 4)dx.$

4. $y = \frac{3}{2}x^{-1} - 5x.$

$dy = -\frac{3}{2}x^{-2}dx - 5\,dx = (-\frac{3}{2}x^{-2} - 5)dx.$

5. $dy = \dfrac{(3x + 1)7\,dx - (7x - 2)(3\,dx)}{(3x + 1)^2} = \dfrac{13\,dx}{(3x + 1)^2}.$

6. $dy = \dfrac{(2x^2 + x + 1)(2x - 3)dx - (x^2 - 3x + 2)(4x + 1)dx}{(2x^2 + x + 1)^2}$

$= \dfrac{7x^2 - 6x - 5}{(2x^2 + x + 1)^2}\,dx.$

7. $dy = 3(3x^2 + 2)^2(6x)dx = 18x(3x^2 + 2)^2 dx.$

8. $dy = -3(2x^4 - x)^{-4}(8x^3 - 1)dx.$

9. $dy = 4\left(\dfrac{1 + x^2}{1 + x}\right)^3 \cdot \dfrac{(1 + x)2x\,dx - (1 + x^2)dx}{(1 + x)^2}$

$= 4\left(\dfrac{1 + x^2}{1 + x}\right)^3 \cdot \dfrac{x^2 + 2x - 1}{(1 + x)^2}\,dx$

$= \dfrac{4(1 + x^2)^3(x^2 + 2x - 1)dx}{(1 + x)^5}.$

10. $dy = \dfrac{(\sqrt{x})(-2x) - (3 - x^2)(\frac{1}{2\sqrt{x}})}{x}\,dx$

$= \dfrac{-4x^2 - 3 + x^2}{2x^{3/2}}\,dx = \dfrac{-3x^2 - 3}{2x^{3/2}}.$

11. $dy = \frac{1}{2}(9 - 3x^2)^{-1/2}(-6x)dx = \dfrac{-3x}{\sqrt{9 - 3x^2}}\,dx.$

12. $dy = x^4\left[\frac{1}{5}(x^2 + 2)^{-4/5}(2x)\right]dx + 4x^3\sqrt[5]{x^2 + 2}\,dx$

$= \left(\dfrac{2x^5}{5(x^2 + 2)^{4/5}} + 4x^3\sqrt[5]{x^2 + 2}\right)dx$

$= \dfrac{2x^5 + 4x^3(x^2 + 2)}{5(x^2 + 2)^{4/5}}\,dx = \dfrac{6x^5 + 8x^3}{5(x^2 + 2)^{4/5}}\,dx.$

13. $dy = \dfrac{1}{2}\left(\dfrac{x - 3}{x + 3}\right)^{-1/2} \cdot \dfrac{(x + 3 - x + 3)}{(x + 3)^2}\,dx$

$= \dfrac{1}{2}\left(\dfrac{x + 3}{x - 3}\right)^{1/2} \cdot \dfrac{6}{(x + 3)^2}\,dx$

$= \dfrac{3}{(x + 3)^2}\left(\dfrac{x + 3}{x - 3}\right)^{1/2} dx$

$= 3(x + 3)^{-3/2}(x - 3)^{-1/2}dx.$

14. $dy = \dfrac{\sqrt{x^2 + 5} - x \cdot \dfrac{1}{2\sqrt{x^2 + 5}}\,2x}{x^2 + 5}\,dx$

$= \dfrac{x^2 + 5 - x^2}{(x^2 + 5)^{3/2}}\,dx = \dfrac{5}{(x^2 + 5)^{3/2}}\,dx.$

15. $dy = \dfrac{(x^2 + 7)\dfrac{3}{2\sqrt{3x + 1}} - \sqrt{3x + 1}\,(2x)}{(x^2 + 7)^2}\,dx$

$= \dfrac{3(x^2 + 7) - (3x + 1)(4x)}{2\sqrt{3x + 1}\,(x^2 + 7)^2}\,dx$

$= \dfrac{-9x^2 - 4x + 21}{2\sqrt{3x + 1} \cdot (x^2 + 7)^2}\,dx.$

16. $dy = \dfrac{\sqrt[3]{x + 1}\,(3x^2) - x^3(\frac{1}{3})(x + 1)^{-2/3}}{(x + 1)^{2/3}}\,dx$

$$= \frac{3(x+1)(3x^2) - x^3}{3(x+1)^{4/3}}\,dx = \frac{9x^3 + 9x^2 - x^3}{3(x+1)^{4/3}}\,dx$$

$$= \frac{x^2(8x-9)}{3(x+1)^{4/3}}\,dx.$$

17. $dy = 3\cos 2x(2\,dx) = 6\cos 2x\,dx.$

18. $dy = -\sin\sqrt{x}\,\left(\frac{1}{2\sqrt{x}}\,dx\right) = -\frac{1}{2\sqrt{x}}\sin\sqrt{x}\,dx.$

19. $dy = 4\tan x(\sec^2 x\,dx) = 4\tan x\sec^2 x\,dx.$

20. $dy = \frac{1}{2}(\sin\pi x)^{-1/2}(\cos\pi x)\,\pi\,dx$

$$= \frac{\pi\cos\pi x}{2\sqrt{\sin\pi x}}\,dx.$$

21. $\frac{d}{dx}\cot u = -\csc^2 u\,\frac{du}{dx}$, so $d(\cot u) = -\csc^2 u\,du.$

22. $\frac{d}{dx}\sec u = \sec u\tan u\,\frac{du}{dx}$, so

$d(\sec u) = \sec u\tan u\,du.$

23. $\frac{d}{dx}\csc u = -\csc u\cot u\,\frac{du}{dx}$, so

$d(\csc u) = -\csc u\cot u\,du.$

24. $D_x\sqrt{u} = D_x u^{1/2} = \frac{1}{2}u^{-1/2}\frac{du}{dx} = \frac{1}{2\sqrt{u}}\frac{du}{dx}$, so

$d(\sqrt{u}) = \frac{1}{2\sqrt{u}}\,du.$

25. $2x\,dx + 2y\,dy = 0$, or $x\,dx + y\,dy = 0.$

26. $18x\,dx = 0 - 8y\,dy$, so $18x\,dx = -8y\,dy$, or

$9x\,dx = -4y\,dy.$

27. $18x\,dx - 32y\,dy = 0$, or $9x\,dx - 16y\,dy = 0.$

28. $\frac{2}{3}x^{-1/3}\,dx + \frac{2}{3}y^{-1/3} = 0$, or $x^{-1/3}\,dx +$

$y^{-1/3}\,dy = 0.$

29. $\frac{1}{3}x^{-2/3}\,dx = 0 - \frac{1}{3}y^{-2/3}\,dy$, or $x^{-2/3}\,dx = -y^{-2/3}dy.$

30. $2x\,dx + x\,dy + y\,dx + 6y\,dy = 0$, or $(2x+y)dx +$

$(x+6y)dy = 0.$

31. $6x^2dx + 15y^2dy = 0 + 8xy\,dy + 4y^2dx - x\,dy - y\,dx,$

or $(6x^2 - 4y^2 + y)dx + (15y^2 - 8xy + x)dy = 0.$

32. $-\frac{1}{2\sqrt{1-x}}\,dx - \frac{1}{2\sqrt{1-y}}\,dy = 0$ or

$\sqrt{1-y}\,dx + \sqrt{1-x}\,dy = 0.$

33. $3x^2dx + 3y^2dy = \frac{1}{3}(x+y)^{-2/3}(dx+dy)$, or

$[9x^2(x+y)^{2/3} - 1]dx + [9y^2(x+y)^{2/3} - 1]dy = 0.$

34. $\cos x\,dx - \sin y\,dy = 0.$

35. $2\tan x\sec^2 x\,dx + 2\tan y\sec^2 y = 0$, or

$\tan x\sec^2 x\,dx + \tan y\sec^2 y\,dy = 0.$

36. $0 - \csc^2 xy(x\,dy + y\,dx) + x\,dy + y\,dx = 0$, or

$(-x\csc^2 xy + x)dy + (-y\csc^2 xy + y)dx = 0.$

37. (a) $\Delta y = 3(x_1 + \Delta x)^2 + 1 - (3x_1^2 + 1)$

$= 3x_1^2 + 6x_1\,\Delta x + 3(\Delta x)^2 + 1 - 3x_1^2 - 1$, so

$\Delta y = 6x_1\Delta x + 3(\Delta x)^2$

$= (6)(1)(0.1) + 3(0.01)$

$= 0.6 + 0.03 = 0.63.$

(b) $f'(x) = 6x$. So $dy = f'(x_1)dx = (6x_1)(\Delta x) =$

$6(1)(0.1) = 0.6.$

(c) $\Delta y - dy = 0.63 - 0.6 = 0.03.$

38. (a) $\Delta y = -5(x_1 + \Delta x)^2 + (x_1 + \Delta x) - (-5x_1^2 + x_1)$

$= -5x_1^2 - 10x_1\Delta x - 5(\Delta x)^2 + x_1 + \Delta x +$

$5x_1^2 - x_1$, so

$\Delta y = -10x_1\Delta x - 5(\Delta x)^2 + \Delta x$

$= -10(2)(0.02) - 5(0.0004) + 0.02$

$= -0.4 - 0.0020 + 0.02 = -0.4020 + 0.02$

$= -0.382.$

(b) $f'(x) = -10x + 1$. So $dy = (-10x_1 + 1)dx = (-10x_1 + 1)\Delta x = (-19)(0.02) = -0.38$.

(c) $\Delta y - dy = -0.382 - (-0.38) = -0.382 + 0.38 = -0.002$.

39. (a) $\Delta y = -2(x_1 + \Delta x)^2 + 4(x_1 + \Delta x) + 1 - (-2x_1^2 + 4x_1 + 1)$
$= -2x_1^2 - 4x_1\Delta x - 2(\Delta x)^2 + 4x_1 + 4\Delta x + 1 + 2x_1^2 - 4x_1 - 1$
$= -4x_1\Delta x - 2(\Delta x)^2 + 4\Delta x$
$= (-4)(2)(0.4) - 2(0.16) + 1.6 = -1.92$.

(b) $f'(x) = -4x + 4$. So $dy = (-4x_1 + 4)dx = (-4x_1 + 4)\Delta x = (-4)(0.4) = -1.6$.

(c) $\Delta y - dy = (-1.92) - (-1.6) = -0.32$.

40. (a) $\Delta y = 2(x_1 + \Delta x)^3 + 5 - 2x_1^3 - 5$
$= 2x_1^3 + 6x_1^2(\Delta x) + 6x_1(\Delta x)^2 + (\Delta x)^3 + 5 - 2x_1^3 - 5$
$= 0.30 - 6(0.0025) + 0.000125$
$= 0.300125 - 0.0150 = 0.285125$.

(b) $f'(x) = 6x^2$. So $dy = 6x_1^2dx = 6x_1^2(\Delta x) = 6(1)(0.05) = 0.30$.

(c) $\Delta y - dy = 0.285125 - 0.30 = -0.14875$.

41. (a) $\Delta y = \dfrac{9}{\sqrt{x_1 + \Delta x}} - \dfrac{9}{\sqrt{x_1}} = \dfrac{9}{\sqrt{9 - 1}} - \dfrac{9}{\sqrt{9}}$
$= \dfrac{9}{\sqrt{8}} - 3 \approx 3.182 - 3 = 0.182$.

(b) $f'(x) = -\dfrac{9}{2\sqrt{x^3}}$. So $dy = -\dfrac{9}{2\sqrt{x_1^3}}$. $dx = -\dfrac{9}{2\sqrt{x_1^3}}(\Delta x) = -\dfrac{9}{2\sqrt{729}}(-1) = \dfrac{1}{6}$.

(c) $\Delta y - dy = \dfrac{9}{\sqrt{8}} - 3 - \dfrac{1}{6} \approx 0.015$.

42. (a) $\Delta y = \dfrac{3}{x_1 + \Delta x + 4} - \dfrac{3}{x_1 + 4}$
$= \dfrac{3}{3 - 2 + 4} - \dfrac{3}{3 + 4} = \dfrac{3}{5} - \dfrac{3}{7} \approx 0.17$.

(b) $f'(x) = \dfrac{-3}{(x + 4)^2}$. So $dy = \dfrac{-3}{(x_1 + 4)^2}dx = \dfrac{-3}{(x_1 + 4)^2}\Delta x = -\dfrac{3}{7^2}(-2) = \dfrac{6}{7^2} = \dfrac{6}{49} \approx 0.122$.

(c) $\Delta y - dy = \dfrac{6}{35} - \dfrac{6}{49} \approx -0.04898$.

43. (a) $\Delta y = 4\cos(x_1 + \Delta x) - 4\cos x_1 = 4\cos = 4\cos(\frac{\pi}{3} + 0.01) - 4\cos\frac{\pi}{3} = -0.0347$.

(b) $y' = -4\sin x$. $dy = -4\sin x_1\, dx = -4\sin x_1\,\Delta x = (-4\sin\frac{\pi}{3})(0.01) = -0.03464$.

(c) $\Delta y - dy = 4\cos(\frac{\pi}{3} + 0.01) - 2 + 0.02\sqrt{3} = -9.9 \times 10^{-5}$.

44. (a) $\Delta y = -\sec(x_1 + \Delta x) + \sec x_1$
$= -\sec(0 + 0.07) + \sec(0) = -\sec(0.07) + 1$
$= -0.0000007463$.

(b) $y' = -\sec x \tan x$. $dy = -\sec x_1 \tan x_1\, dx = -\sec 0 \tan 0\, \Delta x = 0$.

(c) $\Delta y - dy = -0.0000007463 - 0 = -0.0000007463$.

45. Let $y = \sqrt{x}$, $x_1 = 9$, and $dx = \Delta x = 0.06$.
$\Delta y = \sqrt{x_1 + \Delta x} - \sqrt{x_1} = \sqrt{9.06} - \sqrt{9} = \sqrt{9.06} - 3$,
so $\sqrt{9.06} = 3 + \Delta y$.
$dy = \dfrac{1}{2\sqrt{x_1}}dx = \dfrac{1}{2\sqrt{9}}(.06) = 0.01$,
so $\sqrt{9.06} \approx 3 + .01 = 3.01$.

46. Let $y = \sqrt{x}$, $x_1 = 49$, and $dx = \Delta x = -0.2$.
$\sqrt{48.8} = 7 + \Delta y$.
$dy = \dfrac{1}{2\sqrt{x_1}}dx = \dfrac{1}{2\sqrt{49}}(-0.2) = \dfrac{-0.2}{14} = -0.0142857143$,
so $\sqrt{48.8} = 7 - 0.0142857143 \approx 6.986$.

47. $y = x^3$, $x_1 = 3$, $dx = \Delta x = 0.07$.
$(3.07)^3 = 3^3 + \Delta y = 27 + \Delta y$.
$dy = 3x_1^2dx = 3(9)(0.07) = 1.89$,
so $(3.07)^3 = 27 + 1.89 = 28.89$.

48. $y = \frac{1}{x}$, $x_1 = 2$, $dx = \Delta x = -0.02$.

$\frac{1}{1.98} = \frac{1}{2} + \Delta y$.

$y' = -\frac{1}{x_1^2}\ dx = -\frac{1}{4}(-0.02) = 0.005$,

so $\frac{1}{1.98} = 0.5 + 0.005 = 0.505$.

49. $y = x^2 + 2x - 3$, $x_1 = 1$, $\Delta x = dx = 0.07$.

$(1.07)^2 + 2(1.07) - 3 = 1^2 + 2 \cdot 1 - 3 + \Delta y = \Delta y$.

$dy = (2x_1 + 2)dx = (2 + 2)(0.07) = 0.28$, so

$(1.07)^2 + 2(1.07) - 3 = 0.28$.

50. $y = \frac{1}{\sqrt[5]{x}}$, $x_1 = 32$, $\Delta x = dx = -1$.

$\frac{1}{\sqrt[5]{31}} = \frac{1}{\sqrt[5]{32}} + \Delta y = \frac{1}{2} + \Delta y$.

$dy = -\frac{1}{5}x^{-6/5}\ dx = -\frac{1}{5}(32)^{-6/5}(-1) = +\frac{1}{320}$,

so $\frac{1}{\sqrt[5]{31}} = \frac{1}{2} + \frac{1}{320} = 0.503185$.

51. $y = \sqrt[4]{x}$, $x_1 = 16$, $\Delta x = dx = -1$.

$\sqrt[4]{15} = \sqrt[4]{16} + \Delta y = 2 + \Delta y$.

$dy = \frac{1}{4}x_1^{-3/4}\Delta x = \frac{1}{4}\ 16^{-3/4}(-1) = -\frac{1}{32}$, so

$\sqrt[4]{15} = 2 - \frac{1}{32} = 1.96875$.

52. $y = \sqrt[3]{x}$, $x_1 = .000064$, $\Delta x = dx = -0.000001$.

$\sqrt[3]{0.000063} = \sqrt[3]{0.000064} + \Delta y = 0.04 + \Delta y$.

$dy = \frac{1}{3}x_1^{-2/3}\ dx = \frac{1}{3}(0.000064)^{-2/3}(-0.000001) =$
-0.000208, so $\sqrt[3]{0.000063} = 0.04 - 0.000208 =$
0.04021.

53. Let $y = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$, $x_1 = 9$, $\Delta x = dx = 1$.

$10^{-\frac{1}{2}} = 9^{-\frac{1}{2}} + \Delta y = \frac{1}{3} + \Delta y$.

$dy = -\frac{1}{2}x_1^{-3/2}dx = -\frac{1}{2}(9)^{-3/2}(1) = -\frac{1}{57}$,

so $10^{-\frac{1}{2}} = \frac{1}{3} - \frac{1}{57} = 0.31579$.

54. $y = \sin x$, $x_1 = \frac{\pi}{6}$, $\Delta x = dx = 0.01$.

$\sin(\frac{\pi}{6} + 0.01) = \sin(\frac{\pi}{6}) + \Delta y = \frac{1}{2} + \Delta y$.

$dy = \cos x\ dx = (\cos\frac{\pi}{6})(0.01) = 0.00866$,

so $\sin(\frac{\pi}{6} + 0.01) = \frac{1}{2} + 0.00866 = 0.50866$.

55. $\cos 61^\circ = \cos 1.064650844$

Let $y = \cos x$, $x_1 = \frac{\pi}{3}$, $\Delta x = 0.0174532928$.

$\cos 61^\circ = \cos\frac{\pi}{3} + \Delta y = 0.5 + \Delta y$.

$dy = -\sin x\ dx = (-\sin\frac{\pi}{3})(0.0174532928)$
$= -0.015114995$, so

$\cos 61^\circ \approx 0.5 - 0.15114995 \approx 0.4849$.

56. $\tan 44^\circ = \tan 0.7679448709$. $y = \tan x$, $x_1 = \frac{\pi}{4}$,
$\Delta x = dx = -0.0174532925$.

$\tan 44^\circ = \tan 45^\circ + \Delta y = 1 + \Delta y$.

$dy = \sec^2 x\ dx = (\sec^2\frac{\pi}{4})\ \Delta x = -0.034906585$, so

$\tan 44^\circ = 1 - 0.034906585 = 0.965093415$.

57. $V(r) = \frac{4}{3}\pi r^3$. Now, $V(r_1 + \Delta r) - V(r_1) \approx V'(r_1)\cdot\Delta r$
where $\Delta r = dr$. Let $r_1 = 3$, $\Delta r = \frac{3}{32}$. So
$V(3 + \frac{3}{32}) - V(3) \approx 4\pi(r_1)^2 \cdot \Delta r = 4\pi(9)(\frac{3}{32})$
$= \frac{27\pi}{8}$. Thus, the volume of the shell is
approximately $\frac{27\pi}{8} \approx 10.6$ cubic inches.

58. Take Δr very small so that the volume ΔV of the shell between the sphere of radius r and the concentric sphere of radius $r + \Delta r$ is given approximately by $\Delta V \approx 4\pi r^2\Delta r$ (since $V(r + \Delta r) - V(r) = \Delta V \approx dV = V'(r)\Delta r = 4\pi r^2\Delta r$). Thus, $\frac{\Delta V}{\Delta r} \approx$

$4\pi r^2$. If A is the area of the surface, then the volume of the spherical shell is also given approximately by $\Delta V \approx A \cdot \Delta r$, so that $\frac{\Delta V}{\Delta r} \approx A$. Both these approximations become better and better as $\Delta r \to 0$, so that $4\pi r^2 = \lim_{\Delta r \to 0} \Delta V = A$.

59. $dV \approx V'(10) \cdot \Delta a$ where $V = a^3$.
$V'(10) \cdot \Delta a = 3(10)^2(\pm 0.02) = 3(100)(\pm 0.02)$
$= \pm 6$.
An approximate upper bound for the error is 6 cubic centimeters.

60. (a) $A(\text{of annulus}) = \pi(5.03)^2 - \pi(5)^2$
$= \pi(25.3009) - 25\pi$
$\approx 0.3009\pi \approx 0.9453$ square meters.
(b) $A(\text{of annulus}) \approx A'(r)\,\Delta r = 2\pi r \Delta r$
$= 2\pi(5)(0.03)0.3\pi$
≈ 0.94248
square meters.
(c) Error $= 0.9453 - 0.94248 = 0.00282$ square meters.

61. If r is the radius of the base and $h = 2$ m $= 200$ cm is the height, then the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h = \frac{200}{3}\pi r^2 \text{cm}^3$. The approximate increase in the volume is therefore given by $dV = \frac{400}{3}\pi r dr$, where $r = 100$ cm and $dr = \Delta r = 5$ cm. Thus, $\Delta V \approx dV = \frac{400}{3}\pi(100)(5) = \frac{200{,}000\pi}{3} \approx 209{,}440 \text{ cm}^3 \approx 0.21 \text{ m}^3$.

62. $s = \frac{1}{3}t^3 - 2t + 3$, so that $ds = (t^2 - 2)dt$ and $\Delta s \approx ds = (2^2 - 2)(2 \cdot 1 - 2) = 0.2$ m.

63. The volume of a cylindrical rod of length 30 cm and radius r cm is given by $V = 30\pi r^2$. Thus, $\Delta V \approx dV = 60\pi r dr = 60\pi(2.34)(0.01) = 1.404\pi \approx$ 4.41 cm^3.

64. $dT = \dfrac{2\pi \frac{dL}{g}}{2\sqrt{\frac{L}{g}}} = \dfrac{\pi dL}{g\sqrt{\frac{L}{g}}}$, so that

$$\frac{dT}{T} = \frac{\pi dL}{g\sqrt{\frac{L}{g}}\left(2\pi\sqrt{\frac{L}{g}}\right)} = \frac{1}{2}\frac{dL}{L}.$$ Therefore,

$\frac{\Delta T}{T} \approx \frac{1}{2}\frac{\Delta L}{L}$, so that $\frac{\Delta L}{L} \approx 2\frac{\Delta T}{T} = 2\frac{3}{(24)(60)} = \frac{1}{240}$
$\approx 0.42\%$

65. $dF = k\left(-\frac{2}{x^3}\right)dx$

$$\frac{dF}{F} = \frac{\frac{-2k}{x^3}}{\frac{k}{x^2}}dx$$

$$\frac{dF}{F} = -\frac{2}{x}dx$$

$$\frac{\Delta F}{F} \approx -\frac{2}{x}\Delta x$$

$$100\left(\frac{\Delta F}{F}\right) \approx -2\left(\frac{\Delta x}{x}\right)100.$$

So $\frac{\Delta F}{F} \cdot 100 \approx -2(2) = -4\%$, or about 4 percent decrease.

66. (a) $P = 10x - C = 10x - \frac{x^3}{15{,}000} + \frac{3}{100}x^2 - 11x - 75$

$$= \frac{-x^3}{15{,}000} + \frac{3x^2}{100} - x - 75$$

(b) $dP = \left(\frac{-x^2}{5000} + \frac{3x}{50} - 1\right)dx$

(c) $\Delta P \approx dP = \left[\frac{-(350)^2}{5000} + \frac{3(350)}{50} - 1\right](5)$

$$= -\frac{(350)^2}{1000} + 105 - 5$$
$$= -122.50 + 105 - 5$$
$$= -22.50.$$

P decreases by approximately \$22.50.

67. Taking differentials on each side, we get
$V^{1.7}\,dP + 1.7V^{0.7}dV \cdot P = 0$
So $\dfrac{V^{1.7}\,dP}{V^{1.7}P} + \dfrac{P(1.7)V^{0.7}dV}{V^{1.7}P} = 0$

Hence, $\frac{dP}{P} + \frac{1.7dV}{V} = 0.$

68. From Theorem 1, we have

$f(x_1 + \Delta x) - f(x_1) - f'(x_1)\Delta x = \Delta x\, \epsilon$

then

$\frac{f(x_1 + \Delta x) - f(x_1) - f'(x_1)\Delta x}{\Delta x} = \epsilon$

Let $\Delta x = x - x_1$. As $x \to x_1$, $\Delta x \to 0$

so we have

$\frac{f(x) - [f(x_1) - f'(x_1)(x - x_1)]}{\Delta x} = \epsilon$

So $\lim\limits_{x \to x_1} \frac{f(x) - g(x)}{x - x_1} = 0$

where $g(x) = f(x_1) - f'(x_1)(x - x_1) = f(x_1) + x_1 f'(x_1) - f'(x_1)x = a + bx$ is a linear function.

69. If the measured value is $x_1 = (31.4)\frac{\pi}{180}$ radians with an error Δx, $|\Delta x| \le (0.05)\frac{\pi}{180}$ radian, then the error in the calculated value of the sine of the angle is $\Delta y = \sin(x_1 + \Delta x) - \sin x_1 \approx dy = \cos x_1\, \Delta x$. Hence, $|\Delta y| \approx |dy| = |\cos x_1\, \Delta x| = |\cos x_1||\Delta x| \le |\cos x_1|\,(0.05)\frac{\pi}{180} \approx 7.45 \times 10^{-4}$.

Problem Set 4.2, page 257

1. $D_x(4x^3 - 3x^2 + x - 1) = 12x^2 - 6x + 1.$

2. $D_x\left(\frac{1 + x^2}{1 - x^2}\right) = \frac{(1 - x^2)(2x) - (1 + x^2)(-2x)}{(1 - x^2)^2}$
$= \frac{4x}{1 - 2x^2 + x^4}.$

3. $D_u(-\frac{1}{2}\cos u^2) = -\frac{1}{2}(-\sin u^2)(2u) = u \sin u^2.$

4. $D_t(\frac{1}{4}t^4 - t^3 + \frac{3}{2}t^2 - t + 753) = t^3 - 3t^2 + 3t - 1 = (t - 1)^3.$

5. $\int(3x^2 - 4x - 5)dx = \frac{3x^3}{3} - \frac{4x^2}{2} - 5x + C = x^3 - 2x^2 - 5x + C.$

6. $\int(x^3 - 3x^2 + 2x - 4)dx = \frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} - 4x + C = \frac{x^4}{4} - x^3 + x^2 - 4x + C.$

7. $\int(2t^3 - 4t^2 - 5t + 6)dt = \frac{2t^4}{4} - \frac{4t^3}{3} - \frac{5t^2}{2} + 6t + C$
$\frac{t^4}{2} - \frac{4t^3}{3} - \frac{5t^2}{2} + 6t + C.$

8. $\int(2 + 3y^2 - 8y^3)dy = 2y + \frac{3y^3}{3} - \frac{8y^4}{4} + C = 2y + y^3 - 2y^4 + C.$

9. $\int(3u^4 - 2u^3 - u - 1)du = \frac{3u^5}{5} - \frac{2u^4}{4} - \frac{u^2}{4} - u + C = \frac{3u^5}{5} - \frac{u^4}{2} - \frac{u^2}{2} - u + C.$

10. $\int(\frac{2}{3}z^5 - \frac{3}{5}z^2 + \frac{4}{7}z - \frac{1}{9})dz = \frac{2}{3}\frac{z^6}{6} - \frac{3}{5}\frac{z^3}{3} + \frac{4}{7}\frac{z^2}{2} - \frac{1}{9}z + C$
$= \frac{z^6}{9} - \frac{z^5}{5} + \frac{2z^2}{7} - \frac{1}{9}z +$

11. $\int(4x^{-2} - 3x^{-4} + 1)dx = \frac{4x^{-1}}{-1} - \frac{3x^{-3}}{-3} + x + C$
$= -4x^{-1} + x^{-3} + x + C.$

12. $\int(2w^{-2} + 5w - 3)dw = \frac{2w^{-1}}{-1} + \frac{5w^2}{2} - 3w + C$
$= -2w^{-1} + \frac{5}{2}w^2 - 3w + C.$

13. $\int(2t^4 + 5 + 7t^{-2})dt = \frac{2t^5}{5} + 5t + \frac{7t^{-1}}{-1} + C$
$= \frac{2t^5}{5} + 5t - 7t^{-1} + C.$

14. $\int(2x^5 + 10x^3 - x^2 - 5)dx = \frac{2x^6}{6} + \frac{10x^4}{4} - \frac{x^3}{3} - 5x + C$
$= \frac{x^6}{3} + \frac{5x^4}{2} - \frac{x^3}{3} - 5x + C.$

15. $\int(t^2 + 3t + t^{-2})dt = \frac{t^3}{3} + \frac{3t^2}{2} + \frac{t^{-1}}{-1} + C$
$= \frac{t^3}{3} + \frac{3t^2}{2} - t^{-1} + C.$

16. $\int(x^{\frac{1}{2}} - x^{-4})dx = \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{-3}}{-3} + C = \frac{2}{3}x^{3/2} + \frac{x^{-3}}{3} + C.$

17. $\int(2x + 10x^3 - x^{-2} - 5)dx = \frac{2x^2}{2} + \frac{10x^4}{4} - \frac{x^{-1}}{-1} - 5x +$

$= x^2 + \frac{5x^4}{2} + x^{-1} - 5x + C.$

18. $\int (x^2 + x + 1)dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C.$

19. $\int (16t^4 + 24t^2 + 9)dt = \frac{16t^5}{5} + 8t^3 + 9t + C.$

20. $\int (49y^{-6} - 56y^{-3} + 16)dy = \frac{49y^{-7}}{-7} - \frac{56y^{-4}}{-4} + 16y + C$
$= -7y^{-7} + 14y^{-4} + 16y + C.$

21. $\int (w^{3/2} - 2w^{\frac{1}{2}})dw = \frac{w^{5/2}}{\frac{5}{2}} - \frac{2w^{3/2}}{\frac{3}{2}} + C = \frac{2}{5}w^{5/2} -$
$\frac{4}{3}w^{3/2} + C = 2w\sqrt{w}(\frac{1}{5}w - \frac{2}{3}) + C.$

22. $\int (3u^{4/3} + 11u^{1/3})du = \frac{3u^{7/3}}{\frac{7}{3}} + \frac{11u^{4/3}}{\frac{4}{3}} + C$
$= \frac{9}{7}u^{7/3} + \frac{33}{4}u^{4/3} + C.$

23. $\int (25x^{5/2} - x^{-\frac{1}{2}})dx = \frac{25x^{7/2}}{\frac{7}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{50}{7}x^{7/2} -$
$2x^{\frac{1}{2}} + C = 2\sqrt{x}(\frac{25}{7}x^3 - 1) + C.$

24. $\int (\sqrt{2}x^{\frac{1}{2}} + 2x^{3/2} + x^{-\frac{1}{2}})dx = \frac{\sqrt{2}x^{3/2}}{\frac{3}{2}} + \frac{2x^{5/2}}{\frac{5}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{2\sqrt{2}}{3}x^{3/2} + \frac{4}{5}x^{5/2} + 2x^{\frac{1}{2}} + C.$

25. $\int (x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}})dx = \frac{x^{3/2}}{\frac{3}{2}} - 2x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$
$= \frac{2}{3}x^{3/2} - 2x + 2x^{\frac{1}{2}} + C.$

26. $\int (t^{8/3} + 2t^{5/3} - 3t^{-1/3})dt = \frac{t^{11/3}}{\frac{11}{3}} + \frac{2t^{8/3}}{\frac{8}{3}} - \frac{3t^{2/3}}{\frac{2}{3}} + C =$
$\frac{3}{11}t^{11/3} + \frac{3}{4}t^{8/3} - \frac{9}{2}t^{2/3} + C.$

27. $\int (2\cos u + 4\sin u)du = 2\sin u - 4\cos u + C.$

28. $\int (3\sec^2 u - 4\csc^2 u)du = 3\tan u - 4\cot u + C.$

29. $\int (3\sec x\tan x - 2\sec^2 x)dx = 3\sec x - 2\tan x + C.$

30. $\int (2\sin t - 3\cos t + 11) = -2\cos t - 3\sin t +$
$11t + C.$

31. $\int (2\csc y\cot y - 7\csc^2 y)dy = -2\csc y +$
$7\cot y + C.$

32. $\int \sec^2 x\,dx = \tan x + C.$

33. $\int (5\csc^2 t + 7\sec^2 t + 4) = -5\cot t + 7\tan t +$
$4t + C.$

34. $\int (\frac{5\cos u}{\sin u}\cdot\frac{1}{\sin u} + \frac{4}{\sin^2 u})du =$
$\int (5\cot u\csc u + 4\csc^2 u) =$
$-5\csc u - 4\cot u + C.$

35. $\int (4\sec^2 x - \csc^2 x)dx = 4\tan x + \cot x + C.$

36. $\int (\sec^2 x - 1)dx = \tan x - x + C.$

37. $\int (x^{2/3} - 4\sin x + 5\cos x)dx = \frac{x^{5/3}}{\frac{5}{3}} + 4\cos x$
$+ 5\sin x + C$
$= \frac{3}{5}x^{5/3} + 4\cos x$
$+ 5\sin x + C.$

38. $\int 4(\csc^2 x - 1)dx = 4(-\cot x - x) + C.$

39. $\int (2x^{1/2} + 3\sec^2 x - 4\csc x \cot x)\,dx = \frac{4}{3}x^{3/2} + 3\tan x + 4\cot x + C.$

40. $\int \frac{1}{2}(1 + \cos x)dx = \frac{1}{2}(x + \sin x) + C.$

41. 1. $D_x(\frac{x^{n+1}}{n+1} + c) = \frac{(n+1)x^n}{n+1} + 0 = x^n.$

2. $D_x\left[a\int f(x)dx\right] = a\,D_x\int f(x)dx = af(x).$

3. $D_x\left[\int f(x)dx + \int g(x)dx\right] = D_x\int f(x)dx + D_x\int g(x)dx$
$= f(x) + g(x).$

4. $D_x\left[a_1\int f_1(x)dx + a_2\int f_2(x)dx + \cdots + a_m\int f_m(x)dx\right]$
$= D_x\left[a_1\int f_1(x)dx\right] + D_x\left[a_2\int f_2(x)dx\right] + \cdots$
$+ D_x[a_m\int f_m(x)dx]$
$= a_1 D_x\int f_1(x)dx + a_2 D_x\int f_2(x)dx + \cdots$
$+ a_m D_x\int f_m(x)dx$
$= a_1 f_1(x) + a_2 f_2(x) + \cdots + a_m f_m(x).$

42. A polynomial is a sum of terms of the form ax^n and $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C.$ Since the antiderivative of a sum is the sum of the antiderivatives, it follows that the antiderivative of a polynomial is again a polynomial.

43. 1. $Du(-\cos u) = -(-\sin u) = \sin u.$

2. $Du(\sin u) = \cos u.$

3. $Du(\tan u) = \sec^2 u.$

4. $Du(-\cot u) = -Du\cot u = -(-\csc^2 u) = \csc^2 u.$

5. $Du(\sec u) = \sec u \tan u.$

6. $Du(-\csc u) = -(-\csc u \cot u) = \csc u \cot u.$

44. $g'(x) = D_x x^{-2} = -2x^{-3} = f(x)$ for $x \neq 0$. Also, for $x \neq 0$, $D_x(\frac{1 - 2x^2}{x^2}) = D_x(x^{-2} - 2) = -2x^{-3} = f(x)$. It follows that $D_x h(x) = f(x)$ for $x \neq 0$. Thus, both g and h are antiderivatives of f. Since $h(x) = g(x)$ for $x > 0$ and $h(x) \neq g(x)$ for $x < 0$, there cannot exist a constant C such that $h = g + C$. This does not contradict Theorem 2 since the domain of h is not an open interval.

45. (a) Let $f(x) = x$. Then $\int f(x)dx = \int xdx = \frac{x^2}{2} + C$ but $f(x)\int dx = f(x)(x + C) = x(x + C) = x^2 + Cx$. Thus $\int f(x)dx \neq f(x)\int dx$.

(b) Let $f(x) = x$ and $g(x) = x$. Then $\int f(x)g(x)dx = \int x^2dx = \frac{x^3}{3} + C$ but $\left[\int f(x)dx\right]\left[\int g(x)dx\right] = \left[\frac{x^2}{2} + C_1\right]\left[\frac{x^2}{2} + C_2\right] \neq \frac{x^3}{3} + C.$

(c) Let $f(x) = x$ and $g(x) = x$. Then $\int \frac{f(x)}{g(x)}dx = \int dx = x + C$, but $\frac{\int f(x)dx}{\int g(x)dx} = \frac{\frac{x^2}{2} + C_1}{\frac{x^2}{2} + C_2} \neq x + C$

46. Suppose that g and h are two antiderivatives of the same function f on an open interval I. Then $g'(x) = f(x)$ and $h'(x) = f(x)$ for all x in I. Let $G = g - h$, so that $G'(x) = g'(x) - h'(x) = f(x) - f(x) = 0$ for all x in I. By Theorem 1, $G(x) = C$, a constant, for all x in I. Hence, $g(x) - h(x) = C$ for all x in I; that is, g and h differ by a constant.

Problem Set 4.3, page 263

1. Let $u = 4x + 3$, so that $du = 4\,dx$, $dx = \frac{1}{4}du$. Thus, $\int (4x + 3)^4dx = \int u^4(\frac{1}{4}du) = \frac{1}{4}\int u^4du = \frac{1}{4}\frac{u^5}{5} + C = \frac{1}{20}(4x + 3)^5 + C.$

2. Here, $u = 4t^2 + 7$, $du = 8t\,dt$, $tdt = \frac{1}{8}du$. So, $\int t(4t^2 + 7)^9dt = \int u^9 \cdot \frac{1}{8}du = \frac{1}{8}\frac{u^{10}}{10} + C = \frac{1}{80}(4t^2 + 7)^{10} + C.$

3. Here $u = 4x^2 + 15$, $du = 8xdx$, $xdx = \frac{1}{8}du.$

Thus, $\int x\sqrt{4x^2+15}\,dx = \int \sqrt{u}\,\frac{1}{8}\,du = \frac{1}{8}\int u^{\frac{1}{2}}\,du =$

$\frac{1}{8}\left[\frac{u^{3/2}}{(\frac{3}{2})}\right] + C = \frac{(4x^2+15)^{3/2}}{12} + C.$

4. $u = 4 - 3x^2$, $du = -6x\,dx$, so $x\,dx = -\frac{1}{6}\,du$.

Thus, $\int \frac{3x\,dx}{(4-3x^2)^8} = \int \frac{3(-\frac{1}{6}\,du)}{u^8} = -\frac{1}{2}$

$\int u^{-8}\,du = -\frac{1}{2}\cdot\frac{u^{-7}}{(-7)} + C = \frac{1}{14}(4-3x^2)^{-7} + C.$

5. $u = 5s^2 + 16$, $du = 10s\,ds$, $s\,ds = \frac{1}{10}\,du$.

Thus, $\int \frac{s\,ds}{\sqrt[3]{5s^2+16}} = \int \frac{\frac{1}{10}\,du}{\sqrt[3]{u}} = \frac{1}{10}\int u^{-1/3}\,du =$

$(\frac{1}{10})\,\frac{u^{2/3}}{(\frac{2}{3})} + C = \frac{3}{20}(5s^2+16)^{2/3} + C.$

6. $u = 4t^2 + 2t + 6$, $du = (8t+2)dt$.

Thus, $\int \frac{(8t+2)dt}{(4t^2+2t+6)^{17}} = \int \frac{du}{u^{17}}$

$= \int u^{-17}\,du = \frac{u^{-16}}{-16} + C$

$= -\frac{1}{16}(4t^2+2t+6)^{-16} + C.$

7. $u = 1 - x^{3/2}$, $du = -\frac{3}{2}x^{\frac{1}{2}}\,dx$, $\sqrt{x}\,dx = -\frac{2}{3}\,du$.

Thus, $\int (1-x^{3/2})^{5/3}\sqrt{x}\,dx = \int u^{5/3}(-\frac{2}{3}\,du)$

$= -\frac{2}{3}\int u^{5/3}\,du$

$= (-\frac{2}{3})\,\frac{u^{8/3}}{(\frac{8}{3})} + C = -\frac{1}{4}(1-x^{3/2})^{8/3} + C.$

8. $u = x - 3$, $du = dx$, $u^2 = x^2 - 6x + 9$

Thus, $\int (x^2-6x+9)^{11/3}\,dx = \int (u^2)^{11/3}\,du$

$= \int u^{22/3}\,du = \frac{u^{25/3}}{(\frac{25}{3})} + C$

$= \frac{3}{25}(x-3)^{25/3} + C.$

9. Let $u = 4x^3 + 1$, so that $du = 12x^2dx$ and $x^2dx =$ $\frac{1}{12}\,du$.

Thus, $\int \frac{x^2dx}{(4x^3+1)^7} = \int \frac{\frac{1}{12}\,du}{u^7} =$

$\frac{1}{12}\int u^{-7}\,du = (\frac{1}{12})(\frac{u^{-6}}{-6}) + C = \frac{-1}{72(4x^3+1)^6} + C.$

10. Let $u = x^3 + 3x$, so that $du = (3x^2+3)\,dx =$ $3(x^2+1)\,dx$ and $(x^2+1)dx = \frac{1}{3}\,du$.

Thus, $\int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx =$

$\int \frac{\frac{1}{3}\,du}{\sqrt{u}} = \frac{1}{3}\int u^{-\frac{1}{2}}\,du$

$= \frac{1}{3}\cdot\frac{u^{\frac{1}{2}}}{(\frac{1}{2})} + C = \frac{2}{3}\sqrt{x^3+3x} + C.$

11. Let $u = 5t^3 + 3t - 2$, so that $du = (15t^2+3)$ $dt = 3(5t^2+1)dt$ and $(5t^2+1)dt = \frac{1}{3}\,du$.

Therefore, $\int (5t^2+1)\sqrt[4]{5t^3+3t-2}\,dt =$

$\int \frac{1}{3}\sqrt[4]{u}\,du = \frac{1}{3}\int u^{\frac{1}{4}}\,du = \frac{1}{3}\cdot\frac{u^{5/4}}{(\frac{5}{4})} + C = \frac{4}{15}$ $(5t^3+3t-2)^{5/4} + C.$

12. Let $u = 1 + \frac{1}{2t}$, so that $du = -\frac{1}{2t^2}\,dt$ and $-2du$ $= \frac{1}{t^2}\,dt$. Thus, $\int \frac{\sqrt[3]{1+\frac{1}{2t}}}{t^2}\,dt = \int (-2)\sqrt[3]{u}\,du$

$= -2\int u^{1/3}\,du = -2\,\frac{u^{4/3}}{(\frac{4}{3})} + C = -\frac{3}{2}(1+\frac{1}{2t})^{4/3} + C.$

13. Let $u = 6x^3 - 9x + 1$, so that $du = (18x^2-9)dx$ $= 9(2x^2-1)dx$ and $(2x^2-1)dx = \frac{1}{9}\,du$. Thus,

$\int \frac{(2x^2-1)dx}{(6x^3-9x+1)^{3/2}} = \int \frac{\frac{1}{9}\,du}{u^{3/2}}$

$= \frac{1}{9}\int u^{-3/2}\,du = \frac{1}{9}\,\frac{u^{-\frac{1}{2}}}{(-\frac{1}{2})} + C$

$= \frac{-2}{9\sqrt{u}} + C = \frac{-2}{9\sqrt{6x^3 - 9x + 1}} + C.$

14. Let $u = 1 + \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\,dx$ and $\frac{1}{\sqrt{x}}\,dx = 2\,du$. Thus, $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int \sqrt{u}\,(2du) = 2\int u^{1/2}\,du = 2\,\frac{u^{3/2}}{(\frac{3}{2})} + C = \frac{4}{3}(1+\sqrt{x})^{3/2} + C.$

15. Let $u = x + \frac{5}{x}$, so that $du = (1 - \frac{5}{x^2})dx = \frac{x^2 - 5}{x^2}\,dx$. Thus, $\int (x + \frac{5}{x})^{21}\,\frac{x^2-5}{x^2}\,dx = \int u^{21}\,du = \frac{u^{22}}{22} + C = \frac{1}{22}(x + \frac{5}{x})^{22} + C.$

16. Let $u = 7x - 3$, so that $du = 7dx$ and $dx = \frac{1}{7}\,du$. Thus, $\int (49x^2 - 42x + 9)^{6/7}\,dx = \int (u^2)^{6/7}\,(\frac{1}{7}\,dx)$ $= \frac{1}{7}\int u^{12/7}\,dx = \frac{1}{7}\cdot\frac{u^{19/7}}{(\frac{19}{7})} + C = \frac{1}{19}(7x-3)^{19/7} + C.$

17. Let $u = 5 - x$, so that $du = -dx$ and $x = 5 - u$. Thus,

$\int x\sqrt{5-x}\,dx = \int (5-u)\sqrt{u}\,(-du)$

$= -\int (5u^{1/2} - u^{3/2})\,du = \int (u^{3/2} - 5u^{1/2})\,du$

$= \frac{u^{5/2}}{\frac{5}{2}} - 5\cdot\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{5}(5-x)^{5/2} - \frac{10}{3}(5-x)^{3/2} + C.$

18. Let $u = 1 + x$, so that $du = dx$ and $x = u - 1$.

Therefore, $x^2 = u^2 - 2u + 1$ and $\int x^2\sqrt{1+x}\,dx$

$= \int (u^2 - 2u + 1)u^{1/2}\,du = \int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$

$= \frac{u^{7/2}}{\frac{7}{2}} - 2\cdot\frac{u^{5/2}}{\frac{5}{2}} + \frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{7}(1+x)^{7/2} - \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2} + C.$

19. Let $u = t + 1$, so that $du = dt$ and $t = u - 1$.

Therefore, $\int \frac{t\,dt}{\sqrt{t+1}} = \int \frac{(u-1)\,du}{\sqrt{u}} = \int (u-1)u^{-1/2}\,du$

$= \int (u^{1/2} - u^{-1/2})\,du = \frac{u^{3/2}}{\frac{3}{2}} - \frac{u^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}(t+1)^{3/2}$

$-2(t+1)^{1/2} + C.$

20. Let $u = 2 - y$, $du = -dy$, $y = 2 - u$. So $\int \frac{y+2}{\sqrt[3]{2-y}}\,dy$

$= \int \frac{(2-u)+2}{u^{1/3}}\,(-du) = -\int (4u^{-1/3} - u^{2/3})\,du =$

$-\frac{4u^{2/3}}{\frac{2}{3}} + \frac{u^{5/3}}{\frac{5}{3}} + C = -6(2-y)^{2/3} + \frac{3}{5}(2-y)^{5/3}$

$+ C.$

21. Let $u = 2 - x$, $du = -dx$, $x = 2 - u$. So

$\int \frac{2x\,dx}{(2-x)^{2/3}} = \int \frac{2(2-u)(-du)}{u^{2/3}} =$

$-\int (4u^{-2/3} - 2u^{1/3})\,du = -12u^{1/3} + \frac{6}{4}u^{4/3} + C$

$= \frac{3}{2}(2-x)^{4/3} - 12(2-x)^{1/3} + C.$

22. Let $u = 1 + x$, $du = dx$, $x = u - 1$. So

$\int (u - 1 + 2)^2\,u^{1/2}\,du = \int (u+1)^2\,u^{1/2}\,du =$

$\int (u^{5/2} + 2u^{3/2} + u^{1/2})\,du = \frac{2}{7}u^{7/2} + \frac{2u^{5/2}}{\frac{5}{2}} +$

$\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{7}(1+x)^{7/2} + \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2}$

$+ C.$

23. Let $u = 3x^2 + 5$, $du = 6x\,dx$, $x\,dx = \frac{1}{6}\,du$, $x^2 = \frac{u-5}{3}$. So $\int \sqrt[3]{3x^2+5}\cdot x^2\cdot x\,dx =$

$\int u^{1/3}\,(\frac{u-5}{3})(\frac{1}{6}\,du) = \frac{1}{18}\int (u^{4/3} - 5u^{1/3})\,du$

$= \frac{1}{18} \cdot \left[\frac{3u^{7/3}}{7} - \frac{15}{4}u^{4/3} + C\right] = \frac{1}{42}(3x^2 + 5)^{7/3} -$

$\frac{5}{24}(3x^2 + 5)^{4/3} + C.$

24. Let $u = x^3 + 1$, $du = 3x^2dx$, $x^2dx = \frac{1}{3}du$, $x^3 = u - 1$.

So $\int \sqrt[4]{x^3 + 1} \cdot x^3 \cdot x^2dx = \int u^{\frac{1}{4}} \cdot (u - 1)\frac{1}{3}du =$

$\frac{1}{3}\int (u^{5/4} - u^{\frac{1}{4}})du = \frac{1}{3}(\frac{u^{9/4}}{\frac{9}{4}} - \frac{u^{5/4}}{\frac{5}{4}}) + C =$

$\frac{4}{27}(x^3 + 1)^{9/4} - \frac{4}{15}(x^3 + 1)^{5/4} + C.$

25. Let $u = t + 4$, $du = dt$, $t = u - 4$, $t^2 = u^2 - 8u + 16$.

So $\int \frac{u^2 - 8u + 16}{u^{\frac{1}{2}}}du = \int (u^{3/2} - 8u^{\frac{1}{2}} + 16u^{-\frac{1}{2}})du =$

$\frac{2}{5}u^{5/2} - \frac{16}{3}u^{3/2} + 32u^{\frac{1}{2}} + C =$

$\frac{2}{5}(t + 4)^{5/2} - \frac{16}{3}(t + 4)^{3/2} + 32(t + 4)^{\frac{1}{2}} + C.$

26. Let $u = 3 - y$, $du = -dy$, $y = 3 - u$. So

$\int \frac{y\,dy}{\sqrt{3 - y}} = -\int \frac{3 - u}{\sqrt{u}}du = -\int (3u^{-\frac{1}{2}} - u^{\frac{1}{2}})du$

$= -\frac{3u^{\frac{1}{2}}}{\frac{1}{2}} + \frac{u^{3/2}}{\frac{3}{2}} + C = -6(3 - y)^{\frac{1}{2}} + \frac{2}{3}(3 - y)^{3/2} + C.$

27. Let $u = 35x$, $du = 35dx$, $dx = \frac{1}{35}du$.

$\int 2\sin 35x\,dx = \int 2\sin u(\frac{1}{35})du = \frac{2}{35}\int \sin u\,du =$

$= \frac{2}{35}(-\cos u) + C = -\frac{2}{35}\cos 35x + C.$

28. Let $u = 5x$, so $du = 5\,dx$. Let $v = 7x$, so $dv = 7\,dx$.

Thus, $\int 7\sin 5x\,dx + \int 3\cos 7x\,dx = \int 7\sin u(\frac{1}{5}du) +$

$\int 3\cos v(\frac{1}{7}dv) = \frac{7}{5}\int \sin u\,du + \frac{3}{7}\int \cos v\,dv =$

$\frac{7}{5}(-\cos u) + \frac{3}{7}\sin v + C = -\frac{7}{5}\cos 5x + \frac{3}{7}\sin 7x + C.$

29. Let $u = 16x - 1$, $du = 16\,dx$.

$\int 8\cos(16x - 1)dx = \int 8\cos u(\frac{1}{16}du)$

$= \frac{1}{2}\int \cos u\,du$

$= \frac{1}{2}\sin u + C = \frac{1}{2}\sin(16x - 1) + C.$

30. Let $u = 8 - 3x$, $du = -3\,dx$.

$\int 5\cos(8 - 3x)dx = \int 5\cos u(-\frac{1}{3}du)$

$= -\frac{5}{3}\int \cos u\,du$

$= -\frac{5}{3}\sin u + C = -\frac{5}{3}\sin(8 - 3x) + C.$

31. Let $u = 11x$, $du = 11dx$.

$\int \sec^2 11x\,dx = \int \sec^2 u(\frac{1}{11}du) = \frac{1}{11}\int \sec^2 u\,du =$

$\frac{1}{11}\tan u + C = \frac{1}{11}\tan 11x + C.$

32. Let $u = 5x$, $du = 5\,dx$.

$\int (-\csc^2 5x)dx = \int (-\csc^2 u)\frac{1}{5}du = \frac{1}{5}\int (-\csc^2 u)du$

$= \frac{1}{5}(\cot u) + C = \frac{1}{5}\cot 5x + C.$

33. Let $u = 3t$, $du = 3\,dt$.

$\int \frac{dt}{\sin^2 3t} = \int \frac{\frac{1}{3}du}{\sin^2 u} = \frac{1}{3}\int \csc^2 u\,du = \frac{1}{3}(-\cot u) + C$

$= -\frac{1}{3}\cot 3t + C.$

34. Let $u = 5y$, $du = 5\,dy$.

$\int \frac{dy}{\cos^2 5y} = \int \frac{\frac{1}{5}du}{\cos^2 u} = \frac{1}{5}\int \sec^2 u\,du = \frac{1}{5}\tan u + C$

$= \frac{1}{5}\tan 5y + C.$

35. Let $u = 2y + 1$, $du = 2\,dy$.

$$\int \sec(2y + 1)\tan(2y + 1)dy = \int \sec u \tan u(\tfrac{1}{2})\,du$$

$$= \frac{1}{2}\int \sec u \tan u\,du$$

$$= \frac{1}{2}\sec u + C = \frac{1}{2}\sec(2y + 1) + C.$$

36. Let $u = 3t + 7$, $du = 3\,dt$.

$$\int \tan^2(3t + 7)dt = \int \tan^2 u(\tfrac{1}{3}\,du) = \frac{1}{3}\int \tan^2 u\,du$$

$$= \frac{1}{3}\int (\sec^2 u - 1)du$$

$$= \frac{1}{3}(\tan u - u) + C = \frac{1}{3}(\tan(3t + 7) - 3t - 7) + C.$$

37. Let $u = \frac{t}{5}$, $du = \frac{1}{5}\,dt$

$$\int (-\sec\frac{t}{5}\tan\frac{t}{5}\,dt) = \int (-\sec u \tan u)5\,du$$

$$= -5\int \sec u \tan u\,du$$

$$= -5\sec u + C = -5\sec\frac{t}{5} + C.$$

38. Let $u = 10z$, $du = 10\,dz$

$$\int \csc 10z \cot 10z\,dz = \int \csc u \cot u(\tfrac{1}{10})du$$

$$= \frac{1}{10}\int \csc u \cot u\,du$$

$$= \frac{1}{10}(-\csc u) + C = -\frac{1}{10}\csc 10z + C.$$

39. $u = \sin x$, $du = \cos x\,dx$.

$$\int \cos x \cos(\sin x)dx = \int \cos u\,du = \sin u + C$$

$$= \sin(\sin x) + C.$$

40. Let $u = 10x^4$, $du = 40x^3 dx$.

$$\int x^3 \sec 10x^4 \tan 10x^4 dx = \int \sec u \tan u(\tfrac{1}{40})du$$

$$= \frac{1}{40}\int \sec u \tan u\,du$$

$$= \frac{1}{40}\sec u + C = \frac{1}{40}\sec 10x^4 + C.$$

41. Let $u = 7x^4$, $du = 28x^3 dx$.

$$\int x^3 \csc^2 7x^4 dx = \int \csc^2 u(\tfrac{1}{28}\,du) = \frac{1}{28}\int \csc^2 u\,du$$

$$= \frac{1}{28}(-\cot u) + C$$

$$= -\frac{1}{28}\cot u + C = -\frac{1}{28}\cot 7x^4 + C.$$

42. Let $u = \sqrt{x + 1}$, $du = \dfrac{1}{2\sqrt{x + 1}}\,dx$.

$$\int \frac{\sin\sqrt{x + 1}}{\sqrt{x + 1}}\,dx = \int \sin u(2\,du) = \int 2\sin u\,du$$

$$= 2\int \sin u\,du$$

$$= 2(-\cos u) + C = -2\cos\sqrt{x + 1} + C.$$

43. Let $u = 2 + \cos x$, $du = -\sin x\,dx$.

$$\int \frac{\sin x\,dx}{(2 + \cos x)^2} = \int \frac{1}{u^2}(-du) = -\int u^{-2}\,du$$

$$= -\left[\frac{u^{-1}}{-1}\right] + C$$

$$= \frac{1}{u} + C = \frac{1}{2 + \cos x} + C.$$

44. Let $u = 3 + 2\tan x$, $du = 2\sec^2 x\,dx$.

$$\int \frac{\sec^2 x\,dx}{(3 + 2\tan x)^3} = \int \frac{1}{u^3}(\tfrac{1}{2}\,du) = \frac{1}{2}\int u^{-3}du$$

$$= \frac{1}{2}\left[\frac{u^{-2}}{-2}\right] + C$$

$$= -\frac{1}{4u^2} + C = -\frac{1}{4(3 + 2\tan x)^2} + C.$$

45. Let $u = 5 + \sin 2y$, $du = 2\cos 2y\,dy$.

$$\int \cos 2y\sqrt{5 + \sin 2y}\,dy = \int u^{1/2}(\tfrac{1}{2}\,du) = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\left[\frac{u^{3/2}}{\frac{3}{2}}\right] + C$$

$$= \frac{1}{3}u^{3/2} + C = \frac{1}{3}(5 + \sin 2y) + C.$$

46. Let $u = 4 + \sec 3t$, $du = 3 \sec 3t \tan 3t\, dt$.

$$\int \tan 3t \sec 3t \sqrt{4 + \sec 3t}\, dt = \int u^{1/2}(\tfrac{1}{3}\, du)$$

$$= \frac{1}{3}\int u^{1/2}\, du = \frac{1}{3}\left[\frac{u^{3/2}}{\frac{3}{2}}\right] + C$$

$$= \frac{2}{9}u^{3/2} + C = \frac{2}{9}(4 + \sec 3t)^{3/2} + C.$$

47. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\, dx$.

$$\int \frac{\cot \sqrt{x} \csc \sqrt{x}}{\sqrt{x}}\, dx = \int \cot u \csc u (2\, du)$$

$$= 2\int \cot u \csc u\, du.$$

$$= 2\left[-\csc u\right] + C = -2\csc(\sqrt{x}) + C.$$

48. Let $u = \sin 2x$, $du = 2 \cos 2x\, dx$.

$$\int (\sin 2x)^{-1/3} \cos 2x\, dx = \int u^{-1/3}(\tfrac{1}{2}\, du)$$

$$= \frac{1}{2}\int u^{-1/3}\, du = \frac{1}{2}\left[\frac{u^{2/3}}{\frac{2}{3}}\right] + C$$

$$= \frac{3}{4}u^{2/3} + C = \frac{3}{4}(\sin 2x)^{2/3} + C.$$

49. Let $u = 1 - 5 \sec 3\theta$, $du = -15 \sec 3\theta \tan 3\theta\, d\theta$.

$$\int \frac{\sec 3\theta \tan 3\theta\, d\theta}{\sqrt{1 - 5 \sec 3\theta}} = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{15}\, du\right)$$

$$= -\frac{1}{15}\int u^{-1/2} du = -\frac{1}{15}\left[\frac{u^{1/2}}{\frac{1}{2}}\right] + C$$

$$= -\frac{2}{15}u^{1/2} + C = -\frac{2}{15}(1 - 5\sec 3\theta)^{1/2} + C.$$

50. Let $u = \dfrac{\sqrt{x}}{2}$, $du = \dfrac{1}{4\sqrt{x}}\, dx$.

$$\int \frac{\cot \frac{\sqrt{x}}{2} \csc^2 \frac{\sqrt{x}}{2}}{\sqrt{x}}\, dx = \int \cot u \csc^2 u (4\, du) =$$

$$4\int \cot u \csc^2 u\, du$$

Let $w = \cot u$, $dw = -\csc^2 u\, du$.

$$4\int \cot u \csc^2 u\, du = 4\int w(-dw) = -2w^2 + C$$

$$= -2\cot^2 u + C = -2\cot^2 \frac{\sqrt{x}}{2} + C.$$

51. (a) $u = \sin x$, $du = \cos x\, dx$.

$$\int \sin x \cos x\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$$

(b) $\displaystyle\int \sin x \cos x\, dx = \int \frac{1}{2}\sin 2x\, dx$

Let $u = 2x$, $du = 2\, dx$. So $\displaystyle\int \frac{1}{2}\sin 2x\, dx$

$$= \int \frac{1}{2}\sin u(\tfrac{1}{2}\, du) = \frac{1}{4}\int \sin u\, du = \frac{1}{4}(-\cos u) + C$$

$$= -\frac{1}{4}\cos 2x + C$$

(c) $\displaystyle\frac{\sin^2 x}{2} = \frac{\frac{1}{2}(1 - \cos 2x)}{2} = \frac{1}{4}(1 - \cos 2x)$

$$= \frac{1}{4} - \frac{1}{4}\cos 2x.$$

52. $\displaystyle\int \sin mx \cos nx\, dx$

$$= \int [\tfrac{1}{2}\sin(m + n)x + \tfrac{1}{2}\sin(m - n)x]dx$$

$$= \frac{1}{2}\int \sin(m + n)x\, dx + \frac{1}{2}\int \sin(m - n)x\, dx.$$

Let $u = (m + n)x$, $du = (m + n)dx$. Let $v = (m - n)x$, $dv = (m - n)dx$. Then

$$\frac{1}{2}\int \sin(m + n)x\, dx + \frac{1}{2}\int \sin(m - n)x\, dx$$

$$= \frac{1}{2}\int \sin u \frac{1}{m + n}\, du + \frac{1}{2}\int \sin v \frac{1}{m - n}\, dv$$

$$= \frac{1}{2(m + n)}\int \sin u\, du + \frac{1}{2(m - n)}\int \sin v\, dv$$

$$= \frac{1}{2(m + n)}(-\cos u) + \frac{1}{2(m - n)}(-\cos v)$$

$$= \frac{-\cos(m + n)x}{2(m + n)} - \frac{\cos(m - n)x}{2(m - n)}.$$

53. (a) Let $u = x + 1$, $du = dx$, $x = u - 1$.

$$\int \frac{x\, dx}{\sqrt{x + 1}} = \int \frac{(u - 1)}{\sqrt{u}}\, du = \int (u^{1/2} - u^{-1/2})du$$

$$= \frac{u^{3/2}}{\frac{3}{2}} - \frac{u^{1/2}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C$$

$$= \frac{2}{3}\sqrt{x+1}(x-2) + C.$$

(b) $u = \sqrt{x+1}$, $du = \frac{1}{2\sqrt{x+1}}\,dx$,

$u^2 = x + 1$, $x = u^2 - 1$.

$$\int \frac{x\,dx}{\sqrt{x+1}} = \int (u^2 - 1)2\,du = 2\int (u^2 - 1)du$$

$$= 2\left[\frac{u^3}{3} - u\right] + C$$

$$= \frac{2}{3}u^3 - 2u + C = \frac{2}{3}(\sqrt{x+1})^3 - 2\sqrt{x+1} + C$$

$$= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C = \frac{2}{3}\sqrt{x+1}\,(x-2) + C.$$

54. $f'(x) = \frac{2}{(1+x)^2}$. Let $u = 1 + x$, $du = dx$.

$$f(x) = \int \frac{2}{(1+x)^2}\,dx = \int \frac{2}{u^2}\,du = \int 2u^{-2}\,du$$

$$= \frac{2u^{-1}}{-1} + C = -\frac{1}{2u} + C = -\frac{1}{1+x} + C.$$ Now

$f(0) = -\frac{1}{1+0} + C = 0$, so $-1 + C = 0$ or $C = 1$.

Therefore, $f(x) = -\frac{1}{1+x} + 1 = \frac{x}{1+x}$.

55. (a) $u = 5x - 1$, $du = 5\,dx$, $x = \frac{u+1}{5}$.

$$\int x^2\sqrt{5x-1}\,dx = \int \left(\frac{u+1}{5}\right)^2 \sqrt{u}\left(\frac{1}{5}\,du\right)$$

$$= \int \frac{u^{5/2} + 2u^{3/2} + u^{1/2}}{125}\,du$$

$$= \frac{1}{125}\left[\frac{u^{7/2}}{\frac{7}{2}} + \frac{2u^{5/2}}{\frac{5}{2}} + \frac{u^{3/2}}{\frac{3}{2}}\right] + C$$

$$= \frac{1}{125}\left(\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{125}\left(\frac{2}{7}(5x-1)^{7/2} + \frac{4}{5}(5x-1)^{5/2} + \frac{2}{3}(5x-1)^{3/2}\right) + C.$$

(b) $u = \sqrt{5x-1}$, $du = \frac{5}{2\sqrt{5x-1}}\,dx$,

$u^2 = 5x - 1$, $x = \frac{u^2+1}{5}$.

$$\int x^2\sqrt{5x-1}\,dx = \int x^2\sqrt{5x-1}\;\frac{2}{5}\sqrt{5x-1}\,du$$

$$= \frac{2}{5}\int \left(\frac{u^2+1}{5}\right)^2 u^2\,du = \frac{2}{125}\int (u^6 + 2u^4 + u^2)\,du$$

$$= \frac{2}{125}\left(\frac{u^7}{7} + \frac{2u^5}{5} + \frac{u^3}{3}\right) + C$$

$$= \frac{2}{125}\left[\frac{(5x-1)^{7/2}}{7} + \frac{2(5x-1)^{5/2}}{5} + \frac{(5x-1)^{3/2}}{3}\right]$$

$+ C.$

56. Let $u = 1 + x$, $du = dx$. $g(x) = \int (1+x)^{-2}dx$

$$= \int u^{-2}du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{x+1} + C.$$

$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty}\left(-\frac{1}{x+1} + C\right) = 0 + C = 0$; thus,

$C = 0$, and so $g(x) = -\frac{1}{x+1}$. The same result

occurs for $\lim_{x\to-\infty} g(x)$.

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1. $y = \int dy = \int (5x^4 + 3x^2 + 1)dx = x^5 + x^3 + x + C.$

2. $y = \int dy = \int (20x^3 - 6x^2 + 17)dx = 5x^4 - 2x^3 + 17x$
$+ C.$

3. $y = \int dy = \int \left(\frac{6}{x^2} + 15x^2 + 10\right)dx = \frac{-6}{x} + 5x^3 + 10x$
$+ C.$

4. $y = \int dy = \int y'dx = \int \frac{(x^2 - 4)^2}{2x^2}\,dx =$

$\int \frac{x^4 - 8x^2 + 16}{2x^2}\, dx =$

$\int (\frac{x^2}{2} - 4 + \frac{8}{x^2})\, dx = \frac{x^3}{6} - 4x - \frac{8}{x} + C.$

5. $y = \int dy = \int \sqrt{7x^3}dx = \sqrt{7} \int x^{3/2} dx = \sqrt{7}\, \frac{x^{5/2}}{(\frac{5}{2})} + C =$ $\frac{2\sqrt{7}}{5} x^{5/2} + C.$

6. $y = \int dy = \int (5t + 12)^3\, dt = \int u^3(\frac{1}{5}\, du) = \frac{u^4}{20} + C$ $= \frac{(5t + 12)^4}{20} + C,$

where $u = 5t + 12$ and $du = 5\, dt$.

7. $ds = (t^{-2} + \sin t)dt$

$s = \int (t^{-2} + \sin t)dt = \frac{t^{-1}}{-1} - \cos t + C = \frac{-1}{t} -$ $\cos t + C.$

8. $du = (\theta^{1/3} - \csc^2 5\theta)d\theta;$

$u = \int \theta^{1/3} d\theta - \int \csc^2 5\theta\, d\theta.$

Let $u = 5\theta$, $du = 5\, d\theta$. $\int \theta^{1/3} d\theta - \int \csc^2 5\theta\, d\theta =$

$\frac{\theta^{4/3}}{\frac{4}{3}} + C_1 - \int \csc^2 u(\frac{1}{5}\, du) = \frac{3}{4}\, \theta^{4/3} + \frac{\cot u}{5} + C_1 + C_2.$

$= \frac{3}{4}\, \theta^{4/3} + \frac{\cot 5\theta}{4} + C.$

9. $y = \int (5 - 3x)dx = 5x - \frac{3}{2} x^2 + C.$ Putting $y = 4$ and $x = 0$, we obtain $4 = C$; hence, $y = 5x - \frac{3}{2} x^2$ $+ 4.$

10. $y = \int (3x^2 + x)dx = x^3 + \frac{x^2}{2} + C.$ Putting $y = -2$ and $x = 1$, we obtain $-2 = 1 + \frac{1}{2} + C$, so that $C =$ $-\frac{7}{2}$. Thus, $y = x^3 + \frac{x^2}{2} - \frac{7}{2}$.

11. $y = \int (t^3 + t^{-2})dt = \frac{t^4}{4} + \frac{t^{-1}}{-1} + C = \frac{t^4}{4} - \frac{1}{t} + C.$ Putting $y = 1$ and $t = -2$, we obtain $1 = \frac{16}{4} + \frac{1}{2} +$ C, so that $C = -\frac{7}{2}$. Therefore, $y = \frac{t^4}{4} - \frac{1}{t} - \frac{7}{2}$.

12. $y = \int (\sqrt{x} + 2)dx = \frac{2}{3}\, x^{3/2} + 2x + C.$ Putting $y =$ 5 and $x = 4$, we obtain $5 = \frac{2}{3}(8) + 8 + C$, so that $C = \frac{-25}{3}$. Therefore, $y = \frac{2}{3}\, x^{3/2} + 2x - \frac{25}{3}$.

13. Let $u = \frac{x}{2}$, $du = \frac{1}{2}\, dx$. $y = \int 3 \sin \frac{x}{2}\, dx$

$= \int 3 \sin u(2\, du) = 6 \int \sin u\, du = 6(-\cos u) + C$

$= -6 \cos \frac{x}{2} + C.$

When $x = \frac{\pi}{3}$, $y = 1$. We have $1 = -6 \cos \frac{\pi}{6} + C$

$= -6(\frac{\sqrt{3}}{2}) + C,$

or $C = 1 + 3\sqrt{3}$;

so $y = -6 \cos \frac{x}{2} + 1 + 3\sqrt{3}$.

14. Let $u = 6t$, $du = 6\, dt$. $s = \int \sec^2 6t\, dt$

$= \int \sec^2 u(\frac{1}{6})du = \frac{1}{6} \int \sec^2 u\, du = \frac{1}{6} \tan u + C$

$= \frac{1}{6} \tan 6t + C$

When $t = 0$, $s = -1$.

$-1 = \frac{1}{6} \tan 0 + C = C$

Thus, $s = \frac{1}{6} \tan 6t - 1$.

15. $y = \int x(x^2 - 3)^4 dx = \int u^4(\frac{1}{2}\, du) = \frac{u^5}{10} + C$ $= \frac{(x^2 - 3)^5}{10} + C$; $u = x^2 - 3$, $du = 2x\, dx$.

16. Let $u = 3x^2 + 2x + 1$; $du = (6x + 2)dx$.

$dy = \frac{6x + 2}{(3x^2 + 2x + 1)^5}\, dx,$

so that $y = \int \frac{(6x + 2)dx}{(3x^2 + 2x + 1)^5} = \int \frac{du}{u^5} = \frac{u^{-4}}{-4} + C =$

$\frac{1}{4(3x^2 + 2x + 1)^4} + C.$

17. Let $u = x^3 + 7$, $du = 3x^2 dx$.

$dy = \frac{x^2 dx}{\sqrt{x^3 + 7}}$, $y = \int \frac{x^2 dx}{\sqrt{x^3 + 7}} = \int \frac{\frac{1}{3}\, du}{\sqrt{u}}$

$= \frac{1}{3} \int u^{-1/2} du = \frac{1}{3}\, \frac{u^{1/2}}{(\frac{1}{2})} + C$

$= \frac{2}{3} \sqrt{x^3 + 7} + C.$

18. $s = \int (t + 1)^2 t^3 dt = \int (t^2 + 2t + 1)t^3 dt$
$= \int (t^5 + 2t^4 + t^3)dt$
$= \frac{t^6}{6} + \frac{2t^5}{5} + \frac{t^4}{4} + C.$

19. $\frac{dy}{y^2} = \frac{dx}{\sqrt{2x + 1}}$, so that $\int y^{-2} dy = \int \frac{dx}{\sqrt{2x+1}}$, or
$\frac{y^{-1}}{(-1)} = \int \frac{\frac{1}{2} du}{\sqrt{u}}.$

$u = 2x + 1$, $du = 2dx$. Thus, $\int \frac{\frac{1}{2} du}{\sqrt{u}} =$
$= \frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \frac{u^{\frac{1}{2}}}{(\frac{1}{2})} + C = \sqrt{2x + 1} + C.$

Therefore, $-\frac{1}{y} = \sqrt{2x + 1} + C$, so that $y =$

$\frac{-1}{\sqrt{2x + 1} + C}.$

20. $\int (y^2 - \sqrt{y})dy = \int (x^2 + \sqrt{x})dx$, so that $\frac{y^3}{3} - \frac{2y^{3/2}}{3}$
$= \frac{x^3}{3} - \frac{2x^{3/2}}{3} + C$, or $y^3 - 2y^{3/2} = x^3 - 2x^{3/2} +$
3C. Since C is a constant, so is 3C; hence, we rewrite 3C simply as C. Thus, $y^3 - 2y^{3/2} = x^3 -$
$2x^{3/2} + C.$

21. $\frac{5y^3 dy}{\sqrt[3]{y^4 + 7}} = x dx$, so that $\int \frac{5y^3 dy}{\sqrt[3]{y^4+7}} = x dx$. In the first integral, let $u = y^4 + 7$, so that $du =$
$4y^3 dy$ and $5y^3 dy = \frac{5}{4} du$. Thus,

$\int \frac{\frac{5}{4} du}{\sqrt[3]{u}} = \int x dx$, or $\frac{5}{4} \int u^{-1/3} du = \frac{x^2}{2} + C.$

Therefore, $\frac{5}{4} \frac{u^{2/3}}{(\frac{2}{3})} = \frac{x^2}{2} + C$, or $\frac{15}{8}(y^4 + 7)^{2/3}$
$= \frac{x^2}{2} + C$. Multiplying by 8 and replacing the constant 8C by C, we obtain $15(y^4 + 7)^{2/3} = 4x^2 +$
C.

22. $\frac{y dy}{\sqrt{10y^2 + 1}} = x^3 dx$, so that $\int \frac{y dy}{\sqrt{10y^2 + 1}} = \int x^3 dx.$
In the first integral, let $u = 10y^2 + 1$, so that

$du = 20y dy$. Thus, $\int \frac{\frac{1}{20} du}{\sqrt{u}} = \int x^3 dx$, or $\frac{1}{20}$

$\int u^{-\frac{1}{2}} du = \frac{x^4}{4} + C$. Therefore, $\frac{1}{20} \frac{u^{\frac{1}{2}}}{(\frac{1}{2})} = \frac{x^4}{4} + C,$

or $\frac{1}{10} \sqrt{10y^2 + 1} = \frac{x^4}{4} + C$. Multiplying by 20 and replacing the constant 20C by C, we obtain

$2\sqrt{10y^2 + 1} = 5x^4 + C.$

23. $\cos 3y\, dy = \sin 2x\, dx,$
$\int \cos 3y\, dy = \int \sin 2x\, dx.$
$\frac{\sin 3y}{3} + C_1 = \frac{-\cos 2x}{2} + C_2,$
or $2 \sin 3y + 3 \cos 2x = C.$

24. $\csc x \cos y\, dx = -\tan x \tan y\, dy.$
$\frac{\csc x}{\tan x} dx = -\frac{\tan y}{\cos y} dy.$
$\csc x \cot x\, dx = -\tan y \sec y\, dy.$
$\int \csc x \cot x\, dx = -\int \tan y \sec y\, dy.$
$-\csc x + C_1 = -\sec y + C_2.$
$\sec y - \csc x = C.$

25. $y^2 dx = \csc x\, dy.$
$\frac{1}{\csc x} dx = \frac{1}{y^2} dy$, or $\sin x\, dx = y^{-2} dy.$
$\int \sin x\, dx = \int y^{-2} dy.$
$-\cos x + C_1 = \frac{y^{-1}}{-1} + C_2.$
$\frac{1}{y} - \cos x = C.$

26. $\cos^2 3t \sin 4s\, ds = \cos^2 4s\, dt.$
$\frac{\sin 4s}{\cos^2 4s} ds = \sec^2 3t\, dt.$
$\int \frac{\sin 4s}{\cos^2 4s} ds = \int \sec^2 3t\, dt.$
$u = \cos 4s$, $v = 3t$,
$du = -4 \sin 4s\, ds$, $dv = 3\, dt.$
So
$\int \frac{1}{u^2}(-\frac{1}{4} du) = \int \sec^2 v(\frac{1}{3} dv).$
$-\frac{1}{4} \int u^{-2} du = \frac{1}{3} \int \sec^2 v\, dv.$
$-\frac{1}{4} \left[\frac{u^{-1}}{-1}\right] + C_1 = \frac{1}{3} \tan v + C_2.$
$\frac{1}{4u} + C_1 = \frac{1}{3} \tan v + C_2$, or $\frac{1}{4 \cos 4s} -$

$\frac{1}{3}\tan 3t = C.$

27. $(2 - x^{3/2})dx = ydy$, so that $\int (2 - x^{3/2})dx = \int ydy$, or $2x - \frac{x^{5/2}}{(\frac{5}{2})} = \frac{y^2}{2} + C$. Putting $y = 2$ and $x = 9$, we obtain $18 - \frac{486}{5} = 2 + C$. Thus, $C = -\frac{406}{5}$, and $2x - \frac{2x^{5/2}}{5} = \frac{y^2}{2} - \frac{406}{5}$. Multiplying by 10, we obtain $20x - 4x^{5/2} = 5y^2 - 812$.

28. $s = \int \frac{t^2 dt}{\sqrt{t^3 + 1}} = \int \frac{\frac{1}{3}du}{\sqrt{u}} = \frac{1}{3}\int u^{-\frac{1}{2}}du = \frac{1}{3}\frac{u^{\frac{1}{2}}}{(\frac{1}{2})} + C$
$= \frac{2}{3}\sqrt{t^3 + 1} + C$. $u = t^3 + 1$, $du = 3t^2 dt$.
Putting $s = \frac{1}{2}$ and $t = 2$, we obtain $\frac{1}{2} = \frac{2}{3}\sqrt{9} + C$, so that $C = -\frac{3}{2}$. Thus, $s = \frac{2}{3}\sqrt{t^3 + 1} - \frac{3}{2}$.

29. $W = \int (1 - t^{1/3})^3\, t^{-2/3}dt$. Put $u = 1 - t^{1/3}$, so that $du = -\frac{1}{3}t^{-2/3}dt$, and $t^{-2/3}dt = -3du$. Thus, $W = \int u^3(-3du) = -3\int u^3 du = -3\frac{u^4}{4} + C = -\frac{3}{4}(1 - t^{1/3})^4 + C$. Putting $W = -1$ and $t = 8$, we obtain $-1 = -\frac{3}{4}(1 - 2)^4 + C$, so that $C = -\frac{1}{4}$.
Thus, $W = \frac{-3(1 - t^{1/3})^4 - 1}{4}$.

30. $y^{-\frac{1}{2}}dy = x^{-\frac{1}{2}}dx$, so that $\int y^{-\frac{1}{2}}dy = \int x^{-\frac{1}{2}}dx$, or $y^{\frac{1}{2}} - x^{\frac{1}{2}} = C$. Putting $x = 1$ and $y = 4$, we obtain $4^{\frac{1}{2}} - 1^{\frac{1}{2}} = 2 - 1 = 1 = C$. Thus,
$y^{\frac{1}{2}} = x^{\frac{1}{2}} + 1$.

31. $\frac{1}{\csc 2s}ds = -\frac{1}{\sec 3t}dt$, or $\sin 2s\, ds = -\cos 3t\, dt$.
Thus, $\int \sin 2s\, ds = -\int \cos 3t\, dt$, or
$3\cos 2s = 2\sin 3t + C.$
Putting $s = \frac{\pi}{3}$ and $t = \frac{\pi}{2}$, we obtain
$3\cos\frac{2\pi}{3} = 2\sin\frac{3\pi}{2} + C$, or
$3(-\frac{1}{2}) = 2(-1) + C$, so $C = \frac{1}{2}$. Therefore,
$3\cos 2s = 2\sin 3t + \frac{1}{2}$.

32. $\csc y\, dx = -\cos^2 x\, dy$;
$\frac{1}{\cos^2 x}dx = -\frac{1}{\csc y}dy$;
$\sec^2 x\, dx = -\sin y\, dy$.
$\int \sec^2 x\, dx = -\int \sin y\, dy$;
$\tan x = \cos y + C$.
When $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$, so $\tan\frac{\pi}{4} = \cos\frac{\pi}{2} + C$
or $1 = 0 + C$ or $C = 1$. Therefore,
$\tan x = \cos y + 1$.

33. $\frac{dy}{dx} = \int (3x^2 + 2x + 1)dx = x^3 + x^2 + x + C_1$;
hence, $y = \int (x^3 + x^2 + x + C_1)dx = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + C_1 x + C_2$.

34. $y' = \int (5x + 1)^4 dx = \int u^4(\frac{1}{5}du) = \frac{u^5}{25} + C_1 = \frac{(5x + 1)^5}{25} + C_1$. $u = 5x + 1$; $du = 5\,dx$.
$y = \int \left[\frac{(5x + 1)^5}{25} + C_1\right]dx = \frac{1}{25}\int (5x + 1)^5 dx + C_1 x + C_2$
$= \frac{1}{25}\int u^5(\frac{1}{5}du) + C_1 x + C_2 = \frac{1}{125}\frac{u^6}{6} + C_1 x + C_2$
$= \frac{1}{750}(5x + 1)^6 + C_1 x + C_2$.

35. $y' = \int \sqrt[3]{4x + 5}\, dx = \int \sqrt[3]{u}\,(\frac{1}{4}du) = \frac{1}{4}\int u^{1/3}du = \frac{1}{4}\frac{u^{4/3}}{(\frac{4}{3})} + C_1 = \frac{3}{16}(4x + 5)^{4/3} + C_1$.
$u = 4x + 5$, $du = 4\,dx$. $y = \int \left[\frac{3}{16}(4x + 5)^{4/3} + C_1\right]dx$
$= \frac{3}{16}\int u^{4/3}(\frac{1}{4}du) + C_1 x + C_2 = \frac{3}{64}\frac{u^{7/3}}{\frac{7}{3}} + C_1 x + C_2 = \frac{9}{448}(4x + 5)^{7/3} + C_1 x + C_2$.

36. $S' = \int \frac{5dt}{(t+7)^3} = -\frac{5}{2}(t+7)^{-2} + C_1$.

$S = \int \left[(-\frac{5}{2})(t+7)^{-2} + C_1 \right] dt$

$= \frac{5}{2}(t+7)^{-1} + C_1 t + C_2$.

37. $\frac{ds}{dt} = \int (2t^4 + 3)dt = \frac{2}{5}t^5 + 3t + C_1$.

$s = \int (\frac{2}{5}t^5 + 3t + C_1)dt$

$= \frac{1}{15}t^6 + \frac{3}{2}t^2 + C_1 t + C_2$.

38. $y' = \int (x+1)^2 dx = \frac{(x+1)^3}{3} + C_1$.

$y = \int \left[\frac{(x+1)^3}{3} + C_1 \right] dx$

$= \frac{(x+1)^4}{12} + C_1 x + C_2$.

39. $\frac{dy}{dx} = \int 0\, dx = 0 + C_1 = C_1$. $y = \int C_1 dx = C_1 x + C_2$.

40. $D_x y = \int 1\, dx = x + C_1$. $y = \int (x + C_1)dx = \frac{x^2}{2} + C_1 x + C_2$.

41. $\frac{dy}{dx} = \int \cos 2x\, dx = \frac{\sin 2x}{2} + C$.

$y = \int (\frac{\sin 2x}{2} + C_1)dx = \frac{1}{2}\int \sin 2x\, dx + C_1 x + C_2$

$= \frac{1}{2}(\frac{-\cos 2x}{2}) + C_1 x + C_2 = -\frac{1}{4}\cos 2x + C_1 x + C_2$.

42. $y' = \int \sin 3x\, dx = \frac{-\cos 3x}{3} + C_1$.

$y = \int (\frac{-\cos 3x}{3} + C_1)dx = -\frac{1}{3}\int \cos 3x\, dx + C_1 x + C_2$.

$= -\frac{1}{3}\frac{\sin 3x}{3} + C_1 x + C_2 = -\frac{1}{9}\sin 3x + C_1 x + C_2$.

43. $y' = \frac{dy}{dx} = \int (6x+1)dx = 3x^2 + x + C_1$. Putting $y' = 3$ and $x = 0$, we obtain $3 = C_1$; hence, $\frac{dy}{dx} = 3x^2 + x + 3$. Thus, $y = \int (3x^2 + x + 3)dx = x^3 + \frac{x^2}{2} + 3x + C_2$. Putting $y = 2$ and $x = 0$, we obtain $2 = C_2$. Hence, $y = x^3 + \frac{x^2}{2} + 3x + 2$.

44. $y' = \frac{dy}{dx} = \int \sqrt{x}dx = \frac{2}{3}x^{3/2} + C_1$. Putting $y' = 2$ and $x = 9$, we obtain $2 = 18 + C_1$, so that $C_1 = -16$. Thus, $\frac{dy}{dx} = \frac{2}{3}x^{3/2} - 16$. Therefore, $y = \int (\frac{2}{3}x^{3/2} - 16)dx = \frac{4}{15}x^{5/2} - 16x + C_2$. Putting $y = 3$ and $x = 9$, we obtain $3 = \frac{324}{5} - 144 + C_2$, so that $C_2 = \frac{411}{5}$. Hence, $y = \frac{4}{15}x^{5/2} - 16x + \frac{411}{5}$.

45. $\frac{ds}{dt} = \int 2dt = 2t + C_1$, $s = \int (2t + C_1)dt = t^2 + C_1 t + C_2$. Putting $s = 0$ and $t = 1$, we have $0 = 1 + C_1 + C_2$. Putting $s = 0$ and $t = -3$, we obtain $0 = 9 - 3C_1 + C_2$. Solving the two simultaneous equations $C_1 + C_2 = -1$ and $3C_1 - C_2 = 9$ for C_1 and C_2, we obtain $C_1 = 2$ and $C_2 = -3$. Hence, $s = t^2 + 2t - 3$.

46. $\frac{dy}{dx} = \int 3x^2 dx = x^3 + C_1$, $y = \int (x^3 + C_1)dx = \frac{x^4}{4} + C_1 x + C_2$. The side conditions give $C_2 = -1$ and $9 = 4 + 2C_1 - 1$, so that $C_1 = 3$. Hence, $y = \frac{x^4}{4} + 3x - 1$.

47. $y' = \int 3(2+5x)^2 dx = \int 3u^2(\frac{1}{5}du) = \frac{3}{5}(\frac{u^3}{3}) + C_1 = \frac{(2+5x)^3}{5} + C_1$; $u = 2 + 5x$, $du = 5\,dx$. Putting $y' = -1$ and $x = 1$, we obtain $-1 = \frac{7^3}{5} + C_1$, so that $C_1 = -\frac{348}{5}$ and $y' = \frac{(2+5x)^3 - 348}{5}$. Thus, $y = \frac{1}{5}\int \left[(2+5x)^3 - 348\right]dx = \frac{1}{5}\left[\frac{(2+5x)^4}{20} - 348x\right] + C_2 = \frac{(2+5x)^4}{100} - \frac{348x}{5} + C_2$. Putting $y = 2$ and $x = 1$, we obtain $2 = \frac{7^4}{100} - \frac{348}{5} + C_2$, so that $C_2 = \frac{4759}{100}$. Hence, $y = \frac{(2+5x)^4}{100} - \frac{348x}{5} + \frac{4759}{100} = \frac{25}{4}x^4 + 10x^3 + 6x^2 - 68x + \frac{191}{4}$.

48. $\frac{ds}{dt} = \int (5t-4)^{1/4} dt = \int u^{1/4}(\frac{1}{5}du) = \frac{4}{25}u^{5/4} + C_1 = \frac{4}{25}(5t-4)^{5/4} + C_1$, where $u = 5t - 4$

and $du = 5\,dt$. Putting $\frac{ds}{dt} = -3$ and $t = 4$, we obtain $-3 = \frac{4}{25}(32) + C_1$, so that $C_1 = -\frac{203}{25}$.

Hence, $\frac{ds}{dt} = \frac{4(5t-4)^{5/4} - 203}{25}$. $s = \int \frac{4(5t-4)^{5/4} - 203}{25}\,dt = \frac{4}{25}\int (5t-4)^{5/4}dt - \frac{203}{25}t + C_2 = \frac{4}{25}\int u^{5/4}(\frac{1}{5}\,du) - \frac{203}{25}t + C_2 = (\frac{4}{25})(\frac{1}{5})\frac{u^{9/4}}{\frac{9}{4}} - \frac{203}{25}t + C_2 = \frac{16}{1125}(5t-4)^{9/4} - \frac{203}{25}t + C_2$. Putting $s = 2$ and $t = 4$, we obtain $2 = \frac{1}{1125}(512) - \frac{812}{25} + C_2$, so that $C_2 = \frac{38278}{1125}$. Thus,

$s = \frac{16}{1125}(5t-4)^{9/4} - \frac{203}{25}t + \frac{38278}{1125}$.

49. $y' = \int \sin\frac{x}{2}\,dx = -2\cos\frac{x}{2} + C_1$

When $x = \pi$, $y' = 0$. Thus, $0 = -2\cos\frac{\pi}{2} + C_1$ or $0 = 0 + C_1$, so $C_1 = 0$. Therefore,

$y' = -2\cos\frac{x}{2}$, and so

$y = -\int 2\cos\frac{x}{2}\,dx = -4\sin\frac{x}{2} + C.$

When $x = 0$, $y = 2$; so $2 = -4\sin 0 + C$, or $C = 2$. Therefore,

$y = -4\sin\frac{x}{2} + 2.$

50. $\frac{ds}{dt} = \int 2\sec^2 t\tan t\,dt.$

Let $u = \tan t$ and $du = \sec^2 t\,dt$.

$\frac{ds}{dt} = \int 2u\,du = u^2 + C = \tan^2 t + C_1.$

Putting $t = 0$ and $\frac{ds}{dt} = 0$, we obtain $0 = \tan^2 0 + C_1$, or $C_1 = 0$. Thus, $\frac{ds}{dt} = \tan^2 t$.

$s = \int \tan^2 t\,dt = \int(\sec^2 t - 1)dt = \tan t - t + C.$

Putting $t = 0$ and $s = 0$, we obtain $0 = \tan 0 - 0 + C$, or $C = 0$. Thus, $s = \tan t - t$.

51. $\frac{ds}{dt} = t^2 - 8t + 15$, so that $s = \int ds = \int(t^2 - 8t + 15)dt = \frac{t^3}{3} - 4t^2 + 15 + C$. Putting $s = 1$ and $t = 0$ we find that $C = 1$. Hence, $s = \frac{t^3}{3} - 4t^2 + 15t + 1$. When $t = 3$, $s = \frac{3^3}{3} - 4(3)^2 + 15(3) + 1$, so that $s = 19$ meters.

52. $\frac{dv}{dt} = -10$, $v = \int dv = \int(-10)dt = -10t + C_1$. When $t = 0$, $v = 25$ meters/sec.; hence, $C_1 = 25$ and $v = -10t + 25$. The particle comes to rest when $0 = -10t + 25$, that is, when $t = \frac{5}{2}$ sec. Since $\frac{ds}{dt} = v = -10t + 25$, then $s = \int ds = \int(-10t + 25)dt = -5t^2 + 25t + C_2$. When $t = 0$, $s = 0$; hence, $C_2 = 0$ and $s = -5t^2 + 25t$. Putting $t = \frac{5}{2}$, we obtain $s = -5(\frac{5}{2})^2 + 25(\frac{5}{2}) = \frac{125}{4}$ meters.

53. $\frac{dv}{dt} = a$, where a is the constant negative acceleration of the car. Hence, $v = \int dv = \int a\,dt = a\int dt = at + C_1$. The speed of the car when $t = 0$ is 55 miles/hr. $= 55 \times \frac{5280}{3600} = \frac{242}{3}$ ft/sec.; hence, $C_1 = \frac{242}{3}$ and $v = at + \frac{242}{3}$. Thus, $\frac{ds}{dt} = v = at + \frac{242}{3}$, so that $s = \int ds = \int(at + \frac{242}{3})dt = \frac{at^2}{2} + \frac{242}{3}t + C_2$. Since $s = 0$ when $t = 0$, it follows that $C_2 = 0$ and $s = \frac{at^2}{2} + \frac{242}{3}t$. Let T be the time required to stop the car. When $t = T$, $v = 0$ so that $0 = aT + \frac{242}{3}$, $T = -\frac{242}{3a}$. When $t = T$, $s = 200$ so that $200 = \frac{a}{2}T^2 + \frac{242}{3}T = \frac{a}{2}(\frac{242}{3a})^2 + \frac{242}{3}(-\frac{242}{3a}) = -\frac{29,282}{9a}$. It follows that $a = -\frac{14,641}{900}$ feet/sec.^{2}.

(a) $T = -\frac{242}{3a} = -\frac{242}{3}(-\frac{900}{14,641}) = \frac{600}{121} \approx 4.96$ seconds.

(b) Let t_1 be the time required to slow the car to 25 miles/hr. $= \frac{25 \times 5280}{3600} = \frac{110}{3}$ ft./sec.

From the equation $v = at + \frac{242}{3}$, we have $\frac{110}{3} = at_1 + \frac{242}{3}$, so that $t_1 = -\frac{44}{a} = (-44)(-\frac{900}{14,641}) = \frac{3600}{1331}$ seconds. The distance moved by the car during t_1 seconds is given by $s = \frac{at_1^2}{2} + \frac{242}{3}t_1 = \frac{1}{2}(-\frac{14,641}{900})(\frac{3600}{1331})^2 + (\frac{242}{3})(\frac{3600}{1331}) = \frac{19,200}{121} \approx 158.68$ feet.

54. Establish a vertical s axis with origin at the surface of the earth and pointing upward. The velocity of the stone is given by $v = \frac{ds}{dt}$. If g is the acceleration of gravity, then $\frac{dv}{dt} = -g$, so that $v = \int(-g)dt = -gt + v_o$, where v_o (the constant of integration) represents the velocity when $t = 0$. Thus, $v_o = 30$ m./sec. Since $\frac{ds}{dt} = -gt + v_o$, then $s = \int(-gt + v_o)dt = -\frac{gt^2}{2} + v_ot + s_o$, where s_o (the constant of integration) is the value of s when $t = 0$. Here $s_o = 20$ meters.

(a) When the stone strikes the ground, $s = 0$, so that $0 = -\frac{gt^2}{2} + v_ot + s_o$. Solving this quadratic equation for t, we obtain $t = \frac{v_o \pm \sqrt{v_o^2 + 2gs_o}}{g}$. Since $t > 0$ when the stone strikes the ground, we reject the minus sign; hence, $t = \frac{30 + \sqrt{900 + 2(9.8)(20)}}{9.8} = \frac{30 + 2\sqrt{323}}{9.8} \approx 6.73$ seconds.

(b) When the stone is at its maximum height, $v = 0$, so that $0 = -gt + v_o$ and $t = \frac{v_o}{g}$. At this instant, $s = -\frac{gt^2}{2} + v_ot + s_o = -\frac{g}{2}(\frac{v_o}{g})^2 + v_o(\frac{v_o}{g}) + s_o = \frac{v_o^2}{2g} + s_o = \frac{900}{2(9.8)} + 20 \approx 65.92$ meters.

(c) When the stone hits the ground, $t = \frac{v_o + \sqrt{v_o^2 + 2gs_o}}{g}$ and the velocity is given by $v = -gt + v_o = -g(\frac{v_o + \sqrt{v_o^2 + 2gs_o}}{g}) + v_o = -\sqrt{v_o^2 + 2gs_o} = -\sqrt{900 + 2(9.8)(20)} = -2\sqrt{323} \approx -35.94$ m./sec.

55. Establish a vertical s axis with the origin at the surface of the earth and pointing upward. Let v be the velocity of the binoculars at time t. Here $\frac{dv}{dt} = -g$, so that $v = \int(-g)dt = -gt + v_o$, where v_o is the velocity of the binoculars when $t = 0$. Thus, $v_o = 10$ ft./sec. Since $\frac{ds}{dt} = v = -gt + v_o$, then $s = \int(-gt + v_o)dt = \frac{-gt^2}{2} + v_ot + s_o$, where s_o is the value of s when $t = 0$. Thus, $s_o = 100$ feet. The binoculars strike the ground when $s = 0$, that is, when $0 = -\frac{gt^2}{2} + v_ot + s_o$, or, $gt^2 - 2v_ot - 2s_o = 0$. Solving this quadratic equation, we obtain $t = \frac{v_o \pm \sqrt{v_o^2 + 2gs_o}}{g}$. Since $t > 0$ when the binoculars hit the ground, we must reject the negative sign.

(a) $t = \frac{v_o + \sqrt{v_o^2 + 2gs_o}}{g} = \frac{10 + \sqrt{100 + 2(32)(100)}}{32} = \frac{10 + 10\sqrt{65}}{32} \approx 2.83$ seconds.

(b) When the binoculars strike the ground, $v = -gt + v_o = -g(\frac{v_o + \sqrt{v_o^2 + 2gs_o}}{g}) + v_o = -\sqrt{v_o^2 + 2gs_o} = -\sqrt{100 + 2(32)(100)} = -10\sqrt{65} \approx -80.62$ ft./sec.

56. We proceed as in Problem 55, except that $s_o = 0$. Thus, the velocity when the projectile strikes the canoneer is given by $v = -\sqrt{v_o^2 + 2gs_o} = -\sqrt{v_o^2} = -v_o$.

57. We proceed as in Problem 55, taking v_o to be the initial velocity with which the binoculars are thrown upward and $s_o = 0$. Thus, the s coordinate of the binoculars at time t is given by $s = \frac{-gt^2}{2}$

$+ v_o t$. The s coordinate of the balloon at time t is given by $s_b = 6 + t$. In order for the binoculars to reach the balloon, we must be able to solve the equation $6 + t = -\frac{gt^2}{2} + v_o t$ for a positive value of t; that is, we must be able to solve the quadratic equation $\frac{g}{2}t^2 + (1 - v_o)t + 6 = 0$ for a positive value of t. The solution is $t = \frac{-(1 - v_o) \pm \sqrt{(1 - v_o)^2 - 12g}}{g}$, provided that $(1 - v_o)^2 - 12g \geq 0$; that is, provided that $|1 - v_o| \geq \sqrt{12g} = 2\sqrt{3g}$. Since we require $t > 0$, we must also have $-(1 - v_o) > 0$; that is, $v_o > 1$. Therefore, $|1 - v_o| = v_o - 1$, and the requirement is $v_o \geq 1 + 2\sqrt{3g} = 1 + 2\sqrt{3(9.8)} \approx 11.84$ m./s.

58. (a) $v = \int a\,dt = at + v_o$, so that

(b) $s = \int \frac{ds}{dt}\,dt = \int (at + v_o)dt = \frac{at^2}{2} + v_o t + s_o$.

59. Set up the s axis as in Problem 55 and let v be the velocity of the stone and s the s coordinate of the stone at time t. Here $s_o = h$ feet and $v_o = 0$. Thus, $s = -\frac{gt^2}{2} + (0)t + h = -4.9t^2 + h$. If the stone hits the ground when $t = T$, then $0 = -4.9T^2 + h$, so that $h = 4.9T^2$.

60. $\frac{d^2y}{dt^2} = k\left[(A - y)\frac{dy}{dt} + (B + y)(-\frac{dy}{dt})\right] = 0$,

so $A - y - B - y = 0$, or

$y = \frac{1}{2}(A - B)$.

Trypsin is being formed most rapidly at the time t when $y = \frac{1}{2}(A - B)$.

Problem Set 4.5, page 279

1. $y = \int (1 - 3x)dx = x - \frac{3}{2}x^2 + C$. Putting $y = 4$ and $x = -1$, we obtain $4 = -1 - \frac{3}{2}(-1)^2 + C$, so that $C = \frac{13}{2}$. Therefore, $f(x) = x - \frac{3}{2}x^2 + \frac{13}{2}$.

2. $y = \int (x^2 + 1)dx = \frac{x^3}{3} + x + C$. Putting $y = 5$ and $x = -3$, we obtain $5 = -\frac{27}{3} - 3 + C$, so that $C = 17$. Therefore, $f(x) = \frac{x^3}{3} + x + 17$.

3. $\frac{dy}{y^2} = \frac{dx}{x^2}$, $\int \frac{dy}{y^2} = \int \frac{dx}{x^2}$, $-\frac{1}{y} = -\frac{1}{x} + C$, $y = \frac{1}{\frac{1}{x} - C}$.

Putting $y = 1$ and $x = 2$, we obtain $1 = \frac{1}{\frac{1}{2} - C}$, so that $\frac{1}{2} - C = 1$ and $C = -\frac{1}{2}$. Therefore, $f(x)$

$= \frac{1}{\frac{1}{x} + \frac{1}{2}} = \frac{2x}{2 + x}$.

4. $\frac{dy}{y^2} = 2x\,dx$, $\int \frac{dy}{y^2} = \int 2x\,dx$, $-\frac{1}{y} = x^2 + C$, $y = \frac{-1}{x^2 + C}$. Putting $y = 1$ and $x = 0$, we obtain $1 = \frac{-1}{C}$, so that $C = -1$. Therefore, $f(x) = \frac{-1}{x^2 - 1} = \frac{1}{1 - x^2}$.

5. $dy = -3\cos 3x\,dx$;

$\int dy = y = \int (-3\cos 3x\,dx) = -\sin 3x + C$

$x = \frac{\pi}{3}$, $y = 2$. $2 = -\sin \pi + C$, or $C = 2$

Therefore, $y = -\sin 3x + 2$.

6. $u = \sec x$, $du = \sec x \tan x\,dx$.

$y = \int dy = \int 2\tan^2 x \sec x\,dx$.

$= \int 2u\,du = u^2 + C$

$= \sec^2 x + C$

When $x = 0$, $y = 1$; so $1 = \sec^2 0 + C = 1 + C$,

or $C = 0$. Therefore, $y = \sec^2 x$.

7. $y\frac{dy}{dx} = x$, or $y\,dy = x\,dx$. $\int y\,dy = \int x\,dx$, or $y^2 = x^2 + C$. Putting $x = 0$ and $y = 1$, we find that $C = 1$. Thus, $y^2 - x^2 = 1$.

8. The equation of the normal line to the curve at (x,y) is $Y - y = \frac{-1}{dy/dx}(X - x)$. Putting $Y = 0$ and solving for X, we find that the coordinate of Q is given by $X = y\frac{dy}{dx} + x$. Hence, the constant distance is $\left|y\frac{dy}{dx} + x - x\right|$, or $\left|y\frac{dy}{dx}\right|$. It follows that $y\frac{dy}{dx} = c$, a constant. When $x = 0$ and $y = 3$, we have $\frac{dy}{dx} = \tan\frac{\pi}{4} = 1$, so $3(1) = c$. Therefore, $y\frac{dy}{dx} = 3$, $y\,dy = 3dx$, $\frac{y^2}{2} = 3x + C$. When $x = 0$, we have $y = 3$; hence, $\frac{9}{2} = C$. Thus, $\frac{y^2}{2} = 3x + \frac{9}{2}$ or $y^2 = 6x + 9$.

9. $dW = Fds = 2sds$, $W = \int 2sds = s^2 + C$, $W = 0$ when $s = s_0 = 1$; hence, $0 = 1 + C$, $C = -1$. Hence, $W = s^2 - 1$. Putting $s = s_1 = 5$, we obtain $W = 5^2 - 1 = 24$ joules.

10. $dW = Fds = 400\,s\sqrt{1 + s^2}\,ds$, $W = \int 400\,s\sqrt{1 + s^2}\,ds = 400\int s\sqrt{1 + s^2}\,ds$; $u = 1 + s^2$, $du = 2s\,ds = 400\int\sqrt{u}\,(\frac{1}{2}\,du) = 200\int u^{1/2}\,du = (200)\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{400}{3}(1 + s^2)^{3/2} + C$. $W = 0$ when $s = s_0 = 0$; hence, $0 = \frac{400}{3}(1 + 0^2)^{3/2} + C$, $C = -\frac{400}{3}$. Therefore, $W = \frac{400}{3}\left[(1 + s^2)^{3/2} - 1\right]$. Putting $s = s_1 = 3$, we obtain $W = \frac{400}{3}\left[10^{3/2} - 1\right] \approx 4083.04$ joules.

11. $dW = Fds = \sqrt{s - 1}\,ds$, $W = \int\sqrt{s - 1}\,ds = \frac{2}{3}(s - 1)^{3/2} + C$. When $s = s_0 = 1$, $W = 0$ so $0 = \frac{2}{3}(0)^{3/2} + C$; hence, $C = 0$ and $W = \frac{2}{3}(s - 1)^{3/2}$. Putting $s = s_1 = 10$, we obtain $W = \frac{2}{3}\,9^{3/2} = 18$ joules.

12. $dW = Fds = (1 + s)^{2/3}\,ds$, $W = \int(1 + s)^{2/3}\,ds = \frac{3}{5}(1 + s)^{5/3} + C$. $W = 0$ when $s = s_0 = -7$; hence, $0 = (-\frac{3}{5})\,6^{5/3} + C$, $C = \frac{3}{5}(6^{5/3})$. Putting $s = s_1 = 7$, we obtain $W = \frac{3}{5}(8^{5/3}) + C = \frac{3}{5}(8^{5/3}) + \frac{3}{5}(6^{5/3}) = \frac{3}{5}\left[32 + 6^{5/3}\right] \approx 31.09$ joules.

13. $dW = Fds = \sin\frac{s}{2}\,ds$, $W = \int\sin\frac{s}{2}\,ds = -2\cos\frac{s}{2} + C$.
$W = 0$ when $s = s_0 = 0$; hence,
$0 = -2\cos 0 + C$, or $C = +2$.
When $s = s_1 = \pi$, $W = -2\cos\frac{\pi}{2} + 2 = 2$ joules.

14. $dW = Fds = \sin s\cos s\,ds$, $W = \int\sin s\cos s\,ds$
Let $u = \sin s$ and $du = \cos s\,ds$. $\int\sin s\cos s\,ds = \int u\,du = \frac{u^2}{2} + C = \frac{\sin^2 s}{2} + C$
When $s = s_0 = 0$, $W = 0$; hence,
$0 = \frac{\sin^2 0}{2} + C$, or $C = 0$.
When $s = s_1 = \frac{\pi}{2}$, $W = \frac{\sin^2\frac{\pi}{2}}{2} = \frac{1}{2}$ joule.

15. $F = ks$, so $k = \frac{F}{s} = \frac{40}{0.5} = 80$. Therefore, $W = k\frac{b^2}{2} = 80\frac{(0.5)^2}{2} = 10$ joules.

16. (a) $F_a = k(a - c)$ and $F_b = k(b - c)$. Therefore, $F_b - F_a = k(b - c) - k(a - c) = kb - kc - ka + kc = kb - ka = k(b - a)$, and so $k = \frac{F_b - F_a}{b - a}$
(b) $\frac{F_a}{F_b} = \frac{k(a - c)}{k(b - c)} = \frac{a - c}{b - c}$, so $(b - c)F_a = (a - c)F_b$, $bF_a - cF_a = aF_b - cF_b$, $cF_b - cF_a = aF_b - bF_a$, and $c(F_b - F_a) = aF_b - bF_a$.
(c) $W_{ab} = W_{cb} - W_{ca} = \frac{1}{2}k(b - c)^2 - \frac{1}{2}k(a - c)^2 =$

$\frac{1}{2}k\left[(b - c)^2 - (a - c)^2\right] = \frac{1}{2} \cdot \frac{F_b - F_a}{b - a}\left[(b - c) + (a - c)\right]\left[(b - c) - (a - c)\right] = \frac{1}{2}\frac{F_b - F_a}{b - a}(a + b - 2c)(b - a) = \frac{1}{2}(F_b - F_a)(a + b - 2c) =$
$\frac{1}{2}(aF_b + bF_b - 2cF_b - aF_a - bF_a + 2cF_a) =$
$\frac{1}{2}\left[aF_b + bF_b - aF_a - bF_a - 2c(F_b - F_a)\right] =$
$\frac{1}{2}\left[aF_b + bF_b - aF_a - bF_a - 2(aF_b - bF_a)\right]$ by part (b). Therefore, $W_{ab} = \frac{1}{2}(bF_b - aF_b + bF_a - aF_a) = \frac{1}{2}(b - a)(F_a + F_b)$.

17. At the start, the spring is stretched $\frac{300}{150} = 2$ inches. $W = \int F ds = \int ks ds = \frac{ks^2}{2} + C$. Since $W = 0$ when $s = 2$, then $0 = \frac{k(2)^2}{2=} + C$, so that $C = -2k = -300$. Hence, $W = \frac{ks^2}{2} - 300$. Thus, when $s = 8$, $W = \frac{150(8)^2}{2} - 300 = 4500$ inch-lbs. $= 375$ ft-lbs.

18. When the bucket is s meters high, $t = \frac{s}{2}$ seconds have elapsed and $1 \cdot \frac{s}{2} = \frac{s}{2}$ kilograms of sand have run out through the hole in the bottom. Hence, when the bucket is s meters high, the bucket and sand together weigh $12 + g(30 - \frac{s}{2})$ newtons. Hence, $W = \int (12 + 30g - \frac{g}{2}s)ds = (12 + 30g)s - \frac{g}{4}s^2 + C$. Because $W = 0$ when $s = 0$, we have $C = 0$. The last of the sand runs out when $\frac{s}{2} = 30$, that is, when $s = 60$ m. The total work done is $W = (12 + 30g)(60) - \frac{g}{4}(60^2) = 720 + 900g = 720 + 900(9.8) = 9540$ joules.

19. Let s be the distance from the center of the sphere to the particle, so that the force F on the particle is given by $F = GMms^{-2}$. Thus, $W = \int F ds = \int GMms^{-2} ds = -GMms^{-1} + C$. Because $W = 0$ when $s = r$, $C = GMmr^{-1}$. Therefore, when $s = R$, $W = -GMmR^{-1} + C = -GMmR^{-1} + GMmr^{-1} = GMm(r^{-1} - R^{-1})$.

20. (a) $\lim_{R \to \infty} GMm(\frac{1}{r} - \frac{1}{R}) = \frac{GMm}{r}$.

(b) For escape, we must have $\frac{1}{2}mN^2 \geq \frac{GMm}{r}$ or $N^2 \geq \frac{2GM}{r}$, that is, $N \geq \sqrt{\frac{2GM}{r}}$.

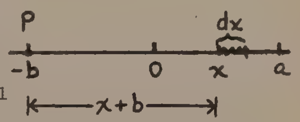
(c) For the earth, N escape $=$
$\sqrt{\frac{2(6.672 \times 10^{-11})(5.983 \times 10^{24})}{6.371 \times 10^6}}$
$= 1.119 \times 10^4$ m/s.

(d) For the moon, N escape $=$
$\sqrt{\frac{2(6.672 \times 10^{-11})(7.347 \times 10^{22})}{1.738 \times 10^6}}$
$= 2.375 \times 10^3$ m/s.

21. $W = GMm(\frac{1}{r} - \frac{1}{R}) =$
$(6.672 \times 10^{-11})(5.98 \times 10^{24})(1)$
$(\frac{1}{6.37 \times 10^6} - \frac{1}{3.80 \times 10^8}) =$
6.16×10^7 joules.

22. For a black hole, N escape $> C$, that is, $\sqrt{2GM/r} > C$, or $2GM/r > C^2$. So, for a black hole, $r < 2GM/C^2$.

23. The (linear) density of the mass distributed on the interval $[0,a]$ is $\frac{M}{a}$ kilograms per meter. The mass dM in the interval $[x, x + dx]$ in the accompanying figure is given by $dM = \frac{M}{a}dx$ kilograms and the gravitational force exerted by dM on P is therefore given by $dF = \frac{G(1)dM}{(x + b)^2} = \frac{GM}{a} \cdot \frac{dx}{(x + b)^2}$. Hence,



$F = \int \frac{GM}{a} \cdot \frac{dx}{(x + b)^2} = \frac{GM}{a}\int (x + b)^{-2}dx = \frac{GM}{a}\frac{(x + b)^{-1}}{(-1)} + C = C - \frac{GM}{a(x + b)}$. Since $F = 0$ when $x = 0$, then $C = \frac{GM}{ab}$; hence, $F = \frac{GM}{ab} - \frac{GM}{a(x + b)}$.
Putting $x = a$, we find that the total gravitation-

al force is given by $F = \frac{GM}{ab} - \frac{GM}{a(a+b)} = \frac{GM}{a}$

$(\frac{1}{b} - \frac{1}{a+b}) = \frac{GM}{a}(\frac{a}{b(a+b)}) = \frac{GM}{b(a+b)}$ newtons.

24. The mass dm on the interval $\left[w - 1, w - 1 + dw\right]$ is given by $dm = Mdw$. By the result of Problem 23 with $a = 1$ and $b = 1 - w$, the force dF on this mass is given by

$$dF = \frac{GM}{(1-w)(1+1-w)}\,dm = \frac{GM^2dw}{(1-w)(2-w)}.$$

Hence, $(1 - w)(2 - w)dF = GM^2dw$.

25. $C'(x) = 5 + 8x^{-\frac{1}{2}}$, so $C(x) = \int(5 + 8x^{-\frac{1}{2}})dx = 5x + 8\frac{x^{\frac{1}{2}}}{(1/2)} + K = 5x + 16\sqrt{x} + K$.

(a) When $x = 100$, $C = \$1200$, so $1200 = 5(100) + 16\sqrt{100} + K = 760 + K$, $K = 1200 - 660 = 540$. Therefore, $C(x) = 5x + 16\sqrt{x} + 540$ dollars.

(b) $P(x) = R(x) - C(x) = 21x - (5x + 16\sqrt{x} + 540) = 16x - 16\sqrt{x} - 540$ dollars.

26. Because $C'(x)$ is a quadratic function of x, it follows that $C(x)$ is a cubic function of x, say $C(x) = ax^3 + bx^2 + cx + C_0$. Now $C'(x) = 3ax^2 + 2bx + c$ and $C''(x) = 6ax + 2b$. Because I is the most efficient production level, $C''(I) = 6aI + 2b = 0$, so $b = -3aI$ and $C(x) = ax^3 - 3aIx^2 + cx + C_0$. Now, the average production cost per unit at production level x is $\bar{c}(x) = ax^2 - 3aIx + c + C_0/x$; hence, $A = aI^2 - 3aI^2 + c + C_0/I = -2aI^2 + c + C_0/I$. Also, $B = C'(I) = 3aI^2 - 6aI^2 + c = -3aI^2 + c$. Solving the last two equations simultaneously for a and c, we find that

$$a = \frac{A - B}{I^2} - \frac{C_0}{I^3},\ c = 3A - 2B - \frac{3C_0}{I}.$$

Therefore,

$$C(x) = (\frac{A - B}{I^2} - \frac{C_0}{I^3})x^3 + (\frac{3C_0}{I^2} - \frac{3A - 3B}{I})x^2 + (3A - 2B - \frac{3C_0}{I})x + C_0.$$

27. $C = \int 30x^{-2/3}dx = 90\sqrt[3]{x} + K$. When $x = 8$ (thousand cans), $C = 2600$, so $2600 = 90(2) + K$, $K = 2420$.

(a) $C_0 = K = \$2420$

(b) $C(125) = 90\sqrt[3]{125} + 2420 = 90(5) + 2420 = \$2870$.

28. In Problem 26, we take $C_0 = 5000$, $I = 20{,}000$, $A = 15$, $B = 10.75$. Then $C(x) = ax^3 + bx^2 + cx + C_0$ with $a = \frac{A - B}{I^2} - \frac{C_0}{I^3} = 10^{-3}$, $b = \frac{3C_0}{I^2} - \frac{3A - 3B}{I} = -6 \times 10^{-4}$, $c = 3A - 2B - \frac{3C_0}{I} = 22.75$. Thus, $C(x) = \frac{x^3}{10^3} - \frac{6x^2}{10^4} + 22.75x + 5000$.

29. $P = R - C = R - 600 - 60x$, $\frac{dP}{dx} = \frac{dR}{dx} - 60 = 400 - 8x - 60$; hence, $\frac{dP}{dx} = 340 - 8x$. The critical value of x is therefore $x = 42.5$. Profit is maximized for 42,500 cassette tapes.

30. $R = \int(40 - \frac{x}{500})dx = 40x - \frac{x^2}{10^3}$, assuming that $R(0) = 0$.

(a) $P(x) = R(x) - C(x)$. Using Problem 28, we find that $P(x) = -\frac{x^3}{10^3} - \frac{4x^2}{10^4} + 17.25x - 5000$.

Using, say, Newton's method to solve the equation $P(x) = 0$, we find break-even production levels of 292 and 25,888 razors.

(b) For maximum profit we require $P'(x) = 0$, that is, $-\frac{3x^2}{10^3} - \frac{8x}{10^4} + 17.25 = 0$.

Solving this equation by using the quadratic formula, and rejecting the negative solution, we

obtain x = 14,103 razors.

31. (a) $R = \int (13 - \frac{x}{40})dx = 13x - \frac{x^2}{80} + K$. Since R = 0 when x = 0, then K = 0, so that $R = 13x - \frac{x^2}{80}$.

(b) $P = R - C = 13x - \frac{x^2}{80} - 3.5x - 100 = 9.5x - \frac{x^2}{80} - 100$.

(c) $\frac{dP}{dx} = 9.5 - \frac{x}{40}$; therefore x = 380 is a critical value.

(d) The cost per subscription is therefore

$$\frac{R}{x} = \frac{13x - \frac{x^2}{80}}{x} = 13 - \frac{x}{80} = 13 - \frac{380}{80} = \$8.25 \text{ per}$$

month.

32. In Problem 28, production cost per razor is $\bar{c}(x) = \frac{x^2}{10^8} - \frac{6x}{10^4} + 22.75 + \frac{5000}{x}$ and $\bar{c}'(x) = \frac{2x}{10^8} - \frac{6}{10^4} - \frac{5000}{x^2}$. Solving the equation $\bar{c}'(x) = 0$ by using, say, Newton's method, we find that x = 30,273 razors yields minimum production cost per razor.

33. (a) A critical value of x for the total revenue is given by 1030 − x = 0 or x = 1030. Since $R''(x) = -1 < 0$, an absolute maximum occurs at x = 1030.

(b) $P = R - C$; $\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} = 1030 - x - (700) = 300 - x$, so that x = 330 is a critical point. ($\frac{d^2P}{dx^2} = -1$, so an absolute maximum occurs at x = 330).

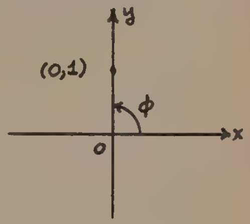
(c) $R = \int (1030 - x)dx = 1030x - \frac{x^2}{2} + K$. When x = 0, R = 0, so K = 0. Thus, $R = 1030x - \frac{x^2}{2}$. The price per dinner is

$$\frac{R}{1000x} = \frac{1030x - \frac{x^2}{2}}{1000x} = \frac{1030}{1000} - \frac{x}{2000}.$$ When x = 330, the price per dinner is $\frac{103}{100} - \frac{330}{2000} = \$0.865$.

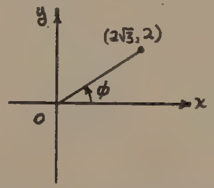
Problem Set 4.6, page 288

1. $\omega^2 = 9$, so $\omega = 3$. We seek a solution y = f(t) for which f(0) = 0 and f'(0) = 3. Take $P = (f(0), \frac{f'(0)}{\omega}) = (0, \frac{3}{3}) = (0,1)$

$A = 1$, $\phi = \frac{\pi}{2}$; thus, $y = f(t) = \cos(3t - \frac{\pi}{2})$.



2. $\omega^2 = 2$, so $\omega = \sqrt{2}$. We seek a solution y = f(t) for which $f(0) = 2\sqrt{3}$, $f'(0) = 2\sqrt{2}$. Take $P = (f(0), \frac{f'(0)}{\omega}) = (2\sqrt{3}, \frac{2\sqrt{2}}{2}) = (2\sqrt{3}, 2)$.

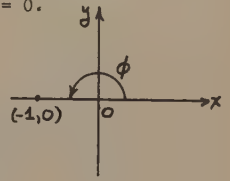


$A = \sqrt{12 + 4} = 4$, $\phi = \frac{\pi}{6}$. So $y = f(t) = \cos(\sqrt{2}\, t - \frac{\pi}{6})$.

3. $\omega^2 = 3$, so $\omega = \sqrt{3}$. We seek a solution y = f(t) for which f(0) = −1, f'(0) = 0. Take $P = (f(0), f'(0)/\omega) = (-1,0)$.

$A = 1$, $\phi = \pi$.

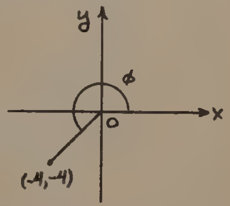
So $y = f(t) = \cos(\sqrt{3}t - \pi)$.



4. $\omega^2 = 5$, so $\omega = \sqrt{5}$. We seek a solution y = f(x) for which f(0) = −4 f'(0) $= -4\sqrt{5}$. Take $P = (f(0), \frac{f'(0)}{\omega}) = (-4, \frac{-4\sqrt{5}}{\sqrt{5}}) = (-4,4)$.

$A = \sqrt{(-4)^2 + (-4)^2} = 4\sqrt{2}$, $\phi = \frac{5\pi}{4}$.

So $y = f(x) = 4\sqrt{2}\cos(\sqrt{5}x - \frac{5\pi}{4})$.



5. $\omega^2 = 1$, so $\omega = 1$. We seek a solution y = f(t) for

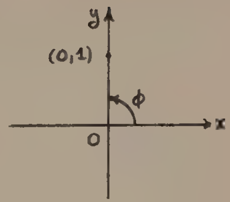
which $f(0) = 0, f'(0) = 1.$

Take $P = \left(f(0), \frac{f'(0)}{\omega}\right)$

$= (0, \frac{1}{1}) = (0,1).$

$A = 1, \phi = \frac{\pi}{2}$; so

$y = f(t) = \cos(t - \frac{\pi}{2}).$



6. $\omega^2 = 4$, so $\omega = 2$. We seek a solution $y = f(x)$ for which $f(0) = 4, f'(0) = 6.$

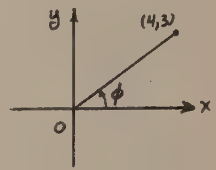
Take $P = \left(f(0), \frac{f'(0)}{\omega}\right)$

$= (4, \frac{6}{2}) = (4,3).$

$A = 5$; $\tan \phi = \frac{3}{4}$ so

$\phi \approx 0.644$. Thus,

$y = f(x) = 5 \cos(2t - 0.644).$



7. Problem 1:

(a) $\nu = \frac{\omega}{2\pi} = \frac{3}{2\pi}$ Hz. (b) $T = \frac{1}{\nu} = \frac{2\pi}{3}$ seconds.

Problem 3:

(a) $\nu = \frac{\omega}{2\pi} = \frac{\sqrt{3}}{2\pi}$ Hz. (b) $T = \frac{1}{\nu} = \frac{2\pi}{\sqrt{3}}$ seconds

8. $0 \le \omega t - \phi \le 2\pi$, or

$\phi \le \omega t \le 2\pi + \phi$, or

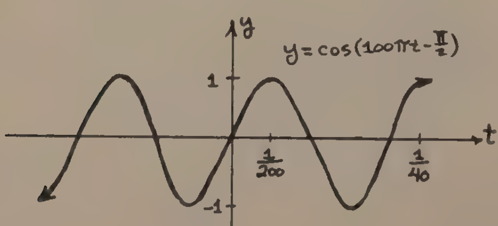
$\frac{\phi}{\omega} \le t \le \frac{2\pi + \phi}{\omega} = \frac{2\pi}{\omega} + \frac{\phi}{\omega}.$

Period $= \frac{2\pi}{\omega} + \frac{\phi}{\omega} - \frac{\phi}{\omega} = \frac{2\pi}{\omega} = \frac{1}{\frac{\omega}{2\pi}} = \frac{1}{\nu} = T.$

9. $\nu = 50 = \frac{\omega}{2\pi}$, so $\omega = 100\pi$. $\frac{\phi}{\omega} = \frac{\frac{\pi}{2}}{100\pi} = \frac{1}{200}.$

$y = A\cos(\omega t - \phi) = \cos(100\pi t - \frac{\pi}{2}).$

$0 \le 100\pi t - \frac{\pi}{2} \le 2\pi$, so $\frac{1}{200} \le t \le \frac{1}{40}.$

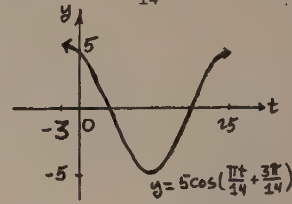


10. $\nu = \frac{1}{28} = \frac{\omega}{2\pi}$, so $\omega = \frac{2\pi}{28} = \frac{\pi}{14}$. $\frac{\phi}{\omega} = \frac{\frac{-3\pi}{14}}{\frac{\pi}{14}} = -3$

$y = 5\cos(\frac{\pi}{14}t + \frac{3\pi}{14}).$

$0 \le \frac{\pi}{14}t + \frac{3\pi}{14} \le 2\pi,$

so $-3 \le t \le 25.$



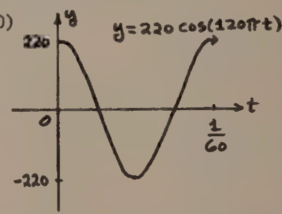
11. $\nu = \frac{\omega}{2\pi} = 60$, so $\omega = 120\pi$. $\frac{\phi}{\omega} = \frac{0}{\omega} = 0.$

$y = 220\cos(120\pi t - 0)$

$= 220 \cos 120\pi t.$

$0 \le 120\pi t \le 2\pi$, so

$0 \le t \le \frac{1}{60}.$

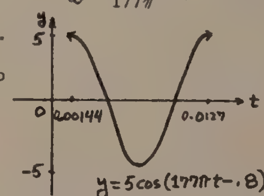


12. $\nu = \frac{\omega}{2\pi} = 88.5$, $\omega = 177\pi$. $\frac{\phi}{\omega} = \frac{0.8}{177\pi} \approx 0.00144$

$y = 5\cos(177\pi t - 0.8).$

$0 \le 177\pi t - 0.8 \le 2\pi$, so

$0.00144 \le t \le 0.0127.$



13. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{19.6}{0.1}} = \sqrt{196} = 14.$

Find a solution $y = f(t)$ where $f(0) = A_o$

$f'(0) = v_o = 0.$

$P = \left(f(0), \frac{f'(0)}{\omega}\right) = (A_o, 0)$. $A = A_o$, $\phi = 0.$

(a) $y = A\cos(\omega t - \phi) = A_o \cos(14t)$, assuming $\phi = 0.$

(b) $\nu = \frac{\omega}{2\pi} = \frac{14}{2\pi} = \frac{7}{\pi}.$

14. $k = \frac{F}{s} = \frac{40}{0.5} = 80$, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{80}{2}} = \sqrt{40} = 2\sqrt{10}.$

Assuming $\phi = 0$:

(a) $y = A_o \cos(2\sqrt{10}t)$;

(b) $\nu = \frac{\omega}{2\pi} = \frac{2\sqrt{10}}{2\pi} = \frac{\sqrt{10}}{\pi}.$

15. $T = 1$, so $\nu = \frac{1}{T} = 1$, and $\omega = 2\pi\nu = 2\pi$. Thus, assuming $\phi = 0$: (a) $y = 0.8\cos 2\pi t$; (b) $\nu = 1$Hz.

16. $k = \frac{mg}{s} = \frac{(0.5)(9.8)}{0.2} = 24.5$, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{s}} = \sqrt{\frac{9.8}{0.2}}$ $= \sqrt{49} = 7$. Thus, assuming $\phi = 0$:

(a) $y = A_o \cos 7t$;

(b) $\nu = \frac{\omega}{2\pi} = \frac{7}{2\pi}$ Hz.

17. $E = \frac{1}{2}ky^2 + \frac{1}{2}m\left(\frac{dy}{dt}\right)^2$. Now $\omega^2 = \frac{k}{m}$, so $k = m\omega^2$, and $E = \frac{1}{2}m\omega^2y^2 + \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}m\mathcal{E}$; hence, $K = \frac{1}{2}m$.

18. (a) $\frac{d^2s}{dt^2} + g\left(\frac{s}{\ell}\right) = 0$ or $\frac{d^2s}{dt^2} + \omega^2 s = 0$ with $\omega^2 = \frac{g}{\ell}$.

(b) $\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$, so $T = \frac{1}{\nu} = 2\pi\sqrt{\frac{\ell}{g}}$.

19. (a) Taking the derivative on both sides of the equation $L\frac{dI}{dt} + \frac{Q}{C} = 0$ with respect to time, we obtain $L\frac{d^2I}{dt^2} + \frac{1}{C}\frac{dQ}{dt} = 0$, or $L\frac{d^2I}{dt^2} + \frac{1}{C}I = 0$; that is $\frac{d^2I}{dt^2} + \frac{1}{LC}I = 0$. Thus, $\frac{d^2I}{dt^2} + \omega^2 I = 0$ with $\omega = 1/\sqrt{LC}$.

(b) $I = \frac{dQ}{dt}$, so the equation $L\frac{dI}{dt} + \frac{Q}{C} = 0$ can be rewritten $L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$, or $\frac{d^2Q}{dt^2} + \omega^2 Q = 0$ with $= 1/\sqrt{LC}$.

(c) $Q = Q_o \cos\omega t$, $\omega = 1/\sqrt{LC}$.

(d) $I = \frac{dQ}{dt} = -Q_o\omega\sin\omega t$, or $I = Q_o\omega\cos(\omega t + \frac{\pi}{2})$.

(e) $\nu = \frac{\omega}{2\pi} = 1/2\pi\sqrt{LC}$.

20. Let $g = 9.8\ m/s^2$ be the acceleration of gravity on the earth and let g_m be the acceleration of gravity on the moon. Then $T = 2\pi\sqrt{\frac{\ell}{g}}$ is the period on the earth (Problem 18) and $T_m = 2\pi\sqrt{\frac{\ell}{g_M}}$ is the period on the moon. The astronant knows that $T_M = 2.45\ T$, so $2\pi\sqrt{\frac{\ell}{g_M}} = (2.45)\ 2\pi\sqrt{\frac{\ell}{g}}$,
$\sqrt{\frac{\ell}{g_M}} = 2.45\sqrt{\frac{\ell}{g}}$, $\frac{\ell}{g_M} = (2.45)^2\ \frac{\ell}{g}$, $g_M = \frac{g}{(2.45)^2}$ $\approx 1.63\ m/s^2$.

21. $E = \frac{Q^2}{2C} + \frac{LI^2}{2} = \frac{1}{2C}Q^2 + \frac{L}{2}\left(\frac{dQ}{dt}\right)^2 =$ $\frac{L}{2}\left[\left(\frac{dQ}{dt}\right)^2 + \frac{1}{LC}Q^2\right] = \frac{L}{2}\left[\left(\frac{dQ}{dt}\right)^2 + \omega^2Q^2\right]$ $= \frac{L}{2}\mathcal{E}$, so $K = \frac{L}{2}$.

22. $B\cos t + C\sin t = A(\cos t\cos\phi + \sin t\sin\phi)$

$= A\cos\phi\cos t + A\sin\phi\sin t$

If this is valid for all t, then

$B = A\cos\phi$ and $C = A\sin\phi$.

Thus, $B^2 + C^2 = A^2(\cos^2\phi + \sin^2\phi) = A^2$,

so $A = \sqrt{B^2 + C}$.

$\frac{A\sin\phi}{A\cos\phi} = \tan\phi = \frac{C}{B}$, so $\phi = \tan^{-1}\frac{C}{B}$.

23. Given $\frac{d^2y}{dt^2} + \omega^2y = 0$

Suppose $y = y_1$ is a solution

Consider $y = Cy_1$, then

$\frac{dy}{dt} = C\frac{dy_1}{dt}$ and $\frac{d^2y}{dt^2} = C\frac{d^2y_1}{dt^2}$.

Substituting we have

$C\frac{d^2y_1}{dt^2} + \omega^2Cy_1 = C\left(\frac{d^2y_1}{dt^2} + \omega^2y_1\right) = C(0) = 0$

since $y = y_1$ is a solution. Therefore

$y = Cy_1$ is also a solution of the harmonic oscillator equation.

24. $\frac{dy}{dt} = -A\omega\sin(\omega t - \phi) + f_p'(t)$ and

$\frac{d^2y}{dt^2} = -A\omega^2\cos(\omega t - \phi) + f_p''(t)$. Therefore,

$\frac{d^2y}{dt^2} = \omega^2y = -A\omega^2\cos(\omega t - \phi) + f_p''(t) +$

$A\omega^2\cos(\omega t - \phi) + \omega^2f_p(t)$

$= f_p''(t) + \omega^2f_p(t) = F(t)$.

25. Let $y_p = \frac{B}{\omega^2}$. Then $\frac{dy_p}{dt} = 0$, $\frac{d^2y_p}{dt} = 0$, and $\frac{d^2y_p}{dt^2} + \omega^2y_p = \omega^2\frac{B}{\omega^2} = B$. Using the result of Problem 24, $y = A\cos(\omega t - \phi) + \frac{B}{\omega^2}$ provides a solution of the given inhomogeneous equation.

26. Let $y = f(t)$ be any solution of $\frac{d^2y}{dt^2} + \omega^2 y = F(t)$. Then $\frac{d^2}{dt^2}\left[y - f_p(t)\right] = \frac{d^2y}{dt^2} - \frac{d^2}{dt^2} f_p(t)$ and $\frac{d^2}{dt^2}\left[y - f_p(t)\right] + \omega^2\left[y - f_p(t)\right] =$
$\frac{d^2y}{dt^2} + \omega^2 y - \left[\frac{d^2}{dt^2} f_p(t) + \omega^2 f_p(t)\right] =$
$F(t) - F(t) = 0$. By Theorem 5, there exist constants A and ϕ such that $y - f_p(t) = A\cos(\omega t - \phi)$. Hence, $y = A\cos(\omega t - \phi) + f_p(t)$.

27. We have $\theta = \omega t - \theta_0$, where θ_0 is the value of θ when $t = 0$, and $y = A\sin\theta = A\cos(\frac{\pi}{2} - \theta)$
$= A\cos(\theta - \frac{\pi}{2}) = A\cos(\omega t - \theta_0 - \frac{\pi}{2})$
$= A\cos(\omega t - \phi)$ with $\phi = \frac{\pi}{2} - \theta_0$.

28. By the results of Problems 25 and 26,
$y = A\cos(\omega t - \phi) + \frac{B}{\omega^2}$.

29. As we saw in the solution to Problem 27, $\phi = \frac{\pi}{2} - \theta_0$. Therefore, the phase angle is the complement of the initial value of θ.

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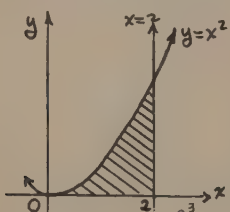
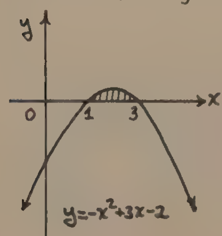
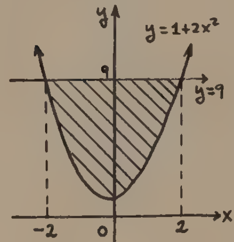
1. 

$dA = \ell\, dx = y\, dx = x^2 dx$;
$A = \int x^2 dx = \frac{x^3}{3} + C$.
When $x = 0$, $A = 0$, so $C = 0$. So $A = \frac{x^3}{3}$
When $x = 2$, $A = \frac{2^3}{3} = \frac{8}{3}$ square units.

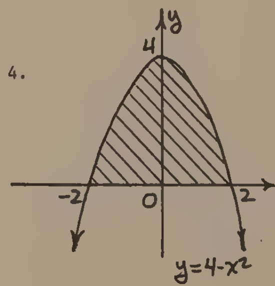
2. 

$dA = \ell\, dx$ $dA = y\, dx$
$= (-x^2 + 3x + 2)dx$;
$A = \int(-x^2 + 3x + 2)dx$
$= -\frac{x^3}{3} + \frac{3x^2}{2} + 2x + C$.

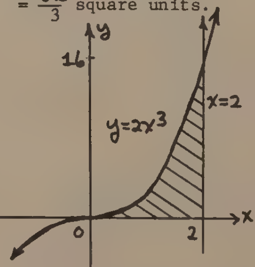
When $x = 1$, $A = 0$, so $0 = -\frac{1}{3} + \frac{3}{2} + 2 + C$ or $C = -\frac{19}{6}$. Therefore,
$A = -\frac{x^3}{3} + \frac{3x^2}{2} + 2x - \frac{19}{6}$. When $x = 3$, $A = \frac{25}{6}$ square units.

3. 

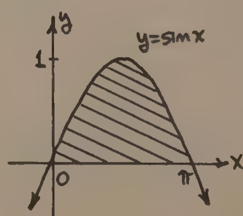
$A = \int \ell\, dx = \int\left[9-(1+2x^2)\right]dx$
$= \int(8 - 2x^2)\, dx =$
$8x - \frac{2}{3}x^3 + C$.
$A = 0$ when $x = -2$, so $0 = 8(-2) - \frac{2}{3}(-2)^3 + C = -\frac{32}{3} + C$, and $C = \frac{32}{3}$. Therefore, $A = 8x - \frac{2}{3}x^3 + \frac{32}{3}$.
When $x = 2$, $A = 16 - \frac{16}{3} + \frac{32}{3} = \frac{64}{3}$ square units.

4. 

Find the area in the first quadrant and then double the result.
$dA = \ell\, dx = (4 - x^2)dx$ so
$A = \int(4 - x^2)dx =$
$4x - \frac{x^3}{3} + C$.
When $x = 0$, $A = 0$, so $C = 0$. So $A = 4x - \frac{x^3}{3}$.
When $x = 2$, $A = 8 - \frac{8}{3} = \frac{16}{3}$. Desired area $= 2(\frac{16}{3})$
$= \frac{32}{3}$ square units.

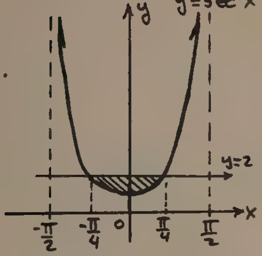
5. 

$dA = \ell\, dx = 2x^3 dx$;
$A = \int 2x^3 dx = x^3 + C$.
When $x = 0$, $A = 0$, so $C = 0$. Therefore, $A = x^3$. When $x = 2$,
$A = 2^3 = 8$ square units.

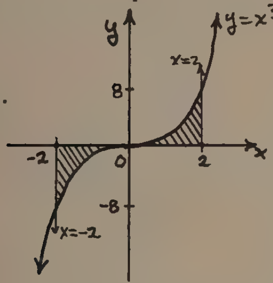
6. 

$dA = \ell\, dx = \sin x\, dx$;
$A = \int \sin x\, dx = -\cos x + C$.
When $x = 0$, $A = 0$, so $C = 1$. So $A = -\cos x + 1$

When $x = \pi$ $A = -\cos \pi + 1 = 2$ square units.

7. 

$\sec^2 x = 2$, $\sec x = \pm\sqrt{2}$, $x = \pm\frac{\pi}{4}$. $\ell = 2 - \sec^2 x$ $A = \int \ell\, dx = \int (2-\sec^2 x)dx = 2x - \tan x + C$. When $x = -\frac{\pi}{4}$, $A = 0$, so $0 = -\frac{\pi}{2} - \tan(-\frac{\pi}{4}) + C$ or $C = \frac{\pi}{2} - 1$. When $x = \frac{\pi}{4}$, $A = \frac{\pi}{2} - \tan\frac{\pi}{4} + \frac{\pi}{2} - 1 = \pi - 2$ square units.

8. 

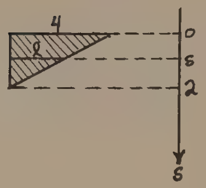
Using symmetry, find the area in the first quadrant, then double that result. $dA = \ell\, dx = x^3dx$; $A = \int x^3dx = \frac{x^4}{4} + C$. When $x = 0$, $A = 0$, so $C = 0$. Thus, $A = \frac{x^4}{4}$. When $x = 2$, $A = 4$, so desired area is $2(4) = 8$ square units.

9. $dA = \ell\cdot ds$; $\ell = 4$ so that $A = 4s + C$. $A = 0$ when $s = 0$, so $A = 4s$. When $s = 2$, $A = (4)(2) = 8$ square units; $A = \ell\cdot w = (4)(2) = 8$ square units.

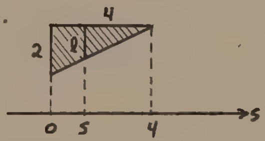
10. Notice that the area is the same as that of Problem 9, since the cross-sectional length ℓ is the same at level s, and $A = 0$ when $s = 0$, while we get total area A when $s = 2$. Hence, $A = 8$ square units; $A = \ell\cdot w = (4)(2) = 8$ square units.

11. 

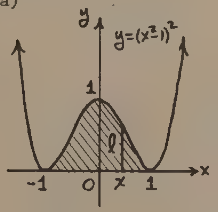
By similar triangles, $\frac{\ell}{4} = \frac{s}{2}$, so $\ell = 2s$. Now $dA = \ell ds = 2sds$; $A = s^2 + C$. $A = 0$ when $s = 0$, so that $C = 0$ and $A = s^2$. When $s = 2$, the total area is $A = (2)^2 = 4$ square units; $A = \frac{1}{2}b\cdot h = \frac{1}{2}\cdot 2\cdot 4 = 4$ square units.

12. 

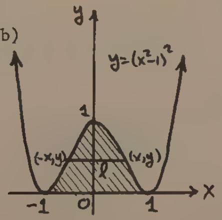
By similar triangles, $\frac{\ell}{4} = \frac{2-s}{2}$, $\ell = 4 - 2s$. $dA = (4-2s)ds$, $A = 4s - s^2 + C$. $A = 0$ when $s = 0$, so that $C = 0$. When $s = 2$, we get the total area $A = 8 - 4 = 4$ square units; $A = \frac{1}{2}b\cdot h = \frac{1}{2}\cdot 4\cdot 2 = 4$ square units.

13. 

By similar triangles, $\frac{\ell}{2} = \frac{4-s}{4}$, $\ell = 2 - \frac{s}{2}$. $dA = \ell ds$ so that $A = 2s - \frac{s^2}{4} + C$ and $A = 0$ when $s = 0$, so $C = 0$. When $s = 4$, we get the total area $A = 8 - 4 = 4$ square units. $A = \frac{1}{2}\cdot 2\cdot 4 = 4$ square units.

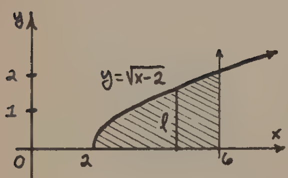
14. (a) 

$dA = \ell dx$ where $\ell = y = (x^2 - 1)^2$. $A = \int (x^2-1)^2dx = \int(x^4 - 2x^2 + 1)dx$, $A = \frac{x^5}{5} - \frac{2}{3}x^3 + x + C$. $A = 0$ when $x = -1$, so $0 = -\frac{1}{5} + \frac{2}{3} - 1 + C$, $C = \frac{8}{15}$. Hence, $A = \frac{x^5}{5} - \frac{2}{3}x^3 + x + \frac{8}{15}$. The total area A can be found when $x = 1$: $A = \frac{1}{5} - \frac{2}{3} + 1 + \frac{8}{15}$, so $A = \frac{16}{15}$ square units.

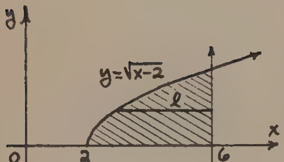
(b) 

$dA = \ell dy$, where $\ell = 2x$, and since $\ell = 0$ when $y = 1$, $x = \sqrt{1-\sqrt{y}}$. Hence, $dA = 2\sqrt{1-\sqrt{y}}\,dy$. Let $u = 1 - \sqrt{y}$, so that du =

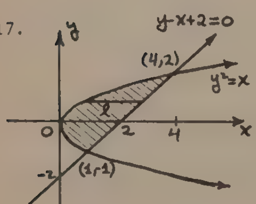
$-\frac{1\,dy}{2\sqrt{y}} = \frac{-dy}{2(1-u)}$.
$A = 2\int\sqrt{1-\sqrt{y}}\,dy = 2\int u^{\frac{1}{2}}\,2(u-1)du =$
$4\int(u^{3/2} - u^{\frac{1}{2}})du = 4[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}] + C =$
$4\left[\frac{2}{5}(1-\sqrt{y})^{5/2} - \frac{2}{3}(1-\sqrt{y})^{3/2}\right] + C.$ Since $A =$ 0 when $y = 0$, $C = -4\left[\frac{2}{5} - \frac{2}{3}\right] = \frac{16}{15}$, and so $A = 4\left[\frac{2}{5}(1-\sqrt{y})^{5/2} - \frac{2}{3}(1-\sqrt{y})^{3/2}\right] + \frac{16}{15}$. The total area is obtained when $y = 1$. Hence, $A = 4\cdot 0 + \frac{16}{15}$, and $A = \frac{16}{15}$ square units, as was found in part (a).

15. 

$dA = \ell\,dx$, where $\ell = \sqrt{x-2}$. So $A = \int\sqrt{x-2}\,dx$. Let $u = x - 2$, $du = dx$. So $A = \int u^{\frac{1}{2}}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x-2)^{3/2} + C$. When $x = 2$, $A = 0$. So $0 = 0 +$ C, and $C = 0$. When $x = 6$, we get the total area $A = \frac{2}{3}(6-2)^{3/2} = (\frac{2}{3})(8) = \frac{16}{3}$ square units.

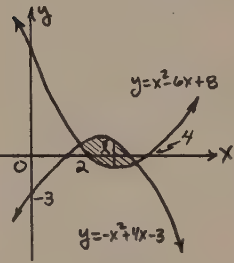
16. 

$dA = \ell\,dy$, where $\ell = 6 - x = 6 - (y^2 + 2) = 4 - y^2$. $A = \int(4 - y^2)dy = 4y - \frac{y^3}{3} + C$, where $A = 0$ when $y = 0$, so $C = 0$. Now if $y = 2$, we get the total area $A = (4)(2) - \frac{8}{3} = \frac{16}{3}$ square units.

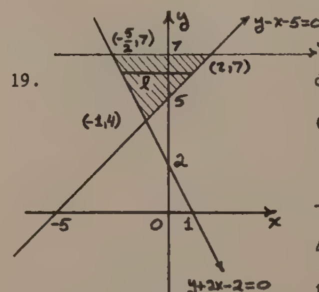
17. 

$dA = \ell\,dy$, where $\ell = (y + 2) - y^2$. $A = \int(y + 2 - y^2)dy = \frac{y^2}{2} + 2y - \frac{y^3}{3} + C$. $A = 0$ when $y = -1$, so that $0 = \frac{1}{2} - 2 + \frac{1}{3} + C$ and $C = \frac{7}{6}$. The total area is obtained when $y = 2$, that is, $A = \frac{4}{2} + 4 - \frac{8}{3} + \frac{7}{6}$ $= \frac{9}{2}$ square units.

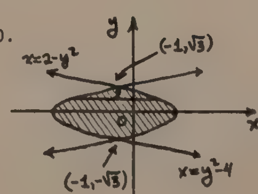
18. $dA = \ell\,dx = \left[(-x^2 + 4x - 3) - (x^2 - 6x + 8)\right]dx.$ So $A = \int(-2x^2 + 10x - 11)dx$, $A = \frac{-2x^3}{3} + 5x^2 - 11x + C$. Now, we can find the points of intersection of the curves by solving the given equations simultaneously for x, that is, by solving $2x^2 - 10x + 11 = 0$. So the x co-ordinates of the points of intersection are given by $x = \frac{5 \pm\sqrt{3}}{2}$. $A = 0$ when $x = \frac{5-\sqrt{3}}{2}$, so that



$C = \frac{2(5-\sqrt{3})^3}{24} - \frac{5(5-\sqrt{3})^2}{4} + 11\,\frac{(5-\sqrt{3})}{2}$. We get the total area when $x = \frac{5+\sqrt{3}}{2}$, so that $A = -\frac{2(5+\sqrt{3})^3}{24} + \frac{5(5+\sqrt{3})^2}{4} - \frac{11(5+\sqrt{3})}{2} + \frac{2(5-\sqrt{3})^3}{24} - \frac{5}{4}(5-\sqrt{3})^2 + \frac{11}{2}(5-\sqrt{3})$ and so $A = \frac{1}{12}\cdot(5-\sqrt{3})^3 - \frac{1}{12}(5+\sqrt{3})^3 + \frac{5}{4}\left[(5+\sqrt{3})^2 - (5-\sqrt{3})^2\right] - 11\sqrt{3} = \sqrt{3}$ square units.

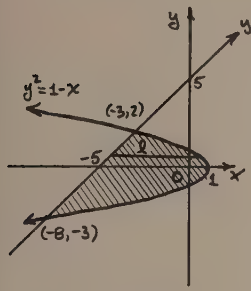
19. 

$dA = \ell\,dy$, where $\ell = (y - 5) - (1 - \frac{y}{2})$. $A = \int(\frac{3y}{2} - 6)dy$. $A = \frac{3y^2}{4} - 6y + C$. $A = 0$ when $y = 4$, so that $C = 12$. We get total area when $y = 7$, so that $A = \frac{3}{4}(49) - 42 + 12$, and so $A = 6.75$ square units.

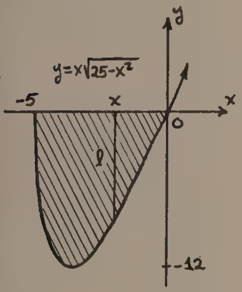
20. 

$dA = \ell\,dy$, where $\ell = (2 - y^2) - (y^2 - 4)$, so that $A = \int(-2y^2 + 6)dy$, $A = -\frac{2}{3}y^3 + 6y + C$; and $A = 0$ when $y = -\sqrt{3}$ so that $C = 4\sqrt{3}$. We get the

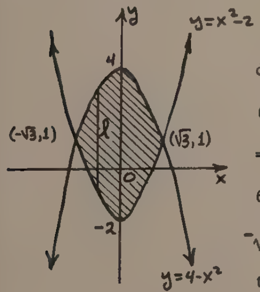
total area when $y = \sqrt{3}$, so that $A = -\frac{2}{3}(\sqrt{3})^3 + 6\sqrt{3} + 4\sqrt{3} = 8\sqrt{3}$

21. 

$dA = \ell\, dy$, where $\ell = (1 - y^2) - (y - 5)$, so that $A = \int (-y^2 - y + 6)dy$ and $A = \frac{-y^3}{3} - \frac{y^2}{2} + 6y + C$. A = 0 when y = −3, so that $C = \frac{27}{2}$. We get the total area when y = 2, so that $A = -\frac{8}{3} - \frac{4}{2} + 12 + \frac{27}{2} = \frac{125}{6}$ square units.

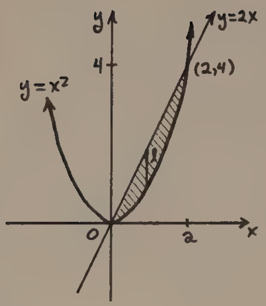
22. 

$dA = \ell\, dx$, where $\ell = -x\sqrt{25 - x^2}$, and $A = \int \left[-x\sqrt{25 - x^2}\right] dx$. Let $u = 25 - x^2$, $du = -2x\, dx$, so $-x\, dx = \frac{1}{2}\, du$. Now, $A = \frac{1}{2}\int u^{\frac{1}{2}}\, du = \frac{1}{3}u^{3/2} + C$, and $A = \frac{1}{3}(25 - x^2)^{3/2} + C$. A = 0 when x = −5 so that $C = \frac{1}{3}(25 - 25)^{3/2} = 0$. The total area is obtained when x = 0, so that $A = \frac{1}{3}(25)^{3/2}$ and so $A = \frac{125}{3}$ square units.

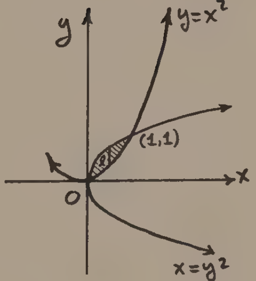
23. 

$dA = \ell\, dx$, where $\ell = (4 - x^2) - (x^2 - 2)$. So $A = \int (-2x^2 + 6)dx = -\frac{2}{3}x^3 + 6x + C$. A = 0 when $x = -\sqrt{3}$, so that $C = 4\sqrt{3}$. The total area is obtained when $x = \sqrt{3}$, so that $A = -\frac{2}{3}(\sqrt{3})^3 + 6\sqrt{3} + 4\sqrt{3} = 8\sqrt{3}$ square units.

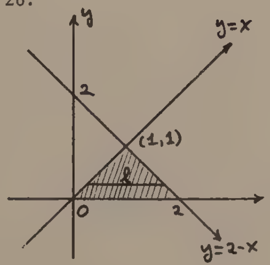
24. $dA = \ell \cdot dx$, where $\ell = 2x - x^2$. $A = \int (2x - x^2)dx =$



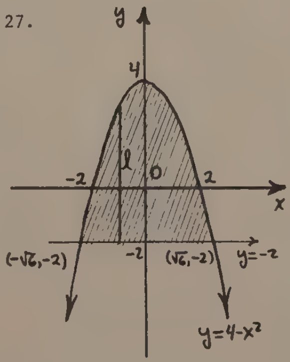
$x^2 - \frac{x^3}{3} + C$. When x = 0, A = 0, so C = 0. The total area is obtained when x = 2, so we have $A = 2^2 - \frac{2^3}{3} = 4 - \frac{8}{3} = \frac{4}{3}$ square units.

25. 

$dA = \ell \cdot dx$, where $\ell = \sqrt{x} - x^2$ so that $A = \int(\sqrt{x} - x^2)dx$, $A = \frac{2}{3}x^{3/2} - \frac{x^3}{3} + C$. A is 0 when x = 0, so C = 0. We get the total area when x = 1, so $A = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ square unit.

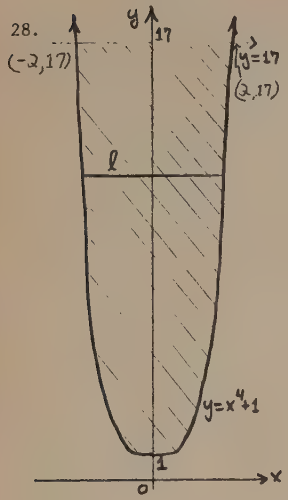
26. 

$dA = \ell \cdot dy$, where $\ell = (2 - y) - y$ and so $A = \int (2 - 2y)dy = 2y - y^2 + C$. A = 0 when y = 0, so C = 0. We get the total area when y = 1, so A = 2 − 1 = 1 square unit.

27. 

$dA = \ell \cdot dx$, where $\ell = (4 - x^2) - (-2)$ so that $A = \int (6 - x^2)dx = 6x - \frac{x^3}{3} + C$. A = 0 when $x = -\sqrt{6}$. So $C = 4\sqrt{6}$. We get the total area when $x = \sqrt{6}$, so that $A = 6\sqrt{6} - \frac{(\sqrt{6})^3}{3} + 4\sqrt{6} = 8\sqrt{6}$ square units.

28.



$dA = \ell \cdot dy$, where $\ell = 2(y - 1)^{1/4}$, so $A = \int 2(y - 1)^{1/4}dy$. Let $u = y - 1$, $du = dy$. Now $A = \int 2u^{1/4}\,du = 2 \cdot \frac{4}{5}u^{5/4} + C$; $A = \frac{8}{5}(y - 1)^{5/4} + C$. Since $A = 0$ when $y = 2$, $C = -\frac{8}{5}$. We get total area when $y = 17$. Hence, $A = \frac{8}{5}(2)^5 - \frac{8}{5}$ and so $A = \frac{248}{5}$ square units.

Review Problem Set, Chapter 4, page 295

1. $dy = 3x^2dx - dx = (3x^2 - 1)dx.$

2. $dy = x(-2 \sin 2x\,dx) + \cos 2x\,dx$
$= (-2x \sin 2x + \cos 2x)dx.$

3. $dy = \dfrac{(2x + 1)2x\,dx - (x^2 + 5)\,2dx}{(2x + 1)^2}$
$= \dfrac{2x^2 + 2x - 10}{(2x + 1)^2}\,dx$

4. $dy = \frac{1}{2}(4 - x^2)^{-1/2}(-2x)dx = \dfrac{-x\,dx}{\sqrt{4 - x^2}}.$

5. $dy = 3.2 \cot x(-\csc^2 x\,dx) = -6 \cot x \csc^2 x\,dx.$

6. $dy = \sec \sqrt{x} \tan \sqrt{x}(\frac{1}{2}x^{-1/2}\,dx) = \dfrac{1}{2\sqrt{x}} \sec \sqrt{x} \tan \sqrt{x}\,dx$

7. $3x^2dx + 3y^2dy - 6(x\,dy + y\,dx) = 0,$
$(3x^2 - 6y)dx + (3y^2 - 6x)dy = 0,$

$(x^2 - 2y)dx + (y^2 - 2x)dy = 0.$

8. $4x^3dx + 4y^3dy - 4(x\,dy + y\,dx) = 0,$
$(4x^3 - 4y)dx + (4y^3 - 4x)dy = 0,$

$(x^3 - 4y)dx + (y^3 - x)dy = 0.$

9. $2 \sec x(\sec x \tan x)dx + 2 \csc y(-\csc y \cot y)dy =$

$\sec^2 x \tan x\,dx - \csc^2 y \cot y\,dy = 0.$

10. $\sec^2 xy(x\,dy + y\,dx) + (x\,dy + y\,dx) = 0,$
$(x \sec^2 xy + x)dy + (y \sec^2 xy + y)dx = 0,$

$x(\sec^2 xy + 1)dy + y(\sec^2 xy + 1)dx = 0.$

11. $\cos \pi xy\,[\pi(x\,dy + y\,dx)] = 12(\dfrac{x\,dy - y\,dx}{x^2}),$
$\pi x^2\cos \pi xy(x\,dy + y\,dx) = 12(x\,dy - y\,dx),$
$(\pi x^3\cos \pi xy - 12x)dy + (\pi x^2 y \cos \pi xy + 12y)dx = 0.$

12. $(\sec^2\sqrt{xy})(\frac{1}{2})(xy)^{-1/2}(x\,dy + y\,dx) = 2x\,dx - 2y\,dy,$
$\dfrac{\sec^2\sqrt{xy}}{\sqrt{xy}}(x\,dy + y\,dx) = 4x\,dx - 4y\,dy,$
$(\dfrac{x \sec^2\sqrt{xy}}{\sqrt{xy}} + 4y)dy + (\dfrac{y \sec^2\sqrt{xy}}{\sqrt{xy}} - 4x)dx = 0.$

13. $\Delta y = [(x + \Delta x)^2 + 1] - [x^2 + 1] = x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1 = 2x\Delta x + (\Delta x)^2.$
(a) $\Delta y = 2(2)(0.01) + (0.01)^2 = 0.0401.$
(b) $dy = (2x)dx = 2x\Delta x = 2(2)(0.01) = 0.04.$
(c) $\Delta y - dy = 0.0001.$
$\left|\dfrac{\Delta y - dy}{\Delta y}\right| 100\% = 0.25\%.$

14. (a) $\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{1.23} - \sqrt{1} \approx 1.109 - 1 = 0.109.$
(b) $dy = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{(2)(1)}(0.23) = 0.115.$
(c) $\Delta y - dy = (\sqrt{1.23} - 1) - (0.115) \approx -0.0059;$
$\left|\dfrac{\Delta y - dy}{\Delta y}\right| 100\% = -5.46\%.$

15. (a) $\Delta y = (x + \Delta x)^3 - x^3 = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$
$\Delta y = 3(4)(0.02) + 6(0.02)^2 + (0.02)^3$, so $\Delta y = 0.242408.$

(b) $dy = 3x^2 dx = 3(4)(\Delta x) = 12(0.02) = 0.24.$

(c) $\Delta y - dy = 0.242408 - 0.24 = 0.002408.$
$\left|\frac{\Delta y - dy}{\Delta y}\right| 100\% = 0.99\%.$

16. (a) $\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = \frac{x - x - \Delta x}{(x + \Delta x)(x)} = -\frac{\Delta x}{x(x + \Delta x)}$
$= -\frac{(-0.5)}{2(1.5)} = \frac{1}{6} \approx 0.167.$

(b) $dy = -\frac{1}{x^2}(dx) = -\frac{1}{x^2}\Delta x = -\frac{1}{(2)^2}(-0.5) = \frac{1}{8} = 0.125.$

(c) $\Delta y - dy - \frac{1}{6} - \frac{1}{8} = \frac{1}{24} \approx 0.042.$
$\left|\frac{\Delta y - dy}{\Delta y}\right| 100\% = 25\%.$

17. (a) $\Delta y = 2\sin(x + \Delta x) - 2\sin x$
$= 2\sin(\frac{\pi}{6} + 0.01) - 2\sin\frac{\pi}{6} = 2(0.50864) - 2(\frac{1}{2}) = 0.01727$

(b) $dy = 2\cos x\, dx = 2\cos\frac{\pi}{6}(0.01) = \frac{2\sqrt{3}}{2}(0.01)$
$= 0.017321.$

(c) $\left|\frac{\Delta y - dy}{\Delta y}\right| 100\% = \left|\frac{0.010727 - 0.017321}{0.01727}\right| 100\%$
$= 0.29\%.$

18. (a) $\Delta y = -\csc(x + \Delta x) + \csc x = -\csc(\frac{\pi}{2} + 0.06) + \csc\frac{\pi}{2}$
$= -(1.00180) + 1 = 0.00180$

(b) $dy = \csc x \cot x = \csc\frac{\pi}{2}\cot\frac{\pi}{2} = (1)0 = 0$

(c) $\left|\frac{\Delta y - dy}{\Delta y}\right| 100\% = \left|\frac{0.00180 - 0}{0.00180}\right| 100\% = 100\%.$

19. Let $y = \sqrt{x}$, $x_1 = 36$, $\Delta x = 0.1.$
$\Delta y = f(x + \Delta x) - f(x) = f(36.1) - f(36)$
$= f(36.1) - 6 = \sqrt{36.1} - 6.$
$\Delta y \approx dy = \frac{1}{2\sqrt{x}}\, dx \approx \frac{1}{2\sqrt{36}}(0.1) = .008\overline{3},$
so $\sqrt{36.1} = 6 + \Delta y \approx 6 + 0.008\overline{3} = 6.0083.$

20. $y = f(x) = x^2 + 2x - 3$, $x = -3$, $\Delta x = -0.02$,
$f(-3) = 0.$
$\Delta y = f(x + \Delta x) - f(x) = f(-3.02) - f(-3).$
$\Delta y \approx dy = (2x + 2)\Delta x = -4(-0.02) = 0.08$, so
$f(-3.02) = f(-3) + \Delta y \approx 0 + 0.08 = 0.08.$

21. $y = f(x) = \cos x$, $x = \frac{\pi}{3}$, $\Delta x = 0.1.$
$\Delta y = f(x + \Delta x) - f(x) = f(\frac{\pi}{3} + 0.1) - f(\frac{\pi}{3}) = \cos(\frac{\pi}{3} + 0.1) - \frac{1}{2}.$
$\Delta y \approx dy = -\sin x\, dx = (-\sin\frac{\pi}{3})(0.1) = -\frac{\sqrt{3}}{2}(0.1)$
$= -0.08660$, so
$\cos(\frac{\pi}{3} + 0.1) = \frac{1}{2} + \Delta y \approx 0.5 + (-0.08660) = 0.4134.$

22. $y = \cot x$, $x = 45^\circ$, $\Delta x = -1^\circ.$
$\Delta y = f(x + \Delta x) - f(x) = \cot(45^\circ - 1^\circ) - \cot(45^\circ)$
$= \cot 44^\circ - 1.$
$\Delta y \approx dy = -\csc^2 x\, dx = -\csc^2 45^\circ(-1) = 2$, so
$\cot 44^\circ = 1 + \Delta y \approx 3.$

23. $f(x) = y = \frac{1}{x}$, $x = 1$, $\Delta x = 0.02.$
$\Delta y = f(x + \Delta x) - f(x) = f(1.02) - f(1)$
$= \frac{1}{1.02} - 1.$
$\Delta y \approx dy = -\frac{1}{x^2}\, dx = -\frac{1}{1}(0.02) = -0.02$, so
$\frac{1}{1.02} = 1 + \Delta y = 1 - 0.02 = 0.98.$

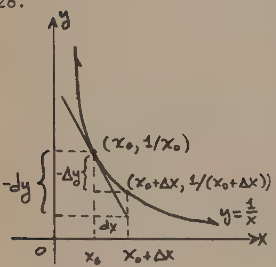
24. $y = f(x) = \sqrt[4]{x}$, $x = 0.0016$, $\Delta x = -0.0001.$
$\Delta y = f(x + \Delta x) - f(x) = f(0.0015) - f(0.0016)$
$= \sqrt[4]{0.0015} - 0.2.$
$\Delta y \approx dy = \frac{1}{4}x^{-3/4}\, dx = \frac{1}{4}(1)(-0.0001) = -0.000025$, so
$\sqrt[4]{0.0015} = 0.2 + (-0.000025) = 0.199975.$

25. Let $y = x^2$, so that y is the area of a square of side x. Then $dy = 2x\,dx$. Hence, $\Delta y \approx dy = 2(100)(2) = 400$ square meters.

26. $\Delta V = \frac{4}{3}\pi(r + \Delta r)^3 - \frac{4}{3}\pi r^3$. Assume $\Delta r = dr$, so that $\Delta V \approx dV = 4\pi r^2 dr = 4\pi r^2 \Delta r$ cubic units.

27. $V = x^3$. Assume $\Delta x = dx$, so that $\Delta V \approx dV = 3x^2 dx = 3x^2 \Delta x$ cubic units.

28.



$\Delta y = \dfrac{1}{x_0 + \Delta x} - \dfrac{1}{x_0} = \dfrac{-\Delta x}{x_0(x_0 + \Delta x)}$,

$dy = \dfrac{-1}{x_0^2}\, dx = \dfrac{-\Delta x}{x_0^2}$.

Since Δx is small, $x_0(x_0 + \Delta x) \approx x_0^2$; hence, $\Delta y \approx dy$. Put $x_0 = 100$, $\Delta x = dx = 2$. Here, $\frac{1}{x_0} = 0.01$, so that $\frac{1}{102} = \frac{1}{100} + \Delta y \approx \frac{1}{100} - \frac{\Delta x}{x_0^2} = 0.01 - \frac{2}{100^2} = 0.0098$. Error $= \frac{1}{102} - (\frac{1}{100} - \frac{2}{100^2}) \approx 0.000004$.

29. $A = \pi r^2$, so $dA = 2\pi r\,dr$, $dA = 2\pi(2.1)(\pm 0.05)$, $dA = \pm(4.2)\pi(0.05) \approx \pm 0.66$ square centimeter. The maximum possible error in the area is approximately ± 0.66 square centimeter.

30. $dp = -115\cos(1.24t)(1.24)dt$
$= -142.6\cos(1.24t)dt$,
so $\Delta p \approx -142.6\cos(1.24t)\Delta t$.

31. $\int(3x^4 + 4x^2 + 11)dx = \frac{3x^5}{5} + \frac{4x^3}{3} + 11x + C$.

32. $\int(4x^3 + 3x^2 - x + 91)dx = x^4 + x^3 - \frac{x^2}{2} + 91x + C$.

33. $\int 3t^{4/3}\,dt = \frac{9}{7}t^{7/3} + C$.

34. Let $u = 1 + 2t$, then $du = 2dt$ and $dt = \frac{1}{2}du$. So $\int(1 + 2t)^5 dt = \int u^5(\frac{1}{2}du) = \frac{u^6}{12} + C = \frac{(1 + 2t)^6}{12} + C$.

35. Let $u = 3t + 9$, $du = 3dt$ and $dt = \frac{1}{3}du$. Hence, $\int \sqrt[7]{3t + 9}\,dt = \int u^{1/7}(\frac{1}{3}du) = \frac{7}{24}u^{8/7} + C = \frac{7}{24}(3t + 9)^{8/7} + C$.

36. Let $u = x^3 - 1$, $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3}du$. $\int x^2(x^3 - 1)^{40}\,dx = \int u^{40}(\frac{1}{3}du) = \frac{u^{41}}{123} + C = \frac{(x^3 - 1)^{41}}{123} + C$.

37. Let $u = x^3 + 8$, $du = 3x^2 dx$, $x^2 dx = \frac{1}{3}du$. $\int x^2(x^3 + 8)^{17}\,dx = \int u^{17}(\frac{1}{3}du) = \frac{u^{18}}{54} + C = \frac{(x^3 + 8)^{18}}{54} + C$.

38. Let $u = x^2 + 4$, $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$. $\int x(x^2 + 4)^{-1/3}dx = \int u^{-1/3}(\frac{1}{2}du) = \frac{1}{2}\,\frac{u^{2/3}}{\frac{2}{3}} + C = \frac{3}{4}(x^2 + 4)^{2/3} + C$.

39. Let $u = x^8 + 13$, $du = 8x^7 dx$, $x^7 dx = \frac{1}{8}du$. $\int \frac{x^7 dx}{\sqrt[5]{x^8 + 13}} = \int \frac{\frac{1}{8}du}{u^{1/5}} = \frac{1}{8}(\frac{u^{4/5}}{\frac{4}{5}}) + C = \frac{5}{32}(x^8 + 13)^{4/5} + C$.

40. Let $u = \sqrt{x} - 3$, $du = \frac{1}{2\sqrt{x}}\,dx$, so $\frac{1}{\sqrt{x}}\,dx = 2\,du$.

$\int \frac{(\sqrt{x} - 3)^{44}}{\sqrt{x}}\,dx = \int u^{44} \cdot 2\,du = \frac{2u^{45}}{45} + C =$

$\frac{2}{45}(\sqrt{x} - 3)^{45} + C.$

41. Let $u = 7 + x$, $x = u - 7$, $du = dx$. $\int x\sqrt{7 + x}\,dx$

$= \int (u - 7)\,u^{½}du = \int (u^{3/2} - 7u^{½})\,du = \frac{2u^{5/2}}{5}$

$- \frac{14}{3}u^{3/2} + C = \frac{2}{5}(7 + x)^{5/2} - \frac{14}{3}(7 + x)^{3/2} + C.$

42. Let $u = t + 5$, $t = u - 5$, $du = dt$. $\int \frac{3t\,dt}{\sqrt{t + 5}}$

$= \int \frac{3(u - 5)du}{u^{½}} = 3\int (u^{½} - 5u^{-½})\,du =$

$3\left[\frac{u^{3/2}}{\frac{3}{2}} - \frac{5u^{½}}{\frac{1}{2}} + C\right] =$

$3\left[\frac{2}{3}(t + 5)^{3/2} - 10(t + 5)^{½} + C\right].$

43. Let $u = x^3 + 1$, $du = 3x^2dx$. $\int x^5\sqrt{x^3 + 1}\,dx$

$= \int x^3 \cdot x^2\sqrt{x^3 + 1}\,dx = \int (u - 1)\sqrt{u}\,\frac{1}{3}\,du$

$= \frac{1}{3}\int (u^{3/2} - u^{½})\,du$

$= \frac{1}{3}\;\frac{u^{5/2}}{\frac{5}{2}} - \frac{u^{3/2}}{\frac{3}{2}} + C$

$= \frac{1}{3}\left[\frac{2}{5}(x^3 + 1)^{5/2} - \frac{2}{3}(x^3 + 1)^{3/2}\right] + C$

$= \frac{2}{45}(x^3 + 1)^{3/2}[3(x^3 + 1) - 5] + C =$

$\frac{2}{45}(x^3 + 1)^{3/2}(3x^3 - 2) + C.$

44. Let $u = x^3 + 1$, $du = 3x^2dx$. $\int \frac{x^8dx}{\sqrt{x^3 + 1}}$

$= \int \frac{x^6 \cdot x^2dx}{\sqrt{x^3 + 1}} = \int \frac{(u - 1)^2\,\frac{1}{3}\,du}{\sqrt{u}} =$

$\frac{1}{3}\int \frac{u^2 - 2u + 1}{\sqrt{u}}\,du = \frac{1}{3}\int (u^{3/2} - 2u^{½} + u^{-½})\,du$

$= \frac{1}{3}\left[\frac{u^{5/2}}{\frac{5}{2}} - \frac{2u^{3/2}}{\frac{3}{2}} + \frac{u^{½}}{\frac{1}{2}}\right] + C$

$= \frac{1}{3}\left[\frac{2}{5}(x^3 + 1)^{5/2} - \frac{4}{3}(x^3 + 1)^{3/2} + 2(x^3 + 1)^{½}\right] + C.$

45. $\int (3\cos x - 2\sin x)dx = 3(\sin x) - 2(-\cos x) + C$

$= 3\sin x + 2\cos x + C.$

46. Let $u = \tan 2x$, $du = 2\tan 2x\sec 2x$,

$v = 2x$, $dv = 2\,dx$.

$\int 2\sec 2x\tan 2x\,dx + \int \sec^2 2x\,dx = \int du + \int \frac{1}{2}\sec^2 v\,dv$

$= u + \frac{1}{2}\tan v + C$

$= \tan 2x + \frac{1}{2}\tan 2x + C.$

47. Let $u = 3x$, $du = 3\,dx$. $\int 2\sin 3x\,dx$

$= \int 2\sin u\left(\frac{1}{3}\,du\right) = \frac{2}{3}\int \sin u\,du = \frac{2}{3}(-\cos u) + C =$

$-\frac{2}{3}\cos 3x + C.$

48. Let $v = 3u^2$, $dv = 6u\,du$. $\int u\sin 3u^2du$

$= \int (\sin v)\left(\frac{1}{6}\,dv\right) = \frac{1}{6}\int \sin v\,dv = -\frac{1}{6}\cos v + C$

$= -\frac{1}{6}\cos 3u^2 + C.$

49. Let $u = 4t$, $du = 4\,dt$.

$\int 2\,dt - 3\int \cos 4t\,dt$

$= 2t + C' - 3\int \cos u\left(\frac{1}{4}\,du\right) = 2t - \frac{3}{4}\sin u + C$

$= 2t - \frac{3}{4}\sin 4t + C.$

50. Let $u = 4x^2 - 1$, $du = 8x\,dx$.

$\int x\cos(4x^2 - 1)dx$

$= \int \cos u\left(\frac{1}{8}\,du\right) = \frac{1}{8}\int \cos u\,du$

$= \frac{1}{8}\sin u + C = \frac{1}{8}\sin(4x^2 - 1).$

51. Let $u = x^2$, $du = 2x\,dx$.

$\int x\sec x^2\tan x^2dx = \int \sec u\tan u\left(\frac{1}{2}\,du\right)$

$= \frac{1}{2}\int \tan u\sec u\,du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec x^2 + C.$

52. Let $u = \tan\theta$, $du = \sec^2\theta\, d\theta$.

$\int \sqrt{\tan\theta}\sec^2\theta\, d\theta$

$= \int \sqrt{u}\, du = \frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{3}\tan^{3/2}\theta + C.$

53. Let $u = 2 + 3\cot\beta$, $du = -3\csc^2\beta\, d\beta$.

$\int (2 + 3\cot\beta)^{3/2}\csc^2\beta\, d\beta$

$= \int \frac{1}{3}u^{3/2}(-du) = -\frac{u^{5/2}}{3(\frac{5}{2})} + C = -\frac{2}{15}$

$(2 + 3\cot\beta)^{5/2} + C.$

54. Let $u = \sin x$, $du = \cos x\, dx$. $\int \frac{\cos x}{\sin^2 x}dx$

$= \int \frac{1}{u^2}du = \int u^{-2}du = \frac{u^{-1}}{-1} + C = -\frac{1}{\sin x} + C.$

55. Let $w = 1 - \sin v$, $dw = -\cos v\, dv$.

$\int \frac{\cos v\, dv}{(1 - \sin v)^4}$

$= \int \frac{(-dw)}{w^4} = -\int w^{-4}dw = -\frac{w^{-3}}{-3} + C$

$= \frac{1}{3}(1 - \sin v)^{-3} + C.$

56. Let $u = \sin x$, $du = \cos x\, dx$.

$\int \csc^2(\sin x)\cos x\, dx$

$= \int \csc^2 u\, du = -\cot u + C = -\cot(\sin x)dx.$

57. Let $u = \sec 3x + 8$, $du = 3\sec 3x\tan 3x\, dx$.

$\int \frac{\sec 3x\tan 3x}{(\sec 3x + 8)^{10}} = \int \frac{\frac{1}{3}du}{u^{10}} = \frac{1}{3}\int u^{-10}du$

$= \frac{1}{3}\frac{u^{-9}}{-9} + C = -\frac{1}{27}(\sec 3x + 8)^{-9} + C.$

58. Let $w = 5 + \cot u$, $dw = -\csc^2 u\, du$. $\int \frac{\csc^2 u\, du}{\sqrt{5 + \cot u}}$

$= \int \frac{(-dw)}{\sqrt{w}} = -\int w^{-\frac{1}{2}} = \frac{-w^{\frac{1}{2}}}{\frac{1}{2}} + C =$

$-2(5 + \cot u)^{\frac{1}{2}} + C.$

59. Let $u = ax + b\sin x$, $du = (a + b\cos x)dx$.

$\int \frac{a + b\cos x}{(ax + b\sin x)^4}dx$

$= \int \frac{du}{u^4} = \int u^{-4}du = \frac{u^{-3}}{-3} + C = -\frac{1}{3}(ax + b\sin x)^{-3} + C.$

60. $\int \frac{dt}{\sin t(\sin t + \cos t)} = \int \frac{dt}{\sin^2 t(1 + \cot t)}$

$= \int \frac{\csc^2 t\, dt}{1 + \cot t}.$

Let $u = 1 + \cot t$, $du = -\csc^2 t\, dt$. $\int \frac{\csc^2 t\, dt}{1 + \cot t}$

$= \int \frac{(-du)}{u} = -\ln|u| + C = -\ln|1 + \cot t| + C.$

61. $y = \int (2x + 1)dx = x^2 + x + C.$

62. $y = \int (3x^2 - 14x + 8)dx = x^3 - 7x^2 + 8x + C.$

63. Let $u = 3 - t$, $du = -dt$.

$s = \int \frac{1}{(3 - t)^2}dt = \int -\frac{du}{u^2} = \frac{-u^{-1}}{-1} + C = \frac{1}{3 - t} + C.$

64. $y = \int \frac{1 + x}{\sqrt{x}}dx = \int (x^{-\frac{1}{2}} + x^{\frac{1}{2}})dx = 2x^{\frac{1}{2}} + \frac{2}{3}x^{3/2} + C.$

65. $y = \int (1 - x^{3/2})^{15}\sqrt{x}\, dx$. Let $u = 1 - x^{3/2}$,

$du = -\frac{3}{2}x^{\frac{1}{2}}dx$, $-\frac{2}{3}du = x^{\frac{1}{2}}dx$. So $\int (1 - x^{3/2})^{15}\cdot$

$\sqrt{x}\, dx = \int u^{15}(-\frac{2}{3}du) = -\frac{2}{3.16}u^{16} + C,$

$y = -\frac{1}{24}(1 - x^{3/2})^{16} + C.$

66. $y = \int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}}dx$. Let $u = 1 + \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$,

So $\frac{1}{x}$ dx = 2 du. So $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int 2u^{\frac{1}{2}}du$

$= 2 \cdot \frac{2}{3} u^{3/2} + C.$ So $y = \frac{4}{3}(1+\sqrt{x})^{3/2} + C.$

67. $\frac{dy}{dx} = \int (3 - 2x + 6x^2)dx = 3x - x^2 + 2x^3 + C_1,$
$y = \int (3x - x^2 + 2x^3 + C_1)dx = \frac{3x^2}{2} - \frac{x^3}{2} + \frac{x^4}{2} +$
$C_1x + C_2.$

68. $\frac{dy}{dx} = \int (1-x)^{-4}\,dx.$ Let $u = 1 - x$, $du = -dx$.
Hence, $\int (1-x)^{-4}dx = \int (-u^{-4})\,du = \frac{-u^{-3}}{-3} + C.$
$\frac{dy}{dx} = \frac{(1-x)^{-3}}{3} + C_1$, $y = \int (\frac{(1-x)^{-3}}{3} + C_1)\,dx.$
Let $u = 1 - x$ again. $\int \frac{(1-x)^{-3}}{3}\,dx = \int \frac{(-u^{-3})}{3}$
$du = \frac{u^{-2}}{6} + C.$ $y = \frac{(1-x)^{-2}}{6} + C_1x + C_2.$

69. $\sqrt{y}\,dx = -\sqrt{x}\,dy,$
$\frac{dx}{\sqrt{x}} = -\frac{1}{\sqrt{y}}\,dy.$
$\int x^{-\frac{1}{2}}\,dx = -\int y^{-\frac{1}{2}}\,dy$
$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C' = -\frac{y^{\frac{1}{2}}}{\frac{1}{2}} + C''.$

$\sqrt{x} + \sqrt{y} = C.$

70. $\sqrt{x^2+1}\,y\,dy = -x\,dx$, $y\,dy = \frac{-x}{\sqrt{x^2+1}}\,dx.$

$\int y\,dy = \int \frac{-x}{\sqrt{x^2+1}}\,dx.$ Let $u = x^2 + 1$, $du = 2x\,dx$.

$\frac{y^2}{2} + C' = \int \frac{-\frac{1}{2}\,du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}du =$

$-\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C'' = -(x^2+1)^{\frac{1}{2}} + C''$, so

$y^2 = -2(x^2+1)^{\frac{1}{2}} + C.$

71. $y = \int (x - 3\cos x)dx = \frac{x^2}{2} - 3\sin x + C.$

72. $\frac{dy}{dx} = \int (4x^2 - \sin 2x)dx = \frac{4x^3}{3} + \frac{\cos 2x}{2} + C_1.$

$y = \int (\frac{4x^3}{3} + \frac{\cos 2x}{2} + C_1dx$
$= \frac{4x^4}{3.4} + \frac{\sin 2x}{2 \cdot 2} + C_1x + C_2$
$= \frac{x^4}{3} + \frac{\sin 2x}{4} + C_1x + C_2.$

73. $y = \int (\sqrt{x} - \sec^2 x)dx = \frac{x^{3/2}}{\frac{3}{2}} - \tan x + C$
$= \frac{2}{3}x^{3/2} - \tan x + C.$

74. $s = \int (\cos 3t - \csc^2 3t)dt = \frac{\sin 3t}{3} + \frac{\cot 3t}{3} + C.$

75. $\sin x \cos^2 y\,dx = -\cos^2 x\,dy,$

$\frac{\sin x\,dx}{\cos^2 x} = -\frac{dy}{\cos^2 y} = -\sec^2 y\,dy.$

$\int \frac{\sin x\,dx}{\cos^2 x} = \int (-\sec^2 y\,dy) = -\tan y + C'.$

Let $u = \cos x$, $du = -\sin x\,dx$. $\int \frac{\sin x\,dx}{\cos^2 x}$

$= \int \frac{-du}{u^2} = -\int u^{-2}du = -\frac{u^{-1}}{-1} + C' = \frac{1}{u} + C' = \frac{1}{\cos x}$

$+ C'.$ So $\frac{1}{\cos x} + C' = -\tan y + C'',$

$\sec x = -\tan y + C.$

76. $\sec t\,ds = -(1+s)^2dt =$
$\frac{ds}{(1+s)^2} = \frac{dt}{\sec t} = \cos t\,dt.$
$\int \frac{ds}{(1+s)^2} = \int \cos t\,dt = \sin t + C'.$
Let $u = 1 + s$, $du = ds$. $\int \frac{ds}{(1+s)^2} = \int \frac{du}{u^2} = \int u^{-2}du$
$= \frac{u^{-1}}{-1} + C'' = -\frac{1}{u} + C'' = -\frac{1}{1+s} + C''.$ So
$-\frac{1}{1+s} + C'' = \sin t + C'$, or $-\frac{1}{1+s} = \sin t + C.$

77. $y = \int (2x^3 + 2x + 1)dx = \frac{x^4}{2} + x^2 + x + C.$ Since
$y = 0$ when $x = 0$, $C = 0$. $y = \frac{x^4}{2} + x^2 + x.$

78. $y = \int x^{-1/3}dx = \frac{3}{2}x^{2/3} + C.$ Now $0 = \frac{3}{2}(1)^{2/3} + C,$
so $C = -\frac{3}{2}$. $y = \frac{3}{2}x^{2/3} - \frac{3}{2}.$

79. $y = \int \frac{x}{\sqrt{1-x^2}}\,dx$. Let $u = 1 - x^2$, $du = -2x\,dx$, $x\,dx = -\frac{1}{2}\,du$. So $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{u^{1/2}} = -\frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + C = -u^{1/2} + C$. $y = -(1-x^2)^{1/2} + C$.
Since $-1 = -1 + C$, then $C = 0$. $y = -\sqrt{1-x^2}$.

80. $y = \int x^2(1+x^3)^{10}\,dx$. Let $u = 1 + x^3$, $du = 3x^2\,dx$, $x^2\,dx = \frac{1}{3}\,du$. So $\int x^2(1+x^3)^{10}\,dx = \int \frac{1}{3}u^{10}\,du = \frac{u^{11}}{33} + C$. $y = \frac{(1+x^3)^{11}}{33} + C$. Since $2 = \frac{(1)^{11}}{33} + C$, then $C = \frac{65}{33}$. $y = \frac{(1+x^3)^{11}}{33} + \frac{65}{33}$.

81. $\frac{dy}{dx} = \int (x^3+1)\,dx = \frac{x^4}{4} + x + C_1$. $y = \int (\frac{x^4}{4} + x + C_1)\,dx = \frac{x^5}{20} + \frac{x^2}{2} + C_1x + C_2$. Now, $0 = 0 + C_2$, so $C_2 = 0$. Also, $1 = 0 + 0 + C_1$, so $C_1 = 1$. $y = \frac{x^5}{20} + \frac{x^2}{2} + x$.

82. $\frac{dy}{dx} = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C_1$. $y = \int (\frac{x^{-2}}{-2} + C_1)\,dx = \frac{1}{2}x^{-1} + C_1x + C_2$. Since $y = 2$, when $x = 1$, $2 = \frac{1}{2} + C_1 + C_2$; since $y' = 1$ when $x = 1$, $1 = -\frac{1}{2} + C_1$, so $C_1 = \frac{3}{2}$. Now $2 = \frac{1}{2} + \frac{3}{2} + C_2$, and $C_2 = 0$. $y = \frac{1}{2x} + \frac{3}{2}x$.

83. $s = \int (t - 3\sin t)\,dt = \frac{t^2}{2} + 3\cos t + C$.
Now, $0 = 0 + 3\cos 0 + C = 3 + C$,
so $C = -3$.
$s = \frac{t^2}{2} + 3\cos t - 3$.

84. $\sec y\,dx = -\csc x\,dy$, $\frac{dx}{-\csc x} = \frac{dy}{\sec y}$,
$-\sin x\,dx = \cos y\,dy$.
$\int (-\sin x)\,dx = \int \cos y\,dx$ or $\cos x = \sin y + C$.
Now $\cos 0 = \sin\frac{\pi}{2} + C$, or $1 = 1 + C$, so $C = 0$.
Therefore, $\cos x = \sin y$.

85. $y = \int x\sqrt{x^2+5}\,dx$. Let $u = x^2 + 5$, $du = 2x\,dx$. $y = \int x\sqrt{x^2+5}\,dx$
$= \int \sqrt{u}\,\frac{1}{2}\,du = \frac{1}{2}\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{1}{3}(x^2+5)^{3/2} + C$.
Since $y = 6$ when $x = 2$,
$6 = \frac{1}{3}(4+5)^{3/2} + C = \frac{1}{3}3^3 + C = 9 + C$, or $C = -3$.
Thus, $y = \frac{1}{3}(x^2+5)^{3/2} - 3$.

86. $\frac{dy}{dx} = x\csc^2 x^2$. $y = \int x\csc^2 x^2\,dx$. Let $u = x^2$, $du = 2x\,dx$. $\int x\csc^2 x^2\,dx = \int \csc^2 u(\frac{1}{2}\,du)$
$= \frac{1}{2}(-\cot u) + C = -\frac{1}{2}\cot x^2 + C$.
When $x = \sqrt{\frac{\pi}{2}}$, $y = 3$, so $3 = -\frac{1}{2}\cot\frac{\pi}{2} + C = C$.
Thus, $y = -\frac{1}{2}\cot x^2 + 3$.

87. (a) Suppose such a function exists. Then $f(x) = \int (3x^2+1)\,dx$, and $f(x) = x^3 + x + C$. If $f(0) = 0$, then $C = 0$; but if $f(1) = 3$, then $C = 1$. This is impossible; so there is no such function f satisfying the conditions given.
(b) $f'(x) = \int (3x^2+1)\,dx = x^3 + x + C_1$. $f(x) = \int (x^3 + x + C_1)\,dx = \frac{x^4}{4} + \frac{x^2}{2} + C_1x + C_2$. If $f(0) = 0$, then $C_2 = 0$; if $f(1) = 3$, then $3 = \frac{1}{4} + \frac{1}{2} + C_1$, and so $C_1 = \frac{9}{4}$. Hence, $f(x) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{9}{4}x$, and $f(0) = 0$ and $f(1) = 3$. In general, a first-order differential equation can be made to satisfy one boundary condition, since there is one constant of integration; whereas a second-order differential equation can be made to satisfy two boundary conditions, since there are two constants of integration.

88. $y = \int (|x| + |x - 1| + |x - 2|)dx$. If $u = x - 1$, then $du = dx$ and if $u = x - 2$, then $du = dx$; hence, $y = x \cdot \frac{|x|}{2} + (x - 1) \cdot \frac{|x - 1|}{2} + (x - 2) \frac{|x - 2|}{2} + C$. $y = 1$, when $x = 0$, so that we can solve for C. $1 = 0 + (-1)(\frac{1}{2}) + (-2)(1) + C$, $C = \frac{7}{2}$. $y = x \cdot \frac{|x|}{2} + (x - 1)\frac{|x - 1|}{2} + (x - 2) \frac{|x - 2|}{2} + \frac{7}{2}$.

89. $W = \int F ds = \int (3s - 1)ds = \frac{3}{2} s^2 - s + C$. When $s = \frac{1}{3}$, $W = 0$, so $0 = \frac{3}{2}(\frac{1}{3})^2 - \frac{1}{3} + C = -\frac{1}{6} + C$, and it follows that $C = \frac{1}{6}$. When $s = 6$, we have $W = \frac{3}{2}(6)^2 - 6 + \frac{1}{6} = \frac{289}{6}$ joules.

90. $W = \int F ds = \int \cos 2s\, ds = \frac{1}{2} \sin 2s + C$. When $s = 0$, $W = 0$, so $0 = \frac{1}{2} \sin 0 + C$, and it follows that $C = 0$. When $s = \frac{\pi}{4}$, $W = \frac{1}{2} \sin \frac{\pi}{2} = \frac{1}{2}$ joule.

91. $W = \int F ds = \int \sec^2 \frac{s}{2}\, ds = 2 \tan \frac{s}{2} + C$. When $s = 0$, $W = 0$, so $0 = 2 \tan 0 + C$, and it follows that $C = 0$. When $s = \frac{2\pi}{3}$, $W = 2 \tan \frac{\pi}{3} = 2\sqrt{3}$ joules.

92. $W = \int F ds = \int \sqrt{1 + \sqrt{s}}\, ds$. Let $u = \sqrt{1 + \sqrt{s}}$, so that $u^2 = 1 + \sqrt{s}$ and $2u\, du = \frac{ds}{2\sqrt{s}}$. Now, $\sqrt{s} = u^2 - 1$, so $ds = 2\sqrt{s}(2u\, du) = 4(u^2 - 1)\, u\, du = 4(u^3 - u)du$. Thus, $W = \int \sqrt{1 + \sqrt{s}}\, ds = \int u[4(u^3 - u)]\, du = 4\int (u^4 - u^2)\, du = 4(\frac{u^5}{5}) - 4(\frac{u^3}{3}) + C$. When $s = 0$, $u = \sqrt{1} = 1$ and $W = 0$, so that $0 = \frac{4}{5} - \frac{4}{3} + C = -\frac{8}{15} + C$, and it follows that $C = \frac{8}{15}$. When $s = 1$, $u = \sqrt{2}$ and $W = \frac{4}{5}(\sqrt{2})^5 - \frac{4}{3}(\sqrt{2})^3 + \frac{8}{15} = \frac{8(\sqrt{2} + 1)}{15}$ joules.

93. $F = ks$. $4000 = \frac{1}{2} k$ (2 tons is 4000 pounds and 6 inches is $\frac{1}{2}$ foot), so $8000 = k$. $W = \int F ds = \int ks\, ds = \int 8000 \cdot ds = 4000s^2 + C$. When $s = 0$, $W = 0$. So $C = 0$. $W = 4000s^2$. When $s = \frac{1}{2}$, $W = 4000(\frac{1}{4}) = 1000$ foot - pounds.

94. $s = \frac{1}{2} at^2 + v_0 t + s_0$. $v_0 = 44$; $s_0 = 0$ when $v_0 = 44$. $v = at + v_0$, so $t = \frac{v - v_0}{a}$. $s = \frac{1}{2} a \frac{(v - v_0)^2}{a^2} + v_0 (\frac{v - v_0}{a}) = \frac{v^2 - 2v \cdot v_0 + v_0^2}{2a} + \frac{2v \cdot v_0 - 2v_0^2}{2a}$. $s = \frac{v^2 - v_0^2}{2a}$, so $a = \frac{v^2 - v_0^2}{2s}$. When $s = 500$, $v = 0$; $a = \frac{0 - (44)^2}{1000} = -1.94$ meters per second per second.

95. Let t_1 be the time it takes to run 15 meters. Let t_2 be the time it takes to run 85 meters. $t_1 + t_2 = 10 - 15 = \frac{1}{2} at_1^2 + v_0 t_1 + s_0$. But $v_0 = 0$ and $s_0 = 0$. So $15 = \frac{1}{2} at_1^2$. The velocity is $\frac{ds}{dt} = v = at_1$ and is maintained for the next 85 meters. Hence, $85 = (at_1) \cdot t_2$ so $t_1 = \frac{85}{at_2}$. Now $t_1 = 10 - t_2$. Substituting: $15 = \frac{1}{2} a (10 - t_2) \frac{85}{at_2}$ and $\frac{15}{85} = \frac{1}{2}(\frac{10}{t_2} - 1)$. $t_2 = \frac{170}{23}$ seconds. $t_1 = \frac{60}{23}$ seconds. Hence, $a = \frac{8.5}{t_2 \cdot t_1} \approx 4.4083$ meters per second per second.

96. $s = \frac{1}{2} at^2 + v_0 t + s_0$ and $s = 0$, $v_0 = 72$. $v = \frac{ds}{dt} = at + 72$. $v = 0$ when $t = 4$; $0 = 4a + 72$. So $a = -18$ kilometers per second. Again, using $v = at + v_0$ when $v_0 = 96$, we have $0 = -18t + 96$, so $t = \frac{96}{18} = 5.33$ seconds.

97. (a) $s = \frac{1}{2} gt^2 + v_0 t + s$. $v_0 = 96$, $s_0 = 0$, $s = 256$. Hence, $256 = 16t^2 + 96t$, so $0 = 16t^2 + 96t$

$-\ 256$; $0 = t^2 + 6t - 16$; $0 = (t + 8)(t - 2)$; $t = 2$ seconds.

(b) $v = \dfrac{ds}{dt} = gt + v_o = 32t + 96$. When $t = 2$, $v = 160$ feet per second.

98. Let $y = s^* - s$. When $t = 0$, $y = k$. Now, $\dfrac{dy}{dt} = \dfrac{ds^*}{dt} - \dfrac{ds}{dt}$, so, when $t = 0$, $\dfrac{dy}{dt} = v_o^* - v_o$. Also, $\dfrac{d^2y}{dt^2} = \dfrac{d^2s^*}{dt^2} - \dfrac{d^2s}{dt^2} = a^* - a$, a constant. Thus, $\dfrac{dy}{dt} = \int (a^* - a)dt = (a^* - a)t + C_o$. When $t = 0$, we have $v_o^* - v_o = (a^* - a)(0) + C_o$, so $C_o = v_o^* - v_o$, and $\dfrac{dy}{dt} = (a^* - a)t + (v_o^* - v_o)$. Thus, $y = \int \left[(a^* - a)t + (v_o^* - v_o)\right] dt = \frac{1}{2}(a^* - a)t^2 + (v_o^* - v_o)t + C_1$. When $t = 0$, we have $k = \frac{1}{2}(a^* - a)(0)^2 + (v_o^* - v_o)(0) + C_1$, so $C_1 = k$, and $y = \frac{1}{2}(a^* - a)t^2 + (v_o^* - v_o)t + k$.

(a) Suppose $a^* < a$, so that $a^* - a < 0$. Solving the equation $0 = \frac{1}{2}(a^* - a)t^2 + (v_o^* - v_o)t + k$ using the quadratic formula, and noting that $k > 0$, we find that the particles collide when t has the positive value

$$\frac{(v_o^* - v_o) + \sqrt{(v_o^* - v_o)^2 + 2k(a - a^*)}}{a - a^*}$$

(b) If $a^* - a$, we have $y = (v_o^* - v_o)t + k$, so that $y = 0$ if and only if $v_o^* \neq v_o$ and $t = \dfrac{k}{v_o - v_o^*}$. Since t must be positive for collision to occur, it follows that collision will occur if and only if $v_o > v_o^*$.

(c) Suppose $a^* > a$. For collision to occur, one of the two roots

$$\frac{v_o - v_o^* \pm \sqrt{(v_o - v_o^*)^2 - 2k(a^* - a)}}{a^* - a}$$

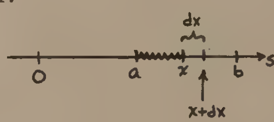
must be positive. This requires that $(v_o - v_o^*)^2 > 2k(a^* - a)$ and $v_o > v_o^*$.

99. $W = \int F ds$, where s denotes the distance in meters through which the weight has been raised. The weight and cable weigh $2000 + 10(50 - s)$ newtons, so $W = \int \left[2000 + 10(50 - s)\right] ds = \int (2500 - 10s)ds = 2500s - 5s^2 + C$. When $s = 0$, $W = 0$, and it follows that $C = 0$. When $s = 50$, $W = 2500(50) - 5(50)^2 = 112{,}500$ joules.

100. (a) $\dfrac{dW}{dt} = \dfrac{EdQ}{dt} = EI$.

(b) $W = \int EI\, dt$, but $\dfrac{I}{C} = \dfrac{dE}{dt}$, so $I = \dfrac{dE}{dt} \cdot C$. Hence, $W = \int E\left(\dfrac{dE}{dt}\right) \cdot C \cdot dt = \int E \cdot C \cdot dE = \dfrac{E^2}{2} C + K$. Now $W = 0$ when $E = 0$, so $K = 0$. Therefore, $W = \dfrac{1}{2} \cdot C \cdot E^2$.

101.



The linear density of mass distributed on the s axis is $\dfrac{m}{b - a}$ kilograms per meter. Let F be the force on the particle at the origin due to the mass on the interval from a to x where $a \leq x \leq b$. If dx represents an infinitesimal increment in x, then the infinitesimal force of attraction of the mass in the interval from x to $x + dx$ will be $dF = \dfrac{GM \frac{m}{b - a} dx}{x^2}$. $F = GM \dfrac{m}{b - a} \int x^{-2} dx = \dfrac{GMm}{b - a}\left(-\dfrac{1}{x}\right) + C$. $F = 0$ when $x = a$, so $C = \dfrac{GMm}{a(b - a)}$ and $F = \dfrac{GMm}{a(b - a)} - \dfrac{GMm}{x(b - a)}$.

The total force is given by $F = \dfrac{GMm}{a(b - a)} - \dfrac{GMm}{b(b - a)} = \dfrac{GMm}{b - a}\left(\dfrac{1}{a} - \dfrac{1}{b}\right) = \dfrac{GMm}{ab}$ newtons.

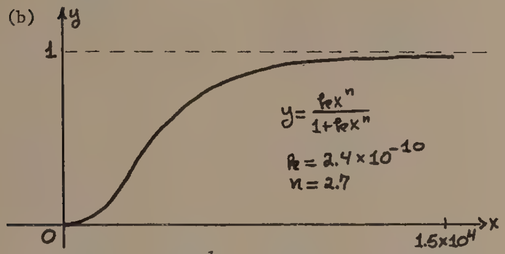
102. (a) $(1 + kx^n)^2 dy - n\, kx^{n-1}\, dx = 0$. Separating variables, we have

$$dy = \frac{n\, kx^{n-1}\, dx}{(1 + kx^n)^2},$$

so $y = \displaystyle\int \frac{n\, kx^{n-1}\, dx}{(1 + kx^n)^2}$.

Let $u = 1 + kx^n$, so that $du = nkx^{n-1}dx$ and $y = \int \frac{du}{u^2} = \int u^{-2}du = \frac{-1}{u} + C = \frac{-1}{1 + kx^n} + C$. Since $y = 0$ when $x = 0$, we have $C = 1$. Therefore, $y = 1 - \frac{1}{1 + kx^n}$, or $y = \frac{kx^n}{1 + kx^n}$.

(b)



(c) $\frac{dy}{dx} = \frac{nkx^{n-1}}{(1 + kx^n)^2}$, $\frac{d^2y}{dx^2} =$

$$\frac{(1+kx^n)^2[n(n-1)kx^{n-2}] - nkx^{n-1} \cdot 2(1+kx^n)(nkx^{n-1})}{(1 + kx^n)^4}$$

$$= \frac{nkx^{n-2}\left[(n - 1) - k(n + 1)x^n\right]}{(1 + kx^n)^3},$$

so the inflection occurs when $(n - 1) - k(n + 1) \cdot x^n = 0$; that is, when $x = \left(\frac{n - 1}{k(n + 1)}\right)^{1/n}$. For $n = 2.7$, $k = 2.4 \times 10^{-10}$, we obtain $x \approx 2.74 \times 10^3$ and $y \approx 0.315$.

103. (a) $R = \int (10 - \frac{x}{12,500})dx = 10x - \frac{x^2}{25,000} + K$. Because $R = 0$ when $x = 0$, we have $K = 0$. $C = \int 6dx = 6x + C_0$.

(b) Because $C = \$400$ when $x = 0$, we have $C_0 = 400$ and $C = 6x + 400$.

(c) $\frac{dR}{dx} = 0$ when $10 - \frac{x}{12,500} = 0$; that is, when $x = 125,000$.

(d) $P = R - C = 10x - \frac{x^2}{25,000} - 6x - 400 = 4x - \frac{x^2}{25,000} - 400$, $\frac{dP}{dx} = 4 - \frac{x}{12,500} = 0$ when $x = 50,000$.

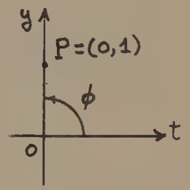
(e) When x pairs are manufactured, the price per pair is $\frac{R}{x} = 10 - \frac{x}{25,000}$. When $x = 50,000$, $\frac{R}{x} = 10 - 2 = 8$ dollars per pair.

104. (a) $C = \int \frac{700}{\sqrt{x}} dx = \frac{700\, x^{1/2}}{\frac{1}{2}} + K$. $C = 500$ when $x = 0$, so $K = 500$. $C = 1400\sqrt{x} + 500$.

(b) $P = R - C$. We want $P' = R' - C' = 0$. $P' = \left[x(-\frac{1}{1000}) + 27 - \frac{x}{1000}\right] - \frac{700}{\sqrt{x}} = 0$. We want x such that $27 - \frac{x}{500} - \frac{700}{\sqrt{x}} = 0$. When $x = 10,000$, $27 - \frac{10,000}{500} - \frac{700}{\sqrt{10,000}} = 27 - 20 - \frac{700}{100} = 27 - 20 - 7 = 0$. Hence, $x = 10,000$ maximizes the profit.

(c) The total revenue for $x = 10,000$ is $R = (10,000)(27 - \frac{10,000}{1000}) = (10,000)(17) = 170,000$. The cost should be $\frac{170,000}{10,000} = \$17$ per sweater.

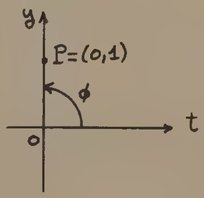
105. $\omega^2 = 25$, so $\omega = 5$



Find a solution $y = f(t)$ for which $f(0) = 0$ and $f'(0) = 5$

Take $P = (f(0),\ f'(0)/\omega)$
$= (0, \frac{5}{5}) = (0,1)$. $A = 1$, $\phi = \frac{\pi}{2}$;
so $y = 1 \cos(5t - \frac{\pi}{2}) = \cos(5t - \frac{\pi}{2})$.

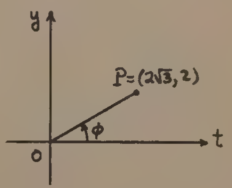
106. $\omega^2 = 7$, so $\omega = \sqrt{7}$



Find a solution $y = f(t)$ for which $f(0) = 0$ and $f'(0) = \sqrt{7}$.

Take $P = (f(0), f'(0)/\omega)$
$= (0, \frac{\sqrt{7}}{\sqrt{7}}) = (0,1)$
$A = 1$, $\phi = \frac{\pi}{2}$; so $y = 1 \cos(\sqrt{7}t - \frac{\pi}{2})$
$= \cos(\sqrt{7}t - \frac{\pi}{2})$.

107. $\omega^2 = 36$, so $\omega = 6$



Find a solution $y = f(t)$ for which $f(0) = 2\sqrt{3}$, $f'(0) = 12$.

Take $P = (f(0), f'(0)/\omega)$

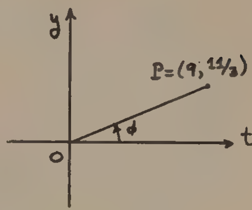
$= (2\sqrt{3}, \frac{12}{6}) =$

$(2\sqrt{3}, 2)$

$A = \sqrt{12 + 4} = 4, \quad \phi = \frac{\pi}{6}$;

so $y = 4 \cos(6t - \frac{\pi}{6})$.

108. $\omega^2 = 9$, so $\omega = 3$



Find a solution $s = f(t)$ for which $f(0) = 9$, $f'(0) = 11$.

Take $P = (f(0), f'(0)/\omega)$

$= (9, \frac{11}{3})$.

$A = \sqrt{81 + \frac{121}{9}} = \sqrt{\frac{950}{9}} = \frac{5}{3}\sqrt{38}$;

$\tan\phi = \frac{11}{27}$, so $\phi \approx 22.17 \approx 0.386876$.

So

$y = \frac{5}{3}\sqrt{38}\cos(3t - 0.386876)$.

109. $\omega^2 = 100$, so $\omega = 10$. Want a solution $y = f(t)$ for which $f(0) = 0.02$ and $f'(0) = 0.24$. Take $P = (f(0), \frac{f'(0)}{\omega}) = (0.02, 0.024)$. Thus, $A = \sqrt{(0.02)^2 + (0.024)^2} \approx 0.031$ and $\phi = \tan^{-1}\frac{0.024}{0.02} \approx 0.876$.

(a) $y = 0.031 \cos(10t - 0.876)$

(b) $\omega = \frac{10}{2\pi} \approx 1.59$ Hz

(c) $T = \frac{1}{\nu} \approx 0.628$ sec.

110. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.5}{0.5}} = \sqrt{9} = 3$.

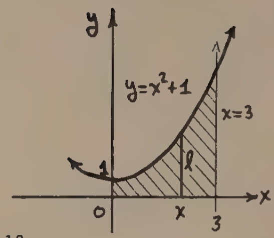
(a) $y = 0.2 \cos 3t$.

(b) $\nu = \frac{\omega}{2\pi} = \frac{3}{2\pi} \approx 0.477$ Hz.

111. $dA = \ell\, dx$, $\ell = y$;

so $dA = y\, dx = (x^2 + 1)dx$.

$A = \int (x^2 + 1)dx = \frac{x^3}{3} + x + C$



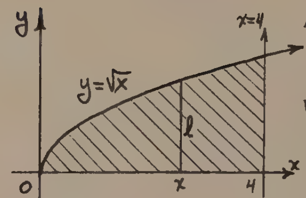
When $x = 0$, $A = 0$ so $C = 0$

$A = \frac{x^3}{3} + x$. Therefore, when $x = 3$,

$A = \frac{3^3}{3} + 3 = 12$ square units

112. $dA = \ell\, dx$, $\ell = y$;

so $dA = y\, dx = \sqrt{x}\, dx$.

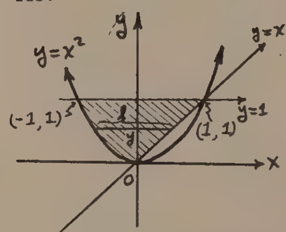


$A = \int \sqrt{x}\, dx = \frac{x^{3/2}}{\frac{3}{2}} + C$.

When $x = 0$, $A = 0$, so $C = 0$. $A = \frac{2}{3}x^{3/2}$.

When $x = 4$, $A = \frac{2}{3}(4)^{3/2} = \frac{2}{3} \cdot 8 = \frac{16}{3}$ square units.

113.



$\ell = y - (-\sqrt{y}) = y + \sqrt{y}$.

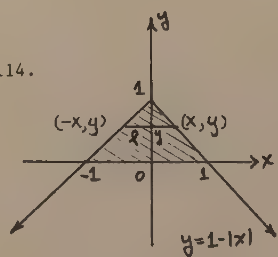
$A = \int \ell\, dy = \int (y + \sqrt{y})dy = \frac{y^2}{2} + \frac{y^{3/2}}{\frac{3}{2}} + C$.

When $y = 0$, $A = 0$, so $C = 0$.

$A = \frac{y^2}{2} + \frac{2}{3}y^{3/2}$. Thus, when $y = 1$

$A = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ square units.

114.



$\ell = 2x$; $|x| = 1 - y$. For $x \geq 0$, $\ell = 2(1 - y)$.

Thus,

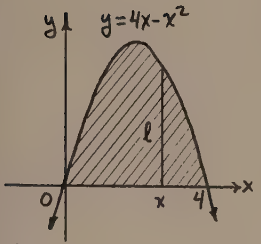
$A = \int \ell\, dy = \int (2)(1 - y)\cdot dy = 2(y - \frac{y^2}{2}) + C$.

When $y = 0$, $A = 0$, so $C = 0$.

So $A = 2y - y^2$. Therefore,

When $y = 1$, $A = 2 - 1 = 1$ square unit.

115. $A = \int \ell\, dx = \int y\, dx =$



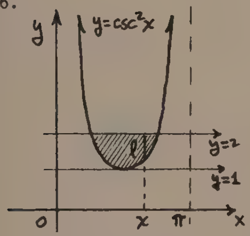
$\int (4x - x^2)dx = 2x^2 - \frac{x^3}{3}$ $+ C$.

When $x = 0, A = 0$, so $C = 0$

$A = 2x^2 - \frac{x^3}{3}$. Thus,

when $x = 4$, $A = 2(4)^2 - \frac{4^3}{3} = 32 - \frac{64}{3} = \frac{32}{3}$ square units.

116.



$\csc^2 x = 2$ so $\csc x = \pm\sqrt{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\ell = 2 - y = 2 - \csc^2 x$

$A = \int \ell\, dx = \int (2 - \csc^2 x)dx$

$= x + \cot x + C$

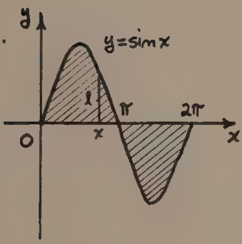
When $x = \frac{\pi}{4}$, $A = 0$, so

$0 = \frac{\pi}{4} + \cot(\frac{\pi}{4}) + C = \frac{\pi}{4} + 1 + C$. So $C = -\frac{\pi}{4} - 1$

$A = x + \cot x - \frac{\pi}{4} - 1$

When $x = \frac{3\pi}{4}$, $A = \frac{3\pi}{4} + \cot\frac{3\pi}{4} - \frac{\pi}{4} - 1 = \frac{\pi}{2} - 2$.

117.



By symmetry, find area from 0 to π and double result.

$\ell = y$.

$A = \int \ell\, dx = \int y\, dx =$

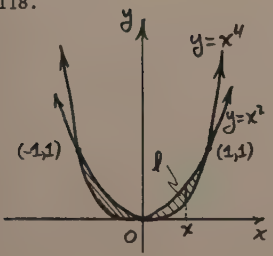
$\int \sin x\, dx = -\cos x + C$.

When $x = 0$, $A = 0$, so $0 = -1 + C$, or $C = 1$.

$A = 1 - \cos x$.

When $x = \pi$, $A = 1 - \cos \pi = 2$. Thus, the area of cycle = 4 square units.

118.



Find area in first quadrant and double result.

$\ell = x^2 - x^4$.

$A = \int \ell\, dx = \int (x^2 - x^4)dx$

$= \frac{x^3}{3} - \frac{x^5}{5} + C$.

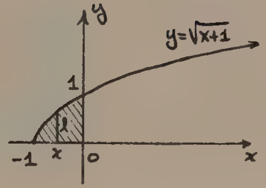
When $x = 0$, $A = 0$, so $C = 0$.

$A = \frac{x^3}{3} - \frac{x^5}{5}$. Therefore,

when $x = 1$, $A = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$.

So desired area is $2(\frac{2}{15}) = \frac{4}{15}$ square unit.

119.



$A = \int \ell\, dx = \int y\, dx =$

$\int \sqrt{x+1}\, dx = \frac{(x+1)^{3/2}}{\frac{3}{2}}$

$+ C$.

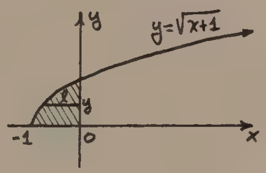
When $x = -1$, $A = 0$, so

$0 = C$. $A = \frac{2}{3}(x+1)^{3/2}$.

When $x = 0$, $A = \frac{2}{3}(1)^{3/2}$

$= \frac{2}{3}$ square units.

120.



$y = \sqrt{x+1}$, so

$y^2 = x + 1$, or

$x = y^2 - 1$.

$A = \int \ell\, dy = \int x\, dy =$

$\int (y^2 - 1)dy$

$= \frac{y^3}{3} - y + C$.

When $y = 0$, $A = 0$, so $C = 0$

$A = \frac{y^3}{3} - y$.

When $y = 1$, $A = \frac{1}{3} - 1 = \frac{2}{3}$ square units.

5 THE DEFINITE INTEGRAL

Problem Set 5.1, page 306

1. $\sum_{k=1}^{6} (2k + 1) = (2\cdot 1 + 1) + (2\cdot 2 + 1) + (2\cdot 3 + 1) + (2\cdot 4 + 1) + (2\cdot 5 + 1) + (2\cdot 6 + 1) = 3+5+7+9+11+13 = 48.$

2. $\sum_{k=1}^{5} 7k^2 = 7(1)^2 + 7(2)^2 + 7(3)^2 + 7(4)^2 + 7(5)^2 = 7+28+63+112+175 = 385.$

3. $\sum_{j=1}^{4} \sin \frac{\pi}{j} = \sin \pi + \sin \frac{\pi}{2} + \sin \frac{\pi}{3} + \sin \frac{\pi}{4}$

$= 0 + 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{3} + \sqrt{2}}{2}.$

4. $\sum_{j=3}^{7} \frac{j}{j - 2}$

$= \frac{3}{3 - 2} + \frac{4}{4 - 2} + \frac{5}{5 - 2} + \frac{6}{6 - 2} + \frac{7}{7 - 2}$

$= 3 + 2 + \frac{5}{3} + \frac{6}{4} + \frac{7}{5} = \frac{574}{60} = \frac{287}{30}.$

5. $\sum_{k=2}^{7} \frac{k}{k(k - 1)}$

$= \frac{1}{2(2-1)} + \frac{1}{3(3-1)} + \frac{1}{4(4-1)} + \frac{1}{5(5-1)} + \frac{1}{6(6-1)}$

$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}.$

6. $\sum_{i=-1}^{3} 3^i = 3^{-1} + 3^0 + 3^1 + 3^2 + 3^3$

$= \frac{1}{3} + 1 + 3 + 9 + 27 = \frac{121}{3}.$

7. $\sum_{j=0}^{3} \frac{1}{j^2 + 3} = \frac{1}{0^2+3} + \frac{1}{1^2+3} + \frac{1}{2^2+3} + \frac{1}{3^2+3}$

$= \frac{1}{3} + \frac{1}{4} + \frac{1}{7} + \frac{1}{12} = \frac{17}{21}.$

8. $\sum_{k=-1}^{3} \frac{k}{k + 2}$

$= \frac{-1}{-1 + 2} + \frac{0}{0 + 2} + \frac{1}{1 + 2} + \frac{2}{2 + 2} + \frac{3}{3 + 2}$

$= -1 + 0 + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} = \frac{13}{30}.$

9. $\sum_{k=1}^{19} (5a_k + 3b_k)$

$= 5 \sum_{k=1}^{19} a_k + 3 \sum_{k=1}^{19} b_k$

$= 5(23) + 3(99) = 412.$

10. $\sum_{k=1}^{19} (2a_k - 3b_k + 4c_k - 5)$

$= 2 \sum_{k=1}^{19} a_k - 3 \sum_{k=1}^{19} b_k + 4 \sum_{k=1}^{19} c_k - 5(19)$

$= 2(23) - 3(99) + 4(-14) - 5(19) = -402.$

11. $\sum_{k=1}^{50} (2k + 3)$

$= 2 \sum_{k=1}^{50} k + \sum_{k=1}^{50} 3 = \frac{2(50)(51)}{2} + 50(3)$

$= 2{,}700.$

12. $\sum_{i=1}^{30} i(i - 1) = \sum_{i=1}^{30} (i^2 - i)$

$= \sum_{i=1}^{30} i^2 - \sum_{i=1}^{30} i$

$= \frac{30(31)(61)}{6} - \frac{30(31)}{2} = 8{,}990.$

13. $\sum_{k=1}^{100} 5^k = \left(\sum_{k=0}^{100} 5^k\right) - 5^0$

$= \frac{1 - 5^{101}}{1 - 5} - 1 = \frac{5^{101} - 1 - 4}{4} = \frac{5^{101} - 5}{4}.$

14. $\sum_{k=0}^{100} (5^{k+1} - 5^k) = \sum_{k=0}^{100} 5^{k+1} - \sum_{k=0}^{100} 5^k$

$= 5 \sum_{k=0}^{100} 5^k - \sum_{k=0}^{100} 5^k = (5-1) \sum_{k=0}^{100} 5^k$

$= 4 \cdot \frac{(5^{101} - 1)}{5 - 1} = 5^{101} - 1.$

Notice that the terms add in such a way that successive terms cancel and we can tell the result without the formula:
$\sum_{k=0}^{100} (5^{k+1} - 5^k) = (5^1 - 5^0) + (5^2 - 5^1) +$
$(5^3 - 5^2) + \ldots + (5^{100} - 5^{99}) +$
$(5^{101} - 5^{100}) = 5^{101} - 5^0 = 5^{101} - 1.$

15. $\sum_{k=1}^{n} k(k + 1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$

$= \frac{n(n + 1)(2n + 1)}{6} + \frac{n(n + 1)}{2}$

$= \frac{n(n + 1)[2n + 1 + 3]}{6} = \frac{n(n + 1)(2n + 4)}{6}$

$= \frac{n(n + 1)(n + 2)}{3}.$

16. $\sum_{k=1}^{100} \left(\frac{1}{k} - \frac{1}{k + 1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$

$+ \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{99} - \frac{1}{100}\right) + \left(\frac{1}{100} - \frac{1}{101}\right)$

$= 1 - \frac{1}{101} = \frac{100}{101}.$

17. $\sum_{k=1}^{n-1} k^2 = \frac{(n - 1)(n - 1 + 1)[2(n - 1) + 1]}{6}$

$= \frac{(n - 1)(n)(2n - 1)}{6}.$

18. $\sum_{k=1}^{n} (a_k - a_{k-1}) = (a_1 - a_0) + (a_2 - a_1)$

$+ (a_3 - a_2) + \ldots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1})$

$= a_n - a_0.$

19. $\sum_{k=1}^{n} (k - 1)^2$. Put $j = k - 1$, so that

$\sum_{k=1}^{n} (k - 1)^2 = \sum_{j=0}^{n-1} j^2 = \sum_{j=1}^{n-1} j^2.$

The latter equality holds since $j^2 = 0$ when $j = 0$. Now, $\sum_{j=1}^{n-1} j^2$

$$= \frac{(n-1)[(n-1)+1][2(n-1)+1]}{6}$$

$$= \frac{(n-1)(n)(2n-1)}{6}.$$ Therefore,

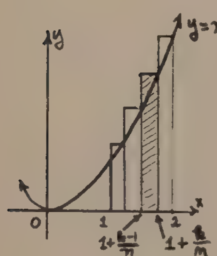
$$\sum_{k=1}^{n} (k-1)^2 = \frac{(n-1)(n)(2n-1)}{6}.$$

20. $$\sum_{j=1}^{100} \frac{1}{10^j} = \sum_{j=1}^{100} \left(\frac{1}{10}\right)^j$$

$$= \frac{1 - (\frac{1}{10})^{101}}{1 - (\frac{1}{10})} - \frac{1}{10^0}$$

$$= \frac{10(1 - \frac{1}{10^{101}})}{9} - 1 = \frac{1 - \frac{1}{10^{100}}}{9}$$

$$= \frac{10^{100} - 1}{9(10^{100})}.$$

21. 

Divide interval from [1,2] into n congruent subintervals each of length $\frac{2-1}{n} = \frac{1}{n}$. Area of k^{th} circumscribed rectangle is $\frac{1}{n}(1 + \frac{k}{n})^2$ so $A \approx \sum_{k=1}^{n} \frac{1}{n}(1 + \frac{k}{n})^2$. Area of the k^{th} inscribed rectangle is $\frac{1}{n}\left[1 + \frac{k-1}{n}\right]^2$.

$A \approx \sum_{k=1}^{n} \frac{1}{n}(1 + \frac{k-1}{n})^2$. Now,

$$\sum_{k=1}^{n} \frac{1}{n}(1 + \frac{k}{n})^2 = \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right)$$

$$= 1 + \frac{2}{n^2}\,\frac{(n)(n+1)}{2} + \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}.$$

Thus, $1 + \frac{n+1}{n} + \frac{2n^2 + 3n + 1}{6n^2} \geq A$.

Now $\sum_{k=1}^{n} \frac{1}{n}(1 + \frac{k-1}{n})^2 = \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{2(k-1)}{n^2} + \frac{(k-1}{3}\right.$

$$= 1 + \frac{2}{n^2}\sum_{\ell=1}^{n-1} \ell + \frac{1}{n^3}\sum_{\ell=1}^{n-1} 1^2$$

$$= \frac{2}{n^2}\,\frac{(n-1)n}{2} + \frac{1}{n^3}\,\frac{(n-1)(n)(2n-1)}{6}$$

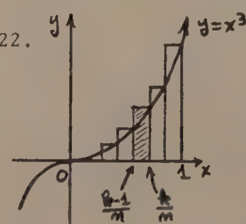
$$= 1 + \frac{n-1}{n} + \frac{2n^2 - 3n + 1}{6n^2} \leq A.$$ So,

$$1 + \frac{n-1}{n} + \frac{2n^2-3n+1}{6n^2} \leq A \leq 1 + \frac{n+1}{n} + \frac{2n^2+3n+1}{6n^2}$$

or $\frac{7}{3} - \frac{3}{2n} + \frac{1}{6n^2} \leq A \leq \frac{7}{3} + \frac{3}{2n} + \frac{1}{6n^2}$

as $n \to \infty$ both $\frac{7}{3} - \frac{3}{2n} + \frac{1}{6n^2}$ and

$\frac{7}{3} + \frac{3}{2n} + \frac{1}{6n^2} \to \frac{7}{3}$. Thus, $A = \frac{7}{3}$.

22. 

Divide interval [0,1] into n congruent parts $\frac{1-0}{n} = \frac{1}{n}$. Area of the k^{th} circumscribed rectangle is $\frac{1}{n}(\frac{k}{n})^3$. So $A \approx \sum_{k=1}^{n} \frac{1}{n}(\frac{k}{n})^3$. Area of k^{th} inscribed rectangle is $\frac{1}{n}\left(\frac{k-1}{n}\right)^3$.

So $A \approx \sum_{k=1}^{n} \frac{1}{n}\left(1 + \frac{k-1}{n}\right)^2$.

Now $\sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^3 = \frac{1}{n^4} \sum_{k=1}^{n} k^3$

$= \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \frac{(n+1)^2}{4n^2}$ and

$\sum_{k=1}^{n} \frac{1}{n}\left(\frac{k-1}{n}\right)^3 = \frac{1}{n^4} \sum_{k=1}^{n} (k-1)^3$

$= \frac{1}{n^4} \sum_{\ell=1}^{n-1} \ell^3 = \frac{1}{n^4} \frac{(n-1)^2 n^2}{4} = \frac{(n-1)^2}{4n^2}$.

Thus, $\frac{(n-1)^2}{4n^2} \le A \le \frac{(n+1)^2}{4n^2}$

or $\frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2} \le A \le \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} \longrightarrow \frac{1}{4}$.

As $n \to \infty$, both $\frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2}$ and

$\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} \to \frac{1}{4}$. Therefore, $A = \frac{1}{4}$.

23. $S = 1 + 2 + 3 + \ldots + (n-1) + n$, or

$S = n + (n-1) + (n-2) + \ldots + 2 + 1$.

Adding we get,

$2S = (n+1) + (n+1) + \ldots + (n+1)$ n terms

$2S = n(n+1)$, or $S = \frac{n(n+1)}{2}$.

24. We want to show that $\sum_{k=1}^{n} k^2$

$= \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$. If $n = 1$,

$\sum_{k=1}^{n} 1^2 = 1$. But $\frac{1(1+1)(2+1)}{6} = \frac{2\cdot 3}{6} = 1$.

So the equality holds when $n = 1$. Now,

assume $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and prove

$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$. $\sum_{k=1}^{n+1} k^2$

$= \sum_{k=1}^{n} k^2 + (n+1)^2$

$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$

$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$

$= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$

and we are done.

25. $\sum_{k=1}^{n} (b_k - b_{k-1}) = (b_1 - b_0) + (b_2 - b_1)$

$+ (b_3 - b_2) + \ldots + (b_n - b_{n-1})$

$= -b_0 + (b_1 - b_1) + (b_2 - b_2) + \ldots + (b_{n-1} - b_{n-1}) + b_n$

$= b_n - b_0$.

26. We want to show that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

for all $n \geq 1$. When $n = 1$, $\sum_{k=1}^{1} k^3 = 1^3 = 1$.

But $\frac{1^2(1+1)^2}{4} = \frac{1\cdot 2^2}{4} = 1$. So the equality

holds for $n = 1$. Assume $\sum_{k=1}^{n} k^3$

$= \frac{n^2(n+1)^2}{4}$ and show $\sum_{k=1}^{n+1} k^3$

$= \frac{(n+1)^2(n+2)^2}{4}$. $\sum_{k=1}^{n+1} k^3$

$= \left(\sum_{k=1}^{n} k^3\right) + (n+1)^3$

$= \frac{(n^2)(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2[n^2+4(n+1)]}{4}$

$= \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4}$.

27. Let $b_k = k^2$, then $b_k - b_{k-1} = k^2 - (k-1)^2$

$= k^2 - k^2 + 2k - 1 = 2k - 1$. Then from Problem 25,

$\sum_{k=1}^{n} (b_k - b_{k-1}) = \sum_{k=1}^{n} (2_{k-1}) = b_n - b_0$

$= n^2 - 0^2 = n^2$.

28. Show $\sum_{k=0}^{n} C^k = \frac{1-C^{n+1}}{1-C}$, $C \neq 0,1$ for all $n \geq 1$. When $n = 1$, $\sum_{k=0}^{n} C^k = C^0 + C^1 = 1 + C$. But $\frac{1-C^2}{1-C} = 1 + C$.

So the equality holds for $n = 1$. Assume $\sum_{k=0}^{n} C^k = \frac{1-C^{n+1}}{1-C}$ and show $\sum_{k=0}^{n+1} C^k = \frac{1-C^{n+2}}{1-C}$. $\sum_{k=0}^{n+1} C^k = \sum_{k=0}^{n} C^k + C^{n+1} = \frac{1-C^{n+1}}{1-C} + C^{n+1} = \frac{1-C^{n+2}}{1-C}$ and we are done.

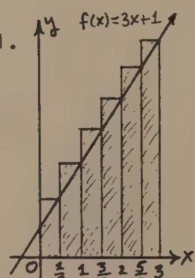
29. By the result of Problem 27, $\sum_{k=1}^{m} (2k-1) = n^2$. Hence, $2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^2$, $2\sum_{k=1}^{n} k - n = n^2$, $2\sum_{k=1}^{n} k = n^2 + n$, $\sum_{k=1}^{n} k = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$.

30. We want to show that $\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|$ for all $n \geq 1$. When $n = 1$, equality holds, since $|a_1| = |a_1|$. Suppose that $\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|$. We will show $\left|\sum_{k=1}^{n+1} a_k\right| \leq \sum_{k=1}^{n+1} |a_k|$. $\left|\sum_{k=1}^{n+1} a_k\right| = \left|\left(\sum_{k=1}^{n} a_k\right) + a_{n+1}\right| \leq \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}|$ (by the triangle inequality) $\leq \sum_{k=1}^{n} |a_k| + |a_{n+1}|$ (by our hypothesis) $= \sum_{k=1}^{n+1} |a_k|$. Hence, $\left|\sum_{k=1}^{n+1} a_k\right| \leq \sum_{k=1}^{n+1} |a_k|$ and we are done.

31. $S = \sum_{k=0}^{n} C^k = 1 + C + C^2 + \ldots + C^n$, $C \neq 0,1$. $SC = C + C^2 + \ldots + C^n + C^{n+1}$. Thus, $S - SC = 1 - C^{n+1}$, $S(1-C) = 1 - C^{n+1}$, $S = \frac{1-C^{n+1}}{1-C}$.

Problem Set 5.2, page 315

1.

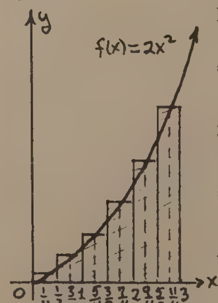


$\sum_{k=1}^{6} f(c_k)\Delta x_k$

$= [f(c_1)+f(c_2)+f(c_3)+f(c_4)+f(c_5)+f(c_6)]\cdot\frac{1}{2} = \left\{[3(\tfrac{1}{2})+1] + [3(1)+1] + [3(\tfrac{3}{2})+1] + [3(2)+1] + [3(\tfrac{5}{2})+1] + [3(3)+1]\right\}\cdot\frac{1}{2}$

$= \left[\frac{5}{2} + 4 + \frac{11}{2} + 7 + \frac{17}{2} + 10\right]\cdot\frac{1}{2} = \frac{75}{4}$.

2.

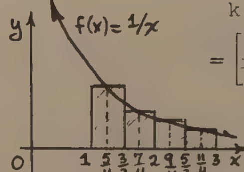


$\sum_{k=1}^{6} f(c_k)\Delta x_k$

$= \frac{1}{2}\cdot\left[f(\tfrac{1}{4}) + f(\tfrac{3}{4}) + f(\tfrac{5}{4}) + f(\tfrac{7}{4}) + f(\tfrac{9}{4}) + f(\tfrac{11}{4})\right]$

$= \frac{1}{2}\left[\frac{1}{8} + \frac{9}{8} + \frac{25}{8} + \frac{49}{8} + \frac{81}{8} + \frac{121}{8}\right] = \frac{1}{2}\left(\frac{286}{8}\right) = \frac{143}{8}$.

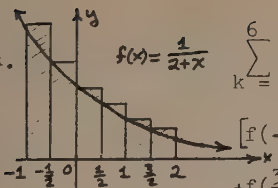
3.



$\sum_{k=1}^{4} f(c_k)\Delta x_k$

$= \left[f(\tfrac{5}{4}) + f(\tfrac{7}{4}) + f(\tfrac{9}{4}) + f(\tfrac{11}{4})\right]\cdot\frac{1}{2} = \left(\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11}\right)\cdot\frac{1}{2}$

$= \frac{3776}{3465} \approx 1.09$.

4. 

$\sum_{k=1}^{6} f(c_k)\,\Delta x_k = \left[f(-1)+f(-\tfrac{1}{2})+f(0)+f(\tfrac{1}{2})+f(1)+f(\tfrac{3}{2})\right]\cdot\frac{1}{2}$

$= (\frac{1}{1}+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\frac{2}{7})\cdot\frac{1}{2} = \frac{669}{420} \approx 1.59.$

5. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$. Since we are using right endpoints, $c_1 = \frac{3}{n}$, $c_2 = \frac{6}{n}$, $c_3 = \frac{9}{n}$, ... $c_n = \frac{3n}{n}$, so that $c_k = \frac{3k}{n}$. Here, $\frac{b-a}{n} = \frac{3}{n} = \Delta x_k$.

$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} 2(\frac{3k}{n})\cdot\frac{3}{n} = \frac{18}{n^2}$.

$\sum_{k=1}^{n} k = \frac{18}{n^2}\cdot\frac{n(n+1)}{2} = 9(\frac{n^2}{n^2}+\frac{n}{n^2}) = 9(1+\frac{1}{n}).$

So $\int_0^3 2x\,dx = \lim_{n\to+\infty} 9(1+\frac{1}{n}) = 9.$

6. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$. Since we are using left endpoints, $c_1 = 0$, $c_2 = \frac{3}{n}$, $c_3 = \frac{6}{n}$, . . . , $c_n = 3 - \frac{3}{n} = \frac{3n-3}{n}$, so that $c_k = \frac{3(k-1)}{n}$. Here, $\frac{b-a}{n} = \frac{3}{n} = \Delta x_k$. So $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$

$= \sum_{k=1}^{n} 2(\frac{3(k-1)}{n})\frac{3}{n} = \frac{18}{n^2}\sum_{k=1}^{n}(k-1)$

$= \frac{18}{n^2}\left[(\sum_{k=1}^{n} k) - n\right] = \frac{18}{n^2}(\frac{n(n+1)}{2} - \frac{2n}{2})$

$= 9(\frac{n^2}{n^2} - \frac{n}{n^2}) = 9(1-\frac{1}{n})$. So $\int_0^3 2x\,dx$

$= \lim_{n\to\infty} 9(1-\frac{1}{n}) = 9.$

7. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$. Since we are using left endpoints, $c_1 = 4$, $c_2 = 4+\frac{3}{n}$, ... , $c_n = 4+\frac{3(n-1)}{n}$, so that $c_k = 4+\frac{3(k-1)}{n}$. Here, $\frac{b-a}{n} = \frac{3}{n} = \Delta x_k$.

So $\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n}\left[2(4+\frac{3(k-1)}{n})-6\right]\frac{3}{n}$

$= \frac{6}{n}\sum_{k=1}^{n}\left[1+\frac{3(k-1)}{n}\right] = \frac{6}{n}\left[n+\frac{3(n-1)}{2}\right]$

$= 6 + 9\,\frac{n-1}{n} = 6 + 9(1-\frac{1}{n}) = 15 - \frac{9}{n}.$

So $\int_4^7 (2x-6)\,dx = \lim_{n\to\infty}(15-\frac{9}{n}) = 15.$

8. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$. Let $c_1 = 1+\frac{2}{n}$, $c_2 = 1+\frac{4}{n}$, ... , $c_n = 1+\frac{2n}{n}$, and $c_k = 1+\frac{2k}{n}$. Here, $\frac{b-a}{n} = \frac{2}{n} = \Delta x_k$.

So $\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n}\left[9-(1+\frac{2k}{n})^2\right]\frac{2}{n}$

$= \frac{2}{n}\sum_{k=1}^{n}(8 - \frac{4k}{n} - \frac{4k^2}{n^2})$

$= \frac{2}{n}\cdot 8n - \frac{8}{n^2}(\frac{(n)(n+1)}{2}) - \frac{8}{n^3}(\frac{(n)(n+1)(2n+1)}{6})$

$= 16 - 4 - \frac{8}{2n} - \frac{8}{3} - \frac{4}{n} - \frac{4}{3n^2}$

$= \frac{28}{3} - \frac{8}{2n} - \frac{4}{n} - \frac{4}{3n^2}$. Hence, $\int_1^3 (9-x^2)\,dx$

$= \lim_{n\to+\infty}(\frac{28}{3} - \frac{8}{2n} - \frac{4}{n} - \frac{4}{3n^2}) = \frac{28}{3}.$

9. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$. Let $c_1 = -2+\frac{1}{n}$, $c_2 = -2+\frac{2}{n}$, ... $c_n = -2+\frac{n}{n}$; hence, $c_k = -2+\frac{k}{n}$. Here, $\frac{b-a}{n} = \frac{1}{n} = \Delta x_k$.

So $\sum_{k=1}^{n} f(c_k)\,\Delta x_k =$

$\sum_{k=1}^{n}\left[(-2+\frac{k}{n})^2 - (-2+\frac{k}{n}) - 2\right]\frac{1}{n}$

$= \frac{1}{n}\sum_{k=1}^{n}\left[4 - \frac{4k}{n} + \frac{k^2}{n^2} + 2 - \frac{k}{n} - 2\right]$

$= \frac{1}{n}\left[4n - \frac{5}{n}\,\frac{n(n+1)}{2} + \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$

$= \frac{11}{6} - \frac{2}{n} + \frac{1}{6n^2}$. Hence, $\int_{-2}^{-1}(x^2 - x - 2)dx$

$= \lim_{n\to+\infty}\left(\frac{11}{6} - \frac{2}{n} + \frac{1}{6n^2}\right) = \frac{11}{6}$.

10. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$. Let

$c_1 = \frac{2}{n}$, $c_2 = \frac{4}{n}$, ... , $c_n = \frac{2n}{n}$; hence,

$c_k = \frac{2k}{n}$. Here, $\frac{b-a}{n} = \frac{2}{n} = \Delta x_k$. So,

$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n}\left((\frac{2k}{n})^3 + 2\right)\frac{2}{n}$

$= \frac{16}{n^4}\left(\sum_{k=1}^{n} k^3\right) + 4 = \frac{16}{n^4}\,\frac{n^2(n+1)^2}{4} + 4$

$= \frac{4(n^2+2n+1)}{n^2} + 4 = 4 + \frac{8}{n} + \frac{4}{n^2} + 4.$

Hence, $\int_0^2 (x^3+2)dx = \lim_{n\to+\infty}\left(8 + \frac{8}{n} + \frac{4}{n^2}\right) = 8.$

11. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$. Let

$c_1 = 0$, $c_2 = \frac{2}{n}$, $c_3 = \frac{4}{n}$, ... , $c_n = 2 - \frac{2}{n}$

$= \frac{2n-2}{n}$; hence, $c_k = \frac{2k-2}{n}$. Here $\frac{b-a}{n}$

$= \frac{2}{n} = \Delta x_k$. So $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$

$= \sum_{k=1}^{n}\left[(\frac{2k-2}{n})^3 + 2\right]\frac{2}{n} = \frac{2}{n}\left(\sum_{k=1}^{n}\frac{8(k-1)^3}{n^3}\right) + 4$

$= \frac{16}{n^4}\left(\sum_{j=0}^{n-1} j^3\right) + 4 = \frac{16}{n^4}\left(\sum_{j=1}^{n-1} j^3\right) + 4$

$= \frac{16}{n^4}\,\frac{(n-1)^2(n)^2}{4} + 4 = \frac{4}{n^2}(n^2 - 2n + 1) + 4$

$= 8 - \frac{8}{n} + \frac{4}{n^2}$. Hence, $\int_0^2 (x^3+2)dx$

$= \lim_{n\to+\infty}\left(8 - \frac{8}{n} + \frac{4}{n^2}\right) = 8.$

12. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$.

$\frac{b-a}{n} = \frac{1}{n} = \Delta x_k$, $c_1 = -2 + \frac{1}{n}$, $c_2 = -2 + \frac{2}{n}$,

... , $c_n = -2 + \frac{n}{n}$; hence, $c_k = -2 + \frac{k}{n}$.

So, $\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n}\left[4 - (-2+\frac{k}{n})^2\right]\frac{1}{n}$

$= \frac{1}{n}\sum_{k=1}^{n}\left(4 - 4 + \frac{4k}{n} - \frac{k^2}{n^2}\right) = \frac{4}{n^2}\sum_{k=1}^{n} k -$

$\frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{4}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$

$= \frac{2}{n}(n+1) - \frac{1}{6n^2}(n+1)(2n+1)$

$= 2(1 + \frac{1}{n}) - \frac{1}{6}(2 + \frac{3}{n} + \frac{1}{n^2}) = \frac{5}{3} + \frac{5}{2n} + \frac{1}{6n^2}$

Hence, $\int_{-2}^{-1}(4 - x^2)dx = \lim_{n\to+\infty}\left(\frac{5}{3} + \frac{5}{2n} + \frac{1}{6n^2}\right) = \frac{5}{3}.$

13. We want to find $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$.

$\frac{b-a}{n} = \frac{3}{n} = \Delta x_k$. $c_1 = -3$, $c_2 = -3 + \frac{3}{n}$,

$c_3 = -3 + \frac{6}{n}$, ... , $c_n = -3 + \frac{3}{n}(n-1)$.

Hence, $c_k = -3 + \frac{3(k-1)}{n}$. So $\sum_{k=1}^{n} f(c_k)\Delta x_k$

$= \sum_{k=1}^{n}\left[1 - 2(-3 + \frac{3(k-1)}{n})^2\right]\frac{3}{n}$

$= \frac{3}{n}\sum_{k=1}^{n}\left[-17 + \frac{36}{n}(k-1) - \frac{18}{n^2}(k-1)^2\right]$

$$= \frac{3}{n}\left[-17 + \frac{36}{n}\frac{(n-1)(n)}{2} - \frac{18}{n^2}\frac{(n-1)(n)(2n-1)}{6}\right]$$

$$= -51 + 54\left(1 - \frac{1}{n}\right) - 9\left(2 - \frac{3}{n} + \frac{1}{n^2}\right)$$

$= -15 - \frac{27}{n} - \frac{9}{n^2}$. Hence, $\int_{-3}^{0} (1 - 2x^2)dx$

$$= \lim_{n\to+\infty} \left(-15 - \frac{27}{n} - \frac{9}{n^2}\right) = -15.$$

4. (a) Find $\sum_{k=1}^{n} f(c_k)\Delta x_k$.

$b - a = \frac{1}{n} = \Delta x_k$. $c_1 = 1$, $c_2 = 1 + \frac{1}{n}$,

$c_3 = 1 + \frac{2}{n}$, ... , $c_n = 1 + \frac{n-1}{n}$. So

$c_k = 1 + \frac{k-1}{n}$. $\sum_{k=1}^{n} f(c_k)\Delta x_k$

$$= \sum_{k=1}^{n}\left[\left(1 + \frac{k-1}{n}\right)^2 - 4\left(1 + \frac{k-1}{n}\right) + 2\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left[-1 - \frac{2}{n}(k-1) + \frac{(k-1)^2}{n^2}\right]$$

$$= \frac{1}{n}\left[-n - \frac{2}{n}\frac{(n-1)n}{2} + \frac{(n-1)n(2n-1)}{6n^2}\right]$$

$$= -1 - \left(1 - \frac{1}{n}\right) + \frac{1}{6}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right) =$$

$= -\frac{5}{3} + \frac{1}{2n} + \frac{1}{6n^2}$. Hence, $\int_1^2 (x^2-4x+2)dx$

$$= \lim_{n\to+\infty} \left(-\frac{5}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = -\frac{5}{3}.$$

(b) $c_1 = 1 + \frac{1}{n}$, $c_2 = 1 + \frac{2}{n}$, ... , $c_n = 1 + \frac{n}{n}$.

So $c_k = 1 + \frac{k}{n}$. $\sum_{k=1}^{n} f(c_k)\Delta x_k$

$$= \sum_{k=1}^{n}\left[\left(1 + \frac{k}{n}\right)^2 - 4\left(1 + \frac{k}{n}\right) + 2\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left[-1 - \frac{2k}{n} + \frac{k^2}{n^2}\right]$$

$$= \frac{1}{n}\left[-n - \frac{2}{n}\frac{n(n+1)}{2} + \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= -1 - \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$= -\frac{5}{3} - \frac{1}{2n} + \frac{1}{6n^2}$. Hence,

$\int_1^2 (x^2 - 4x + 2)dx$

$$= \lim_{n\to+\infty} \left(-\frac{5}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = -\frac{5}{3}.$$

15. It exists since $f(x) = \frac{1}{x}$ is continuous on $[1, 1{,}000]$.

16. It does not exist, since $f(x) = \frac{1}{x}$ is not defined at $x = 0$.

17. It exists since $f(x) = |x|$ is continuous on $[-1,1]$.

18. It does not exist, since $f(x) = \frac{x+1}{\sqrt{x}}$ is not defined at $x = 0$.

19. It exists, since $f(x) = x^4$ is continuous on $[0,1]$.

20. It exists since $f(x) = [\![x]\!]$ is piecewise continuous and bounded on $[1,100]$.

21. It exists since f is piecewise continuous and bounded on $[0,3]$.

22. It exists since f is piecewise continuous and bounded on $[0,2]$.

23. It does not exist since $f(x) = \tan x$ is not defined at $x = \frac{\pi}{2}$.

24. It exists since $f(x) = \sec x$ is continuous on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

25. It exists since $f(x) = \cos 2x$ is continuous on $[0, \pi]$.

26. It exists since $f(x) = \sin |x|$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

27. It does not exist since $f(x) = \tan x$ is not defined at $x = \frac{\pi}{2}$.

28. It exists since $f(x) = \sin^2 x$ is continuous on $[-\pi, \pi]$.

29. $\int_3^0 2x\,dx = -\int_0^3 2x\,dx = -9.$

30. $\int_3^1 (9-x^2)dx = -\int_1^3 (9-x^2)dx = -\frac{28}{3}.$

31. $\int_{-1}^{-2} (x^2-x-2)dx = -\int_{-2}^{-1}(x^2-x-2)dx = -\frac{11}{6}.$

32. $\int_{-1}^{-2} (4-x^2)dx = -\int_{-2}^{-1}(4-x^2)dx = -\frac{5}{3}.$

33. $\int_3^3 (x^4 - 2x^3 + x^2 - x + 5)dx = 0.$

34. $\int_{\pi}^{\pi} \tan x\,dx = 0.$

35. Let $[x_0,x_1], [x_1,x_2], \ldots, [x_{n-1},x_n]$ be a partition of $[1,2]$ and put $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_k = x_k - x_{k-1}, \ldots, \Delta x_n = x_n - x_{n-1}$. Augment this partition by choosing c_k with $x_{k-1} \leq c_k \leq x_k$ for $k = 1,2,\ldots,n$. The corresponding Riemann sum is $\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} 7\,\Delta x_k = 7\sum_{k=1}^{n} \Delta x_k$. Note that $\sum_{k=1}^{n} \Delta x_k = \Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots + \Delta x_n = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = 2 - 1 = 1$. Hence, $\sum_{k=1}^{n} f(c_k)\Delta x_k = 7(1) = 7$, so that

$$\int_1^2 7\,dx = \lim_{\|\mathcal{P}\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = 7.$$

Geometrically, it simply means that a rectangle of height 7 with base 1 has area 7 square units.

36. Here $\Delta x_k = \frac{1}{10}$, $c_1 = 1$, $c_2 = 1 + \frac{1}{10}$, $c_3 = 1 + \frac{2}{10}, \ldots$, and $c_n = 1 + \frac{9}{10}$; hence, $c_k = 1 + \frac{k-1}{10}$. So $\int_1^2 \frac{1}{x}\,dx$

$$\approx \sum_{k=1}^{10} \frac{1}{c_k}\Delta x_k = \sum_{k=1}^{10} \frac{1}{1+\frac{k-1}{10}}\left(\frac{1}{10}\right) = \sum_{k=1}^{10} \frac{1}{10+k-1} = \sum_{k=1}^{10} \frac{1}{9+k} \approx 0.719.$$

37. With $f(x) = 1$ for all values of x, we have $\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} 1\,\Delta x_k = \sum_{k=1}^{n} \Delta x_k = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = b - a$. Thus,

$$\int_a^b 1\,dx = \lim_{\|\mathcal{P}\|\to 0} (b-a) = b - a.$$

38. Suppose $L_1 \neq L_2$ and let $\varepsilon = \frac{1}{2}|L_1 - L_2|$. Then there exists $\delta > 0$ such that $\left|\sum_{k=1}^{n} f(c_k)\Delta x_k - L_1\right| < \varepsilon$ and $\left|\sum_{k=1}^{n} f(c_k)\Delta x_k - L_2\right| < \varepsilon$ whenever $\|\mathcal{P}\| < \delta$. Select an augmented partition with norm less than δ and let $S = \sum_{k=1}^{n} f(c_k)\Delta x_k$ be its corresponding Riemann sum. Then $|S - L_1| < \varepsilon$ and $|S - L_2| < \varepsilon$. Therefore, $2\varepsilon = |L_1 - L_2| = |L_1 - S + S - L_2| \leq |L_1 - S| + |S - L_2| = |L_1 - S| + |L_2 - S| < \varepsilon + \varepsilon = 2\varepsilon$; that is, $2\varepsilon < 2\varepsilon$. This is impossible, so there cannot be two such different numbers L_1 and L_2.

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1. $\int_3^4 2dx = 2(4 - 3) = 2(1) = 2.$

2. $\int_{-5}^4 (7 + \pi)dx = (7 + \pi)\left[4 - (-5)\right] = 9(7+\pi).$

3. $\int_{-2}^7 (-dx) = \int_{-2}^7 (-1)dx = (-1)\left[7-(-2)\right]$

$= (-1)9 = -9.$

4. $\int_{0.5}^{0.75} (1 + \sqrt{2} - \sqrt{3})dx = (1+\sqrt{2}-\sqrt{3})\left[\frac{3}{4} - \frac{1}{2}\right]$

$= \frac{1 + \sqrt{2} - \sqrt{3}}{4}.$

5. $\int_2^1 dx = -\int_1^2 1dx = -1(2 - 1) = -1.$

6. $\int_{-2}^{-4} (-4)dx = -\int_{-4}^{-2} (-4)\, dx$

$= \int_{-4}^{-2} 4dx = 4\left[-2 - (-4)\right] = 8.$

7. $\int_{-\pi}^{\pi} 2dx = 2\left[\pi - (-\pi)\right] = 4\pi.$

8. $\int_{\pi}^{\pi} 2dx = 0.$

9. $\int_1^3 5xdx = 5\int_1^3 xdx = 5 \cdot \frac{1}{2}(3^2 - 1^2)$

$= \frac{5}{2}(8) = 20.$

10. $\int_3^{-2}(-3x)dx = -\int_{-2}^3(-3x)dx = 3\int_{-2}^3 xdx$

$= 3 \cdot \frac{1}{2}\left[3^2 - (-2)^2\right] = \frac{3}{2}(9 - 4) = \frac{15}{2}.$

11. $\int_5^1 (-2x)dx = - {}_1^5(-2x)dx = 2\int_1^5 xdx$

$= 2 \cdot \frac{1}{2}\left[5^2 - 1^2\right] = 24.$

12. $\int_1^{-1} (4 - 3x)\, dx = -\int_{-1}^1 (4 - 3x)\, dx$

$= \int_{-1}^1 (3x - 4)\, dx = 3\int_{-1}^1 xdx - \int_{-1}^1 4dx$

$= 3 \cdot \frac{1}{2}\left[1^2 - (-1)^2\right] - 4\left[1 - (-1)\right]$

$= 0 - 4(2) = -8.$

13. $\int_{-2}^3 (2x + 1)\, dx = 2\int_{-2}^3 xdx + \int_{-2}^3 1\, dx$

$= 2 \cdot \frac{1}{2}\left[3^2 - (-2)^2\right] + 1\left[3 - (-2)\right]$

$= (9 - 4) + (5) = 10.$

14. $\int_{\frac{2}{3}}^{\frac{3}{4}} (\frac{x}{2} - 1)\, dx = \frac{1}{2}\int_{\frac{2}{3}}^{\frac{3}{4}} xdx - \int_{\frac{2}{3}}^{\frac{3}{4}} 1dx$

$= \frac{1}{2} \cdot \frac{1}{2}\left[(\frac{3}{4})^2 - (\frac{2}{3})^2\right] - 1\left[\frac{3}{4} - \frac{2}{3}\right]$

$= \frac{1}{4}(\frac{9}{16} - \frac{4}{9}) - 1(\frac{1}{12}) = \frac{-31}{576}.$

15. $\int_1^2 (x + x^2)\, dx = \int_1^2 xdx + \int_1^2 x^2dx$

$= \frac{1}{2}(2^2 - 1^2) + \frac{1}{3}(2^3 - 1^3) = \frac{1}{2}(3) + \frac{1}{3}(7)$

$= \frac{23}{6}.$

16. $\int_2^1 (x^2 - 1)dx = -\int_1^2 (x^2 - 1)\, dx$

$= \int_1^2 (1 - x^2)\, dx = \int_1^2 1dx - \int_1^2 x^2dx$

$= 1(2 - 1) - \frac{1}{3}(2^3 - 1^3) = 1 - \frac{1}{3}(7) = -\frac{4}{3}.$

17. $\int_{-2}^3(3x^2 - 2x + 1)dx = 3\int_{-2}^3 x^2dx - 2\int_{-2}^3 xdx$

$+ \int_{-2}^3 1dx = 3 \cdot \frac{1}{3}\left[3^3-(-2)^3\right]$

$- 2 \cdot \frac{1}{2}\left[3^2 - (-2)^2\right] + 1\left[3 - (-2)\right]$

$= 27 + 8 - (9 - 4) + 5 = 35.$

18. $\int_{-2}^{-3}(-2x^2 + 4x + 5)dx = -\int_{-3}^{-2}(-2x^2+4x+5)dx$

$= \int_{-3}^{-2} (2x^2 - 4x - 5)\, dx$

$= 2\,\frac{1}{3}\left[(-2)^3-(-3)^3\right]-4\cdot\frac{1}{2}\left[(-2)^2-(-3)^2\right]$

$- 5\left[-2-(-3)\right] = \frac{2}{3}(-8+27)-2(4-9)-5(1) = \frac{53}{3}.$

19. $\int_3^{-2}(3x - 1)(2x + 3)\, dx$

$= -\int_{-2}^3 (6x^2 + 7x - 3)\, dx$

$= -6\,\frac{1}{3}\left[3^3 - (-2)^3\right]-7\cdot\frac{1}{2}\left[3^2-(-2)^2\right]+3\left[3-(-2)\right]$

$= -2(35) - \frac{7}{2}(5) + 3(5) = -\frac{145}{2}.$

20. $\int_1^{-1} (2x+1)^2dx = -\int_{-1}^1 (4x^2+4x+1)dx$

$= -4\,\frac{1}{3}\left[1^3-(-1)^3\right]-4\cdot\frac{1}{2}\left[1^2-(-1)^2\right]-1\left[1-(-1)\right]$

$= -\frac{4}{3}(2) - 2(0) - 1(2) = -\frac{14}{3}.$

21. $\int_3^{-2} 4x(2x-7)dx = -\int_{-2}^{3} (8x^2-28x)dx$

$= -8\cdot\frac{1}{3}\left[3^3-(-2)^3\right] + 28\cdot\frac{1}{2}\left[3^2-(-2)^2\right]$

$= -\frac{8}{3}(35) + 14(5) = -\frac{70}{3}.$

22. $\int_1^0 \frac{x^2-25}{x-5}\,dx = -\int_0^1 (x+5)dx$

$= -\frac{1}{2}\left[1^2 - 0^2\right] - 5(1-0)$

$= -\frac{1}{2} - 5 = -\frac{11}{2}.$

23. $\int_{-1}^{0} xdx + \int_0^1 xdx = \int_{-1}^{1} xdx$

$= \frac{1}{2}\left[1^2 - (-1)^2\right] = \frac{1}{2}(0) = 0.$

24. $\int_{-1}^{a} xdx = \int_a^1 xdx = \int_{-1}^{1} xdx = 0$. (see Problem 23).

25. $\int_0^{\pi} (2x-1)dx + \int_{\pi}^{4} (2x-1)\,dx$

$= \int_0^4 (2x-1)\,dx = 2\cdot\frac{1}{2}\left[4^2 - 0^2\right] - 1(4-0)$

$= 16 - 4 = 12.$

26. $\int_{-1}^{a} x^2dx - \int_1^a x^2dx$

$= \int_{-1}^{a} x^2dx - \left[-\int_a^1 x^2dx\right]$

$= \int_{-1}^{a} x^2dx + \int_a^1 x^2dx = \int_{-1}^{1} x^2dx$

$= \frac{1}{3}\left[1^3 - (-1)^3\right] = \frac{1}{3}(2) = \frac{2}{3}.$

27. $\int_{-2}^{1} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_0^1 |x|\,dx$

$= \int_{-2}^{0} (-x)dx + \int_0^1 xdx$

$= -\frac{1}{2}\left[0^2-(-2)^2\right] + \frac{1}{2}\left[1^2 - 0^2\right]$

$= -\frac{1}{2}(-4) + \frac{1}{2} = \frac{5}{2}.$

28. $\int_{-1}^{2} |x-1|\,dx = \int_{-1}^{1} |x-1|\,dx + \int_1^2 |x-1|\,dx$

$= \int_{-1}^{1} (1-x)\,dx + \int_1^2 (x-1)\,dx$

$= 1\left[1-(-1)\right] - \frac{1}{2}\left[1^2-(-1)^2\right] + \frac{1}{2}\left[2^2-1^2\right] - 1\left[2-1\right]$

$= 2 - \frac{1}{2}(0) + \frac{1}{2}(3) - 1 = \frac{5}{2}.$

29. $\int_{-1}^{2} |x^3|\,dx = \int_{-1}^{0} |x^3|\,dx + \int_0^2 |x^3|\,dx$

$= \int_{-1}^{0} (-x^3)\,dx + \int_0^2 x^3dx$

$= -\frac{1}{4}\left[0^4-(-1)^4\right] + \frac{1}{4}\left[2^4-0^4\right]$

(using $\int_a^b x^3dx = \frac{1}{4}(b^4 - a^4)$)

$= -\frac{1}{4}(-1) + \frac{1}{4}(16)$

$= \frac{17}{4}.$

30. $\int_3^{-2} [x]\,dx = -\int_{-2}^{3} [x]\,dx$

$= -\left[\int_{-2}^{-1} [x]\,dx + \int_{-1}^{0} [x]\,dx + \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx\right] = -\left[\int_{-2}^{-1} (-2)\,dx + \int_{-1}^{0} (-1)\,dx + \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx\right]$

$= -\left[(-2)\left[-1-(-2)\right] + (-1)\left[0-(-1)\right] + 0 + 1(2-1) + 2(3-2)\right]$

$= -\left[-2 - 1 + 1 + 2\right] = 0.$

31. $\int_0^{\frac{\pi}{2}} (3\cos x - 2\sin^2 x)\,dx$

$= 3\int_0^{\frac{\pi}{2}} \cos x\,dx - 2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$= 3(1) - 2(\frac{\pi}{4}) = 3 - \frac{\pi}{2}.$

32. $\int_0^{\frac{\pi}{2}} \cos^2 xdx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)dx$

$= \int_0^{\frac{\pi}{2}} 1\,dx - \int_0^{\frac{\pi}{2}} \sin^2 xdx$

$= 1(\frac{\pi}{2} - 0) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$

33. $\int_{\frac{\pi}{2}}^{0} 3\cos x\,dx = -3\int_0^{\frac{\pi}{2}} \cos x\,dx = -3(1) = -3.$

34. $\int_0^{\frac{\pi}{2}} \cos 2x\,dx = \int_0^{\frac{\pi}{2}} (1-2\sin^2 x)dx$

$= \int_0^{\frac{\pi}{2}} 1dx - 2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$= \frac{\pi}{2}(1-0) - 2(\frac{\pi}{4}) = \frac{\pi}{2} - \frac{\pi}{2} = 0.$

35. $\int_0^4 f(x)dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx$

$= \int_0^2 2x^2dx + \int_2^4 4xdx$

$= 2\cdot\frac{1}{3}\left[2^3-0\right] + 4\cdot\frac{1}{2}\left[4^2-2^2\right]$

$= \frac{2}{3}(8) + 2(12) = \frac{88}{3}.$

36. $\int_{-1}^1 f(x)dx = \int_{-1}^0 f(x)dx + \int_0^1 f(x)dx$

$= \int_{-1}^0 1dx + \int_0^1 1\,dx$

$= 1(0-(-1)) + 1(1-0) = 1+1 = 2.$

37. $\int_{-2}^3 f(x)dx = \int_{-2}^0 f(x)dx + \int_0^3 f(x)dx$

$= \int_{-2}^0 (1-x)dx + \int_0^3 (1+x)dx$

$= 1\left[0-(-2)\right] - \frac{1}{2}\left[0^2-(-2)^2\right] + 1(3-0) + \frac{1}{2}(3^2-0^2)$

$= 2 - \frac{1}{2}(-4) + 3 + \frac{9}{2} = \frac{23}{2}.$

38. $\int_0^1 f(x)dx = \int_0^1 (x+1)dx$

$= \frac{1}{2}\left[1^2 - 0^2\right] + 1(1-0) = \frac{1}{2} + 1 = \frac{3}{2}.$

39. (a) For $0 \le x \le 1$, $x \le 1$, so that

$\int_0^1 xdx \le \int_0^1 1\cdot dx$ by the comparison theorem.

(b) $\int_1^2 x^2dx < \int_1^2 xdx$ does <u>not</u> hold since $x < x^2$ for $1 \le x \le 2$.

(c) $\sin x \ge 0$ for $0 \le x \le \pi$ so

$\int_0^{\pi} \sin x\,dx \ge \int_0^{\pi} 0dx = 0$ by the comparison theorem.

(d) $\frac{1}{1+x^2} \ge 0$ for all $0 \le x \le 1$, so by the nonnegative theorem, $0 \le \int_0^1 \frac{dx}{1+x^2}$.

(e) $x^5 \ge x^6$ for $0 \le x \le 1$, so by the comparison theorem $\int_0^1 x^6 \le \int_0^1 x^5dx$.

The given inequality does <u>not</u> hold.

(f) For $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le x$ so

$\int_0^{\frac{\pi}{2}} \sin x\,dx \le \int_0^{\frac{\pi}{2}} xdx$ by the comparison theorem.

40. Since $K \le f(x)$, $\int_a^b Kdx \le \int_a^b f(x)dx.$

But $\int_a^b Kdx = K(b-a)$, where $K > 0$ and $b-a > 0$. Hence, $K(b-a) > 0$. So

$0 < \int_a^b Kdx \le \int_a^b f(x)dx$, and

it follows that $0 < \int_a^b f(x)dx.$

41. $\frac{1}{3-1}\int_1^3 (x+5)dx = \frac{1}{2}\left[\frac{x^2}{2} + 5x\right]\Big|_1^3$

$= \frac{1}{2}\left[\frac{9}{2} + 15 - (\frac{1}{2} + 5)\right] = \frac{1}{2}(14) = 7.$

Find c, $1 \le c \le 3$, such that $f(c) = c+5=7$.

$c = 2.$

42. $\frac{1}{-1-(-3)}\int_{-3}^{-1} x^2dx = \frac{1}{2}\left[\frac{x^3}{3}\right]\Big|_{-3}^{-1}$

$= \frac{1}{2}\left[-\frac{1}{3} - (-9)\right] = \frac{13}{3}.$

Find c, $-3 \le c \le -1$, such that $c^2 = \frac{13}{3}$

$c = -\sqrt{\frac{13}{3}}$ (reject $c = \sqrt{\frac{13}{3}}$).

43. $\frac{1}{5-(-1)}\int_{-1}^5 (x^2-2x+3)dx = \frac{1}{6}\left[\frac{x^3}{3} - x^2+3x\right]\Big|_{-1}^5$

$= \frac{1}{6}\left[\frac{125}{3} - 25+15-(-\frac{1}{3} - 1-3)\right] = 6.$

Find c, $-1 \le c \le 5$, such that $c^2 - 2c + 3 = 6$, or $c^2 - 2c - 3 = 0$, $(c - 3)(c + 1) = 0$.

Thus, $c = 3, -1$.

44. $\frac{1}{2-(-3)} \int_{-3}^{2} (x-2)(x+3)dx$

$= \frac{1}{5} \int_{-3}^{2} (x^2+x-6)dx = \frac{1}{5}\left[\frac{x^3}{3} + \frac{x^2}{2} - 6x\right]\Big|_{-3}^{2}$

$= \frac{1}{5}\left[\frac{8}{3} + 2 - 12 - (-9+\frac{9}{2}+18)\right] = -\frac{25}{6}$

Find c, $-3 \le c \le 2$, so that $c^2 + c - 6 = -\frac{25}{6}$

or $c^2 + c - \frac{11}{6} = 0$. $\quad c = \dfrac{-1 \pm \sqrt{1 + \frac{22}{3}}}{2}$

$= -\frac{1}{2} \pm \frac{5\sqrt{3}}{6}$.

45. $\frac{1}{5-(-2)} \int_{-2}^{5} |x|\, dx = \frac{1}{7} \int_{-2}^{0} |x|\, dx + \frac{1}{7}\int_{0}^{5} |x|\, dx$

$= \frac{1}{7} \int_{-2}^{0} (-x)dx + \frac{1}{7} \int_{0}^{5} x\, dx$

$= \frac{1}{7}(- \frac{x^2}{2})\Big|_{-2}^{0} + \frac{1}{7}(\frac{x^2}{2})\Big|_{0}^{5}$

$= \frac{1}{7}(2) + \frac{1}{7}(\frac{25}{2}) = \frac{29}{14}$

Find c, $-2 \le c \le 5$, so that $|c| = \frac{29}{14}$

$c = \frac{29}{14}$ (reject $c = -\frac{29}{14}$).

46. $\frac{1}{b-a} \int_{a}^{b} x dx = \frac{1}{b-a} (\frac{x^2}{2})\Big|_{a}^{b}$

$= \frac{1}{b-a} (\frac{b^2-a^2}{2}) = \frac{b+a}{2}$. Find c, $a \le c \le b$, so that $c = \frac{b+a}{2}$.

47. $\frac{1}{b-a} \int_{a}^{b} (Ax+B)dx = \frac{1}{b-a}\left[\frac{Ax^2}{2} + Bx\right]\Big|_{a}^{b}$

$= \frac{1}{b-a}\left[\frac{Ab^2}{2} + Bb - \frac{Aa^2}{2} - Ba\right]$

$= \frac{1}{b-a}\left[\frac{A}{2}(b^2-a^2)+B(b-a)\right] = \frac{A}{2}(b+a) + B$.

We require that $Ac + B = \frac{A}{2}(b+a) + B$, so $c = \frac{b+a}{2}$.

48. $\frac{1}{a-(-a)} \int_{-a}^{a} x^2 dx = \frac{1}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{2a}\left[\frac{a^3}{3} - (-\frac{a^3}{3})\right]$

$= \frac{1}{2a} \cdot \frac{2a^3}{3} = \frac{a^2}{3}$. Find c, $-a \le c \le a$, so that $c^2 = \frac{a^2}{3}$. $c = \frac{a}{\sqrt{3}}$ or $-\frac{a}{\sqrt{3}}$.

49. $1 + \cos x \ge 0$ for all x so

$\int_{\pi/4}^{3\pi/4} (1+\cos x)dx \ge 0$ by the nonnegative theorem.

50. $0 \le x^4 \le x$ for $0 \le x \le 1$ So

$\int_0^1 x^4 dx \le \int_0^1 x dx$ by the comparison property.

51. $\int_0^{0.8} \tan x\, dx = \int_0^{0.8} \tan t\, dt$ since x and t are dummy variables.

52. $0 \le x^6 \le x^2$ when $0 \le x \le \frac{1}{2}$ so

$\int_0^{\frac{1}{2}} x^6 dx \le \int_0^{\frac{1}{2}} x^2 dx$ by the comparison property. But $\int_0^{\frac{1}{2}} x^2 dx = \int_0^{\frac{1}{2}} t^2 dt$ since variables are dummy variables.

53. $\int_0^{2\pi} \sin x\, dx = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} \sin x\, dx$ by additive property so solving for $\int_{\pi}^{2\pi} \sin x\, dx$ we have

$\int_{\pi}^{2\pi} \sin x\, dx = \int_0^{2\pi} \sin x\, dx - \int_0^{\pi} \sin x\, dx$.

54. Since 1776 is even, $x^{1776} \ge 0$ and $\sqrt{|x|^{1699}}$ is positive or 0. Hence, $x^{1776} + \sqrt{|x|^{1699}} \ge 0$ for any x. So by the nonnegative theorem,

$\int_{-1000}^{1000} \left[x^{1776} + \sqrt{|x|^{1699}}\right] dx \ge 0$.

55. We know that $\int_0^5 \sqrt{x^2 + 1}\, dx$

$= \int_0^1 \sqrt{x^2 + 1}\, dx + \int_1^5 \sqrt{x^2 + 1}\, dx$.

So subtracting, $- \int_1^5 \sqrt{x^2 + 1}\, dx$

$= \int_0^1 \sqrt{x^2+1}\,dx - \int_0^5 \sqrt{x^2+1}\,dx.$

Therefore, $\int_5^1 \sqrt{x^2+1}\,dx$

$= \int_0^1 \sqrt{x^2+1}\,dx - \int_0^5 \sqrt{x^2+1}\,dx$, where the left side is justified by Definition 2, part ii, page 315.

56. $\int_{-1}^4 \sqrt[3]{5x^2+3}dx = \int_{-1}^2 \sqrt[3]{5x^2+3}dx + \int_2^4 \sqrt[3]{5x^2+3}\,dx$. Therefore,

$0 = -\int_{-1}^4 \sqrt[3]{5x^2+3}\,dx + \int_2^4 \sqrt[3]{5x^2+3}\,dx + \int_{-1}^2 \sqrt[3]{5x^2+3}\,dx$. By Definition 2 (page 315),

$0 = \int_4^{-1} \sqrt[3]{5x^2+3}\,dx - \int_4^2 \sqrt[3]{5x^2+3}\,dx + \int_{-1}^2 \sqrt[3]{5x^2+3}\,dx$. Now by the symmetric property of equality, $\int_4^{-1} \sqrt[3]{5x^2+3}\,dx - \int_4^2 \sqrt[3]{5x^2+3}\,dx + \int_{-1}^2 \sqrt[3]{5x^2+3}\,dx = 0.$

57. $\int_3^4 \frac{dx}{1+x^2} + \int_4^6 \frac{dt}{1+t^2} = \int_3^6 \frac{dy}{1+y^2}$, since $\int_4^6 \frac{dt}{1+t^2} = \int_4^6 \frac{dx}{1+x^2}$, and $\int_3^6 \frac{dy}{1+y^2}$

$= \int_3^6 \frac{dx}{1+x^2}$. Now $\int_3^4 \frac{dx}{1+x^2} + 0$

$= -\int_4^6 \frac{dt}{1+t^2} + \int_3^6 \frac{dy}{1+y^2}$, and hence,

$\int_3^4 \frac{dx}{1+x^2} + \int_5^5 \frac{dx}{1+x^2} = \int_6^4 \frac{dt}{1+t^2} + \int_3^6 \frac{dy}{1+y^2}.$

58. $\int_a^b \frac{dx}{\sqrt{1+x^2}} + \int_b^c \frac{dy}{\sqrt{1+y^2}} = \int_a^c \frac{dz}{\sqrt{1+z^2}}$, where the name of the variable of integration does not change the function involved. Since $\int_c^a \frac{dz}{\sqrt{1+z^2}} = -\int_a^c \frac{dz}{\sqrt{1+z^2}}$, we have $\int_a^b \frac{dx}{\sqrt{1+x^2}} + \int_b^c \frac{dy}{\sqrt{1+y^2}} + \int_c^a \frac{dz}{\sqrt{1+z^2}}$

$= \int_a^c \frac{dz}{\sqrt{1+z^2}} - \int_a^c \frac{dz}{\sqrt{1+z^2}} = 0.$

59. If $\underline{b < a < c}$, then $\int_b^c f(x)\,dx$

$= \int_b^a f(x)\,dx + \int_a^c f(x)dx$, olving for $\int_a^c f(x)dx$, we have $\int_a^c f(x)dx$

$= -\int_b^a f(x)dx + \int_b^c f(x)dx$

$= \int_a^b f(x)dx + \int_b^c f(x)dx$

If $\underline{b < c < a}$, then $\int_b^a f(x)dx$

$= \int_b^c f(x)dx + \int_c^a f(x)dx$. Solving for $\int_c^a f(x)dx$ we get $\int_c^a f(x)dx$

$= \int_b^a f(x)dx - \int_b^c f(x)dx$ or

$-\int_a^c f(x)dx = -\int_a^b f(x)dx - \int_b^c f(x)dx.$

Multiplying by -1, $\int_a^c f(x)dx$

$= \int_a^b f(x)dx + \int_b^c f(x)dx.$

If $\underline{c < a < b}$, then $\int_c^b f(x)dx$

$= \int_c^a f(x)dx + \int_a^b f(x)dx$ so

$\int_c^a f(x)dx = \int_c^b f(x)dx - \int_a^b f(x)dx$

or $-\int_a^c f(x)dx = -\int_b^c f(x)dx - \int_a^b f(x)dx.$

Thus, $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$

If $\underline{c < b < a}$, then $\int_c^a f(x)dx$

$= \int_c^b f(x)dx + \int_b^a f(x)dx$ so

$-\int_a^c f(x)dx = -\int_b^c f(x)dx - \int_a^b f(x)dx$

or $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$

Suppose, for instance, that b = c, then

$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

becomes $\int_a^b f(x)dx$

$= \int_a^b f(x)dx + \int_b^b f(x)dx$; that is,

$\int_a^b f(x)dx = \int_a^b f(x)dx + 0,$

which is obviously correct.

60. Suppose $a < b$. Let $\mathcal{P}_n$ denote the partition of the interval $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. Thus $\mathcal{P}_n$ consists of the n subintervals $[a, a+\Delta x]$, $[a+\Delta x, a+2\Delta x]$ ... $[a+(n-1)\Delta x, b]$ and the k^{th} subinterval is $[a+(k-1)\Delta x, a+k\Delta x]$ for $k = 1, 2, \ldots, n$. To obtain an augmented partition $\mathcal{P}_n^*$, we choose the numbers $c_1, c_2, \ldots, c_n$ to be the right-hand endpoints of the corresponding subintervals so that $c_k = a+k\Delta x$ for $k = 1, 2, \ldots, n$. Thus, the Riemann sum for $f(x) = x^2$ corresponding to $\mathcal{P}_n^*$ is

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} c_k^2 \Delta x$$

$$= \sum_{k=1}^{n} (a+k\Delta x)^2 \Delta x$$

$$= \frac{b-a}{n} \sum_{k=1}^{n} [a^2 + 2ak\Delta x + k^2\Delta x^2]$$

$$= \frac{b-a}{n} a^2 n + \frac{b-a}{n} . 2a \frac{b-a}{n} \sum_{k=1}^{n} k + \frac{b-a}{n} \left(\frac{b-a}{n}\right)^2 \sum_{k=1}^{n} k^2$$

$$= a^2(b-a) + 2a\frac{(b-a)^2}{n^2}\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= a^2(b-a) + a(b-a)^2 \frac{n+1}{n} + \frac{(b-a)^3}{6}\frac{(n+1)(2n+1)}{n^2}$$

$$= a^2(b-a)+a(b-a)^2(1+\tfrac{1}{n})+\frac{(b-a)^3}{6}(1+\tfrac{1}{n})(2+\tfrac{1}{n}).$$

Thus, $\int_a^b x^2dx = \lim_{n \to +\infty} \sum_{k=1}^{n} f(c_k)\Delta x$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{6}(2)$$

$$= a^2b-a^3+a(b^2-2ba+a^2)+ \frac{b^3-3b^2a+3ba^2-a^3}{3}$$

$$= \frac{b^3}{3} - \frac{a^3}{3}.$$

61. The mean value of F is given by

$$\frac{1}{b-a}\int_a^b ksds = \frac{k}{b-a}\left[\tfrac{1}{2}(b^2 - a^2)\right] = \frac{k}{2}(b+a).$$

We want c, $a \le c \le b$, such that $F(c) = k \cdot c = \frac{k}{2}(b+a)$. Thus, $c = \frac{b+a}{2}$.

62. Suppose f is not the zero function. Then there exists a point x_0 where $f(x_0) > 0$, or if necessary multiplying by -1 will giv such a value. Since f is continuous at x_0, there exists an interval (a,b) around x_0 such that $f(x) \ge \frac{f(x_0)}{2}$ for all x in $[a,b]$. Now by the comparison theorem, $\int_a^b f(x)dx \ge \int_a^b \frac{f(x_0)}{2}dx = \frac{f(x_0)}{2}(b-a) > 0$ since $\frac{f(x_0)}{2} > 0$ and $b - a > 0$. But this is a contradictio to $\int_a^b f(x)dx = 0$ on $[a,b]$. Hence, f is the zero function.

63. Yes. By Problem 61, the average force is $F_{av} = \frac{1}{b-a}\int_a^b ksds = \frac{1}{b-a}W$, so $W = F_{av}\ (b-a)$.

64. $\left|\int_a^b f(x)dx - \int_a^b g(x)dx\right|$

$= \left|\int_a^b (f(x) - g(x))dx\right|$ (linear property)

Now, $\left|\int_a^b (f(x) - g(x))\,dx\right| \le$

$\int_a^b \left|f(x) - g(x)\right| dx$. But

$\left|f(x) - g(x)\right| \leq K$, so that

$\int_a^b \left|f(x) - g(x)\right| dx \leq \int_a^b K dx$.

Furthermore, $\int_a^b K dx = K(b-a)$. Therefore,

$\left|\int_a^b f(x)dx - \int_a^b g(x)dx\right| \leq K \cdot (b-a)$.

5. If $a > b$, then by Theorem 10 there exists a number c on the closed interval between a and b such that $f(c) \cdot (a - b)$

$= \int_b^a f(x)dx$. So $-f(c)\cdot(a-b) = -\int_b^a f(x)dx$.

Therefore, $f(c)\cdot(b-a) = \int_a^b f(x)dx$.

If $a = b$, then $f(c)\cdot(b-a) = 0$ and

$\int_a^b f(x)dx = 0$. So again $f(c)\cdot(b-a)$

$= \int_a^b f(x)dx$.

6. $\left|\int_a^b (f(x)+g(x))dx\right| \leq \int_a^b \left|f(x)+g(x)\right| dx$.

But $\left|f(x)+g(x)\right| \leq \left|f(x)\right| + \left|g(x)\right|$. So

$\int_a^b \left|f(x)+g(x)\right| dx \leq \int_a^b \left|f(x)\right| dx +$

$\int_a^b \left|g(x)\right| dx$ by the comparison and additive theorems. Hence, $\left|\int_a^b (f(x)+g(x))dx\right| \leq$

$\int_a^b \left|f(x)\right| dx + \int_a^b \left|g(x)\right| dx$.

7. (a) $\int_a^x dt = x - a$.

(b) $\int_a^x t dt = \frac{1}{2}(x^2 - a^2)$.

(c) $\int_a^x t^2 dt = \frac{1}{3}(x^3 - a^3)$

8. $\int_a^x f(t)dt = \int_a^x (At^2 + Bt + C)dt$

$= A \int_a^x t^2 dt + B \int_a^x t dt + C \int_a^x dt$

$= A(\frac{1}{3})(x^3-a^3) + B\cdot\frac{1}{2}(x^2-a^2) + C(x-a)$.

So $\frac{d}{dx} \int_a^x f(t)dt = A\cdot\frac{1}{3}\cdot 3x^2 + B\cdot\frac{1}{2}\cdot 2x + C$

$= Ax^2 + Bx + C = f(x)$.

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1. $\int_0^2 3x dx = \frac{3x^2}{2} \Big|_0^2 = 6$.

2. $\int_1^{14} 2 dx = 2x \Big|_1^{14} = 28 - 2 = 26$.

3. $\int_{-1}^{4} (-t)dt = -\frac{t^2}{2} \Big|_{-1}^{4} = -8 + (\frac{1}{2}) = \frac{-15}{2}$.

4. $\int_5^0 (-4u)du = -2u^2 \Big|_5^0 = 0 + 2(25) = 50$.

5. $\int_1^{-3} 5x^4 dx = x^5 \Big|_1^{-3} = (-3)^5 - 1^5 = -244$.

6. $\int_0^{16} z^{\frac{5}{4}} dz = \frac{4}{9} z^{\frac{9}{4}} \Big|_0^{16} = \frac{4}{9}(16)^{\frac{9}{4}} - 0$

$= \frac{4}{9}(2^9) = \frac{2048}{9}$.

7. $\int_2^3 (3x+4)dx = (\frac{3}{2}x^2 + 4x) \Big|_2^3$

$= \left[\frac{3}{2}(9) + 4(3)\right] - \left[\frac{3}{2}(4) + 8\right] = \frac{23}{2}$.

8. $\int_{-3}^{-1} (4-8x+3x^2)dx = (4x-4x^2+x^3) \Big|_{-3}^{-1}$

$= (-4 - 4 - 1) - (-12 - 36 - 27) = 66$.

9. $\int_1^5 (x^3-3x^2+1)dx = (\frac{x^4}{4} - x^3 + x) \Big|_1^5$

$= (\frac{625}{4} - 125 + 5) - (\frac{1}{4} - 1 + 1) = 36$.

10. $\int_1^3 (x-1)(x^2 + x + 1)dx = \int_1^3 (x^3-1)dx$

$= (\frac{x^4}{4} - x) \Big|_1^3 = (\frac{3^4}{4} - 3) - (\frac{1}{4} - 1) = 18$.

11. $\int_0^1 (x^2+2)^2 dx = \int_0^1 (x^4+4x^2+4)dx$

$= (\frac{x^5}{5} + \frac{4x^3}{3} + 4x) \Big|_0^1 = (\frac{1}{5} + \frac{4}{3} + 4) - (0+0+0) = \frac{83}{15}$.

12. $\int_1^5 \frac{x^4-16}{x^2+4} dx = \int_1^5 (x^2 - 4) dx$

$= (\frac{x^3}{3} - 4x) \Big|_1^5 = (\frac{125}{3} - 20) - (\frac{1}{3} - 4) = \frac{76}{3}$.

13. $\int_0^8 (2 - \sqrt[3]{t})^2 dt = \int_0^8 (4-4\sqrt[3]{t}+(\sqrt[3]{t})^2)dt$

$= (4t-3t^{\frac{4}{3}}+ \frac{3}{5}t^{\frac{5}{3}}) \Big|_0^8 = \left[32-3(16)+\frac{3}{5}(32)-0\right] = \frac{16}{5}.$

14. $\int_1^{32} (t^{-\frac{1}{3}} + t^{\frac{1}{15}})dt = (\frac{3}{2}t^{\frac{2}{3}} + \frac{15}{16}t^{\frac{16}{15}}) \Big|_1^{32}$

$= \frac{3}{2}(32)^{\frac{2}{3}} + \frac{15}{16}(32)^{\frac{16}{15}} - \frac{3}{2} - \frac{15}{16}$

$= \frac{3}{2}(32)^{\frac{2}{3}} + \frac{15}{16}(32)^{\frac{16}{15}} - \frac{39}{16}.$

15. $\int_0^{\frac{\pi}{6}} \cos x \, dx = \sin x \Big|_0^{\frac{\pi}{6}} = \sin\frac{\pi}{6} - \sin 0$

$= \frac{1}{2} - 0 = \frac{1}{2}.$

16. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \sin t = -\cos t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{3}} = -\cos\frac{\pi}{3} + \cos(-\frac{\pi}{4})$

$= -\frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2}.$

17. $\int_0^{\frac{\pi}{4}} \sec t \tan t \, dt = \sec t \Big|_0^{\frac{\pi}{4}}$

$= \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1.$

18. $\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \csc^2 y \, dy = -\cot y \Big|_{\frac{\pi}{4}}^{\frac{\pi}{6}}$

$= -\cot\frac{\pi}{6} + \cot\frac{\pi}{4} = -\sqrt{3} + 1 = 1 - \sqrt{3}.$

19. $\int_0^{\frac{\pi}{3}} \sec^2 u \, du = \tan u \Big|_0^{\frac{\pi}{3}} = \tan\frac{\pi}{3} - \tan 0$

$= \sqrt{3} - 0 = \sqrt{3}.$

20. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc z \cot z dz = -\csc z \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= -\csc\frac{\pi}{2} + \csc\frac{\pi}{6} = -1 + 2 = 1.$

21. Let $u = y^3 + 1$, so that $du = 3y^2 dy$ and $y^2 dy = \frac{1}{3}du$. So $\int_0^1 \frac{y^2 dy}{(y^3+1)^5} = \int_1^2 \frac{\frac{1}{3}du}{u^5}$

$= \frac{1}{3} \frac{u^{-4}}{-4} \Big|_1^2 = \frac{-1}{12}(2^{-4} - 1) = \frac{5}{64}.$

22. Let $u = 2x + 3$, so that $du = 2dx$ and $dx = \frac{1}{2}du$. So $\int (2x+3)^{10} dx$

$= \int u^{10} \cdot \frac{1}{2}du = \frac{u^{11}}{22} + C.$ Hence,

$\int_0^1 (2x+3)^{10} dx = \frac{(2x+3)^{11}}{22} \Big|_0^1$

$= \frac{5^{11}}{22} - \frac{3^{11}}{22} = \frac{5^{11} - 3^{11}}{22}.$

23. Let $u = 1 - x$, so that $du = -dx$. So

$\int \sqrt{1-x} \, dx = -\int u^{\frac{1}{2}} du = -\frac{2}{3}u^{\frac{3}{2}} + C$

$= -\frac{2}{3}(1-x)^{\frac{3}{2}} + C.$ Hence, $\int_{-1}^{1} \sqrt{1-x} \, dx$

$= -\frac{2}{3}(1-x)^{\frac{3}{2}} \Big|_{-1}^{1} = -\frac{2}{3}\left[0 - 2^{\frac{3}{2}}\right]$

$= \frac{2}{3}\sqrt{8} = \frac{4}{3}\sqrt{2}.$

24. Let $u = 4 - 3x$, so that $du = -3dx$ and $dx = -\frac{1}{3}du$. So $\int \sqrt{4-3x} \, dx = \int \sqrt{u} \cdot (-\frac{1}{3})du$

$= -\frac{2}{9}u^{\frac{3}{2}} + C.$ Hence, $\int_0^1 \sqrt{4-3x} \, dx$

$= -\frac{2}{9}(4-3x)^{\frac{3}{2}} \Big|_0^1 = -\frac{2}{9}(1^{\frac{3}{2}} - 4^{\frac{3}{2}})$

$= -\frac{2}{9}(-7) = \frac{14}{9}.$

25. Let $u = 1 + x$, so that $du = dx$. So

$\int \frac{dx}{\sqrt{1+x}} = \int \frac{du}{u^{\frac{1}{2}}} = 2u^{\frac{1}{2}} + C = 2(1+x)^{\frac{1}{2}} + C.$

Hence, $\int_{\frac{1}{4}}^{3} \frac{dx}{\sqrt{1+x}} = 2(1+x)^{\frac{1}{2}} \Big|_{\frac{1}{4}}^{3}$

$= 2(2) - 2(\frac{5}{4})^{\frac{1}{2}} = 4 - \sqrt{5}.$

26. Let $u = 4 + x^2$, so that $du = 2xdx$ and $xdx = \frac{1}{2}du$. So $\int \frac{xdx}{(4+x^2)^{\frac{3}{2}}} = \int \frac{\frac{1}{2}du}{u^{\frac{3}{2}}}$

$= \frac{-2}{2}u^{-\frac{1}{2}} + C = -u^{-\frac{1}{2}} + C.$ Hence,

$\int_0^2 \frac{xdx}{(4+x^2)^{\frac{3}{2}}} = -(4+x^2)^{-\frac{1}{2}} \Big|_0^2$

$= -(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{4}}) = \frac{2-\sqrt{2}}{4}.$

27. Let $u = x^3 + 1$, then $du = 3x^2dx$ and $x^2dx = \frac{1}{3}du$. So $\int x^2 \sqrt[3]{x^3+1}\, dx = \int \frac{1}{3}u^{\frac{1}{3}}du$

$= \frac{1}{4}u^{\frac{4}{3}} + C = \frac{(x^3+1)^{\frac{4}{3}}}{4} + C$. Hence,

$\int_0^2 x^2 \sqrt[3]{x^3 + 1}\, dx = \frac{1}{4}(x^3 + 1)^{\frac{4}{3}} \Big|_0^2$

$= \frac{1}{4}(9)^{\frac{4}{3}} - \frac{1}{4} = \frac{1}{4}(9^{\frac{4}{3}} - 1)$.

28. Let $u = x^2 + 6x + 2$, so that $du = (2x+6)dx$

$= 2(x + 3)dx$ and $(x + 3)dx = \frac{1}{2}du$. So

$\int \frac{x + 3}{\sqrt{x^2 + 6x + 2}}\, dx = \int \frac{\frac{1}{2}du}{u^{\frac{1}{2}}} = u^{\frac{1}{2}} + C$

$= (x^2 + 6x + 2)^{\frac{1}{2}} + C$. Hence,

$\int_0^1 \frac{x + 3}{\sqrt{x^2 + 6x + 2}}\, dx = (x^2 + 6x + 2)^{\frac{1}{2}} \Big|_0^1$

$= 9^{\frac{1}{2}} - 2^{\frac{1}{2}} = 3 - \sqrt{2}$.

29. Let $u = x - 6$, so that $du = dx$ and $x = u+6$.

So $\int \frac{x\, dx}{\sqrt{x - 6}} = \int \frac{(u + 6)}{\sqrt{u}}\, du = \int (u^{\frac{1}{2}} + 6u^{-\frac{1}{2}})du$

$= \frac{2}{3}u^{\frac{3}{2}} + 12u^{\frac{1}{2}} + C = \frac{2}{3}(x-6)^{\frac{3}{2}} + 12(x-6)^{\frac{1}{2}} + C$.

Hence, $\int_7^{10} \frac{x\, dx}{\sqrt{x-6}} = [\frac{2}{3}(4)^{\frac{3}{2}} + 12(4)^{\frac{1}{2}}] -$

$(\frac{2}{3} \cdot 1 + 12 \cdot 1) = \frac{50}{3}$.

30. Let $u = x+1$, $du = dx$, $x+2 = u+1$.

$\int_3^0 (x+2)\sqrt{x+1}\, dx = \int_4^1 (u+1)\sqrt{u}\, du$

$= \int_4^1 (u^{\frac{3}{2}} + u^{\frac{1}{2}})du = (\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}) \Big|_4^1$

$= \frac{2}{5} + \frac{2}{3} - (\frac{2}{5} \cdot 4^{\frac{5}{2}} + \frac{2}{3} \cdot 4^{\frac{3}{2}})$

$= \frac{2}{5} + \frac{2}{3} - \frac{64}{5} - \frac{16}{3} = -\frac{256}{15}$.

31. Let $u = 3x$, $du = 3dx$.

$\int_0^{\pi} 2 \sin 3x dx = \int_0^{3\pi} 2 \sin u \,(\frac{1}{3}\, du)$

$= \frac{2}{3} \int_0^{3\pi} \sin u dx = -\frac{2}{3}[\cos u] \Big|_0^{3\pi}$

$= -\frac{2}{3}(\cos 3\pi - \cos 0) = -\frac{2}{3}(-1-1) = \frac{4}{3}$.

32. Let $u = 3t$, $du = 3dt$.

$\int_0^{\frac{\pi}{3}} (2+\cos 3t)dt = \int_0^{\frac{\pi}{3}} 2dt + \int_0^{\frac{\pi}{3}} \cos 3tdt$

$= 2(\frac{\pi}{3}-0) + \int_0^{\pi} \cos u\, (\frac{1}{3}du)$

$= \frac{2\pi}{3} + \frac{1}{3}\int_0^{\pi} \cos udu = \frac{2\pi}{3} + \frac{1}{3}(\sin u) \Big|_0^{\pi}$

$= \frac{2\pi}{3} + \frac{1}{3}(\sin\ - \sin 0)$

$= \frac{2\pi}{3} + \frac{1}{3}(0) = \frac{2\pi}{3}$.

33. Let $u = \frac{\pi x}{4}$, $du = \frac{\pi}{4}dx$.

$\int_1^0 \sec^2 \frac{\pi x}{4}dx = \int_{\frac{\pi}{4}}^0 \sec^2 u(\frac{4}{\pi})du$

$= \frac{4}{\pi}(\tan u \Big|_{\frac{\pi}{4}}^0) = \frac{4}{\pi}(\tan 0 - \tan \frac{\pi}{4})$

$= \frac{4}{\pi}(0 - 1) = -\frac{4}{\pi}$.

34. Let $u = \frac{\pi t}{4}$, $du = \frac{\pi}{4}dt$.

$\int_0^1 \sec \frac{\pi t}{4} \tan \frac{\pi t}{4}\, dt = \int_0^{\frac{\pi}{4}} \sec u \tan u(\frac{4}{\pi})du$

$= \frac{4}{\pi} \sec u \Big|_0^{\frac{\pi}{4}} = \frac{4}{\pi}(\sec \frac{\pi}{4} - \sec 0)$

$= \frac{4}{\pi}(\sqrt{2} - 1)$.

35. Let $u = \frac{\pi t}{3}$, $du = \frac{\pi}{3}\, dt$

$\int_{\frac{1}{2}}^1 \csc \frac{\pi t}{3} \cot \frac{\pi t}{3}\, dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc u \cot u(\frac{3}{\pi}du)$

$= -\frac{3}{\pi} \csc u \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\frac{3}{\pi}(\csc \frac{\pi}{3} - \csc \frac{\pi}{6})$

$= -\frac{3}{\pi}(\frac{2}{\sqrt{3}} - 2) = \frac{2}{\pi}(3 - \sqrt{3})$.

36. Let $u = \frac{\pi}{2} - \frac{\pi}{2}$, $du = \frac{1}{2}\, dx$

$\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \csc^2(\frac{x}{2} - \frac{\pi}{2})dx = \int_{-\frac{\pi}{4}}^{-\frac{3\pi}{8}} \csc^2 u(2du)$

$= -2 \cot u \Big|_{-\frac{\pi}{4}}^{-\frac{3}{8}} = -2\left[\cot(-\frac{3\pi}{8}) - \cot(-\frac{\pi}{4})\right]$

$= -2\left[-\cot \frac{3\pi}{8} + \cot \frac{\pi}{4}\right]$

$= -2\left[-(\sqrt{2} - 1) + 1\right] = 2\sqrt{2} - 4$.

37. Let $u = \sin x$, $du = \cos x\, dx$.

$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x\, dx = \int_0^1 u^2 du$

$$= \frac{u^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

38. Let $u = \sin 2x$, $du = 2\cos 2x\,dx$

$$\int_{\frac{7\pi}{12}}^{\frac{11\pi}{12}} \frac{\cos 2x}{\sin^2 2x}dx = \int_{-\frac{1}{2}}^{-\frac{1}{2}} \frac{\frac{1}{2}\,du}{u^2} = 0.$$

39. Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$.

$$\int_{\frac{\pi}{4}}^{0} \cos^3\theta \sin\theta\,d\theta = \int_{\frac{\sqrt{2}}{2}}^{1} u^3(-du)$$

$$= -\frac{u^4}{4}\Big|_{\frac{\sqrt{2}}{2}}^{1} = -\frac{1}{4}\left(1 - \frac{1}{4}\right) = -\frac{3}{16}.$$

40. $$\int_{-1}^{1}\sqrt{|t|+t}\,dt = \int_{-1}^{0}\sqrt{|t|+t}\,dt + \int_{0}^{1}\sqrt{|t|+t}\,dt$$

$$= \int_{-1}^{0}\sqrt{-t+t}\,dt + \int_0^1\sqrt{2t}\,dt$$

$$= 0 + \left(\sqrt{2}\cdot\frac{2}{3}t^{\frac{3}{2}}\right)\Big|_0^1 = \frac{2\sqrt{2}}{3}.$$

41. $$\int_0^3 |3-x^2|\,dx = \int_0^{\sqrt{3}}(3-x^2)dx + \int_{\sqrt{3}}^{3}(x^2-3)dx$$

$$= \left(3x - \frac{x^3}{3}\right)\Big|_0^{\sqrt{3}} + \left(\frac{x^3}{3} - 3x\right)\Big|_{\sqrt{3}}^{3}$$

$$= \left(3\sqrt{3} - \frac{3\sqrt{3}}{3}\right) - 0 + (9-9) - \left(\frac{3\sqrt{3}}{3} - 3\sqrt{3}\right) = 4\sqrt{3}.$$

42. $$\int_{-1}^{3}\sqrt[3]{2(|x|-x)}\,dx = \int_{-1}^{0}\sqrt[3]{2(|x|-x)}\,dx + \int_0^3\sqrt[3]{2(|x|-x)}\,dx = \int_{-1}^{0}\sqrt{-4x}\,dx + 0$$

$$= -\frac{4}{3}(-x)^{\frac{3}{2}}\Big|_{-1}^{0} = 0 + \left(\frac{4}{3}\cdot 1\right) = \frac{4}{3}.$$

43. $$\int_0^3 y|2-y|\,dy = \int_0^2 y(2-y)dy + \int_2^3 y(y-2)dy$$

$$= \int_0^2(2y-y^2)dy + \int_2^3(y^2-2y)dy$$

$$= \left(y^2 - \frac{y^3}{3}\right)\Big|_0^2 + \left(\frac{y^3}{3} - y^2\right)\Big|_2^3$$

$$= \left(4 - \frac{8}{3}\right) - 0 + \left(\frac{27}{3} - 9\right) - \left(\frac{8}{3} - 4\right)$$

$$= \frac{4}{3} - 0 + 0 + \frac{4}{3} = \frac{8}{3}.$$

44. $$\int_{-1}^{3}[\![x]\!]\,x\,dx = \int_{-1}^{0}-x\,dx + \int_0^1 0\,dx + \int_1^2 1\cdot x\,dx + \int_2^3 2\cdot x\,dx$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + 0 + \frac{x^2}{2}\Big|_1^2 + x^2\Big|_2^3$$

$$= 0 - \left(-\frac{1}{2}\right) + 0 + \left(\frac{4}{2} - \frac{1}{2}\right) + (9-4) = 7.$$

45. $$\int_{-3}^{5} f(x)dx = \int_{-3}^{0}(1-x)^{\frac{3}{2}}dx + \int_0^5 (x+4)^{\frac{1}{2}}\,dx.$$

Let $u = 1-x$, $du = -dx$. Let $v = x+4$, $dv = dx$. $\int -u^{3/2}du = -\frac{2}{5}u^{5/2} + C$;

$\int v^{\frac{1}{2}}dv = \frac{2}{3}v^{\frac{3}{2}} + C$. So,

$$\int_{-3}^{0}(1-x)^{\frac{3}{2}}dx = -\frac{2}{5}(1-x)^{\frac{5}{2}}\Big|_{-3}^{0} = -\frac{2}{5} - \left(-\frac{2}{5}\cdot 32\right)$$

$$= \frac{62}{5}, \text{ and } \int_0^5 (x+4)^{\frac{1}{2}}dx = \frac{2}{3}(x+4)^{\frac{3}{2}}\Big|_0^5$$

$$= \frac{2}{3}(27) - \frac{2}{3}\cdot 8 = \frac{38}{3}. \text{ Hence,}$$

$$\int_{-3}^{5} f(x)dx = \frac{62}{5} + \frac{38}{3} = \frac{376}{15}.$$

46. No, since $f(x) = \frac{1}{x^2}$ does not satisfy the hypothesis of that theorem, inasmuch as f is not defined at 0, and 0 is in the interval from -1 to 1.

47. $$\frac{d}{dx}\int_0^x (t^2+1)dt = x^2 + 1.$$

48. $$\frac{d}{dx}\int_1^x (w^3-2w+1) = x^3 - 2x + 1.$$

49. $$\frac{d}{dx}\int_{-1}^{x}\frac{ds}{1+s^2} = \frac{1}{1+x^2}.$$

50. $$\frac{d}{dx}\left(\int_0^x \frac{ds}{1+s} + \int_2^x \frac{ds}{1+s}\right) = \frac{1}{1+x} + \frac{1}{1+x} = \frac{2}{1+x}$$

51. $$\frac{d}{dx}\int_0^x \sin(t^4)dt = \sin x^4.$$

52. $$\frac{d}{dx}\int_{-\pi}^{x}\sec^3 t\,dt = \sec^3 x.$$

53. $$D_x\int_{-1}^{x}\sqrt{t^2+4}\,dt = \sqrt{x^2+4}.$$

54. $$D_x\int_x^1 (t^3 - 3t + 1)^{10}dt$$

$$= -D_x\int_1^x (t^3 - 3t + 1)^{10}\,dt$$

$$= -(x^3 - 3x + 1)^{10}.$$

55. $$D_x\int_x^1 (w^{10}+3)^{25}dw = -D_x\int_1^x (w^{10}+3)^{25}dw$$

$$= -(x^{10} + 3)^{25}.$$

56. $$D_x\int_x^4 \sqrt[3]{4s^2 + 7}\,ds = -D_x\int_4^x \sqrt[3]{4s^2 + 7}\,ds$$

$$= -\sqrt[3]{4x^2 + 7}.$$

57. $\frac{d}{dx}\left(\int_x^0 \sqrt[3]{t^2+1}\,dt + \int_0^x \sqrt[3]{t^2+1}\,dt\right)$

$= \frac{d}{dx}\left(-\int_0^x \sqrt[3]{t^2+1}\,dt + \int_0^x \sqrt[3]{t^2+1}\,dt\right)$

$= \frac{d}{dx}(0) = 0.$

58. $D_x^2 \int_1^x \frac{1}{1+t^2}\,dt = D_x\left(D_x \int_1^x \frac{1}{1+t^2}\,dt\right)$

$= D_x\left(\frac{1}{1+x^2}\right) = D_x(1+x^2)^{-1} = -1(1+x^2)^{-2}(2x)$

$= \frac{-2x}{(1+x^2)^2}.$

59. Put $u = 3x$, so that $\frac{du}{dx} = 3$. Now

$y = \int_1^u (5t^3+1)^7 dt$, and so $\frac{dy}{dx}$

$= \frac{d}{du}\left[\int_1^u (5t^3+1)^7 dt\right] \cdot \frac{du}{dx} = (5u^3+1)^7 \cdot 3$

$= \left[5(3x)^3+1\right]^7 \cdot 3 = 3(135x^3+1)^7.$

60. Put $u = 5x+1$, so that $\frac{du}{dx} = 5$. Now y

$= \int_1^u \frac{dt}{9+t^2}$, and so $\frac{dy}{dx} = \frac{d}{du}\left[\int_1^u \frac{dt}{9+t^2}\right] \cdot \frac{du}{dx}$

$= \frac{1}{9+u^2} \cdot 5 = \frac{5}{9+(5x+1)^2} = \frac{5}{25x^2+10x+10}.$

61. Put $u = 8x+2$, so that $\frac{du}{dx} = 8$. Now y

$= \int_1^u (w-3)^{15} dw$, and so $\frac{dy}{dx}$

$= \frac{d}{du}\left[\int_1^u (w-3)^{15} dw\right] \cdot \frac{du}{dx} = (u-3)^{15} \cdot 8$

$= (8x+2-3)^{15} \cdot 8 = 8(8x-1)^{15}.$

62. Put $u = x-1$, $\frac{du}{dx} = 1$. So y

$= \int_1^u \sqrt{s^2-1}\,ds$ and $\frac{dy}{dx} = \frac{d}{du}\left[\int_1^u \sqrt{s^2-1}\,ds\right] \cdot \frac{du}{dx}$

$= \sqrt{u^2-1} \cdot 1 = \sqrt{x^2-2x}.$

63. $y = -\int_0^{-x} \sqrt{t+2}\,dt$. Put $u = -x$, so that $\frac{du}{dx} = -1$. Now $y = -\int_0^u \sqrt{t+2}\,dt$, and so

$\frac{dy}{dx} = \frac{d}{du}\left[-\int_0^u \sqrt{t+2}\,dt\right] \cdot \frac{du}{dx} = -\sqrt{u+2} \cdot (-1)$

$= \sqrt{-x+2} = \sqrt{2-x}.$

64. $y = -\int_2^{x^2+1} \sqrt[3]{u-1}\,du$. Put $v = x^2+1$, so that $\frac{dv}{dx} = 2x$. Now $y = -\int_2^v \sqrt[3]{u-1}\,du$,

and $\frac{dy}{dx} = \frac{d}{dv}\left[-\int_2^v \sqrt[3]{u-1}\,du\right] \cdot \frac{dv}{dx}$

$= -\sqrt[3]{v-1} \cdot 2x = -2x\sqrt[3]{x^2}.$

65. $y = \int_x^0 \sqrt[4]{t^4+17}\,dt + \int_0^{3x^2+2} \sqrt[4]{t^4+17}\,dt$,

so that $y = -\int_0^x \sqrt[4]{t^4+17}\,dt + \int_0^u \sqrt[4]{t^4+17}\,dt$,

where $u = 3x^2+2$, $\frac{du}{dx} = 6x$. $\frac{dy}{dx}$

$= -\sqrt[4]{x^4+17} + \frac{d}{du}\left[\int_0^u \sqrt[4]{t^4+17}\,dt\right] \cdot \frac{du}{dx};$

$\frac{dy}{dx} = -\sqrt[4]{x^4+17} + \sqrt[4]{u^4+17} \cdot 6x$. $\frac{dy}{dx}$

$= -\sqrt[4]{x^4+17} + \sqrt[4]{(3x^2+2)^4+17} \cdot 6x$

$= 6x\sqrt[4]{(3x2+2)^4+17} - \sqrt[4]{x^4+17}.$

66. $y = \int_x^0 \sqrt[3]{t^3+1}\,dt + \int_0^{x-x^2} \sqrt{t^3+1}\,dt$. Put

$u = x^3$ and $v = x-x^2$, so that $\frac{du}{dx}$

$= 3x^2$ and $\frac{dv}{dx} = 1-2x$. So y

$= -\int_0^u \sqrt{t^3+1}\,dt + \int_0^v \sqrt{t^3+1}\,dt$. Now

$\frac{dy}{dx} = \frac{d}{du}\left[-\int_0^u \sqrt{t^3+1}\,dt\right] \cdot \frac{du}{dx} + \frac{d}{dv}\left[\int_0^v \sqrt{t^3+1}\,dt\right]$

$\cdot \frac{dv}{dx}$; $\frac{dy}{dx} = -\sqrt{u^3+1}(3x^2) + \sqrt{v^3+1} \cdot (1-2x);$

$\frac{dy}{dx} = (1-2x)\sqrt{(x-x^2)^3+1} - 3x^2\sqrt{x^9+1}.$

67. $\frac{d}{dx}\int_u^v f(t)dt = \frac{d}{dx}\left[\int_u^0 f(t)dt + \int_0^v f(t)dt\right]$

$= \frac{d}{du}\left[-\int_0^u f(t)dt\right] \cdot \frac{du}{dx} + \frac{d}{dv}\left[\int_0^v f(t)dt\right] \cdot \frac{dv}{dx}$

$= f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}.$

68. We want to show that $f(b) = g'+(b)$. Choose Δx small and <u>positive</u> so that $b + \Delta x$ belongs to (b,c). The argument continues in a manner similar to that in the proof of Theorem 1 until we look at $g'+(b)$. Then $g'+(b)$

$= \lim_{\Delta x \to 0^+} \frac{g(b+\Delta x)-g(b)}{\Delta x} = \lim_{x^{\#} \to b} f(x^{\#})$

$= f(b)$, where $x^{\#}$ is on the closed interval $[b, b+x]$ such that $f(x^{\#}) \cdot \Delta x = \int_b^{b+\Delta x} f(t)dt$ and where we use the continuity of f in the last equation. We also want to show that $f(c) = g'_-(b)$. Here we choose Δx small and <u>negative</u> so that $c + \Delta x$ belongs to the interval (b,c). Again the argument is similar until we look at $g'_-(c)$. Then $g'_-(c) =$

$\lim_{\Delta x \to 0^-} \frac{g(c+\Delta x)-g(c)}{\Delta x} = \lim_{x^* \to c} f(x^*) = f(c)$,

where x^* is on the closed interval $[c+\Delta x, c]$ such that $f(x^*) \cdot \Delta x$

$= \int_c^{c+\Delta x} f(t)dt$, and where we use the continuity of f in the last equation.

69. If $a > b$, then $y = \int_a^b f(x)dx$

$= -\int_b^a f(x)dx$. Then using the fundamental theorem of calculus $y = -\left[g(x)\right]_b^a$

$= -(g(a)-g(b)) = g(b)-g(a)$.

If $a = b$, then $\int_a^b f(x)dx = 0$ and $g(b)-g(a) = 0$.

70. Let V be the value of the truck. So $f(t) = -\frac{dV}{dt}$ and $\int_0^t f(x)dx = -\int_0^t \frac{dV}{dt}dt$

$= V(0) - V(t)$. Hence, $\int_0^t f(x)dx$ represents the loss in value of the truck over the period of time t. Now adding the fixed cost of an overhaul, $\int_0^t f(x)dx + K$ is the total cost over the period of time t. Therefore,

$\left[\frac{\int_0^t f(x)dx + K}{t}\right] = t^{-1}\left[\int_0^t f(x)dx + K\right]$

$= g(t)$ is the average cost per month, and so the value of t that minimizes g is just T.

71. $C'(x) = 200 - 30\sqrt{x}$. Increase in cost

$= \int_4^{25} (200-30\sqrt{x})dx = \left(200x - 30 \cdot \frac{2}{3}x^{\frac{3}{2}}\right)\Big|_4^{25}$

$= (5000 - 2500) - (800 - 160) = 1{,}860$.

The total increase in cost would be \$1,860.

72. $f'(x) = |x| - |x-1|$. To find the critical numbers, solve $f'(x) = |x| - |x-1| = 0$ are determined by $|x| - |x-1| = 0$ or $|x| = |x-1|$. $x = \frac{1}{2}$. f': $-$ | $+$ at $\frac{1}{2}$

Relative minimum at $x = \frac{1}{2}$.

No relative maxima. So we must look at the value of f(x) at the endpoints of the interval [-1,2]:

$f(-1) = \int_0^{-1} (|t|-|t-1|)dt = \int_0^{-1} [(-t)-(1-t)]dt$

$= \int_0^{-1} (-1)dt = -t \Big|_0^{-1} = 1$;

$f(2) = \int_0^2 (|t|-|t-1|)dt = \int_0^1 (|t|-|t-1|)dt$

$+ \int_1^2 (|t|-|t-1|)dt = \int_0^1 (2t-1)dt + \int_1^2 dt$

$= (t^2-t)\Big|_0^1 + t\Big|_1^2 = 1$. Thus, an absolute maximum value of 1 occurs at $x = -1, 2$.

73. (a) For $x \leq 0$, $\int_0^x (-t)dt = \frac{-t^2}{2}\Big|_0^x$

$= -\frac{x^2}{2} = \frac{x|x|}{2}$. For $x > 0$, $\int_0^x t\,dt$

$= \frac{t^2}{2}\Big|_0^x = \frac{x^2}{2} = \frac{x|x|}{2}$.

(b) $\frac{d}{dx}\left(\frac{x|x|}{2}\right) = \frac{d}{dx}\int_0^x |t|\,dt = |x|$.

74. For $x < 0$, $\int_0^x |t|^n dt = \int_0^x (-t)^n dt$

$= \frac{-(-t)^{n+1}}{n+1}\Big|_0^x = \frac{-(-x)^{n+1}}{n+1} = \frac{-(-x)^n(-x)}{n+1}$

$= \frac{|x|^n \cdot x}{n+1}$. For $x \geq 0$, $\int_0^x |t|^n dt = \int_0^x t^n dt$

$= \frac{t^{n+1}}{n+1}\Big|_0^x = \frac{x^{n+1}}{n+1} = \frac{|x|^n \cdot x}{n+1}$. Hence, $\int_0^x |t|^n dt$

$= \frac{|x|^n \cdot x}{n+1}$ for all x.

Now, $\frac{d}{dx}\int_0^x |t|^n dt = |x|^n$. So $\frac{|x|^n \cdot x}{n+1}$

is an antiderivative of $|x|^n$.

75. (a) $M = \frac{1}{4-1}\int_1^4 (x^2+1)dx = \frac{1}{3}(\frac{x^3}{3} + x)\Big|_1^4$

$= \frac{1}{3}\left[(\frac{4^3}{3} + 4) - (\frac{1}{3} + 1)\right] = 8.$

(b) $M = \frac{1}{3-1}\int_1^3 (x^3-1)dx = \frac{1}{2}(\frac{x^4}{4} - x)\Big|_1^3$

$= \frac{1}{2}\left[(\frac{3^4}{4} - 3) - (\frac{1}{4} - 1)\right] = 9.$

(c) $M = \frac{1}{9-1}\int_1^9 \sqrt{x}\, dx = \frac{1}{8}(\frac{2}{3}x^{\frac{3}{2}})\Big|_1^9$

$= \frac{1}{8}(\frac{2}{3} \cdot 27 - \frac{2}{3}) = \frac{13}{6}.$

(d) $M = \frac{1}{b-a}\int_a^b (|x| + 1)dx = \frac{1}{b-a}\left[\frac{x|x|}{2} + x\right]\Big|_a^b$

$= \frac{1}{b-a}\left[(\frac{b|b|}{2} + b) - (\frac{a|a|}{2} + a)\right]$

$= \frac{1}{2(b-a)}(b|b| - a|a| + 2b - 2a).$

76. We want to find $\frac{d}{dx}\left[\int_0^{-x} f(t)dt + \int_0^x f(-t)dt\right]$.

Let $u = -x$, so $\frac{du}{dx} = -1$. Hence,

$\frac{d}{du}\left[\int_0^u f(t)dt\right] \cdot \frac{du}{dx} = f(u)\cdot(-1) = f(-x)(-1)$. So

$\frac{d}{dx}\int_0^{-x} f(t)dt + \frac{d}{dx}\int_0^x f(-t)dt = -f(-x) +$

$f(-x) = 0$. Hence, $\int_0^{-x} f(t)dt + \int_0^x f(-t)dt$

is a constant. If we let $x = 0$, then

$\int_0^{-x} f(t)dt = \int_0^0 f(t)dt = 0$, and

$\int_0^0 f(-t)dt = 0$. $0 + 0 = 0$. So the

constant is 0.

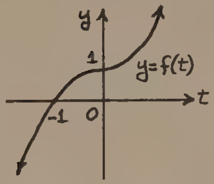
77. For $x \leq 0$, $g(x) = \int_{-1}^x (1-t^2)dt$

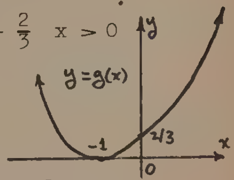
$= (t - \frac{t^3}{3})\Big|_{-1}^x = x - \frac{x^3}{3} + \frac{2}{3}$. For $x > 0$,

$g(x) = \int_{-1}^0 (1-t^2)dt + \int_0^x (1+t^2)dt$

$= (t - \frac{t^3}{3})\Big|_{-1}^0 + (t + \frac{t^3}{3})\Big|_0^x = \frac{2}{3} + x + \frac{x^3}{3}$.

Hence, $g(x) = \begin{cases} x - \frac{x^3}{3} + \frac{2}{3} & x \leq 0 \\ x + \frac{x^3}{3} + \frac{2}{3} & x > 0 \end{cases}$.





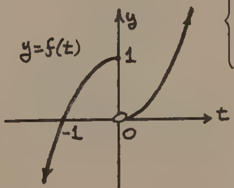
78. For $x \leq 0$, $g(x) = \int_{-1}^x (1-t^2)dt$

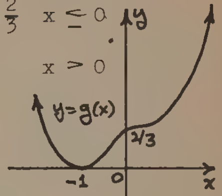
$= (t - \frac{t^3}{3})\Big|_{-1}^x = x - \frac{x^3}{3} + \frac{2}{3}$. For $x > 0$,

$g(x) = \int_{-1}^0 (1-t^2)dt + \int_0^x t^2 dt$

$= (t - \frac{t^3}{3})\Big|_{-1}^0 + \frac{t^3}{3}\Big|_0^x = \frac{2}{3} + \frac{x^3}{3}$.

Hence, $g(x) = \begin{cases} x - \frac{x^3}{3} + \frac{2}{3} & x \leq 0 \\ \frac{2}{3} + \frac{x^3}{3} & x > 0 \end{cases}$.





79. For $-3 \leq x \leq -2$, $g(x) = \int_0^{-1} dt + \int_{-1}^{-2} -2dt$

$+ \int_{-2}^x -3dt = -t\Big|_0^{-1} - 2t\Big|_{-1}^{-2} - 3t\Big|_{-2}^x$

$= 1 + 2 - 3(x+2) = -3x - 3$. For

$-2 \leq x \leq -1$, $g(x) = \int_0^{-1} -dt + \int_{-1}^x -2dt$

$= -t\Big|_0^{-1} - 2t\Big|_{-1}^x = -2x - 1$. For $-1 \leq x \leq 0$,

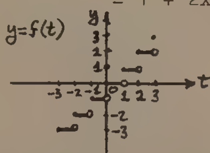
$g(x) = \int_0^x -dt = -t\Big|_0^x = -x$. For $0 \leq x \leq 1$,

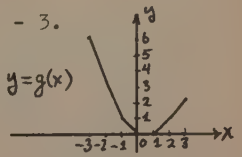
$g(x) = \int_0^x 0dt = 0$; for $1 \leq x \leq 2$,

$g(x) = \int_1^x dt = x - 1$; for $2 \leq x \leq 3$,

$g(x) = \int_1^2 dt + \int_2^x 2dt = t\Big|_1^2 + 2t\Big|_2^x$

$= 1 + 2x - 4 = 2x - 3.$





80. For $x \leq 0$, $g(x) = \int_{-1}^x -2dt = -2t\Big|_{-1}^x$

$= -2x - 2 = -2(x+1)$. For $0 < x \leq 5$,

$$g(x) = \int_{-1}^{0} -2dt + \int_{0}^{x} t\,dt$$

$$= -2t \Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{x} = -2 + \frac{x^2}{2}.$$ For $x > 5$,

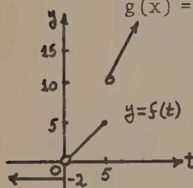
$$g(x) = \int_{-1}^{0} -2dt + \int_{0}^{5} t\,dt + \int_{5}^{x} 2t\,dt$$

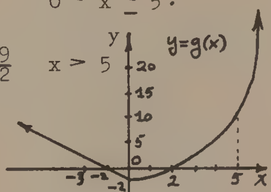
$$= -2t\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{5} + t^2\Big|_{5}^{x}$$

$$= -2 + \frac{25}{2} + x^2 - 25 = x^2 - \frac{29}{2}.$$

Hence,

$$g(x) = \begin{cases} -2(x-1) & x \leq 0 \\ \frac{x^2}{2} - 2 & 0 < x \leq 5. \\ x^2 - \frac{29}{2} & x > 5 \end{cases}$$





81. By the first part of the fundamental theorem, $D_x \int_a^x f(t)dt = f(x)$.

By the second part of the fundamental theorem, $\int_a^x D_t f(t)dt = f(t)\Big|_a^x = f(x)-f(a)$.

Hence, $f(x) = f(x) - f(a) + C$ for $C=f(a)$.

Problem Set 5.5, page 345

1. $T_4 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + \frac{y_4}{2})\Delta x$, where $\Delta x = \frac{1-0}{4} = \frac{1}{4}$, $y_0 = \frac{1}{1+0^2}$, $y_1 = \frac{1}{1+(\frac{1}{4})^2}$, $y_2 = \frac{1}{1+(\frac{1}{2})^2}$, $y_3 = \frac{1}{1+(\frac{3}{4})^2}$, $y_4 = \frac{1}{1+(1)^2}$.

$T_4 = (\frac{1}{2} + \frac{16}{17} + \frac{4}{5} + \frac{16}{25} + \frac{1}{4})(\frac{1}{4}) \approx 0.783$.

Hence, $\int_0^1 \frac{dx}{1+x^2} \approx 0.783$.

2. $T_3 = (\frac{y_0}{2} + y_1 + y_2 + \frac{y_3}{2})\Delta x$, where $\Delta x = \frac{3-1}{3} = \frac{2}{3}$, $y_0 = \frac{1}{1}$, $y_1 = \frac{1}{\frac{5}{3}} = \frac{3}{5}$, $y_2 = \frac{1}{\frac{7}{3}} = \frac{3}{7}$, $y_3 = \frac{1}{3}$. $T_3 = (\frac{1}{2} + \frac{3}{5} + \frac{3}{7} + \frac{1}{6})(\frac{2}{3}) \approx 1.13$

Hence, $\int_1^3 \frac{dx}{x} \approx 1.130$.

3. $T_6 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + \frac{y_6}{2})(\Delta x)$, where $\Delta x = \frac{8-2}{6} = 1$, and $y_k = \frac{1}{1+(2+k)}$.

Hence, $T_6 = (\frac{1}{6} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{18})(1)$ ≈ 1.107. Hence, $\int_2^8 \frac{dx}{1+x} \approx 1.107$.

4. $T_6 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + \frac{y_6}{2})\Delta x$, where $\Delta x = \frac{3-0}{6} = \frac{1}{2}$ and $y_k = \sqrt{9-(\frac{1}{2}k)^2}$.

$T_6 = (\frac{3}{2} + \frac{\sqrt{35}}{2} + \sqrt{8} + \frac{\sqrt{27}}{2} + \sqrt{5} + \frac{\sqrt{11}}{2} + 0)(\frac{1}{2})$ ≈ 6.889. Hence, $\int_0^3 \sqrt{9-x^2}\,dx \approx 6.889$.

5. $T_5 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + y_4 + \frac{y_5}{2})(\Delta x)$, where $\Delta x = \frac{1-0}{5} = \frac{1}{5}$ and $y_k = \frac{1}{\sqrt{1+(\frac{k}{5})^4}}$.

$T_5 = (\frac{1}{2} + \frac{25}{\sqrt{626}} + \frac{25}{\sqrt{641}} + \frac{25}{\sqrt{706}} + \frac{25}{\sqrt{881}} + \frac{1}{2\sqrt{2}})$ $(\frac{1}{5}) \approx 0.925$. Hence, $\int_0^1 \frac{dx}{\sqrt{1+x^4}} \approx 0.925$.

6. $T_4 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + \frac{y_4}{2})(\Delta x)$, where $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $y_k = \frac{1}{1+(\frac{k}{4})^3}$.

$T_4 = (\frac{1}{2} + \frac{64}{65} + \frac{64}{72} + \frac{64}{91} + \frac{1}{4})(\frac{1}{4}) \approx 0.832$.

Hence, $\int_0^1 \frac{dx}{1+x^3} \approx 0.832$.

7. $T_6 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + \frac{y_6}{2})(\Delta x$ where $\Delta x = \frac{8-2}{6} = 1$ and $y_k = \frac{1}{\sqrt[3]{4+(2+k)^2}}$.

$T_6 = (\frac{1}{2\sqrt[3]{8}} + \frac{1}{\sqrt[3]{13}} + \frac{1}{\sqrt[3]{20}} + \frac{1}{\sqrt[3]{29}} + \frac{1}{\sqrt[3]{40}} + \frac{1}{\sqrt[3]{53}} + \frac{1}{2\sqrt[3]{68}})(1) \approx 2.050$.

Hence, $\int_2^8 (4+x^2)^{-\frac{1}{3}}\,dx \approx 2.050.$

8. $T_7 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + \frac{y_7}{2})\Delta x$, where $\Delta x = \frac{3-2}{7} = \frac{1}{7}$ and $y_k = \sqrt{1+(2+\frac{k}{7})^2}$. $T_7 = (\frac{\sqrt{5}}{2} + \frac{\sqrt{274}}{7} + \frac{\sqrt{305}}{7} + \frac{\sqrt{338}}{7} + \frac{\sqrt{373}}{7} + \frac{\sqrt{410}}{7} + \frac{\sqrt{449}}{7} + \frac{\sqrt{10}}{2})(\frac{1}{7})$ ≈ 2.695. Hence, $\int_2^3 \sqrt{1+x^2}\,dx \approx 2.695$.

9. $T_5 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + y_4 + \frac{y_5}{2})\Delta x$, where $\Delta x = \frac{2-1}{5} = \frac{1}{5}$ and $y_k = \frac{1}{(1+\frac{k}{5})\sqrt{1+(1+\frac{k}{5})}}$ $= \frac{1}{(1+\frac{k}{5})\sqrt{2+\frac{k}{5}}}$. $T_5 = (\frac{1}{2\sqrt{2}} + \frac{5}{6\sqrt{\frac{11}{5}}} + \frac{5}{7\sqrt{\frac{12}{5}}} + \frac{5}{8\sqrt{\frac{13}{5}}} + \frac{5}{9\sqrt{\frac{14}{5}}} + \frac{1}{4\sqrt{3}})(\frac{1}{5}) \approx 0.448.$

Hence, $\int_1^2 \frac{dx}{x\sqrt{1+x}} \approx 0.448.$

10. $T_4 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + \frac{y_4}{2})\Delta x$, where $\Delta x = \frac{\pi - \frac{\pi}{2}}{4} = \frac{\pi}{8}$ and $y_k = \frac{\sin(\frac{\pi}{2} + \frac{k\pi}{8})}{\frac{\pi}{2} + \frac{k\pi}{8}}$ $= \frac{\sin(\frac{(4+k)\pi}{8})}{\frac{(4+k)\pi}{8}}$.

$T_4 = (\frac{1}{\pi} + \frac{\sin\frac{5\pi}{8}}{\frac{5\pi}{8}} + \frac{\sin\frac{3\pi}{4}}{\frac{3\pi}{4}} + \frac{\sin\frac{7\pi}{8}}{\frac{7\pi}{8}} + 0)$ $(\frac{\pi}{8}) \approx 0.482$. Thus, $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x}\,dx \approx 0.482.$

11. $T_4 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + \frac{y_4}{2})\Delta x$, where $\Delta x = \frac{\frac{\pi}{4} - 0}{4} = \frac{\pi}{16}$, $y_k = \tan\frac{k\pi}{16}$.

$T_4 = (0 + \tan\frac{\pi}{16} + \tan\frac{\pi}{8} + \tan\frac{3\pi}{16} + \frac{\tan\frac{\pi}{4}}{2})$ $\cdot \frac{\pi}{16} \approx 0.349$. Hence, $\int_0^{\frac{\pi}{4}} \tan x\,dx$ ≈ 0.349.

12. $T_6 = (\frac{y_0}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + \frac{y_6}{2})\Delta x$, where $\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6}$, $y_k = \frac{\sin(\frac{\pi}{6}k)}{1 + \frac{\pi}{6}k}$.

$T_4 = (0 + \frac{\sin\frac{\pi}{6}}{1+\frac{\pi}{6}} + \frac{\sin\frac{\pi}{3}}{1+\frac{\pi}{3}} + \frac{\sin\frac{\pi}{2}}{1+\frac{\pi}{2}} + \frac{\sin\frac{2\pi}{3}}{1+\frac{2\pi}{3}} + \frac{\sin\frac{5\pi}{6}}{1+\frac{5\pi}{6}} + 0)\ \frac{\pi}{6} \approx 0.816$; hence, $\int_0^{\pi} \frac{\sin x}{1+x}\,dx \approx 0.816.$

13. Here $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$. Notice that $f'''(x) = \frac{24x(1-x^2)}{(1+x^2)^4} \geq 0$ for $0 \leq x \leq 1$; hence, f'' is an increasing function on the interval $[0,1]$. Since $f''(0) = -2$ and $f''(1) = \frac{1}{2}$, it follows that the maximum value of $|f''(x)|$ for $0 \leq x \leq 1$ is $|-2| = 2$. Hence, we can take $M = 2$ in Theorem 2. We conclude that $|\text{error}| \leq M\,\frac{(b-a)^3}{12M^2}$ $= (2)\,\frac{1^3}{(12)(16)} = \frac{1}{96} \approx 0.01.$

14. Here $f''(x) = \frac{2}{x^3}$ and f'' is a decreasing function on the interval $[1,3]$. Since $f''(x) > 0$ for $1 \leq x \leq 3$, then the maximum value of $|f''(x)|$ for $1 \leq x \leq 3$ is $f''(1) = 2$. Hence, we can take $M = 2$ in Theorem 2 and conclude that $|\text{error}|$ $\leq M\,\frac{(b-a)^3}{12M^2} = (2)\,\frac{2^3}{(12)(9)} = \frac{4}{27} \approx 0.15.$

15. $T_n \leq \int_a^b f(x)\,dx$ because each trapezoid is contained in the region under the curve.

16. The area of the k^{th} trapezoid is $\Delta x(\frac{y_k + y_{k-1}}{2})$; hence, the sum of the areas of all of the trapezoids is

$$\sum_{k=1}^{n} \Delta x \left(\frac{y_k + y_{k-1}}{2}\right) =$$

$$\frac{(y_1+y_0)+(y_2+y_1)+\dots+(y_{n-1}+y_{n-2})+(y_n+y_{n-1})}{2}\Delta x$$

$$= \left(\frac{y_0}{2} + \frac{y_1+y_1}{2} + \frac{y_2+y_2}{2} + \dots + \frac{y_{n-1}+y_{n-1}}{2} + \frac{y_n}{2}\right)\Delta x$$

$$= \left(\frac{y_0}{2} + y_1 + y_2 + \dots + y_{n-1} + \frac{y_n}{2}\right)\Delta x.$$

17. $S_4 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$, where $\Delta x = \frac{b-a}{2n} = \frac{2}{4} = \frac{1}{2}$ and $y_k = \frac{1}{1 + (-1+\frac{k}{2})^2}$. $S_4 = \frac{1}{6}(\frac{1}{2} + \frac{(4)(4)}{5} + 2 + \frac{(4)(4)}{5} + \frac{1}{2}) \approx 1.567$. Hence, $\int_{-1}^{1} \frac{dx}{1+x^2} \approx 1.567$.

18. $S_8 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8)$, where $\Delta x = \frac{b-a}{2n} = \frac{4}{8} = \frac{1}{2}$ and $y_k = (\frac{k}{2})^2\sqrt{\frac{k}{2} + 1}$. $S_8 = \frac{1}{6}\Big[0 + 4(\frac{1}{4})\sqrt{\frac{3}{2}} + (2)\sqrt{2} + 4(\frac{9}{4})\sqrt{\frac{5}{2}} + 2\cdot 4\sqrt{3} + 4(\frac{25}{4})\sqrt{\frac{7}{2}} + 2(9)(2) + 4(\frac{49}{4})\sqrt{\frac{9}{2}} + 16\sqrt{5}\Big] \approx 42.439$. Hence, $\int_0^4 x^2\sqrt{x+1}\,dx \approx 42.439$.

19. $S_8 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8)$, where $\Delta x = \frac{b-a}{2n} = \frac{8-0}{8} = 1$ and $y_k = \frac{1}{k^3+k+1}$. So $S_8 = \frac{1}{3}(1 + \frac{4}{3} + \frac{2}{11} + \frac{4}{31} + \frac{2}{69} + \frac{4}{131} + \frac{2}{223} + \frac{4}{351} + \frac{1}{521}) \approx 0.909$. Hence, $\int_0^8 \frac{dx}{x^3+x+1} \approx 0.909$.

20. $S_8 = \frac{\Delta x}{3}(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+2y_6+4y_7+y_8)$, where $\Delta x = \frac{b-a}{2n} = \frac{8}{8} = 1$, $y_k = \frac{1}{1 + (2+k)^3}$. So $S_8 = \frac{1}{3}(\frac{1}{9} + \frac{4}{28} + \frac{2}{65} + \frac{4}{126} + \frac{2}{217} + \frac{4}{344} + \frac{2}{513} + \frac{4}{730} + \frac{1}{1001}) \approx 0.116$. Hence, $\int_2^{10} \frac{dx}{1+x^3} \approx 0.116$.

21. $S_6 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$, where $\Delta x = \frac{2-0}{6} = \frac{1}{3}$ and $y_k = (\frac{k}{3})\sqrt{9-(\frac{k}{3})^3}$. So $S_6 = \frac{1}{9}(0 + \frac{4}{3}\sqrt{9-\frac{1}{27}} + \sqrt{9 - \frac{8}{27}} + 4\sqrt{8} + \frac{8}{3}\sqrt{9-\frac{64}{27}} + \frac{20}{3}\sqrt{9-\frac{125}{27}} + 2)$. Hence, $S_6 \approx 4.671$. Hence, $\int_0^2 x\sqrt{9-x^3}\,dx \approx 4.671$.

22. $S_4 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$, where $\Delta x = \frac{2}{4} = \frac{1}{2}$, and $y_k = \frac{1}{\sqrt{1+(\frac{k}{2})^2}}$. $S_4 = \frac{1}{6}(1 + \frac{8}{\sqrt{5}} + \frac{2}{\sqrt{2}} + \frac{8}{\sqrt{13}} + \frac{1}{\sqrt{5}}) \approx 1.443$. Hence, $\int_0^2 \frac{dx}{\sqrt{1 + x^2}} \approx 1.443$.

23. $S_8 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8)$, where $\Delta x = \frac{b-a}{2n} = \frac{2}{8} = \frac{1}{4}$ and $y_k = \sqrt{1+(\frac{k}{4})^4}$. $S_8 = \frac{1}{12}(1 + \frac{\sqrt{257}}{4} + \frac{\sqrt{17}}{2} + \frac{\sqrt{337}}{4} + 2\sqrt{2} + \frac{\sqrt{881}}{4} + \frac{\sqrt{97}}{2} + \frac{\sqrt{2657}}{4} + \sqrt{17}) \approx 3.653$. Hence, $\int_0^2 \sqrt{1+x^4}\,dx \approx 3.653$.

24. $S_8 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8)$, where $\Delta x = \frac{2-0}{8} = \frac{1}{4}$ and $y_k = \sqrt[3]{1-(\frac{k}{4})^2}$. $S_8 = \frac{1}{12}(1+4\sqrt[3]{\frac{15}{16}} + 2\sqrt[3]{\frac{3}{4}} + 4\sqrt[3]{\frac{7}{16}} + 2\cdot 0 + 4\sqrt[3]{-\frac{9}{16}} + 2\sqrt[3]{-\frac{5}{4}} + 4\sqrt[3]{-\frac{33}{16}} + \sqrt[3]{-3}) \approx -0.185$. Hence, $\int_0^2 \sqrt[3]{1 - x^2}\,dx \approx -0.185$

25. $S_6 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$, where $\Delta x = \frac{b-a}{2n} = \frac{\frac{\pi}{2}}{6} = \frac{\pi}{12}$ and $y_k = \sqrt{\cos\frac{k\pi}{12}}$.

$S_8 = \frac{\pi}{36}\left(1 + 4\sqrt{\cos\frac{\pi}{12}} + 2\sqrt{\cos\frac{\pi}{6}} + 4\sqrt{\cos\frac{\pi}{4}}\right.$

$\left. + 2\sqrt{\cos\frac{\pi}{3}} + 4\sqrt{\cos\frac{5\pi}{12}} + 0\right) \approx 1.187.$

Hence, $\int_0^{\frac{\pi}{2}}\sqrt{\cos x}\,dx \approx 1.187.$

26. $S_8 + \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4$

$+ 4y_5 + 2y_6 + 4y_7 + y_8)$, where $\Delta x = \frac{b-a}{2n}$

$= \frac{\frac{\pi}{3} - \frac{\pi}{6}}{8} = \frac{\pi}{48}$ and $y_k = \csc(\frac{\pi}{6} + \frac{\pi}{48}k)$.

$S_8 = \frac{\pi}{144}\left(\csc\frac{\pi}{6} + 4\csc\frac{9\pi}{48} + 2\csc\frac{5\pi}{24}\right.$

$+ 4\csc\frac{11\pi}{48} + 2\csc\frac{\pi}{4} + 4\csc\frac{13\pi}{48}$

$\left. + 2\csc\frac{7\pi}{24} + 4\csc\frac{15\pi}{48} + \csc\frac{\pi}{3}\right) \approx 0.768.$

Hence, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc x\,dx \approx 0.768.$

27. Here $f^{(4)}(x) = \frac{24}{x^5}$, and on the interval $[1,2]$, $\left|f^{(4)}(x)\right| = \frac{24}{x^5}$ takes on its maximum value $N = 24$ when $x = 1$. We want $N \cdot \frac{(b-a)^5}{2880n^4} \leq 0.0001$; that is, $\frac{24}{2880n^4} \leq \frac{1}{10{,}000}$, or $\frac{250}{3} \leq n^4$. The smallest value of n for which this holds is $n = 4$.

28. Here $f^{(4)}(x) = \frac{24}{x^5}$, and on the interval $[2.5, 2.7]$, $\left|f^4(x)\right| = \frac{24}{x^5}$ takes on its maximum value $N = \frac{24}{(2.5)^5}$ when $x = 2.5$.

An upper bound for the error is

$N \cdot \frac{(b-a)^5}{2880(1)^4} = \left[\frac{24}{(2.5)^5}\right]\left[\frac{(0.2)^5}{2880}\right] = \frac{32}{(120)(25)^5}$

$< 3 \times 10^{-8}$. $\quad x = \frac{2.7 - 2.5}{2} = \frac{1}{10}$,

$y_0 = \frac{1}{2.5}$, $y_1 = \frac{1}{2.6}$, $y_2 = \frac{1}{2.7}$, and

$S_2 = \frac{1}{30}\left(\frac{1}{2.5} + \frac{4}{2.6} + \frac{1}{2.7}\right) \approx 0.076961.$

29. Because $\int_0^1\sqrt{1 - x^2}\,dx$ gives the area of $\frac{1}{4}$ of a circle of radius 1. $S_4 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$, where $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $y_k = \sqrt{1 - (\frac{k}{4})^2}$.

$S_4 = \frac{1}{12}(1 + \sqrt{15} + \sqrt{3} + \sqrt{7} + 0) \approx 0.771.$

So $4(0.771) \approx \pi$. Hence, $\pi \approx 3.084$.

30. $S_{10} = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4$

$+ 4y_5 + 2y_6 + 4y_7 + 2y_8 + 4y_9 + y_{10})$,

where $\Delta x = \frac{b-a}{2n} = \frac{1}{10}$ and $y_k = \sqrt{1-(\frac{k}{10})^2}$.

$S_{10} = \frac{1}{30}\left(1 + \frac{4\sqrt{99}}{10} + \frac{2\sqrt{24}}{5} + \frac{4\sqrt{91}}{10} + \frac{2\sqrt{21}}{5}\right.$

$\left. + \frac{4\sqrt{3}}{2} + \frac{2\cdot 4}{5} + \frac{4\sqrt{51}}{10} + \frac{2\cdot 3}{5} + \frac{4\sqrt{19}}{10} + 0\right)$

≈ 0.78175. Hence, $\pi \approx 3.12701$ by this estimation. Notice that our estimate of π is correct only to the first place.

31. (a) $\int_{-a}^{a}(Ax^3 + Bx^2 + Cx + D)\,dx$

$= \frac{2a}{6}\left[Aa^3 + Ba^2 + Ca + D + 4(0+0+0+D) + \right.$

$\left. -Aa^3 + Ba^2 - Ca + D\right] = \frac{a}{3}(2Ba^2 + 6D).$

(b) $\int_{-a}^{a}(Ax^3 + Bx^2 + Cx + D)\,dx$

$= \frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx)\Big|_{-a}^{a}$

$= \frac{Aa^4}{4} + \frac{Ba^3}{3} + \frac{Ca^2}{2} + Da - \frac{Aa^4}{4} + \frac{Ba^3}{3} -$

$\frac{Ca^2}{2} + Da = \frac{2Ba^3}{3} + 2Da = \frac{a}{3}(2Ba^2 + 6D).$

32. Let $f(x) = Ax^3 + Bx^2 + Cx + D$.

$\int_a^b(Ax^3 + Bx^2 + Cx + D)dx$

$= \left(\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx\right)\Big|_a^b = \frac{Ab^4}{4} + \frac{Bb^3}{3} +$

$\frac{Cb^2}{2} + Db - \frac{Aa^4}{4} - \frac{Ba^3}{3} - \frac{Ca^2}{2} - Da$

$= \frac{A}{4}(b^4-a^4) + \frac{B}{3}(b^3-a^3) + \frac{C}{2}(b^2-a^2) + D(b-a)$

$= (b-a)\left[\frac{A}{4}(b^2+a^2)(b+a) + \frac{B}{3}(b^2+ab+a^2) + \frac{C}{2}(a+b) + D\right] = \frac{b-a}{12}\left[3A(b^3+a^2b+b^2a+a^3) + 4B(b^2+ab+a^2) + 6C(b+a) + 12D\right] = \frac{b-a}{12}$
$\left[3AB^3 + 3Aa^2b + 3Ab^2a + 3Aa^3 + 4Bb^2 + 4Bab + 4Ba^2 + 6Cb + 6Ca + 12D\right]$
$= \frac{b-a}{12}\left[2Aa^3 + 2Ba^2 + 2Ca + 2D + 2Ab^3 + 2Bb^2 + 2Cb + 2D + Aa^3 + 2Ba^2 + 4Ca + 8D + Ab^3 + 2Bb^2 + 4Cb + 3Ab^2a + 3Aa^2b + 4Bab\right]$
$= \frac{b-a}{12}\left[2f(a) + 2f(b) + Aa^3 + 2Ba^2 + 4Ca + 6D + Ab^3 + 2Bb^2 + 4Cb + 3Ab^2 + 3Aa^2b + 4Bab\right] = \frac{b-a}{12}\left[2f(a) + 2f(b) + A(a^3 + b^3 + 3ab^2 + 3a^2b + b^3) + 2B(a^2 + 2ab + b^2) + 4C(a+b) + 8D\right] = \frac{b-a}{12}\left[2f(a) + 2f(b) + 8A\left(\frac{a+b}{2}\right)^3 + 8B\left(\frac{a+b}{2}\right)^2 + 8C\left(\frac{a+b}{2}\right) + 8D\right]$
$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$

33. We need only solve the following three equations in three unknowns:

$p = Ac^2 + Bc + C;$

$q = A(c + \Delta x)^2 + B(c + \Delta x) + C;$

$r = A(c + 2\Delta x)^2 + B(c + 2\Delta x) + C.$

$q = Ac^2 + 2Ac\Delta x + A(\Delta x)^2 + Bc + B\Delta x + C$

Subtracting p from q, $q-p = 2Ac\Delta x + A(\Delta x)^2 + B\Delta x$.

$r = Ac^2 + 4Ac\Delta x + 4A(\Delta x)^2 + Bc + 2B\Delta x + C,$

and subtracting p from r, $r - p = 4Ac\Delta x + 4A(\Delta x)^2 + 2B\Delta x$. Now $2(p-q) = 4Ac\Delta x + 2A(\Delta x)^2 + 2B\Delta x$, so $r - p - 2q - 2p = 2A(\Delta x)^2$, and so

$A = \frac{r + p - 2q}{2(\Delta x)^2}$. Hence, $q - p$

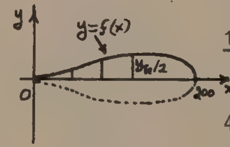
$= \frac{(r + p - 2q)}{x} \cdot c + \frac{r + p - 2q}{2} + B\Delta x.$

So $B = \frac{q - p}{\Delta x} - \left(\frac{r + p - 2q}{(\Delta x)^2}\right)c - \frac{r + p - 2q}{2\Delta x}$

Now $C = p - Ac^2 - Bc =$
$p - \left(\frac{r + p - 2q}{2(\Delta x)^2}\right)\cdot c^2 - \left[\frac{q - p}{\Delta x} - \frac{(r + p - 2q)(c)}{(\Delta x)^2} - \frac{(r + p - 2q)}{2\Delta x}\right]\cdot c.$

With the choices of A, B, and C as above, the graph of $y = Ax^2 + Bx + C$ will pass through the given points.

34. (a)



$\frac{\text{Area}}{2} = \int_0^{200} f(x)dx \approx \frac{10}{3}\left[\frac{y_0}{2} + 4\left(\frac{y_1}{2}\right) + 2\left(\frac{y_2}{2}\right) + 4\left(\frac{y_3}{2}\right) + \ldots + 4\left(\frac{y_{19}}{2}\right) + \frac{y_{20}}{2}\right]$

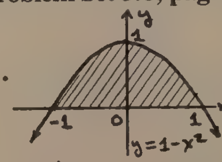
So area $= \frac{10}{3}(4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{18} + 4y_{19})$.

(b) The weight of a slab of water 1 foot high with given cross-sectional area $\doteq (64)\left(\frac{10}{3}\right)(4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{18} + 4y_{19})$ pounds

$= \frac{640}{6000}(4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{18} + 4y_{19})$ tons of freight.

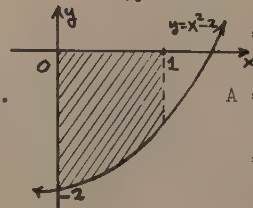
Problem Set 5.6, page 351

1.



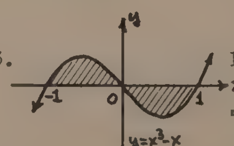
$A = \int_{-1}^{1}(1-x^2)dx = \left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1}$
$= \left(1 - \frac{1}{3}\right)-\left(-1 + \frac{1}{3}\right)$
$= \frac{2}{3}-\left(-\frac{2}{3}\right)=\frac{4}{3}$ square units.

2.



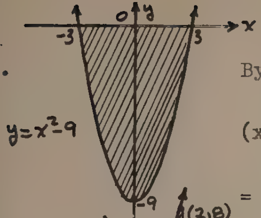
$A = -\int_0^1(x^2-2)dx=-\left(\frac{x^3}{3}-2x\right)\Big|_0^1$
$= -\left[\left(\frac{1}{3} - 2\right)-\left(\frac{0^3}{3}- 0\right)\right]$
$= \frac{5}{3}$ square units.

3.



By symmetry, $A = 2\int_{-1}^{0}(x^3-x)d$
$= 2\left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0}= 2\left[0-\left(\frac{1}{4} - \frac{1}{2}\right)\right]$
$= \frac{1}{2}$ square unit.

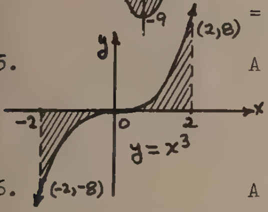
4.



By symmetry, $A = -2\int_0^3 (x^2-9)dx = -2(\frac{x^3}{3} - 9x)\Big|_0^3$

$= -2(9-27-0) = 36$ sq. units.

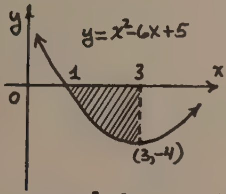
5.



$A = 2\int_0^2 x^3dx = 2\cdot\frac{x^4}{4}\Big|_0^2$

$= 2(\frac{2^4}{4} - 0) = 8$ sq.units.

6.

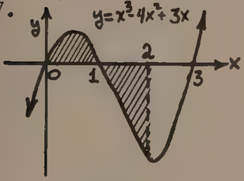


$A = -\int_1^3(x^2 - 6x + 5)dx$

$= -(\frac{x^3}{3} - 3x^2 + 5x)\Big|_1^3$

$= -\left[(9 - 27 + 15) - (\frac{1}{3} - 3 + 5)\right] = \frac{16}{3}$ sq. units.

7.



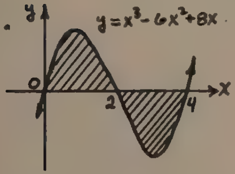
$A = \int_0^1(x^3-4x^2+3x)dx - \int_1^2(x^3 - 4x^2 + 3x)dx$

$= (\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2)\Big|_0^1 - (\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2)\Big|_1^2$

$= (\frac{1}{4} - \frac{4}{3} + \frac{3}{2}) - 0 - \left[(4 - \frac{32}{3} + 6) - (\frac{1}{4} - \frac{4}{3} + \frac{3}{2})\right] = \frac{1}{2} - \frac{8}{3} + 3 - 4 + \frac{32}{3} - 6$

$= \frac{3}{2}$ square units.

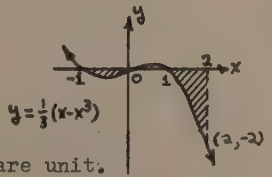
8.



By symmetry, $A = 2\int_0^2(x^3-6x^2+8x)dx = 2(\frac{x^4}{4} - 2x^3 + 4x^2)\Big|_0^2$

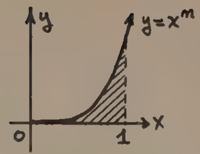
$= 2(4-16+16-0)=8$ sq. units.

9. $A = -\int_{-1}^0 \frac{1}{3}(x-x^3)dx + \int_0^1 \frac{1}{3}(x-x^3)dx - \int_1^2 \frac{1}{3}(x-x^3)dx = -\left[\frac{1}{3}(\frac{x^2}{2} - \frac{x^4}{4})\right]\Big|_{-1}^0 + \left[\frac{1}{3}(\frac{x^2}{2} - \frac{x^4}{4})\right]\Big|_0^1 - \left[\frac{1}{3}(\frac{x^2}{2} - \frac{x^4}{4})\right]\Big|_1^2$



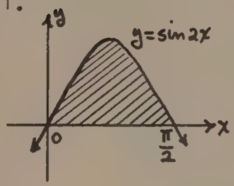
$= \frac{1}{12} + \frac{1}{12} + \frac{3}{4} = \frac{11}{12}$ square unit.

10.



$A = \int_0^1 x^n dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$ square unit.

11.

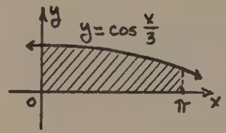


$A = \int_0^{\frac{\pi}{2}} \sin 2x \, dx$

$= \frac{-\cos 2x}{2}\Big|_0^{\frac{\pi}{2}} = \frac{-\cos\pi}{2} + \frac{\cos 0}{2}$

$= \frac{1}{2} + \frac{1}{2} = 1$ square unit.

12.



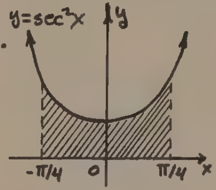
$A = \int_0^{\pi} \cos\frac{x}{3}\, dx$

$= 3\sin\frac{x}{3}\Big|_0^{\pi}$

$= 3\sin\frac{\pi}{3} - 3\sin 0$

$= 3\frac{\sqrt{3}}{2} - 3(0) = \frac{3\sqrt{3}}{2}$ sq. units.

13.



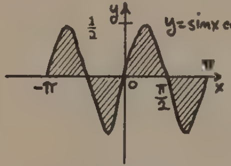
By symmetry $A = 2\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$

$= 2\tan x\Big|_0^{\frac{\pi}{4}} = 2\tan\frac{\pi}{4} - 2\tan 0$

$= 2(1) - 2(0) = 2$ sq. units.

14. $F(x) = \sin x \cos x = \frac{1}{2}(2\sin x\cos x)$

$= \frac{1}{2}\sin 2x$



By symmetry, $A = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2x\, dx$

$= 2(\frac{-\cos 2x}{2})\Big|_0^{\frac{\pi}{2}} = -\cos\pi + \cos 0$

$= -(-1) + 1 = 2$ square units.

15. (a) $A = \int_{-1}^3\left[1+\sqrt{1+x} - (\frac{x+3}{2})\right]dx$

$= \int_{-1}^3(\sqrt{1+x} - \frac{x}{2} - \frac{1}{2})dx$. Let $u = 1 + x$ and $du = dx$, then $\int\sqrt{1+x}\, dx = \int u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}}+c$

$= \frac{2}{3}(1 + x)^{\frac{3}{2}} + c$ so $A = \left[\frac{2}{3}(1+x)^{\frac{3}{2}} - \frac{x^2}{4} - \frac{1}{2}x\right]\Big|_{-1}^3$

$= \frac{2}{3}(4)^{\frac{3}{2}} - \frac{9}{4} - \frac{3}{2} - (0 - \frac{1}{4} + \frac{1}{2})$

$= \frac{16}{3} - \frac{15}{4} - \frac{1}{4} = \frac{4}{3}$ square units.

(b) $A = \int_1^3\left[2y - 3 - (y^2 - 2y)\right] dy$

$= \int_1^3(-y^2 + 4y - 3)\, dy$

$= (-\frac{y^3}{3} + 2y^2 - 3y)\Big|_1^3$

$= -9 + 18 - 9 - (-\frac{1}{3} + 2 - 3) = \frac{4}{3}$ sq.units.

16. (a) $A = \int_{-4}^{0}\left[-\frac{x^2}{4} - (-\sqrt{-4x})\right]$

$= \int_{-4}^{0}\left[-\frac{x^2}{4} + 2\sqrt{-x}\right]dx$

Let $u = -x$ and $du = -dx$, then

$\int\sqrt{-x}\,dx = \int u^{\frac{1}{2}}(-du) = -\frac{2}{3}u^{\frac{3}{2}} + c$

$= -\frac{2}{3}(-x)^{\frac{3}{2}} + c$

So $A = \left[-\frac{x^3}{12} + 2(-\frac{2}{3})(-x)^{\frac{3}{2}}\right]_{-4}^{0}$

$= 0 - \left[\frac{64}{12} - \frac{4}{3}(4)^{\frac{3}{2}}\right] = -(\frac{64}{12} - \frac{32}{3})$

$= \frac{64}{12} = \frac{16}{3}$.

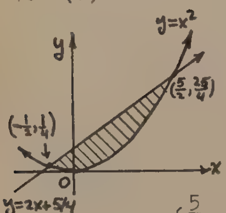
(b) $A = \int_{-4}^{0}\left[-\frac{y^2}{4} - (-\sqrt{-4y})\right]dy$

$= \int_{-4}^{0}(-\frac{y^2}{4} + 2\sqrt{-y})\,dy$

$= \int_{-4}^{0}(-\frac{x^2}{4} + 2\sqrt{-x})\,dx$ from part

(a) $= \frac{16}{3}$ square units.

17. (a)



(b) $x^2 = 2x + \frac{5}{4}$, $4x^2 - 8x - 5 = 0$, $(2x+1)(2x-5) = 0$. $x = -\frac{1}{2}$ or $x = \frac{5}{2}$. The points of intersection are $(-\frac{1}{2}, \frac{1}{4})$ and $(\frac{5}{2}, \frac{25}{4})$.

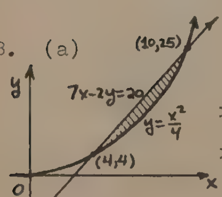
(c) $A = \int_{-\frac{1}{2}}^{\frac{5}{2}}\left[(2x + \frac{5}{4}) - (x^2)\right]dx$

$= (x^2 + \frac{5}{4}x - \frac{x^3}{3})\Big|_{-\frac{1}{2}}^{\frac{5}{2}}$

$= (\frac{25}{4} + \frac{25}{8} - \frac{125}{24}) - (\frac{1}{4} - \frac{5}{8} + \frac{1}{24})$

$= \frac{9}{2}$ square units.

18. (a)



(b) $7x - 2(\frac{x^2}{4}) = 20$, $7x - \frac{x^2}{2} = 20$, $x^2 - 14x + 40 = 0$, $(x-10)(x-4) = 0$; $x = 10$ or $x = 4$. The points of intersection are $(10,25)$ and $(4,4)$.

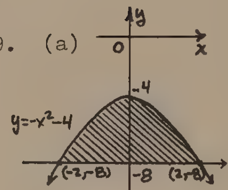
(c) $A = \int_{4}^{10}\left[(\frac{7x-20}{2}) - \frac{x^2}{4}\right]dx$

$= (\frac{7x^2}{4} - 10x - \frac{x^3}{12})\Big|_4^{10}$

$= (7(25) - 100 - \frac{1000}{12}) - (28 - 40 - \frac{64}{12})$

$= 9$ square units.

19. (a)



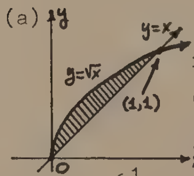
(b) $-x^2 - 4 = -8$, $x^2 = 4$; $x = 2$ or $x = -2$. The points of intersection are $(2,-8)$ and $(-2,-8)$.

(c) $A = 2\int_{0}^{2}\left[(-x^2 - 4) - (-8)\right]dx$

$= 2(-\frac{x^3}{3} + 4x)\Big|_0^2 = 2(-\frac{8}{3} + 8)$

$= \frac{32}{3}$ square units.

20. (a)

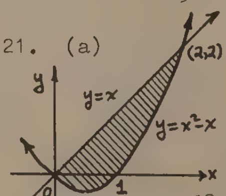


(b) $\sqrt{x} = x$, $x = x^2$, $x^2 - x = 0$; $x = 0$ or $x = 1$. The points of intersection are $(0,0)$ and $(1,1)$.

(c) $A = \int_{0}^{1}(\sqrt{x} - x)dx$

$= (\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2})\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ sq.unit.

21. (a)



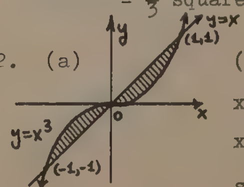
(b) $x^2 - x = x$, $x^2 - 2x = 0$, $x(x-2) = 0$; so $x = 0$ or $x = 2$. The points of intersection are $(0,0)$ and $(2,2)$.

(c) $A = \int_{0}^{2}\left[x - (x^2 - x)\right]dx$

$= (x^2 - \frac{x^3}{3})\Big|_0^2 = 4 - \frac{8}{3}$

$= \frac{4}{3}$ square units.

22. (a)



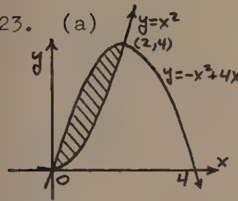
(b) $x^3 = x$, $x^3 - x = 0$, $x(x^2 - 1) = 0$; so $x = 0$, $x = 1$, or $x = -1$. The points of intersection are $(0,0)$, $(1,1)$ $(-1,-1)$

(c) $A = 2\int_0^1 (x - x^3)\,dx$

$= 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right)$

$= \frac{1}{2}$ square unit.

23. (a)



(b) $-x^2+4x=x^2$, $2x^2-4x=0$

$2x^2-4x=0$

$2x(x-2)=0$; $x=0$ or $x=2$. The Points of intersection are $(0,0)$ and $(2,4)$.

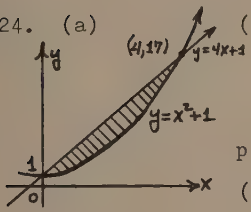
(c) $A = \int_0^2 \left[-x^2 + 4x - x^2\right] dx$

$= \int_0^2 (-2x^2 + 4x)\,dx$

$= \left(-\frac{2}{3}x^3 + 2x^2\right)\Big|_0^2 = -\frac{16}{3} + 8$

$= \frac{8}{3}$ square units.

24. (a)



(b) $x^2+1 = 4x+1$

$x^2-4x = 0$, $x(x-4) = 0$;

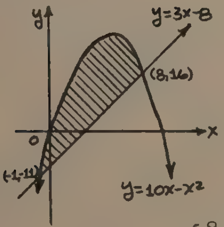
$x = 0$ or $x = 4$. The points of intersection are $(0,1)$ and $(4,17)$.

(c) $A = \int_0^4 \left[4x + 1 - (x^2 + 1)\right] dx$

$= \int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{x^3}{3}\right)\Big|_0^4$

$= 32 - \frac{64}{3} = \frac{32}{3}$ square units.

25. (a)



(b) $10x-x^2 = 3x-8$,

$x^2-7x-8 = 0$,

$(x-8)(x+1) = 0$;

$x = 8$ or $x = -1$. The points of intersection are $(8,16)$ and $(-1,-11)$.

(c) $A = \int_{-1}^8 \left[10x - x^2-(3x - 8)\right] dx$

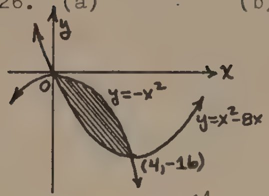
$= \int_{-1}^8 (7x - x^2 + 8)\,dx$

$= \left(\frac{7x^2}{2} - \frac{x^3}{3} + 8x\right)\Big|_{-1}^8$

$= 7(32) - \frac{512}{3} + 64 - \left(\frac{7}{2} + \frac{1}{3} - 8\right)$

$= 296 - \frac{512}{3} - \frac{23}{6} = \frac{729}{6} = \frac{243}{2}$ square units.

26. (a)



(b) $x^2-8x=-x^2$, $2x^2-8x=0$,

$2x(x-4)=0$;

$x = 0$ or $x = 4$. The points of intersection are $(0,0)$ and $(4,-16)$.

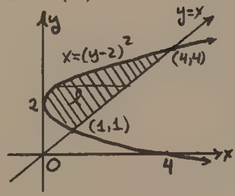
(c) $A = \int_0^4 \left[-x^2 - (x^2 - 8x)\right] dx$

$= \int_0^4 (-2x^2 + 8x)\,dx$

$= \left(-\frac{2x^3}{3} + 4x^2\right)\Big|_0^4 = -\frac{128}{3} + 64$

$= \frac{64}{3}$ square units.

27. (a)



(b) $y = (y-2)^2$, $y=y^2-4y+4$,

$0 = y^2-5y+4$, $0=(y-4)(y-1)$;

$y = 4$ or $y = 1$. The points of intersection are $(4,4)$ and $(1,1)$.

(c) By slicing, taking the reference axis to be the y axis, we have

$A = \int_1^4 \left[y - (y - 2)^2\right] dy$

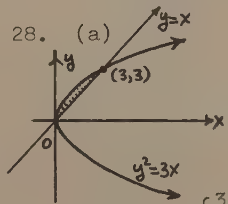
$= \int_1^4 (5y - y^2 - 4)\,dy$

$= \left(\frac{5}{2}y^2 - \frac{y^3}{3} - 4y\right)\Big|_1^4$

$= \left(\frac{5}{2}(16) - \frac{64}{3} - 16\right) - \left(\frac{5}{2} - \frac{1}{3} - 4\right)$

$= \frac{9}{2}$ square units.

28. (a)



(b) $x^2 = 3x$, $x(x-3) = 0$;

$x = 0$ or $x = 3$. The points of intersection are $(0,0)$ and $(3,3)$.

(c) $A = \int_0^3 (\sqrt{3x} - x)\,dx = \left(\sqrt{3}\cdot\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{2}\right)\Big|_0^3$

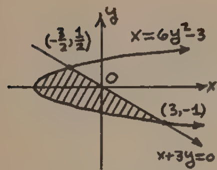
$= \frac{2\sqrt{3}}{3}\cdot 3\sqrt{3} - \frac{9}{2}$

$= 6 - \frac{9}{2} = \frac{3}{2}$ square units.

29. (a)

(b) $-3y = 6y^2-3$, $6y^2+3y-3=0$,

$2y^2+y-1=0$, so $(2y-1)(y+1)$



$= 0$; $y = \frac{1}{2}$ or $y = -1$.

The points of intersection are $(-\frac{3}{2},\frac{1}{2})$ and $(3,-1)$.

(c) By slicing, taking the reference axis to be the y axis, we have

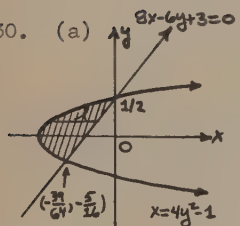
$$A = \int_{-1}^{\frac{1}{2}} \left[(-3y)-(6y^2-3)\right]dy,$$

$$A = (-\tfrac{3}{2}y^2 - 2y^3 + 3y)\Big|_{-1}^{\frac{1}{2}}$$

$$= (-\tfrac{3}{2}(\tfrac{1}{4}) - \tfrac{1}{4} + \tfrac{3}{2})-(-\tfrac{3}{2} + 2 - 3)$$

$$= \tfrac{27}{8} \text{ square units.}$$

30. (a)



(b) $8(4y^2-1)-6y+3=0$, $32y^2-6y-5=0$, $(16y+5)(2y-1) = 0$; $y = -\frac{5}{16}$ or $y = \frac{1}{2}$.

The points of intersection are $(0,\frac{1}{2})$ and $(-\frac{39}{64},-\frac{5}{16})$.

(c) By slicing, taking the reference axis to be the y axis, we have,

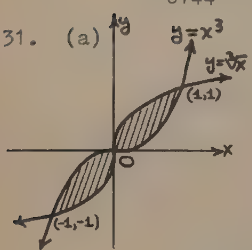
$$A = \int_{-\frac{5}{16}}^{\frac{1}{2}} \left[(\tfrac{6y}{8} - \tfrac{3}{8})-(4y^2 - 1)\right]dy$$

$$= (\tfrac{3y^2}{8} - \tfrac{4}{3}y^3 + \tfrac{5}{8}y)\Big|_{-\frac{5}{16}}^{\frac{1}{2}} =$$

$$\left[\tfrac{3}{8}(\tfrac{1}{4}) - \tfrac{4}{3}(\tfrac{1}{8}) + \tfrac{5}{8}(\tfrac{1}{2})\right] - \left[\tfrac{3}{8}\cdot\tfrac{25}{256} + \tfrac{4}{3}\cdot\tfrac{125}{16^3} - \tfrac{25}{128}\right]$$

$$= \tfrac{2197}{6144} \text{ square units.}$$

31. (a)



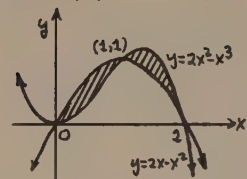
(b) $x^3 = \sqrt[3]{x}$, $x^9 = x$, $x^9-x = 0$, $x(x^8-1) = 0$; $x = 0$ or $x = \pm1$. So $(-1,-1)$, $(0,0)$ and $(1,1)$ are the points of intersection.

(c) $$A = 2 \int_0^1 (\sqrt[3]{x} - x^3)\, dx$$

$$= 2\cdot\left[\tfrac{3}{4}x^{\frac{4}{3}} - \tfrac{x^4}{4}\right]\Big|_0^1 = 2(\tfrac{3}{4} - \tfrac{1}{4})$$

$$= 1 \text{ square unit.}$$

32. (a)



(b) $2x^2-x^3=2x-x^2$, so $x^3-3x^2+2x=0$, $x(x-2)(x-1) =0$; $x=0$, $x=2$, or $x=1$.

The points of intersectio are $(0,0)$, $(2,0)$, and $(1,1$

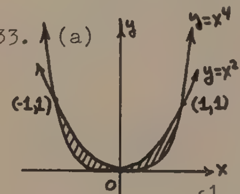
(c) $$A = \int_0^1 \left[(2x-x^2)-(2x^2-x^3)\right] dx + \int_1^2\left[(2x^2-x^3) - (2x-x^2)\right] dx$$

$$A = (x^2-x^3+ \tfrac{x^4}{4})\Big|_0^1 + (x^3 - \tfrac{x^4}{4}-x^2)\Big|_1^2$$

$$A = (1 - 1 + \tfrac{1}{4}) + (8 - \tfrac{16}{4}-4)-(1-\tfrac{1}{4} -1)$$

$$A = \tfrac{1}{2} \text{ square unit.}$$

33. (a)



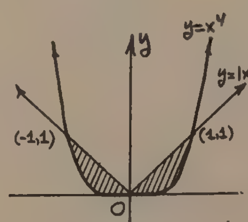
(b) $x^2=x^4$, $x^2(x^2-1)=0$ $x = 0$; $x = 1$ or $x = -1$.

The points of intersectio are $(0,0)$, $(1,1)$, $(-1,1)$.

(c) $$A = 2 \int_0^1(x^2-x^4)dx = 2(\tfrac{x^3}{3} - \tfrac{x^5}{5})\Big|_0^1$$

$$= 2(\tfrac{1}{3} - \tfrac{1}{5}) = \tfrac{4}{15} \text{ square unit.}$$

34. (a)



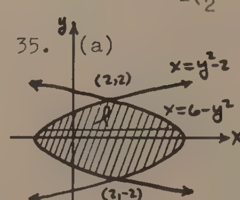
(b) $x^4 = |x|$. For $x > 0$, $x^4-x=0$, $x(x^3-1)=0$; $x=0$ or $x=1$. For $x < 0$, $x^4=-x$ $x(x^3+1)=0$; $x=0$ or $x=-1$.

The points of intersectio are $(0,0)$, $(1,1)$, and $(-1,$

(c) $$A = 2 \int_0^1(x-x^4)dx = 2(\tfrac{x^2}{2} - \tfrac{x^5}{5})\Big|_0^1$$

$$= 2(\tfrac{1}{2} - \tfrac{1}{5}) = \tfrac{3}{5} \text{ square unit.}$$

35. (a)



(b) $y^2-2 = 6 - y^2$, $2y^2=8$, $y^2 = 4$; $y = 2$ or $y = -2$.

The points of intersectio are $(2,2)$ and $(2,-2)$.

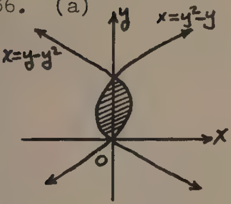
(c) By slicing, taking the reference axis to be the y axis, we have

$$A = 2 \int_0^2\left[(6 - y^2) - (y^2 - 2)\right] dy$$

$= 2(8y - \frac{2}{3}y^3) \Big|_0^2 = 2(16 - \frac{16}{3}$

$= \frac{64}{3}$ square units.

36. (a)



(b) $y^2-y=y-y^2$, $2y^2=2y$, $y^2=y$, $y(y-1)=0$; $y=0$ or $y=1$. The points of intersection are (0,0) and (0,1).

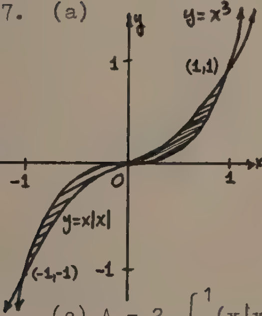
(c) By slicing, taking the reference axis to be the y axis, we have

$A = 2\int_0^1 (y - y^2 - 0)\,dy$

$= 2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right)$

$= \frac{1}{3}$ square unit.

37. (a)



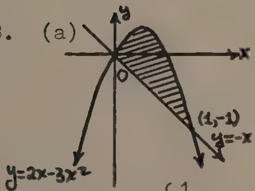
(b) If $x > 0$, $x^2 = x^3$, $x^2(x-1) = 0$; so $x = 0$ or $x = 1$. If $x < 0$, $-x^2 = x^3$, $x^2(x+1) = 0$; so $x = 0$ or $x = -1$. The points of intersection are (0,0), (1,1), and (-1,-1).

(c) $A = 2\int_0^1 (x|x| - x^3)\,dx = 2\int_0^1 (x^2 - x^3)\,dx$

$= 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 2\left(\frac{1}{3} - \frac{1}{4}\right)$

$= \frac{1}{6}$ square unit.

38. (a)



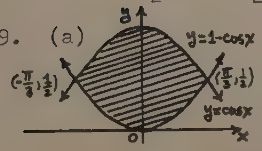
(b) $-x = 2x - 3x^2$, $0 = 3x - 3x^2 = 3x(1-x)$; $x = 0$ or $x = 1$. The points of intersection are (0,0) and (1,-1).

(c) $A = \int_0^1 \left[(2x - 3x^2) - (-x)\right] dx$

$= \int_0^1 (3x - 3x^2)\,dx = \left(\frac{3x^2}{2} - x^3\right)\Big|_0^1$

$= \frac{3}{2} - 1 = \frac{1}{2}$ square unit.

39. (a)



(b) $1-\cos x = \cos x$, $\cos x = \frac{1}{2}$; $x = \frac{\pi}{3}, -\frac{\pi}{3}$. The points of intersection are $(\frac{\pi}{3}, \frac{1}{2})$ and $(-\frac{\pi}{3}, \frac{1}{2})$.

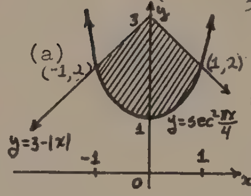
(c) $A = 2\int_0^{\pi/3} \left[\cos x - (1 - \cos x)\right] dx$

$= 2\int_0^{\pi/3} (2\cos x - 1)\,dx$

$= 2(2\sin x - x)\Big|_0^{\pi/3}$

$= 2\left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) = 2\sqrt{3} - \frac{2\pi}{3}$ sq. unit.

40. (a)



(b) For $x \geq 0$, $3 - x = \sec^2 \frac{\pi x}{4}$; so $x = 1$. For $x < 0$, $3 + x = \sec^2\frac{\pi x}{4}$; so $x = -1$. The points of intersection are (1,2) and (-1,2).

(c) $A = 2\int_0^1 \left[3 - x - \sec^2\frac{\pi x}{4}\right]dx$

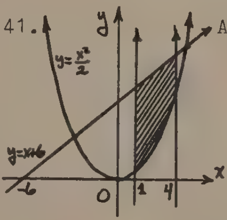
$= 2\left(3x - \frac{x^2}{2} - \frac{4}{\pi}\tan\frac{\pi x}{4}\right)\Big|_0^1$

$= 2\left(3 - \frac{1}{2} - \frac{4}{\pi}\tan\frac{\pi}{4}\right) - 0$

$= 2\left(3 - \frac{1}{2} - \frac{4}{\pi}\right) = 2\left(\frac{5}{2} - \frac{4}{\pi}\right)$

$= 5 - \frac{8}{\pi}$ square units.

41.



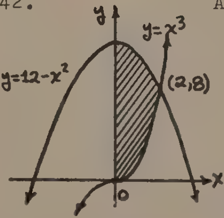
$A = \int_1^4 \left[(x+6) - (\frac{1}{2}x^2)\right] dx$

$= \left(\frac{x^2}{2} + 6x - \frac{x^3}{6}\right)\Big|_1^4$

$= \left(8 + 24 - \frac{32}{3}\right) - \left(\frac{1}{2} + 6 - \frac{1}{6}\right)$

$= 15$ square units.

42.



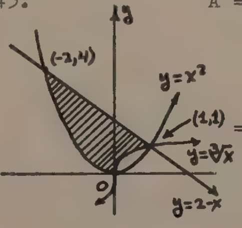
$A = \int_0^2 \left[(12 - x^2) - x^3\right] dx$

$= \left(12x - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^2$

$= 24 - \frac{8}{3} - 4 - 0$

$= \frac{52}{3}$ square units.

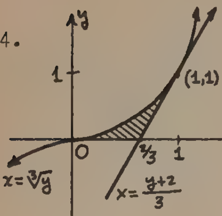
43.



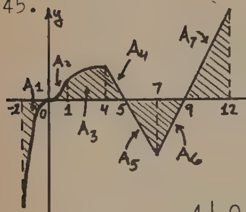
$A = \int_{-2}^0 \left[(2 - x) - x^2\right]dx + \int_0^1 \left[(2 - x) - \sqrt[3]{x}\right]dx$

$= \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-2}^0 + \left(2x - \frac{x^2}{2} - \frac{3}{4}x^{\frac{4}{3}}\right)\Big|_0^1$

$= 0 - \left(-4 - 2 + \frac{8}{3}\right) + 2 - \frac{1}{2} - \frac{3}{4} = \frac{49}{12}$ sq. units.

44. 

$A = \int_0^1\left[\frac{y+2}{3} - \sqrt[3]{y}\right]dy$
$= \left(\frac{y^2}{6}+\frac{2}{3}y - \frac{3}{4}y^{\frac43}\right)\Big|_0^1$
$= \frac16+\frac23-\frac34 = \frac{1}{12}$ sq.unit.

45. 

$A(R) = -\int_{-2}^0 x^3dx + \int_0^1 x^3dx + \int_1^4\sqrt{x}\,dx + \int_4^5(10-2x)dx - \int_5^7(10-2x)dx - \int_7^9(2x-18)dx + \int_9^{12}(2x-18)\,dx;$

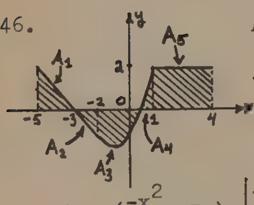
$A(R) = -\frac{x^4}{4}\Big|_{-2}^0 + \frac{x^4}{4}\Big|_0^1 + \frac23 x^{\frac32}\Big|_1^4 + (10x-x^2)\Big|_4^5 - (10x-x^2)\Big|_5^7 - (x^2-18x)\Big|_7^9 + (x^2-18x)\Big|_9^{12};$

$A(R) = (+4) + (\frac14) + (\frac{16}{3}-\frac23) + (25-24) - (-4) - (-4) + (9);$

$A(R) = +4+\frac14+\frac{14}{3}+1+4+4+9 = 22\frac{59}{12}$

$= \frac{323}{12}$ square units.

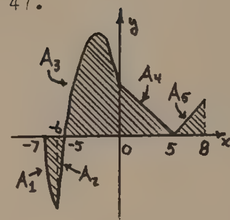
$\int_{-2}^{12} f(x)dx = A_2+A_3+A_4+A_7-A_1 - A_5 - A_6 = \frac14+\frac{14}{3}+1+9-4-4-4 = \frac{35}{12}.$

46. 

$A = \int_{-5}^{-3}(-x-3)dx - \int_{-3}^{-2}(-x-3)dx - \int_{-2}^{\sqrt2-1}(x^2+2x-1)dx + \int_{\sqrt2-1}^{1}(x^2+2x-1)dx + \int_1^4 2dx;$

$A = (\frac{-x^2}{2}-3x)\Big|_{-5}^{-3} - (\frac{-x^2}{2}-3x)\Big|_{-3}^{-2} - (\frac{x^3}{3}+x^2-x)\Big|_{-2}^{\sqrt2-1} + (\frac{x^3}{3}+x^2-x)\Big|_{\sqrt2-1}^{1} + (2x)\Big|_1^4;$ $A = \left[(-\frac92+9)-(-\frac{25}{2}+15)\right] - \left[(-2+6)-(-\frac92+9)\right] - \left[\frac{(\sqrt2-1)^3}{3} + (\sqrt2-1)^2-(\sqrt2-1)-(-\frac83+4+2)\right] + \left[(\frac13+1-1)-(\frac{(\sqrt2-1)^3}{3}+(\sqrt2-1)^2 - (\sqrt2-1))\right] + (8-2) = \frac16(65+16\sqrt2)$ sq. units.

$\int_{-5}^{4} f(x)dx = A_1+A_4+A_5-A_2-A_3 = \frac92.$

47. 

$A = -\int_{-7}^{-6}(x^2+6x-7)dx - \int_{-6}^{-5}(-x^2-4x+5)dx + \int_{-5}^{0}(-x^2-4x+5)dx + \int_0^5(5-x)dx + \int_5^8(x-5)\,dx;$

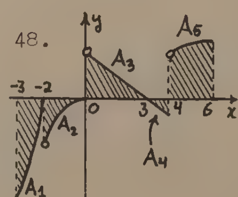
$A = -(\frac{x^3}{3}+3x^2-7x)\Big|_{-7}^{-6} - (-\frac{x^3}{3}-2x^2+5x)\Big| + (-\frac{x^3}{3}-2x^2+5x)\Big|_{-5}^{0} + (5x-\frac{x^2}{2})\Big|_0^5 + (\frac{x^2}{2}-5x)\Big|_5^8;$ $A = -\left[(-\frac{216}{3}+108+42) - (-\frac{343}{3}+147+49)\right] - \left[(\frac{125}{3}-50-25) - (\frac{216}{3}-72-30)\right] + \left[0-(\frac{125}{3}-50-25)\right] + (25-\frac{25}{2}) + \left[(32-40)-(\frac{25}{2}-25)\right];$

$A = 238 - \frac{593}{3}+\frac{34}{2} = \frac{344}{6}$ square units.

$\int_{-7}^{8} f(x) = A_3+A_4+A_5-A_1-A_2$

$= (75-\frac{125}{3})+\frac{25}{2}+\frac92-(46-\frac{127}{3})-(\frac{91}{3}-27)$

$= \frac{130}{3}.$

48. 

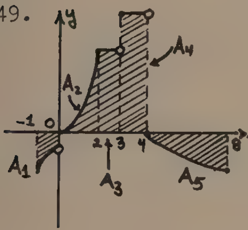
$A = -\int_{-3}^{-2} x\sqrt{x^2-4}\,dx - \int_{-2}^{0}(-x^2)dx + \int_0^3(3-x)dx - \int_3^4(3-x)dx + \int_4^6\sqrt{2x+1}\,dx.$

Let $u = x^2-4$, $du = 2xdx$ to evaluate the first integral. So $\int \frac12 u^{\frac12}du = \frac13 u^{\frac32} + C$. Also, let $v = 2x+1$, $dv = 2dx$ in order to evaluate the last integral. So $\int\sqrt{2x+1}\,dx = \int\frac12 v^{\frac12}dv = \frac13 v^{\frac32}+$ Hence, $A = -\frac13(x^2-4)^{\frac32}\Big|_{-3}^{-2} - (-\frac{x^3}{3})\Big|_{-2}^{0} + (3x-\frac{x^2}{2})\Big|_0^3 - (3x-\frac{x^2}{2})\Big|_3^4 + \frac13(2x+1)^{\frac32}\Big|_4^6;$

$A = -\left[\frac{1}{3}\cdot 0 - \frac{1}{3}\cdot 5^{\frac{3}{2}}\right] + (0 + \frac{8}{3}) + (9 - \frac{9}{2}) - \left[(12 - 8) - (9 - \frac{9}{2})\right] + \left[\frac{1}{3}(13)^{\frac{3}{2}} - \frac{27}{3}\right];$

$A = \frac{5}{3}\sqrt{5} + \frac{8}{3} + \frac{9}{2} + \frac{1}{2} + (\frac{13}{3}\sqrt{13} - 9) = \frac{1}{3}(-4 + 5\sqrt{5} + 13\sqrt{13})$ square units.

$\int_{-3}^{6} f(x)dx = A_3 + A_5 + A_1 - A_2 - A_4 = \frac{1}{3}(-23 - 5\sqrt{5} + 13\sqrt{13}).$

49. 

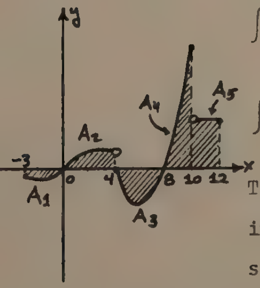
$A = -\int_{-1}^{0}(x^3-1)dx + \int_{0}^{2}x^2dx + \int_{2}^{3}4dx + \int_{3}^{4}6dx - \int_{4}^{8}(-\sqrt{x-4})dx.$ In the last integral, $u = x - 4$, so $du = dx$ and $\int(-u^{\frac{1}{2}})du = -\frac{2}{3}u^{\frac{3}{2}} + C.$

$A = -(\frac{x^4}{4} - x)\Big|_{-1}^{0} + \frac{x^3}{3}\Big|_{0}^{2} + 4x\Big|_{2}^{3} + 6x\Big|_{3}^{4} + \frac{2}{3}(x-4)^{\frac{3}{2}}\Big|_{4}^{8};$ $A = -\left[0 - (\frac{1}{4} + 1)\right] + \frac{8}{3} + (12 - 8) + (24 - 18) + \frac{2}{3}(8 - 0);$

$A = \frac{5}{4} + \frac{8}{3} + 4 + 6 + \frac{16}{3} = \frac{77}{4}$ square units.

$\int_{-1}^{8} f(x)dx = A_2 + A_3 + A_4 - A_1 - A_5 = \frac{8}{3} + 4 + 6 - \frac{5}{4} - \frac{16}{3} = \frac{73}{12}.$

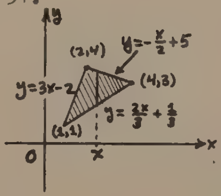
50. $A = -\int_{-3}^{0}\frac{xdx}{(x^2+1)^2} + \int_{0}^{4}\sqrt[3]{2x}dx - \int_{4}^{8}(x^2-12x+32)dx + \int_{8}^{10}(x^2-12x+32)dx + \int_{10}^{12}5dx.$



To evaluate the first integral, let $u = x^2+1$, so $du = 2xdx$, and $\int\frac{xdx}{(x^2+1)^2} = \int\frac{\frac{1}{2}du}{u^2} = -\frac{u^{-1}}{2} + C.$ So

$A = +\frac{1}{2(x^2+1)}\Big|_{-3}^{0} + \sqrt[3]{2}(\frac{3}{4}x^{\frac{4}{3}})\Big|_{0}^{4} - (\frac{x^3}{3} - 6x + 32x)\Big|_{4}^{8} + (\frac{x^3}{3} - 6x^2 + 32x)\Big|_{8}^{10} + 5x\Big|_{10}^{12};$ $A = (\frac{1}{2} - \frac{1}{20}) + \sqrt[3]{2}\,\frac{3}{4}(4^{\frac{4}{3}}) - \left[(\frac{512}{3} - 384 + 256) - (\frac{64}{3} - 96 + 128)\right] + \left[(\frac{1000}{3} - 600 + 320) - (\frac{512}{3} - 384 + 256)\right] + (60 - 50);$ $A \approx \frac{9}{20} + 6.00 + (160 - \frac{448}{3}) + (\frac{488}{3} - 152) + 10 = \frac{2267}{60}$ square units.

$\int_{-3}^{12}f(x)dx = A_2 + A_4 + A_5 - A_1 - A_3 = \frac{311}{20}.$

51. 

The line through (1,1) and (2,4) has equation $y - 1 = 3(x-1)$; the line through (2,4) and (4,3) has equation $y - 4 = -\frac{1}{2}(x-2)$; the line through (1,1) and (4,3) has equation $y - 1 = \frac{2}{3}(x-1)$. Taking the reference axis to be the x axis, we have $A = \int_{1}^{2}\left[(3x-2) - (\frac{2}{3}x+\frac{1}{3})\right]dx + \int_{2}^{4}\left[(-\frac{1}{2}x+5) - (\frac{2}{3}x+\frac{1}{3})\right]dx.$ So,

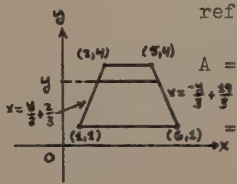
$A = \int_{1}^{2}(\frac{7}{3}x - \frac{7}{3})dx + \int_{2}^{4}(-\frac{7}{6}x + \frac{14}{3})dx$

$= (\frac{7}{6}x^2 - \frac{7}{3}x)\Big|_{1}^{2} + (\frac{-7}{12}x^2 + \frac{14}{3}x)\Big|_{2}^{4}$

$= (\frac{7}{6}(4) - \frac{14}{3}) - (\frac{7}{6} - \frac{7}{3}) + (-\frac{7}{12}(16) + \frac{14}{3}(4))$

$- (\frac{-28}{12} + \frac{28}{3}) = \frac{7}{2}$ square units.

52. Take the y axis to be the reference axis. Then



$A = \int_{1}^{4}\left[(-\frac{y}{3} + \frac{19}{3}) - (\frac{y}{3} + \frac{2}{3})\right]dy = \int_{1}^{4}(-\frac{2y}{3} + \frac{17}{3})dy$

$= (-\frac{y^2}{3} + \frac{17}{3}y)\Big|_{1}^{4}$

$= (-\frac{16}{3} + \frac{68}{3}) - (-\frac{1}{3} + \frac{17}{3}) = 12$ sq. units.

Review Problem Set, Chapter 5, page 353

1. $\sum_{k=1}^{5}(5k+3) = (5\cdot1 + 3)+(5\cdot2 + 3)+(5\cdot3 + 3)$
$+(5\cdot4 + 3)+(5\cdot5 + 3) = 8 + 13 + 18 + 23$
$+ 28 = 90.$

2. $\sum_{i=1}^{3} 5(i+1)^2 = 5(1+1)^2+5(2+1)^2+5(3+1)^2$
$= 5\cdot4 + 5\cdot9 + 5\cdot16 = 145.$

3. $\sum_{k=0}^{4} \frac{k}{k+1} = \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1}$
$= 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} = \frac{163}{60}.$

4. $\sum_{i=0}^{4} \frac{1}{i^2+1} = \frac{1}{0^2+1} + \frac{1}{1^2+1} + \frac{1}{2^2+1} + \frac{1}{3^2+1}$
$= 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{9}{5}.$

5. $\sum_{k=1}^{4} \cos\frac{\pi}{k} = \cos\pi + \cos\frac{\pi}{2} + \cos\frac{\pi}{3} + \cos\frac{\pi}{4}$
$= -1 + 0 + \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2}.$

6. $\sum_{k=0}^{6} \tan\frac{k\pi}{3} = \tan 0 + \tan\frac{\pi}{3} + \tan\frac{2\pi}{3}$
$+ \tan\pi + \tan\frac{4\pi}{3} + \tan\frac{5\pi}{3} + \tan 2\pi$
$= 0 + \sqrt{3} - \sqrt{3} + 0 + \sqrt{3} - \sqrt{3} + 0$
$= 0$

7. $\sum_{k=1}^{n} k(2k-1) = \sum_{k=1}^{n} 2k^2 - \sum_{k=1}^{n} k$
$= \frac{2\cdot n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)\cdot[2(2n+1)-3]}{6}$
$= \frac{n(n+1)(4n-1)}{6}.$

8. $\sum_{j=1}^{n} (6^{j+1} - 6^j) = \sum_{j=1}^{n} 6^j(6-1) = 5\cdot\sum_{j=1}^{n} 6^j$
$= 5\cdot6 \sum_{j=1}^{n} 6^{j-1} = 5\cdot6 \sum_{k=0}^{n-1} 6^k = 5\cdot6\,\frac{1-6^n}{1-6}$
$= -6(1 - 6^n) = 6(6^n - 1).$

9. $\sum_{j=0}^{n} (3^j + 3^{j+1}) = \sum_{j=0}^{n} 3^j(1 + 3)$
$= 4\sum_{j=0}^{n} 3^j = \frac{4(1-3^{n+1})}{1-3}$
$= 2(3^{n+1} - 1).$

10. $\sum_{k=0}^{n}(k+1)^3 = \sum_{j=1}^{n+1} j^3$
$= \frac{(n+1)^2(n+2)^2}{4}.$

11. $2 + 4 + 6 + 8 + \ldots + 2000$
$= 2(1 + 2 + 3 + 4 + \ldots + 1000)$
$= 2\sum_{k=1}^{1000} k = \frac{2(1000)(1001)}{2} = 1{,}001{,}000.$

12. If $n = 1$, then $2(1) - 1 = 1^2$. Suppose $\sum_{k=1}^{n}(2k-1) = n^2$. We want to show that $\sum_{k=1}^{n+1}(2k-1) = (n+1)^2$. Now, $\sum_{k=1}^{n+1}(2k-1)$
$= \sum_{k=1}^{n}(2k-1) + [2(n+1)-1] = n^2+2n+1$
$= (n+1)^2$. Hence, $\sum_{k=1}^{n}(2k-1) = n^2$
for all n.

13. (a) $1 + 3 + 9 + 27 + \ldots + 3^{k-1} + \ldots$
$S = \sum_{k=1}^{14} 3^{k-1}$ cents.

(b) $\sum_{k=1}^{14} 3^{k-1} = \sum_{\ell=0}^{13} 3^\ell = \frac{1-3^{14}}{1-3}$
$= \frac{3^{14}-1}{2} = 2{,}391{,}484$ cents;
in dollars, \$23,914.84.

14. $\sum_{k=1}^{n} \frac{1}{k^2+k} = \sum_{k=1}^{n}(\frac{1}{k} - \frac{1}{k+1})$
$= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5})$
$+ \ldots + (\frac{1}{n-1} - \frac{1}{n}) + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$
$= \frac{n+1-1}{n+1} = \frac{n}{n+1}.$

15. $\sum_{k=1}^{n} f(c_k)\,\Delta x_k = f(\frac{1}{8})\cdot\frac{1}{4} + f(\frac{3}{8})\cdot\frac{1}{4} +$

$f(\frac{5}{8})\cdot\frac{1}{4} + f(\frac{7}{8})\cdot\frac{1}{4} = (\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64})(\frac{1}{4})$

$= \frac{21}{64}$. This Riemann sum is an approximation to the area under the curve $y = x^2$ from $x = 0$ to $x = 1$.

16. (a) $\lim\limits_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) = \int_0^1 f(x)dx.$

(b) $\lim\limits_{n\to+\infty} \frac{\sum_{k=1}^{n} k^5}{n^6} = \lim\limits_{n\to+\infty} \sum_{k=1}^{n} \frac{k^5}{n^5}\cdot\frac{1}{n}$

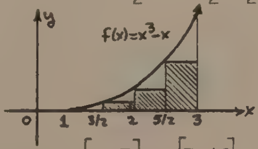
$= \lim\limits_{n\to+\infty} \sum_{k=1}^{n} (\frac{k}{n})^5\cdot\frac{1}{n}$. Here $\Delta x_k = \Delta x$

$= \frac{1}{n}$ and $c_k = \frac{k}{n}$, so that $f(c_k) = (\frac{k}{n})^5$.

Hence, $\lim\limits_{n\to+\infty} \frac{\sum_{k=1}^{n} k^5}{n^6} = \int_0^1 x^5dx.$

17. The augmented partition is $[1,\frac{3}{2}]$, $[\frac{3}{2},2]$, $[2,\frac{5}{2}]$, $[\frac{5}{2},3]$; $c_1 = 1$, $c_2 = \frac{3}{2}$, $c_3 = 2$, $c_4 = \frac{5}{2}$. $\sum_{k=1}^{4} f(c_k)\,\Delta x = f(1)\cdot\frac{1}{2} + f(\frac{3}{2})\cdot\frac{1}{2} + f(2)\cdot\frac{1}{2} + f(\frac{5}{2})\cdot\frac{1}{2}$

$= \frac{21}{2}$.

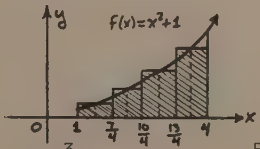


18. The augmented partition is $[1,\frac{7}{4}]$, $[\frac{7}{4},\frac{10}{4}]$, $[\frac{10}{4},\frac{13}{4}]$, $[\frac{13}{4},4]$; $c_1 = \frac{11}{8}$, $c_2 = \frac{17}{8}$, $c_3 = \frac{23}{8}$, $c_4 = \frac{29}{8}$. $\sum_{k=1}^{4} f(c_k)\Delta x =$

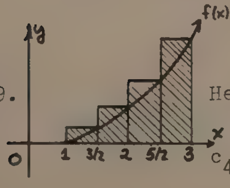
$[f(\frac{11}{8}) + f(\frac{17}{8}) + f(\frac{23}{8}) + f(\frac{29}{8})]\cdot\frac{3}{4}$

$= (\frac{185}{64} + \frac{353}{64} + \frac{593}{64} + \frac{905}{64})(\frac{3}{4}) = \frac{2036}{64}\ \frac{3}{4}$

$= \frac{509\cdot 3}{64} = \frac{1527}{64}.$



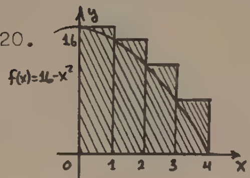
19.



Here, $c_1 = \frac{3}{2}$, $c_2 = 2$, $c_3 = \frac{5}{2}$, $c_4 = 3$. So, $\sum_{k=1}^{4} f(c_k)\,\Delta x$

$= (\frac{15}{8} + 6 + \frac{105}{8} + 24)(\frac{1}{2}) = \frac{45}{2}.$

20.



The augmented partition is $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$. $c_1 = 0$, $c_2 = 1$, $c_3 = 2$, $c_4 = 3$.

$\sum_{k=1}^{4} f(c_k)\,\Delta x = [f(0) + f(1) + f(2) + f(3)]\cdot 1$

$= 16 + 15 + 12 + 7 = 50.$

21. Choose $c_k = 1 + \frac{3(k-1)}{n}$, since $f(x) = x^2+1$ is increasing. $\Delta x = \frac{3}{n}$. So, $\sum_{k=1}^{n} f(c_k)\Delta x$

$= \sum_{k=1}^{n} ([1 + \frac{3(k-1)}{n}]^2 + 1)\frac{3}{n}$

$= \frac{3}{n}\sum_{k=1}^{n} (1+\frac{6(k-1)}{n} + \frac{9(k-1)^2}{n^2} + 1)$

$= \frac{3}{n}\sum_{k=1}^{n} (2 + \frac{6k}{n} - \frac{6}{n} + \frac{9k^2}{n^2} - \frac{18k}{n^2} + \frac{9}{n^2})$

$= \frac{3}{n}\Big[2n + \frac{6}{n}\cdot\frac{n(n+1)}{2} - 6 + \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}$

$- \frac{18}{n^2}\cdot\frac{n(n+1)}{2} + \frac{9}{n^2}\cdot n\Big] = 6 + \frac{9(n+1)}{n}$

$- \frac{18}{n} + \frac{9(n+1)(2n+1)}{2n^2} - \frac{27(n+1)}{n^2} + \frac{27}{n^2}$

$= 24 + \frac{9}{n} - \frac{18}{n} + \frac{27}{2n} + \frac{9}{2n^2} - \frac{27}{n} - \frac{27}{n^2} + \frac{27}{n^2}.$

$\lim\limits_{n\to+\infty} (24 - \frac{36}{n} + \frac{27}{2n} + \frac{9}{2n^2}) = 24$. Hence,

$\int_1^4 (x^2 + 1)\, dx = 24.$

22. Since $3x^2 + 1 > 0$, graph of $y = x^3 + x$ is always increasing; choose $c_k=1+\frac{2(k-1)}{n}$.

$\Delta x = \frac{2}{n}$. So $\sum_{k=1}^{n} f(c_k)\,\Delta x =$

$\sum_{k=1}^{n} \left\{ [1 + \frac{2(k-1)}{n}]^3 + 1 + \frac{2(k-1)}{n} \right\}\frac{2}{n}$

$= \frac{2}{n}\sum_{k=1}^{n} \Big[1 + \frac{3\cdot 2(k-1)}{n} + \frac{3\cdot 4(k-1)^2}{n^2}$

$+ \frac{8(k-1)^3}{n^3} + 1 + \frac{2(k-1)}{n}\Big] =$

$$= \frac{2}{n}\sum_{k=1}^{n}\left[2 + \frac{6k}{n} - \frac{6}{n} + \frac{12k^2}{n^2} - \frac{24k}{n^2} + \frac{12}{n^2}\right.$$

$$\left. + \frac{8k^3}{n^3} - \frac{24k^2}{n^3} + \frac{24k}{n^3} - \frac{8}{n^3} + \frac{2k}{n} - \frac{2}{n}\right]$$

$$= \frac{2}{n}\left[2n + \frac{6}{n}\,\frac{n(n+1)}{2} - \frac{8}{n}n + \frac{12}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right.$$

$$- \frac{24}{n^2}\,\frac{n(n+1)}{2} + \frac{12}{n^2}\cdot n + \frac{8}{n^3}\,\frac{n^2(n+1)^2}{4}$$

$$- \frac{24}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{24}{n^3}\,\frac{n(n+1)}{2} - \frac{8}{n^3}\,n$$

$$\left. + \frac{2}{n}\,\frac{(n)(n+1)}{2}\right] = 4 + 6(1 + \frac{1}{n}) - \frac{16}{n}$$

$$+ 4(2 + \frac{3}{n} + \frac{1}{n^2}) - 24(\frac{1}{n} + \frac{1}{n^2}) + \frac{24}{n^2}$$

$$+ 4(1 + \frac{2}{n} + \frac{1}{n^2}) - 8(\frac{2}{n} + \frac{3}{n^2} + \frac{1}{n^3}) + 24(\frac{1}{n^2} + \frac{1}{n^3})$$

$$- \frac{16}{n^3} + 2(1 + \frac{1}{n}) = s_n$$

$$\lim_{n\to+\infty} S_n = 4 + 6 + 8 + 4 + 2 = 24$$

Hence, $\int_1^3 (x^3 + x)dx = 24.$

23. $f(x) = x^2 + 1$ is increasing, so we choose c_k to be the right endpoint of each subinterval; that is, $c_k = 1 + \frac{3k}{n}$, where $\Delta x = \frac{4-1}{n} = \frac{3}{n}$. So $\sum_{k=1}^{n} f(c_k)\,\Delta x$

$$= \sum_{k=1}^{n}\left[(1 + \frac{3k}{n})^2 + 1\right]\frac{3}{n} = \frac{3}{n}\sum_{k=1}^{n}\left[2 + \frac{6k}{n} + \frac{9k^2}{n^2}\right]$$

$$= \frac{3}{n}\left[2n + \frac{6}{n}\cdot\frac{n(n+1)}{2} + \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{3}{n}\left[5n + 3 + \frac{6n^2+9n+3}{2n}\right] = 15 + \frac{9}{n} + 9 + \frac{27}{2n}$$

$$+ \frac{9}{2n^2}. \quad \lim_{n\to+\infty}(24 + \frac{9}{n} + \frac{27}{2n} + \frac{9}{2n^2}) = 24.$$

Hence, $\int_1^4 (x^2 + 1)dx = 24.$

24. Graph of $y = x^2 - 2x + 3$ is decreasing on $[-1,0]$. $\Delta x = \frac{1}{n}$. Choose $c_k = -1 + \frac{k-1}{n}$.

$$\sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n}\left[(-1 + \frac{k-1}{n})^2 - \right.$$

$$\left. - 2(-1 + \frac{k-1}{n}) + 3\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left[1 - \frac{2(k-1)}{n} + \frac{(k-1)^2}{n^2} + 2 - \frac{2(k-1)}{n}\right.$$

$$\left. + 3\right] = \frac{1}{n}\sum_{k=1}^{n}\left[6 - \frac{4k}{n} + \frac{4}{n} + \frac{k^2}{n^2} - \frac{2k}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{n}\left[6n - \frac{4}{n}\,\frac{n(n+1)}{2} + \frac{4}{n}\cdot n + \frac{1}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right.$$

$$\left. - \frac{2}{n^2}\,\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot n\right]$$

$$= 6 - 2(1 + \frac{1}{n}) + \frac{4}{n} + \frac{1}{6}(2 + \frac{3}{n} + \frac{1}{n^2})$$

$$- (\frac{1}{n} + \frac{1}{n^2}) + \frac{1}{n^2} = S_n.$$

$$\lim_{n\to+\infty} S_n = 6 - 2 + \frac{1}{3} = 4 + \frac{1}{3} = \frac{13}{3}.$$

25. It exists since $f(x) = \frac{[\![x]\!]}{x}$ is piecewise continuous on $[1,100]$.

26. It exists since $f(x) = \frac{1}{[\![x]\!]}$ is piecewise continuous on $[1,2]$.

27. It does not exist since $f(x) = \sqrt{1-x^2}$ is not defined for $x > 1$.

28. It exists since $f(x) = \frac{1}{1+x^2}$ is continuous on $[0,1]$.

29. It does not exist since $\cos\frac{\pi}{2} = 0$ and $\frac{1}{\cos\frac{\pi}{2}}$ is not defined.

30. It exists since f is continuous on $[-1,1]$.

31. It does not exist since $\tan 0 = 0$, so $\cot 0$ is not defined, and $\tan\pi = 0$, so $\cot\pi$ is not defined.

32. It does not exist since $\csc\frac{x}{2}$ is not defined at $x = 0$ and $x = 2\pi$.

33. $\int_4^1 (x^2 + 1)dx = -\int_1^4 (x^2 + 1)dx = -2x.$

34. $\int_3^1 (x^3 + x)dx = -\int_1^3 (x^3 + x)dx = -24.$

35. $\int_{\pi/2}^{\pi/2} \cot x \, dx = 0.$

36. Is not defined since $\tan \frac{\pi}{2}$ does not exist.

37. $f(x) = [\![x]\!]$ is not continuous on $[0,3]$ but $\int_0^3 f(x)\, dx$ does exist since $f(x)$ is piecewise continuous.

38. We will find $\sum_{k=1}^{n} f(c_k)\, \Delta x_k$, where $\Delta x_k = \frac{b-a}{n}$ and $\mathcal{P}_n^*$ consists of $\left[a, a + \frac{b-a}{n}\right]$. $\left[a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}\right], \ldots, \left[a + \frac{(n-1)(b-a)}{n}, b\right]$; $c_1 = a$, $c_2 = a + \frac{b-a}{n}$, $\ldots c_k = a + \frac{k(b-a)}{n}, \ldots c_n = a + \frac{n(b-a)}{n} = b$.

Here $f(x) = K$. Hence, $\sum_{k=1}^{n} f(c_k)\, \Delta x_k$

$= \sum_{k=1}^{n} K \frac{b-a}{n} = \frac{b-a}{n} \sum_{k=1}^{n} \cdot K = \frac{(b-a)}{n} \cdot K$

$= K(b-a)$. Now $\lim_{n \to +\infty} K \cdot (b-a) = K(b-a)$.

Therefore, $\int_a^b K dx = K \cdot (b-a)$.

39. $\int_1^3 (-5) f(x) dx = -5 \int_1^3 f(x) dx = -5(6) = -30.$

40. $\int_3^1 7f(x) dx = 7 \int_3^1 f(x) dx$

$= -7 \int_1^3 f(x) dx = -7(6) = -42.$

41. $\int_3^5 [f(x) + 3g(x)]\, dx = \int_3^5 f(x) dx$

$+ 3 \int_3^5 g(x) dx = 7 + 3(8) = 7 + 2x = 31.$

42. $\int_5^3 [4g(x) - 2f(x)]\, dx = -\int_3^5 [4g(x) - 2f(x)] dx$

$= -4 \int_3^5 g(x) dx + 2 \int_3^5 f(x) dx$

$= -4(8) + 2(7) = -32 + 14 = -18.$

43. $\int_1^5 f(x) dx = \int_1^3 f(x) dx + \int_3^5 f(x) dx$

$= 6 + 7 = 13.$

44. $\int_{\pi/2}^{\pi/2} \cos^2 x dx = 0.$

45. $\int_1^5 h(x) dx = \int_1^3 4f(x) dx$

$+ \int_3^5 [-2g(x)]\, dx = 4 \int_1^3 f(x) dx$

$- 2 \int_3^5 g(x) dx = 4(6) - 2(8) = 8.$

46. $\int_1^3 F(x) dx = \int_1^3 f(x) dx$ since $F(x)$

$= f(x)$ except for two values of x on the interval $[1,3]$. Hence,

$\int_1^3 F(x) dx = 6.$

47. $\int_0^{\pi/2} (4 - 3 \cos^2 x) dx$

$= \int_0^{\pi/2} 4 dx - 3 \int_0^{\pi/2} \cos^2 x dx$

$= 4(\frac{\pi}{2} - 0) - 3(\frac{\pi}{4}) = 2\pi - \frac{3\pi}{4} = \frac{5\pi}{4}.$

48. $\int_5^1 H(x) dx = -\int_1^5 H(x) dx$

$= -\int_1^3 H(x) dx - \int_3^5 H(x) dx$

$= -\int_1^3 [1 + f(x)]\, dx - \int_3^5 [g(x) - 1]\, dx$

$= -\int_1^3 dx - \int_1^3 f(x) dx - \int_3^5 g(x) + \int_3^5 dx$

$= -(3-1) - 6 - 8 + (5-3) = -14.$

49. (a) $\int_1^{10} f(x) dx = \int_1^6 f(x) dx + \int_6^{10} f(x) dx$,

so $\int_1^{10} f(x) dx - \int_6^{10} f(x) dx = \int_1^6 f(x) dx$ is true.

(b) $\int_{-2}^4 g(x) dx = \int_{-2}^3 g(x) dx + \int_3^4 g(x) dx$

so (b) is false.

50. By Theorem 9, Section 5.3, $\left| \int_a^b f(x) dx \right| \le \int_a^b |f(x)| dx$. Since $|f(x)| \le K$ for $a \le x \le b$, Theorem 8, Section 5.3, implies that $\int_a^b |f(x)|\, dx \le \int_a^b K dx = K \int_a^b dx$

$= K(b-a)$. Therefore, $\left| \int_a^b f(x) dx \right| \le K(b-a) = K \cdot |b-a|.$

51. $x \leq x^3$ for $1 \leq x \leq 3$,
so $\int_1^3 x\,dx \leq \int_1^3 x^3 dx$.

52. $x^2 \leq x$ for $0 \leq x \leq 1$,
so $1 + x^2 \leq 1 + x$ and
$\frac{1}{1+x^2} \geq \frac{1}{1+x}$.
Thus, $\int_0^1 \frac{1}{1+x^2}\,dx \geq \int_0^1 \frac{1}{1+x}\,dx$.

53. $x^5 \leq x$ for $0 \leq x \leq 1$.
Thus, $\int_0^1 x^5 dx \leq \int_0^1 x\,dx$.

54. For $\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$, $\cos x \leq \frac{\sin x}{x}$ (Theorem 2, page 48);
so $\int_{\pi/6}^{\pi/3} \cos x\,dx \leq \int_{\pi/6}^{\pi/3} \frac{\sin x}{x}\,dx$.

55. For $0 \leq x \leq \frac{\pi}{4}$, $\sin x \leq \cos x$
(equality at $x = \frac{\pi}{4}$);
so $\int_0^{\pi/4} \sin x\,dx \leq \int_0^{\pi/4} \cos x\,dx$.

56. $0 \leq \left[f(x) - Kg(x)\right]^2 = \left[f(x)\right]^2 - 2Kf(x)g(x) + K^2\left[g(x)\right]^2$. Hence, $\int_a^b (\left[f(x)\right]^2 - 2Kf(x)g(x) + K^2\left[g(x)\right]^2)dx \geq 0$ by the nonnegative theorem. Now $\int_a^b (\left[f(x)\right]^2 - 2Kf(x)g(x) + K^2\left[g(x)\right]^2)dx$
$= \int_a^b \left[f(x)\right]^2 dx - 2K \int_a^b f(x)g(x)dx + K^2 \int_a^b \left[g(x)\right]^2 dx \geq 0$. So
$2K \int_a^b f(x)g(x)dx \leq \int_a^b \left[f(x)\right]^2 dx + K^2 \int_a^b \left[g(x)\right]^2 dx$.

57. $\int_1^3 x\,dx = \frac{x^2}{2}\Big|_1^3 = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$.
$\int_1^3 x^3 dx = \frac{x^4}{4}\Big|_1^3 = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$.
Clearly, $4 < 20$.

58. If $\int_a^b (g(x))^2 dx = 0$, then $g(x) = 0$ on $[a,b]$, so that $\int_a^b f(x)g(x)dx = 0$; and so the inequality holds. Now assume $\int_a^b (g(x))^2 dx \neq 0$. In Problem 56, putting $K = \frac{\int_a^b f(x)g(x)dx}{\int_a^b \left[g(x)\right]^2 dx}$, we have

$$\frac{2\left[\int_a^b f(x)g(x)dx\right]^2}{\int_a^b \left[g(x)\right]^2 dx} \leq \int_a^b \left[f(x)\right]^2 dx + \frac{\left[\int_a^b f(x)g(x)dx\right]^2}{\left[\int_a^b \left[g(x)\right]^2 dx\right]^2} \int_a^b \left[g(x)\right]^2 dx,$$

so that

$$2\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b \left[f(x)\right]^2 dx \cdot \int_a^b \left[g(x)\right]^2 dx + \left[\int_a^b f(x)g(x)dx\right]^2$$

and so

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b \left[f(x)\right]^2 dx \cdot \int_a^b \left[g(x)\right]^2 dx.$$

59. Mean value is given by $\frac{1}{4-0} \int_0^4 (3x+1)dx$
$= \frac{1}{4}\left(\frac{3x^2}{2} + x\right)\Big|_0^4 = \frac{1}{4}(24 + 4 - 0) = 7$.
Find a value of c, $0 \leq c \leq 4$, so that $f(c) = 3c + 1 = 7$, $3c = 6$. $c = 2$.

60. Mean value is given by $\frac{1}{3-0} \int_0^3 (x^2+2)dx$
$= \frac{1}{3}\left(\frac{x^3}{3} + 2x\right)\Big|_0^3 = \frac{1}{3}(9 + 6 - 0) = 5$.
Find a value of c, $0 \leq c \leq 3$, so that $f(c) = c^2+2 = 5$. $c^2 = 3$. $c = \sqrt{3}$ (reject $c = -\sqrt{3}$).

61. Mean value is given by $\frac{1}{2-0} \int_0^2 (4-x^2)dx$
$= \frac{1}{2}\left(4x - \frac{x^3}{3}\right)\Big|_0^2 = \frac{1}{2}\left(8 - \frac{8}{3}\right) = \frac{8}{3}$.
Find a value of c, $c \leq c \leq 2$, so that $f(c) = 4 - c^2 = \frac{8}{3}$, $c^2 = 4 - \frac{8}{3} = \frac{4}{3}$.
$c = \frac{2}{\sqrt{3}}$ (reject $c = -\frac{2}{\sqrt{3}}$).

2. Mean value is given by $\frac{1}{1-(-1)}\int_{-1}^{1}(4+3x^2)dx$

$= \frac{1}{2}(4x + x^3)\Big|_{-1}^{1} = \frac{1}{2}(5 - (-5)) = 5.$

Find a value of c, $-1 \le c \le 1$, so that $f(c) = 4 + 3c^2 = 5$, $3c^2=1$. $c = \pm\frac{1}{\sqrt{3}}$.

3. Mean value is given by $\frac{1}{3-(-1)}\int_{-1}^{3}|x|\,dx$

$= \frac{1}{4}\int_{-1}^{0}(-x)dx + \frac{1}{4}\int_{0}^{3}x\,dx$

$= \frac{1}{4}(-\frac{x^2}{2})\Big|_{-1}^{0} + \frac{1}{4}(\frac{x^2}{2})\Big|_{0}^{3} = \frac{1}{4}(\frac{1}{2}) + \frac{1}{4}(\frac{9}{2})$

$= \frac{1}{4}\,\frac{10}{2} = \frac{5}{4}$. Find a value of c, $-1 \le c \le 3$, so that $f(c) = |c| = \frac{5}{4}$. $c = \frac{5}{4}$.

4. (a) We want $-0.5 \le c \le 1$ and f(c)

$= \frac{1}{1+(0.5)}\int_{-0.5}^{1}x|x|\,dx = \frac{1}{1.5}\left[\int_{-0.5}^{0}(-x^2)dx + \int_{0}^{1}x^2dx\right] = \frac{1}{1.5}\left[(\frac{-x^3}{3})\Big|_{-0.5}^{0} + \frac{x^3}{3}\Big|_{0}^{1}\right]$

$= \frac{1}{1.5}\left[0 - \frac{-(-0.5)^3}{3} + \frac{1}{3}\right] = \frac{1}{1.5}(-\frac{1}{24} + \frac{1}{3})$

$= \frac{2}{3}(\frac{7}{24}) = \frac{7}{36}$. We want $c\cdot|c| = \frac{7}{36}$; that is, $c^2 = \frac{7}{36}$. $c = \frac{\sqrt{7}}{6}$.

5. $\int_{0}^{3}(4x + 3)dx = (2x^2 + 3x)\Big|_{0}^{3}$

$= 18 + 9 = 27.$

6. $\int_{-4}^{0}3y^2dy = y^3\Big|_{-4}^{0} = 0 - (-4)^3 = 64.$

7. $\int_{0}^{4}6\sqrt{x}\,dx = \int_{0}^{4}6\cdot x^{\frac{1}{2}}dx = 6\cdot\frac{2}{3}x^{\frac{3}{2}}\Big|_{0}^{4}$

$= 4x^{\frac{3}{2}}\Big|_{0}^{4} = 4(8-0) = 32.$

8. $\int_{8}^{27}9\sqrt[3]{t}\,dt = 9\cdot\frac{3}{4}t^{\frac{4}{3}}\Big|_{8}^{27} = \frac{27}{4}(27^{\frac{4}{3}} - 8^{\frac{4}{3}})$

$= \frac{27}{4}(81 - 16) = \frac{27}{4}(65) = \frac{1755}{4}.$

9. $\int_{0}^{1}(2t + 3t^2)dt = (t^2 + t^3)\Big|_{0}^{1} = 1+1-0 = 2.$

0. $\int_{0}^{3}(3u-1)(u^2+1)du = \int_{0}^{3}(3u^3-u^2+3u-1)du$

$= (\frac{3u^4}{4} - \frac{u^3}{3} + \frac{3u^2}{2} - u)\Big|_{0}^{3}$

$= \frac{3(81)}{4} - \frac{27}{3} + \frac{27}{2} - 3 - 0 = \frac{297}{4} - 12 = \frac{249}{4}.$

71. $\int_{-1}^{1}(z^2+2)^2dz = \int_{-1}^{1}(z^4+4z^2+4)dz$

$= (\frac{z^5}{5} + \frac{4z^3}{3} + 4z)\Big|_{-1}^{1} = \frac{1}{5} + \frac{4}{3} + 4-(-\frac{1}{5}-\frac{4}{3} - 4)$

$= \frac{166}{15}.$

72. $\int_{0}^{1}5(x-\sqrt{x})^2dx = 5\int_{0}^{1}(x^2-2x^{\frac{3}{2}}+x)dx$

$= 5(\frac{x^3}{3} - 2\cdot\frac{2}{5}x^{\frac{5}{2}} + \frac{x^2}{2})\Big|_{0}^{1} = 5(\frac{1}{3} - \frac{4}{5} + \frac{1}{2})$

$= 5\,(\frac{1}{30}) = \frac{1}{6}.$

73. $\int_{-1}^{3}(x + |x|)dx = \int_{-1}^{0}(x-x)dx + \int_{0}^{3}(x+x)dx$

$= 0 + (x^2)\Big|_{0}^{3} = 9.$

74. $\int_{-1}^{3}|x + 1|\,dx = \int_{-1}^{3}(x + 1)dx$

$= (\frac{x^2}{2} + x)\Big|_{-1}^{3} = (\frac{9}{2} + 3)-(\frac{1}{2} - 1) = 8.$

75. $\int_{-1}^{1}\sqrt{x+1}\,dx$. Let $u = x+1$, $du = dx$;

so $\int_{-1}^{1}\sqrt{x+1}\,dx = \int_{0}^{2}u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}}\Big|_{0}^{2}$

$= \frac{2}{3}(2^{3/2}) = \frac{4}{3}\sqrt{2}.$

76. $\int_{-4}^{4}x^2|x|\,dx = \int_{-4}^{0}(-x^3)dx + \int_{0}^{4}x^3dx$

$= -\frac{x^4}{4}\Big|_{-4}^{0} + \frac{x^4}{4}\Big|_{0}^{4} = 0-(-\frac{256}{4})+\frac{256}{4} = 128.$

77. Let $u = 1 + x^2$, so $du = 2xdx$; $u = 2$ when $x = -1$ and $u = 10$ when $x = 3$.

So $\int_{-1}^{3}\frac{2x}{(1+x^2)^2}dx = \int_{2}^{10}\frac{du}{u^2} = \int_{2}^{10}u^{-2}du$

$= \frac{u^{-1}}{-1}\Big|_{2}^{10} = (-\frac{1}{10}) - (-\frac{1}{2}) = \frac{2}{5}.$

78. Let $u = x^4+1$, $du = 4x^3dx$, so $\frac{1}{4}du = x^3dx$.

$\int x^3\sqrt{x^4+1}\,dx = \frac{1}{4}\int u^{\frac{1}{2}}du = \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + C.$

So $\int_{0}^{1}x^3\sqrt{x^4+1}\,dx = \frac{1}{6}(x^4+1)^{\frac{3}{2}}\Big|_{0}^{1}$

$= \frac{1}{6}(2)^{\frac{3}{2}} - \frac{1}{6} = \frac{1}{6}(\sqrt{8} - 1).$

79. $\int_{0}^{4}(|x-1| + |x-2|)dx = \int_{0}^{1}[(1-x)+(2-x)]\,dx$

$+ \int_{1}^{2}[(x-1)+(2-x)]dx + \int_{2}^{4}(x-1+x-2)dx$

$= (3x - x^2)\Big|_0^1 + x\Big|_1^2 + (x^2 - 3x)\Big|_2^4$

$= 2 + 1 + (16-12) - (4-6)$

$= 3 + 4 + 2 = 9.$

80. Let $u = 3x + 10$, so $du = 3dx$.

$$\int \frac{xdx}{\sqrt{3x+10}} = \int \frac{\frac{u-10}{3} \cdot \frac{1}{3}du}{u^{\frac{1}{2}}}$$

$= \frac{1}{9}\int (u^{\frac{1}{2}} - 10u^{-\frac{1}{2}})du = \frac{1}{9}(\frac{2}{3}u^{\frac{3}{2}} - 20u^{\frac{1}{2}} + C).$

So $\int_0^2 \frac{xdx}{\sqrt{3x+10}} = \frac{1}{9}\left[\frac{2}{3}(3x+10)^{\frac{3}{2}} - 20(3x-10)^{\frac{1}{2}}\right]_0^2$

$= \frac{1}{9}\left[(\frac{2}{3}\cdot 64 - 20\cdot 4) - (\frac{2}{3})(10)^{\frac{3}{2}} + 20\sqrt{10}\right]$

$= \frac{1}{27}(40\sqrt{10} - 112).$

81. $\int_0^{\frac{\pi}{8}} \sin 4x dx = \frac{-\cos 4x}{4}\Big|_0^{\frac{\pi}{8}} = \frac{-\cos\frac{\pi}{2}}{4} + \frac{\cos 0}{4}$

$= -\frac{0}{4} + \frac{1}{4} = \frac{1}{4}.$

82. $\int_0^{\frac{\pi}{4}} \tan u \sec^2 u\, du = \frac{1}{2}\tan^2 u\Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\tan^2\frac{\pi}{4} - \frac{1}{2}\tan^2 0 = \frac{1}{2} - 0 = \frac{1}{2}.$

83. $\int_{\frac{\pi}{6}}^{\pi} \sin y \cos y\, dy = \frac{\sin^2 y}{2}\Big|_{\frac{\pi}{6}}^{\pi}$

$= \frac{1}{2}(\sin^2\pi - \sin^2\frac{\pi}{6})$

$= \frac{1}{2}(0 - (\frac{1}{2})^2) = \frac{1}{2}(-\frac{1}{4}) = -\frac{1}{8}.$

84. Let $u = \tan 2\theta$, so $du = 2\sec^2 2\theta\, d\theta$.

$$\int \frac{\sec^2 2\theta d\theta}{\tan^3 2\theta} = \int \frac{\frac{1}{2}du}{u^3} = \frac{1}{2}\int u^{-3}du = \frac{1}{2}\frac{u^{-2}}{-2} + C$$

$= -\frac{1}{4u^2} + C.$ So $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sec^2 2\theta d\theta}{\tan^3 2\theta}$

$= \left[-\frac{1}{4\tan^2 2\theta}\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} = -\frac{1}{4\tan^2\frac{\pi}{3}} + \frac{1}{4\tan^2\frac{\pi}{6}}$

$= -\frac{1}{4(3)} + \frac{1}{4(\frac{1}{3})} = \frac{2}{3}.$

85. $\int_0^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi}(-\sin x)dx$

$= -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi}$

$-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi$

$= 1 + 1 + 1 + 1 = 4.$

86. $\int_0^{2\pi} \cos|x| dx = \int_0^{2\pi} \cos x\, dx$ since $|x| = x$ when $0 \le x \le 2\pi$. So $\int_0^{2\pi} \cos x\, dx = \sin x\Big|_0^{2\pi}$

$= \sin 2\pi - \sin 0 = 0 - 0 = 0.$

87. $\int_0^3 f(x)dx = \int_0^1 (1-x)dx + \int_1^2 (x^2-1)dx + \int_2^3 (x+1)dx = (x - \frac{x^2}{2})\Big|_0^1 + (\frac{x^3}{3} - x)\Big|_1^2 + (\frac{x^2}{2} + x)\Big|_2^3 = \frac{1}{2} + (\frac{2}{3} - (-\frac{2}{3})) + (\frac{15}{2} - 4) = \frac{16}{3}.$

88. $\int_{-1}^2 g(x)dx = \int_{-1}^0 (-\sqrt{|x|})dx + \int_0^1 \sqrt{x+1}\, dx + \int_1^2 x\sqrt{1+x^2}dx.$ For the last integration, let $u = 1 + x^2$, $du = 2xdx$, $xdx = \frac{1}{2}du$; so $\int x\sqrt{1+x^2}dx = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{3}u^{\frac{3}{2}} + C.$ So $\int_{-1}^2 g(x)dx$

$= \int_{-1}^0 (-\sqrt{-x})dx + \frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^1 + \frac{1}{3}(1+x^2)^{\frac{3}{2}}\Big|_1^2.$

Let $v = -x$, $dv = -dx$; so $\int(-\sqrt{-x})dx = v^{\frac{1}{2}}dv$ $= \frac{2}{3}v^{\frac{3}{2}} + C.$ $\int_{-1}^2 g(x)dx = \frac{2}{3}(-x)^{\frac{3}{2}}\Big|_{-1}^0$ $+ \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3} + \frac{1}{3}(5)^{\frac{3}{2}} - \frac{1}{3}(2)^{\frac{3}{2}}$

$= -\frac{4}{3} + \frac{1}{3}(2)^{\frac{3}{2}} + \frac{1}{3}(5)^{\frac{3}{2}}$

$= \frac{1}{3}(5\sqrt{5} + 2\sqrt{2} - 4).$

89. $D_x\int_3^x (4t+1)^{300}dt = (4x+1)^{300}.$

90. $\frac{d}{dx}\int_2^x (3w^2-7)^{15}dw = (3x^2-7)^{15}.$

91. $g'(x)$ for $g(x) = \int_1^x (8t^{17} + 5t^2 - 13)^{40}dt$ is $(8x^{17} + 5x^2 - 13)^{40}.$

92. $h'(t) = \sqrt{1+t^{16}}.$ $h''(t)$ $= \frac{1}{2}(1+t^{16})^{-\frac{1}{2}}(16t^{15}) = \frac{8t^{15}}{\sqrt{1+t^{16}}}.$

93. $D_x\int_x^{1000} \frac{t^2dt}{\sqrt{t^4+8}} = D_x\left[-\int_{1000}^x \frac{t^2dt}{\sqrt{t^4+8}}\right]$

$= -\frac{x^2}{\sqrt{x^4+8}}$.

4. $\frac{d}{dx}\int_x^0 |w|\,dw = \frac{d}{dx}\left[-\int_0^x |w|\,dw\right] = -|x|$.

5. $g'(t) = -\sqrt{1+t^2} + \sqrt{1+t^2} = 0.$ $g''(t) = 0.$

6. $h'(t) = -\frac{1}{1+t^2} + \frac{1}{1+t^2} = 0.$

7. Let $u = x^2$, $\frac{du}{dx} = 2x$, $D_x \int_1^{x^2} \frac{t^2dt}{1+t^2}$

$= D_u\left[\int_1^u \frac{t^2dt}{1+t^2}\right]\cdot\frac{du}{dx} = \frac{u^2}{1+u^2}\cdot 2x$

$= \frac{x^4}{1+x^4}(2x) = \frac{2x^5}{1+x^4}$.

8. $\frac{d}{dx}\left[\int_{3x+1}^{x^2} \frac{t+\sqrt{t}}{t^3+5}\,dt\right] = \frac{d}{dx}\left[\int_{3x+1}^{0} \frac{t+\sqrt{t}}{t^3+5}\,dt\right.$

$\left.+\int_0^{x^2} \frac{t+\sqrt{t}}{t^3+5}\,dt\right]$. Let $u = 3x+1$, then $\frac{du}{dx}=3$.

Let $v = x^2$, then $\frac{dv}{dx} = 2x$. Then $\frac{d}{du}\left[\int_u^0\right.$

$\left.\frac{t+\sqrt{t}}{t^3+5}\,dt\right]\frac{du}{dx} + \frac{d}{dv}\left[\int_0^v \frac{t+\sqrt{t}}{t^3+5}\right]\frac{dv}{dx} = -\frac{u+\sqrt{u}}{u^3+5}\cdot 3$

$+ \frac{v+\sqrt{v}}{v^3+5}(2x) = -3.\ \frac{(3x+1)+\sqrt{3x+1}}{(3x+1)^3+5} + \frac{x^2+|x|}{x^6+5}.2x.$

9. $D_t \int_{4t+3}^{5t^2+t} \cos(w^5+1)\,dw$

$= D_t \int_{4t+3}^{0} \cos(w^5+1)\,dw + \int_0^{5t^2+t} \cos(w^5+1)\,dw$

Let $u = 4t+3$, so $\frac{du}{dt} = 4$. Let $v = 5t^2+t$, so

$\frac{dv}{dt} = 10t+1$. Then $\left[D_u \int_u^0 \cos(w^5+1)\,dw\right]\frac{du}{dt} +$

$\left[D_v \int_0^v \cos(w^5+1)\,dw\right]\frac{dv}{dt}$

$= \left[-\cos(u^5+1)\right]4 + \left[\cos(v^5+1)\right]\left[10t+1\right]$

$= -4\cos\left[(4t+3)^5+1\right] + (10t+1)\cos\left[(5t^2+t)^5+1\right].$

0. (a) $\frac{d}{dx}\left[\int_a^x f(g(t))\cdot g'(t)\,dt - \int_{g(a)}^{g(x)} f(u)\,du\right]$

$= f(g(x))\cdot g'(x) - f(g(x))\cdot g'(x) = 0.$

(b) Hence, $\int_a^x f(g(t))g'(t)\,dt - \int_{g(a)}^{g(x)} f(u)\,du$

$= C$. When $x = a$, $\int_a^a f(g(t))g'(t)\,dt -$

$\int_{g(a)}^{g(a)} f(u)\,du = 0 - 0 = 0.$ So $C = 0$.

Therefore, $\int_a^b f(g(t))g'(t)\,dt$

$= \int_{g(a)}^{g(b)} f(u)\,du.$

101. (a) $g'(x) = x^2-4x+4 = 0$; that is, $(x-2)^2 = 0$ when $x = 2$. Now the value of g at 2 is $\int_0^2 (t^2-4t+4)\,dt = \left(\frac{t^3}{3} - 2t^2+4t\right)\Big|_0^2$

$= \frac{8}{3} - 8 + 8 = \frac{8}{3}$. At the endpoints: $g(0) = 0$, $g(4) = \int_0^4 (t^2-4t+4)\,dt = \left(\frac{t^3}{3}-2t+4t\right)\Big|_0^4$

$= \frac{64}{3} - 32 + 16 = \frac{16}{3}$. The maximum value of g is $\frac{16}{3}$.

(b) $g'(x) = \sqrt{x} - x = 0$; that is, $\sqrt{x} = x$, $x = x^2$, $x^2 - x = 0$, $x(x-1) = 0$; that is, $x = 0$ or $x = 1$. $g(0) = 0$, $g(1) = \int_0^1$

$(\sqrt{t} - t)\,dt = \left(\frac{2}{3}t^{\frac{3}{2}} - \frac{t^2}{2}\right)\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

The maximum value of g is $\frac{1}{6}$.

102. Since f is bounded on $[a,b]$, there exists K such that $|f(x)| \le K$ for all x in $[a,b]$.

We want to show that $\lim_{x\to c}[g(x)-g(c)] = 0$.

First suppose that $c<x$. Then $|g(x)-g(c)|$

$= \left|\int_a^x f(t)\,dt - \int_a^c f(t)\,dt\right| = \left|\int_c^x f(t)\,dt\right| \le$

$\int_c^x |f(t)|\,dt \le \int_c^x K\cdot dt = K\cdot(x-c)$. But

$\lim_{x\to c^+} K(x-c) = 0$. So $\lim_{x\to c^+} |g(x)-g(c)| = 0$.

Now suppose that $x < c$. By a similar argument we can show that $|g(c)-g(x)| \le K\cdot(c-x)$. Since $\lim_{x\to c^-} K(c-x) = 0$, it follows that $\lim_{x\to c^-} |g(c)-g(x)| = 0$. Hence, $\lim_{x\to c} [g(x) - g(c)] = 0$, and g is continuous on $[a,b]$.

103. False, take $f(x) = x$ and $g(x) = x^2$, $a=0$, $b = 1$. $\int_0^1 x\cdot x^2\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$, but

$\int_0^1 x\,dx \cdot \int_0^1 x^2 dx = \frac{x^2}{2}\Big|_0^1 \cdot \frac{x^3}{3}\Big|_0^1$

$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ and $\frac{1}{4} \neq \frac{1}{6}$.

104. False, let $f(x) = x$ and $g(x) = x^2$ on $[0,1]$. Then, $\int_0^1 x\cdot x^2 dx = \frac{1}{4}$ as we have seen in Problem 103. $f(x) \int_0^1 x^2 dx = x\cdot(\frac{1}{3})$; $g(x) \int_0^1 f(x)dx = x^2(\frac{1}{2})$. But $\frac{1}{4} \neq \frac{x}{3} + \frac{x^2}{2}$ since the left side is a constant and the right side is a variable.

105. False. Let $f(x) = x^2$, $g(x) = x$, $a = 1$, $b = 2$. $\int_a^b \frac{f(x)}{g(x)} dx = \int_1^2 \frac{x^2}{x} dx = \int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$. $\int_1^2 f(x)dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$; $\int_1^2 g(x)dx = \frac{x^2}{2}\Big|_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$. $\frac{\frac{7}{3}}{\frac{3}{2}} = \frac{14}{9}$ and $\frac{14}{9} \neq \frac{3}{2}$.

106. True. Consider the fact that $[f(x)\cdot h(x)]' = f'(x)h(x) + f(x)h'(x)$. Then $\int_a^b [f(x)\cdot h(x)]' dx = \int_a^b f'(x)h(x)dx + \int_a^b f(x)\cdot h'(x)dx$. So, $[f(x)\cdot h(x)]\Big|_a^b - \int_a^b h(x)f'(x)dx = \int_a^b f(x)g(x)dx$.

107. True. $\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$

Let $u = -x$, so $du = -dx$. Then $\int_{-a}^a f(x)dx$ $= \int_a^0 f(-u)(-du) + \int_0^a f(x)dx$ $= \int_a^0 f(u)du + \int_0^a f(x)dx$ (since $f(-u) = -f(u)$) $= -\int_0^a f(u)du + \int_0^a f(x)dx = 0$.

108. True. We look at $D_x\left[\int_a^x f(kt)dt - \frac{1}{k}\int_{ka}^{kx} f(t)dt\right] = f(kx) - \frac{1}{k}\cdot\frac{d}{du}\left[\int_{ka}^{u} f(t)dt\frac{du}{dx}\right]$

$= f(kx) - \frac{1}{k}\cdot f(u)\cdot k = f(kx) - f(kx) = 0$.

Hence, $\int_a^x f(kt)dt - \frac{1}{k}\int_{ka}^{kx} f(t)dt = C$.

If $x = a$, $C = 0$. Therefore,

$\int_a^x f(kt)dt = \frac{1}{k}\int_{ka}^{kx} f(t)dt.$

109. False. Let $g(x) = x$, $a = -1$, $b = 1$. $\left|\int_{-1}^1 x\,dx\right| = [\frac{x^2}{2}]\Big|_{-1}^1 = 0$; whereas $\int_{-1}^1 |x|\,dx$ $= \int_{-1}^0 (-x)dx + \int_0^1 x\,dx = -\frac{x^2}{2}\Big|_{-1}^0 + \frac{x^2}{2}\Big|_0^1$ $= \frac{1}{2} + \frac{1}{2} = 1$.

110. False. Take $f(x) = 1-x$. Then $g(x) = x - \frac{1}{2}x^2$, which is increasing for $x < 1$.

111. $T_4 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)$ $= \frac{1}{2}\cdot\frac{2-0}{4}\left[0 + 2\cdot\frac{1}{2}\sqrt{16 - \frac{1}{8}} + 2\cdot 1\sqrt{16-1} + 2\cdot\frac{3}{2}\sqrt{16 - \frac{27}{8}} + 2\sqrt{16-8}\right] = \frac{1}{4}\left[\sqrt{\frac{127}{8}} + 2\sqrt{15} + 3\sqrt{\frac{101}{8}} + 2\sqrt{8}\right] \approx 7.01$. So $\int_0^2 x\sqrt{16-x^3}dx \approx 7.$

112. $T_4 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3)$ $+ f(x_4))$, where $\Delta x = \frac{1}{4}\cdot f(x_k) = \sqrt{4 + (1+\frac{k}{4})^3}$. So $f(x_0) = 2.236$, $f(x_1) = 2.440$, $f(x_2) = 2.716$, $f(x_3) = 3.059$, $f(x_4) = 3.464$. $T_4 \approx \left[\frac{1}{8}\,2.236 + 2(2.440) + 2(2.716) + 2(3.059) + 3.464\right]$. So $T_4 \approx 2.77$. Hence, $\int_1^2 \sqrt{4 + x^3}dx \approx 2.77$.

113. $T_5 \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3)$ $+ 2f(x_4) + f(x_5))$ where $\Delta x = \frac{10-0}{5} = 2$. $f(x_k) = \sqrt[3]{125 + 8k^3}$, so $T_5 = 1(5 + 2\sqrt[3]{133}$ $+ 2\sqrt[3]{189} + 2\sqrt[3]{341} + 2\sqrt[3]{637} + \sqrt[3]{1125}$ ≈ 68.27. So $\int_0^{10}\sqrt[3]{125+x^3}dx \approx 68.27$.

114. $T_8 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) +$ $2f(x_3) + 2f(x_4) + 2f(x_5) + 2f(x_6) +$ $2f(x_7) + f(x_8))$. $\Delta x = \frac{8-4}{8} = \frac{1}{2}$.

$f(x_k) = \sqrt{64 - (4 + \frac{k}{2})^2}$, so $f(x_0) = \sqrt{48} f(x_1)$ $= \sqrt{64 - (\frac{9}{2})^2} = \sqrt{43.750}$, $f(x_2) = \sqrt{39}$, $f(x_3) = \sqrt{33.750}$, $f(x_4) = \sqrt{28}$, $f(x_5) =$

$= \sqrt{21.750}$, $f(x_6) = \sqrt{15}$, $f(x_7) = \sqrt{7.75}$, $f(x_8) = 0$. $T_8 \approx \frac{1}{4}(\sqrt{48} + 2\sqrt{43.75} + 2\sqrt{39} + 2\sqrt{33.750} + 2\sqrt{28} + 2\sqrt{21.750} + 2\sqrt{15} + 2\sqrt{7.75} + 0) \approx 19.37$. So $\int_4^8 \sqrt{64-x^2}\,dx \approx 19.37$.

15. $T_6 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2(fx_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6))$, where $\Delta x = \dfrac{\frac{\pi}{3} - 0}{6} = \frac{\pi}{18}$. $f(x_k) = \sec(\frac{\pi k}{18})$, so $f(x_0) = \sec 0 = 1$, $f(x_1) = \sec \frac{\pi}{18} = 1.0154$, $f(x_2) = \sec \frac{\pi}{9} = 1.0642$, $f(x_3) = \sec \frac{\pi}{6} = 1.1547$, $f(x_4) = \sec \frac{2\pi}{9} = 1.3054$, $f(x_5) = \sec \frac{\pi 5}{18} = 1.5557$, $f(x_6) = \sec \frac{\pi}{3} = 2$. So $T_6 \approx \frac{\pi}{36}\left[1 + 2(1.0154) + 2(1.0642) + 2(1.1547) + 2(1.3054) + 2(1.5557) + 2\right] = 1.33$. So $\int_0^{\frac{\pi}{3}} \sec x\,dx \approx 1.33$.

16. $T_8 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + 2f(x_6) + 2f(x_7) + f(x_8))$, where $\Delta x = \dfrac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$. $f(x_k) = \sin^3(\frac{k\pi}{16})$, so $f(x_0) = 0$, $f(x_1) = \sin^3\frac{\pi}{16} = 0.0074$, $f(x_2) = \sin^3\frac{\pi}{8} = 0.0560$, $f(x_3) = \sin^3\frac{3\pi}{16} = 0.1715$, $f(x_4) = \sin^3\frac{\pi}{4} = 0.3536$, $f(x_5) = \sin^3\frac{5\pi}{16} = 0.5748$, $f(x_6) = \sin^3\frac{3\pi}{8} = 0.7886$, $f(x_7) = \sin^3\frac{7\pi}{16} = 0.9435$, $f(x_8) = \sin^3\frac{\pi}{2} = 1$. So $T_8 \approx \frac{\pi}{32}[0 + 2(0.0074) + 2(0.0560) + 2(0.1715) + 2(0.3536) + 2(0.5748) + 2(0.7886) + 2(0.9435) + 1] \approx 0.6667$

So $\int_0^{\frac{\pi}{2}} \sin^3 dx \approx 0.6667$.

117. $S_8 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8))$, where $\Delta x = \frac{8-0}{2n} = \frac{8}{8} = 1$. $f(x_0) = 0$, $f(x_1) = \frac{3}{2}$, $f(x_2) = \frac{6}{9} = \frac{2}{3}$, $f(x_3) = \frac{9}{28}$, $f(x_4) = \frac{12}{65}$, $f(x_5) = \frac{15}{126}$, $f(x_6) = \frac{18}{217}$, $f(x_7) = \frac{21}{344}$, $f(x_8) = \frac{24}{513} = \frac{8}{171}$. $S_8 = \frac{1}{3}(0 + 6 + \frac{4}{3} + \frac{9}{7} + \frac{24}{65} + \frac{10}{21} + \frac{36}{217} + \frac{21}{86} + \frac{8}{171}) \approx 3.31$. So $\int_0^8 \frac{3x}{1+x^3}dx \approx 3.31$.

118. $S_4 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4))$, where $\Delta x = \frac{4-0}{2n} = \frac{4}{4} = 1$. $f(x_k) = \sqrt{16-k^2}$. $S_4 = \frac{1}{3}(4 + 4\sqrt{15} + 2\sqrt{12} + 4\sqrt{7} + 0) \approx 12.33$. So $\int_0^4 \sqrt{16-x^2}\,dx \approx 12.33$.

119. $S_6 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6))$, where $\Delta x = \frac{8-2}{2n} = \frac{6}{6} = 1$ and $f(x_k) = \dfrac{2 + k}{\sqrt[3]{3+(2+k)^3}}$

$S_6 = \frac{1}{3}(\frac{2}{\sqrt[3]{11}} + 4\frac{3}{\sqrt[3]{30}} + 2\frac{4}{\sqrt[3]{67}} + 4\frac{5}{\sqrt[3]{128}} + 2\frac{6}{\sqrt[3]{219}} + 4\frac{7}{\sqrt[3]{346}} + \frac{8}{\sqrt[3]{515}})$

$S_6 \approx 5.892$. Hence, $\int_2^8 \frac{x\,dx}{\sqrt[3]{3+x^3}} \approx 5.892$.

120. $S_6 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6))$, where $\Delta x = \frac{5-0}{6} = \frac{5}{6}$ and $f(x_k) = \dfrac{(\frac{5}{6}k)^3}{\sqrt{1+(\frac{5}{6}k)^3}}$.

$$S_6 = \left[\frac{5}{18}(0 + \frac{(4)(\frac{5}{6})^3}{\sqrt{1+(\frac{5}{6})^3}} + \frac{(2)(\frac{10}{6})^3}{\sqrt{1+(\frac{10}{6})^3}} + \frac{(4)(\frac{15}{6})^3}{\sqrt{1+(\frac{15}{6})^3}} + \frac{(2)(\frac{20}{6})^3}{\sqrt{1+(\frac{20}{6})^3}} + \frac{(4)(\frac{25}{6})^3}{\sqrt{1+(\frac{25}{6})^3}} + \frac{5^3}{\sqrt{1+5^3}}\right].$$

$S_6 \approx 21.6$. Hence,

$$\int_0^5 \frac{x^3dx}{\sqrt{1+x^3}} \approx 21.6.$$

121. $S_6 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$, where $\Delta x = \frac{\frac{\pi}{2}}{6} = \frac{\pi}{12}$ and $f(x_k) = \frac{1}{2+\sin\frac{\pi}{12}k}$.

$$S_6 = \frac{\pi}{36}\left[\frac{1}{2+\sin 0} + \frac{4}{2+\sin\frac{\pi}{12}} + \frac{2}{2+\sin\frac{\pi}{6}} + \frac{4}{2+\sin\frac{\pi}{4}} + \frac{2}{2+\sin\frac{\pi}{3}} + \frac{4}{2+\sin\frac{5\pi}{12}} + \frac{1}{2+\sin\frac{\pi}{2}}\right]$$

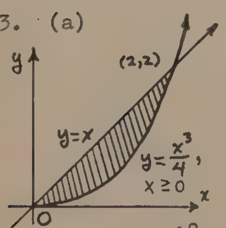
$S_6 \approx 0.6046$ so $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} \approx 0.6046.$

122. $S_8 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)\right]$, where $\Delta x = \frac{1-0}{8} = \frac{1}{8}$ and $f(x_k) = \cos\left[\frac{\pi}{2}\sqrt{\frac{k}{8}}\right]$.

$$S_8 = \frac{1}{24}\left[\cos 0 + 4\cos\frac{\pi}{2\sqrt{8}} + 2\cos\frac{\pi}{4} + 4\cos\frac{\pi}{2}\sqrt{\frac{3}{8}} + 2\cos\frac{\pi}{2\sqrt{2}} + 4\cos\frac{\pi}{2}\sqrt{\frac{5}{8}} + 2\cos\frac{\pi\sqrt{3}}{4} + 4\cos\frac{\pi}{4}\sqrt{\frac{7}{8}} + \cos\frac{\pi}{2}\right]$$

$S_8 \approx 0.4627$ so $\int_0^1 \cos\left(\frac{\pi\sqrt{x}}{2}\right)dx \approx 0.4627.$

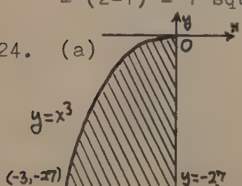
123. (a)



(b) $\frac{1}{4}x^3 = x$, $x^3 - 4x = 0$, $x(x^2-4) = 0$; $x = 0$, $x = 2$, for $x \geq 0$. The points of intersection are $(0,0)$ and $(2,2)$ for $x \geq 0$.

(c) $A = \int_0^2 (x - \frac{1}{4}x^3)dx = \left(\frac{x^2}{2} - \frac{x^4}{16}\right)\Big|_0^2 = (2-1) = 1$ square unit.

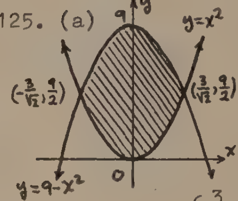
124. (a)



(b) $x^3 = -27$ for $x = -3$, and $x^3 = 0$ for $x = 0$. The points of intersection are $(-3,-27)$ and $(0,0)$.

(c) $A = -\int_{-3}^0 (-27-x^3)dx = -\left(-27x - \frac{x^4}{4}\right)\Big|_{-3}^0 = 0 - \left[-\left(81 - \frac{81}{4}\right)\right] = \frac{243}{4}$ square units.

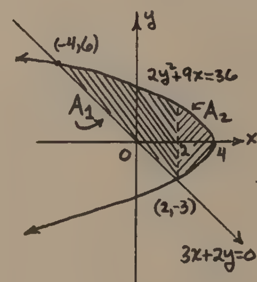
125. (a)



(b) $9-x^2 = x^2$, $9 = 2x^2$, $\frac{9}{2} = x^2$; $x = \pm\frac{3}{\sqrt{2}}$. $(\frac{3}{\sqrt{2}},\frac{9}{2}), (-\frac{3}{\sqrt{2}}, \ldots$ are the points of intersection.

(c) $A = 2\int_0^{\frac{3}{\sqrt{2}}} \left[(9-x^2)-x^2\right]dx = 2\int_0^{\frac{3}{\sqrt{2}}} (9-2x^2)dx = 2\left[9x - \frac{2}{3}x^3\right]\Big|_0^{\frac{3}{\sqrt{2}}} = 18\sqrt{2}$ square units.

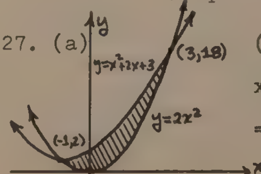
126. (a)



(b) Multiply the linear equation by 3: $9x+6y = 0$ so $9x = -6y$. So $2y^2-6y-36 = 0$, $y^2-3y-18 = 0$, $(y-6)(y+3) = 0$; $y = 6$ or $y = -3$. Hence, for $y = 6$, $x = -4$; for $y = -3$, $x = 2$. The points of intersection are $(2,-3)$ and $(-4,6)$.

(c) $A = A_1 + A_2 = \int_{-4}^2 \left(\sqrt{\frac{36-9x}{2}} - \frac{-3x}{2}\right)dx + 2\int_2^4 \left(\sqrt{\frac{36-9x}{2}}\right)dx$. Let $u = \frac{36-9x}{2}$, $du = -\frac{9}{2}dx$, $dx = -\frac{2}{9}du$. $\int\sqrt{\frac{36-9x}{2}}dx = \int u^{\frac{1}{2}}\left(-\frac{2}{9}\right)du = \frac{-4}{27}u^{\frac{3}{2}} + C$. So $A_1 + A_2 = \left[\frac{-4}{27}\left(\frac{36-9x}{2}\right)^{\frac{3}{2}} + \frac{3x^2}{4}\right]\Big|_{-4}^{2} + 2\cdot\left(\frac{-4}{27}\right)\left(\frac{36-9x}{2}\right)^{\frac{3}{2}}\Big|_2^4 = \left[\frac{-4}{27}(9)^{\frac{3}{2}} + 3\right] - \left[-\frac{4}{27}\cdot 216 + 12\right] - \frac{8}{27}(0-27) = (-4+3) - (-32+12) + 8 = -1 + 20 + 8 = 27.$ The area is 27 square units.

127. (a)



(b) $2x^2 = x^2+2x+3$, $x^2-2x-3 = 0$, $(x-3)(x+1) = 0$; $x=3$ or $x=-1$. The

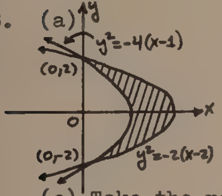
points of intersection are (3,18) and (-1,2).

(c) $A = \int_{-1}^{3}\left[(x^2+2x+3)-(2x^2)\right]dx = \int_{-1}^{3}(-x^2+2x+3)dx = (\frac{-x^3}{3}+x^2+3x)\Big|_{-1}^{3}$

$= (-9+9+9)-(\frac{1}{3}+1-3) = 9-\frac{1}{3}+2$

$= \frac{32}{3}$ square units.

128. (a)



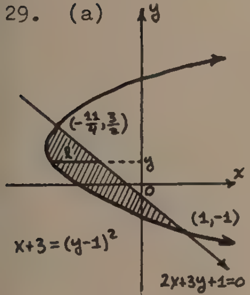
(b) $-4x+4=-2x+4$, $-2x=0$; $x=0$. The points of intersection are (0,2) and (0,-2).

(c) Take the reference axis to be the y axis: $A = 2\int_{0}^{2}\left[(2-\frac{y^2}{2})-(1-\frac{y^2}{4})\right]dy$

$= 2\int_{0}^{2}(1-\frac{y^2}{4})dy = 2(y-\frac{y^3}{12})\Big|_{0}^{2}$

$= 2(2-\frac{8}{12}) = 2(\frac{4}{3}) = \frac{8}{3}$ square units.

129. (a)



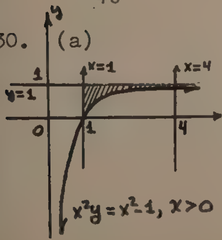
(b) $2\left[(y-1)^2-3\right]+3y+1=0$, $2(y^2-2y-2)+3y+1=0$, $2y^2-y-3=0$, $(2y-3)(y+1)=0$; $y=\frac{3}{2}$ or $y=-1$. For $y=\frac{3}{2}$, $x=-\frac{11}{4}$; for $y=-1$, $x=1$. The points of intersection are (1,-1) and $(-\frac{11}{4},\frac{3}{2})$.

(c) Take the y axis to be the reference axis. $A = \int_{-1}^{3/2}\left[-\frac{(3y+1)}{2}-((y-1)^2-3)\right]dy$

$= \int_{-1}^{3/2}(-y^2+\frac{y}{2}+\frac{3}{2})dy = (-\frac{y^3}{3}+\frac{y^2}{4}+\frac{3}{2}y)\Big|_{-1}^{3/2}$

$= \frac{125}{48}$ square units.

130. (a)



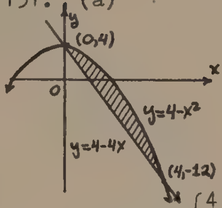
(b) The line x=1 and the curve $x^2y=x^2-1$ intersect at (1,0). The line x=4 and the curve intersect at $(4,\frac{\sqrt{15}}{4})$ and $(4,\frac{-\sqrt{15}}{4})$.

The other points of intersection are (1,1) and (4,1).

(c) $A = \int_{1}^{4}\left[1-(\frac{x^2-1}{x^2})\right]dx = \int_{1}^{4}(1-1+\frac{1}{x^2})dx.$

$A = -\frac{1}{x}\Big|_{1}^{4} = -\frac{1}{4}+1 = \frac{3}{4}$ square unit.

131. (a)



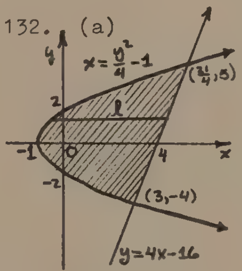
(b) $4-x^2=4-4x$, $x^2-4x=0$; $x=0$ or $x=4$. If $x=0$, $y=4$; if $x=4$, $y=-12$. The points of intersection are (0,4) and (4,-12).

(c) $A = \int_{0}^{4}\left[(4-x^2)-(4-4x)\right]dx$

$= \int_{0}^{4}(-x^2+4x)dx = (-\frac{x^3}{3}+2x^2)\Big|_{0}^{4}$

$= -\frac{64}{3}+32 = \frac{32}{3}$ square units.

132. (a)



(b) $y = 4(\frac{1}{4}y^2-1) = 16$, $y = y^2-20$, $0 = y^2-y-20 = (y-5)(y+4)$; $y=5$ or $y=-4$. The points of intersection are $(\frac{21}{4},5)$ and (3,-4).

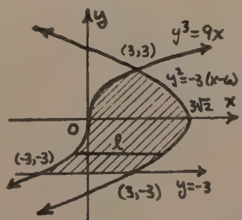
(c) Taking the y axis as the reference axis: $A = \int_{-4}^{5}\left[\frac{y+16}{4}-(\frac{1}{4}y^2-1)\right]dy$

$= \int_{-4}^{5}(-\frac{1}{4}y^2+\frac{y}{4}+5)dy = (-\frac{y^3}{12}+\frac{y^2}{8}+5y)\Big|_{-4}^{5}$

$= (-\frac{125}{12}+\frac{25}{8}+25)-(\frac{64}{12}+\frac{16}{8}-20)$

$= -\frac{189}{12}+\frac{9}{8}+45 = \frac{729}{24} = \frac{243}{8}$ sq. units.

133. (a)



(b) $y^2 = -3(\frac{y^3}{9}-6)$, $0 = -\frac{y^3}{3}-y^2+18$, $0=y^3+3y^2-54$, $0=(y-3)(y^2+6y+18)$; $y=3$. The point of intersection is (3,3) for these two curves.

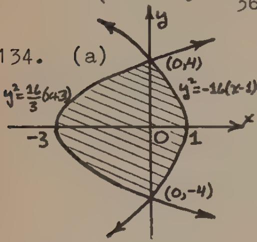
Also (-3,-3) and (3,-3) are points of intersection.

(c) Taking the reference axis as the y axis: $A = \int_{-3}^{3}\left[\left(6 - \frac{y^2}{3}\right)-\left(\frac{y^3}{9}\right)\right]dy$

$= \left(6y - \frac{y^3}{9} - \frac{y^4}{36}\right)\Big|_{-3}^{3} = \left(18 - 3 - \frac{81}{36}\right)$

$- \left(-18 + 3 - \frac{81}{36}\right) = 36 - 6 = 30$ sq. units.

134. (a)



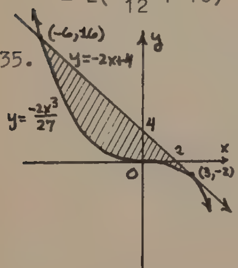
(b) $-16(x-1) = \frac{16}{3}(x+3)$, $-16x + 16 = \frac{16}{3}x + 16$, $0 = \frac{64}{3}x$; $x = 0$. The points of intersection are $(0,4)$ and $(0,-4)$.

(c) Taking the reference axis to be the y axis: $A = 2\int_{0}^{4}\left[\left(1 - \frac{y^2}{16}\right)-\left(\frac{3}{16}y^2 - 3\right)\right]dy$

$= 2\int_{0}^{4}\left(-\frac{y^2}{4} + 4\right)dy = 2\left(-\frac{y^3}{12} + 4y\right)\Big|_{0}^{4}$

$= 2\left(-\frac{64}{12} + 16\right) = 2\left(\frac{32}{3}\right) = \frac{64}{3}$ square units.

135.



$\frac{dy}{dx} = -\frac{2}{9}x^2$. So the slope of the tangent line is -2. The tangent line has equation $y + 2 = -2(x-3)$.

$A = \int_{-6}^{3}\left[(-2x+4)-\left(-\frac{2}{27}x^3\right)\right]dx$

$= \left(\frac{x^4}{54} - x^2 + 4x\right)\Big|_{-6}^{3} = \left(\frac{81}{54} - 9 + 12\right) -$

$\left(\frac{1296}{54} - 36 - 24\right) = -\frac{1215}{54} + 63 =$

$-\frac{45}{2} + \frac{126}{2} = \frac{81}{2}$ square units.

6 APPLICATIONS OF THE DEFINITE INTEGRAL

roblem Set 6.1, page 364

. $V = \int_{-1}^{3}\pi[f(x)]^2dx = \pi\int_{-1}^{3}9x^4dx = 9\pi\frac{x^5}{5}\Big|_{-1}^{3}$

$= \frac{9}{5}\pi\left[243-(-1)\right] = \frac{2196}{5}\pi$ cubic units.

. $V = \int_{1}^{4}\pi[f(x)]^2dx = \pi\int_{1}^{4}9x\,dx = 9\pi\frac{x^2}{2}\Big|_{1}^{4}$

$= \frac{9}{2}\pi(16-1) = \frac{135}{2}\pi$ cubic units.

. $V = \int_{-1}^{3}\pi(\sqrt{9-x^2})^2dx = \pi\int_{-1}^{3}(9-x^2)dx$

$= \pi(9x - \frac{x^3}{3})\Big|_{-1}^{3} = \pi\left[(27-\frac{27}{3})-(-9+\frac{1}{3})\right]$

$= \frac{80}{3}\pi$ cubic units.

. $V = \int_{-2}^{1}\pi(|x|)^2dx = \pi\int_{-2}^{1}x^2dx = \pi\frac{x^3}{3}\Big|_{-2}^{1}$

$= \pi(\frac{1}{3} - \frac{-8}{3}) = 3\pi$ cubic units.

. $V = \int_{0}^{2}\pi(\sqrt{2+x^2})^2dx = \pi\int_{0}^{2}(2+x^2)dx$

$= \pi(2x + \frac{x^3}{3})\Big|_{0}^{2} = \pi\left[(4+\frac{8}{3})-0\right]$

$= \frac{20}{3}\pi$ cubic units.

. $V = \pi\int_{-3}^{2}(|x|-x)^2dx = \pi\int_{-3}^{2}(|x|^2-2|x|x+x^2)dx$

$= \pi\int_{-3}^{0}(x^2+2x^2+x^2)dx + \pi\int_{0}^{2}(x^2-2x^2+x^2)dx$

$= 4\pi\int_{-3}^{0}x^2dx + \pi\cdot 0 = \frac{4}{3}\pi x^3\Big|_{-3}^{0}$

$= \frac{4}{3}\pi\left[0-(-27)\right] = 36\pi$ cubic units.

7. $V = \pi\int_{0}^{\frac{\pi}{4}}\sec^2x\,dx = \pi\tan x\Big|_{0}^{\frac{\pi}{4}}$

$= \pi(\tan\frac{\pi}{4} - \tan 0) = \pi(1-0) = \pi$ cubic units.

8. $V = \pi\int_{0}^{\frac{\pi}{3}}\tan^2x\,dx = \pi\int_{0}^{\frac{\pi}{3}}(\sec^2x-1)dx$

$= \pi(\tan x - x)\Big|_{0}^{\frac{\pi}{3}} = \pi(\tan\frac{\pi}{3} - \frac{\pi}{3} - 0)$

$= \pi\tan\frac{\pi}{3} - \frac{\pi^2}{3} = \pi\sqrt{3} - \frac{\pi^2}{3}$ cubic units.

9. $V = \int_{0}^{8}\pi(g(y))^2dy = \pi\int_{0}^{8}(\sqrt[3]{y})^2dy =$

$\pi\frac{3}{5}y^{\frac{5}{3}}\Big|_{0}^{8} = \frac{3}{5}\pi(8^{\frac{5}{3}} - 0) = \frac{96}{5}\pi$ cubic units.

10. $V = \int_{0}^{4}\pi(y^2)^2dy = \frac{\pi y^5}{5}\Big|_{0}^{4} = \frac{1024}{5}\pi$ cubic units.

11. $V = \int_{0}^{4}\pi(\frac{y^2}{4})^2dy = \frac{\pi}{16}\cdot\frac{y^5}{5}\Big|_{0}^{4} = \frac{\pi}{16}\,\frac{(4)^5}{5}$

$= \frac{64}{5}\pi$ cubic units.

12. $V = \int_{2}^{4}\pi(\sqrt{y-2})^2dy = \pi\int_{2}^{4}(y-2)dy = \pi(\frac{y^2}{2}-2y)\Big|_{2}^{4}$

$= \pi\left[(8-8) - (2-4)\right] = 2\pi$ cubic units.

13. $V = \pi\int_{0}^{8}(y^{\frac{2}{3}})^2dy = \pi\int_{0}^{8}y^{\frac{4}{3}}dy = \pi(\frac{3}{7}y^{\frac{7}{3}})\Big|_{0}^{8}$

$= \frac{384}{7}\pi$ cubic units.

14. $V = \int_{0}^{2}\pi\left[(\frac{y}{2})^{\frac{1}{3}}\right]^2dy = \frac{\pi}{2^{2/3}}\int_{0}^{2}y^{\frac{2}{3}}dy$

$= \frac{\pi}{2^{2/3}}\cdot\frac{3}{5}y^{\frac{5}{3}}\Big|_{0}^{2} = \frac{\pi}{2^{2/3}}\cdot\frac{3}{5}\cdot 2^{\frac{5}{3}}$

$= \pi\cdot\frac{3}{5}\cdot 2 = \frac{6}{5}\pi$ cubic units.

15. $V = \int_0^1 \pi\left(\sqrt{\cos\frac{\pi y}{4}}\right)^2 dy = \int_0^1 \pi\cos\frac{\pi y}{4}\,dy$

$= 4\sin\frac{\pi y}{4}\Big|_0^1 = 4(\sin\frac{\pi}{4} - \sin 0)$

$= 4(\frac{\sqrt{2}}{2}) = 2\sqrt{2}$ cubic units.

16. $V = \int_1^2 \pi\csc^2\frac{\pi y}{6}\,dy = 6\tan\frac{\pi y}{6}\Big|_1^2$

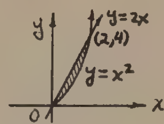
$= 6(\tan\frac{\pi}{3} - \tan\frac{\pi}{6}) = 6(\sqrt{3} - \frac{1}{\sqrt{3}})$

$= 4\sqrt{3}$ cubic units.

17. $V = \pi\int_0^2 \left[(2x)^2 - (x^2)^2\right]dx$

$= \pi\int_0^2 (4x^2 - x^4)\,dx$

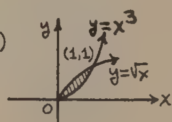
$= \pi(\frac{4x^3}{3} - \frac{x^5}{5})\Big|_0^2$



$= \pi(\frac{32}{3} - \frac{32}{5}) = \frac{64}{15}\pi$ cubic units.

18. $V = \int_0^1 \pi\left[(\sqrt{x})^2 - (x^3)^2\right]dx = \pi\int_0^1 (x - x^6)\,dx$

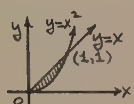
$= \pi(\frac{x^2}{2} - \frac{x^7}{7})\Big|_0^1 = \pi(\frac{1}{2} - \frac{1}{7})$



$= \frac{5\pi}{14}$ cubic units.

19. $V = \int_0^1 \pi\left[(\sqrt{y})^2 - y^2\right]dy$

$= \pi(\frac{y^2}{2} - \frac{y^3}{3})\Big|_0^1$

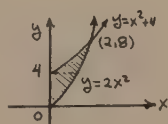


$= \pi(\frac{1}{2} - \frac{1}{3}) = \frac{\pi}{6}$ cubic unit.

20. $V = \int_0^4 \pi(\sqrt{\frac{y}{2}})^2 dy + \int_4^8 \pi\left[(\sqrt{\frac{y}{2}})^2\right.$

$\left. - (\sqrt{y-4})^2\right]dy$

$= \frac{\pi y^2}{4}\Big|_0^4 + \pi(\frac{y^2}{4} - \frac{y^2}{2} + 4y)\Big|_4^8$

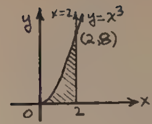


$= \frac{\pi y^2}{4}\Big|_0^4 + \pi(4y - \frac{y^2}{4})\Big|_4^8$

$= 4\pi + \pi\left[(4\cdot 8 - \frac{64}{4}) - (16-4)\right]$

$= 4\pi + \pi(16-12) = 8\pi$ cubic units.

21. $V = \pi\int_0^8 \left[2^2 - (\sqrt[3]{y})^2\right]dy$



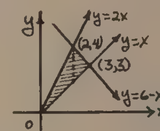
$= \pi(4y - \frac{3}{5}y^{\frac{5}{3}})\Big|_0^8$

$= \pi(32 - \frac{3}{5}\cdot 32) = \frac{64}{5}\pi$ cubic units.

22. $V = \int_0^2 \pi\left[(2x)^2 - (x^2)\right]dx + \int_2^3 \pi\left[(6-x)^2 - x^2\right]dx$

$= \pi\int_0^2 3x^2\,dx + \pi\int_2^3 (36-12x)\,dx$

$= \pi(x^3)\Big|_0^2 + \pi(36x - 6x^2)\Big|_2^3$

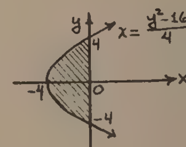


$= 8\pi + \pi\left[(108-54) - (72-24)\right]$

$= 8\pi + \pi(6) = 14\pi$ cubic units.

23. Using symmetry,

$V = 2\pi\int_0^4 (\frac{y^2-16}{4})^2 dy$



$= \frac{\pi}{8}\int_0^4 (y^4 - 32y^2 + 256)\,dy$

$= \frac{\pi}{8}(\frac{y^5}{5} - \frac{32y^3}{3} + 256y)\Big|_0^4$

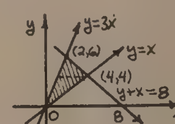
$= \frac{\pi}{8}\left[\frac{4^5}{5} - \frac{32(4^3)}{3} + 256(4)\right] = \frac{\pi}{8}(\frac{8192}{15})$

$= \frac{1024}{15}\pi$ cubic units.

24. $V = \pi\int_4^6 \left[(8-y)^2 - (\frac{y}{3})^2\right]dy + \pi\int_0^4 \left[y^2 - (\frac{y}{3})^2\right]dy$

$= \pi\int_4^6 (64 - 16y + y^2 - \frac{y^2}{9})\,dy + \pi\int_0^4 \frac{8y^2}{9}\,dy$

$= \pi(64y - 8y^2 + \frac{8y^3}{27})\Big|_4^6$



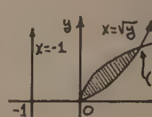
$+ \pi\frac{8y^3}{27}\Big|_0^4$

$= \pi\left[(384 - 288 + \frac{1728}{27}) - (256 - 128 + \frac{512}{27})\right]$

$+ \frac{512}{27}\pi = 32\pi$ cubic units.

25. $V = \pi\int_0^1 \left[(\sqrt{y}+1)^2 - (y^2+1)^2\right]dy$

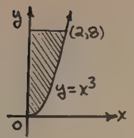
$= \pi\int_0^1 (y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1)\,dy$



$= \pi(-\frac{y^5}{5} - \frac{2y^3}{3} + \frac{y^2}{2} + \frac{4}{3}y^{\frac{3}{2}})\Big|_0^1$

$= \pi(-\frac{1}{5} - \frac{2}{3} + \frac{1}{2} + \frac{4}{3}) = \frac{29}{30}\pi$ cubic units.

26. $V = \pi\int_0^2 (8-x^3)^2 dx$



$= \pi\int_0^2 (64-16x^3+x^6)$

$= \pi(64x-4x^4+\frac{x^7}{7}) \Big|_0^2$

$= \pi(128-64 + \frac{128}{7}) = \frac{576}{7}\pi$ cubic units.

27. $V = \pi\int_0^3 \left\{[3-(2-\sqrt{4-y})]^2-(3-y)^2\right\} dy +$

$\pi\int_3^4 [3-(2-\sqrt{4-y})]^2 - [3-(2+\sqrt{4-y})]^2 dy$

$= \pi\int_0^3 \left[(1+\sqrt{4-y})^2-(3-y)^2\right] dy +$

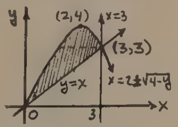
$\pi\int_3^4 \left[(1+\sqrt{4-y})^2-(1-\sqrt{4-y})^2\right] dy$

$= \pi\int_0^3 (1+2\sqrt{4-y}+4-y-9+6y-y^2) dy +$

$\pi\int_3^4 (1+2\sqrt{4-y}+4-y-1+2\sqrt{4-y}-4+y) dy$

$= \pi\int_0^3 (-4+5y-y^2+2\sqrt{4-y}) dy +$

$\pi\int_3^4 4\sqrt{4-y}(dy)$



$= \pi\left[(-4y+\frac{5}{2}y^2-\frac{y^3}{3}-\frac{4}{3}(4-y)^{\frac{3}{2}}) \Big|_0^3 -\right.$

$\left.\frac{8}{3}(4-y)^{\frac{3}{2}} \Big|_3^4\right]$

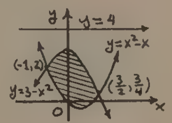
$= \pi\left[-12 + \frac{45}{2} - 9 - \frac{4}{3} + \frac{32}{3} - (0 - \frac{8}{3})\right]$

$= \frac{27}{2}\pi$ cubic units.

28. $V = \pi\int_{-1}^{3/2} \left[(4-x^2+x)^2-(4-3+x^2)^2\right] dx$

$= \pi\int_{-1}^{3/2} (16-7x^2+8x-2x^3+x^4)$

$- (1+2x^2+x^4) dx$



$= \pi\cdot(15x-3x^3+4x^2-\frac{x^4}{2}) \Big|_{-1}^{\frac{3}{2}}$

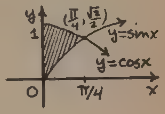
$= \pi\left[(\frac{45}{2} - \frac{81}{8} + 9 - \frac{81}{32}) -\right.$

$\left.(-15 + 3 + 4 - \frac{1}{2})\right] = \pi(\frac{875}{32})$ cubic units

29. $V = \pi\int_0^{\frac{\pi}{4}} \left[\cos^2 x-\sin^2 x\right] dx$

$= \pi\int_0^{\frac{\pi}{4}} \cos 2x dx = \frac{\pi}{2} \sin 2x \Big|_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}(\sin \frac{\pi}{2} - \sin 0)$

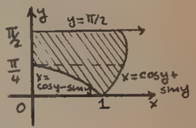


$= \frac{\pi}{2}(1 - 0)$

$= \frac{\pi}{2}$ cubic units.

30. $V = \pi\int_0^{\frac{\pi}{4}} \left[(\cos y+\sin y)^2-(\cos y-\sin y)^2\right] dy$

$+ \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos y + \sin y)^2 dy$



$= \pi\int_0^{\frac{\pi}{4}} 4 \sin y \cos y dy$

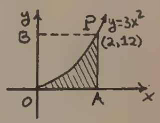
$+ \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + 2 \cos y \sin y) dy$

$= \pi\int_0^{\frac{\pi}{4}} 2\sin 2y dy + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\sin 2y) dy$

$= \pi(-\cos 2y) \Big|_0^{\frac{\pi}{4}} + \pi(y - \frac{1}{2}\cos 2y) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \pi(\frac{3}{2} + \frac{\pi}{4}) = \frac{3\pi}{2} + \frac{\pi^2}{4}$ cubic units.

31. $V = \pi\int_0^2 (3x^2)^2 dx$

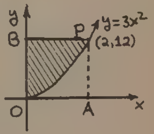


$= \pi\int_0^2 9x^4 dx$

$= \pi(\frac{9}{5})x^5 \Big|_0^2 = \frac{288}{5}\pi$ cubic units.

32. $V = \pi\int_0^2 (12^2 - 9x^4) dx$

$= \pi(144x - \frac{9}{5}x^5) \Big|_0^2$



$= \pi(288 - \frac{288}{5})$

$= \frac{1152}{5}\pi$ cubic units.

33. $V = \pi\int_0^{12} (\sqrt{\frac{y}{3}})^2 dy$

$= \pi(\frac{y^2}{6}) \Big|_0^{12} = 24\pi$ cubic units.

34. $V = \pi\int_0^{12}\left[2^2-\left(\sqrt{\frac{y}{3}}\right)^2\right]dy = \pi\int_0^{12}\left(4-\frac{y}{3}\right)dy$

$= \pi\left(4y-\frac{y^2}{6}\right)\Big|_0^{12} = \pi(48-24)$

$= 24\pi$ cubic units.

35. $V = \pi\int_0^{12}\left(2-\sqrt{\frac{y}{3}}\right)^2dy = \pi\left(4y-\frac{4}{\sqrt{3}}\cdot\frac{2}{3}y^{\frac{3}{2}}+\frac{y^2}{6}\right)\Big|_0^{12}$

$= \pi\left(48-\frac{8}{3\sqrt{3}}\cdot 24\sqrt{3}+24\right) = \pi(72-64)$

$= 8\pi$ cubic units.

36. $V = \pi\int_0^{12}\left[2^2-\left(2-\sqrt{\frac{y}{3}}\right)^2\right]dy = \pi\int_0^{12}\left(\frac{4}{\sqrt{3}}\sqrt{y}-\frac{y}{3}\right)dy$

$= \pi\left(\frac{4}{\sqrt{3}}\cdot\frac{2}{3}y^{\frac{3}{2}}-\frac{y^2}{6}\right)\Big|_0^{12} = \pi\left(\frac{8}{3\sqrt{3}}\,24\sqrt{3}-24\right)$

$= 40\pi$ cubic units.

37. $V = \pi\int_0^2(12-3x^2)^2dx = \pi\int_0^2(144-72x^2+9x^4)dx$

$= \pi\left(144x-24x^3+\frac{9}{5}x^5\right)\Big|_0^2 = \frac{768}{5}\pi$ cubic units

38. $V = \pi\int_0^2\left[12^2-(12-3x^2)^2\right]dx$

$= \pi\int_0^2(72x^2-9x^4)\,dx$

$= \pi\left(24x^3-\frac{9}{5}x^5\right)\Big|_0^2 = \pi\left(192-\frac{288}{5}\right)$

$= \frac{672}{5}\pi$ cubic units.

39. $V = \pi\int_0^{12}\left[\left(a-\sqrt{\frac{y}{3}}\right)^2-(a-2)^2\right]dy$

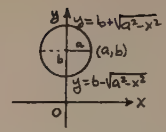
$= \pi\int_0^{12}\left(\frac{y}{3}+4a-4-\frac{2a}{\sqrt{3}}y^{\frac{1}{2}}\right)dy$

$= \pi\left[\frac{y^2}{6}+(4a-4)y-\frac{2a}{\sqrt{3}}\cdot\frac{2}{3}y^{\frac{3}{2}}\right]\Big|_0^{12}$

$= \pi\left[24+(4a-4)12-\frac{4a}{3\sqrt{3}}\cdot 24\sqrt{3}\right]$

$= (16a-24)\pi$ cubic units.

40. We place the center at $(0,b)$ so that the equation of the circle is $x^2+(y-b)^2 = a^2$; that is, $y = b\pm\sqrt{a^2-x^2}$. Using the method of circular rings, we have



$V = \pi\int_{-a}^{a}\left[(b+\sqrt{a^2-x^2})^2 - (b-\sqrt{a^2-x^2})^2\right]dx$

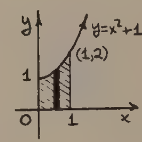
$= \pi\int_{-a}^{a}(b^2+2b\sqrt{a^2-x^2}+a^2-x^2-b^2+2b\sqrt{a^2-x^2}-a^2+x^2)dx = \pi\int_{-a}^{a}4b\sqrt{a^2-x^2}dx$

$= 4\pi b\int_{-a}^{a}\sqrt{a^2-x^2}dx.$ But $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$ represents the area of a semicircle of radius a, and so $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \frac{1}{2}\pi a^2.$

Hence, $V = 4\pi b\left(\frac{1}{2}\pi a^2\right) = 2\pi^2a^2b$ cubic units

Problem Set 6.2, page 368

1.



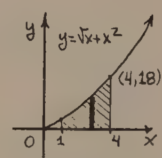
$V = 2\pi\int_0^1 x(x^2+1)dx$

$= 2\pi\int_0^1(x^3+x)dx$

$= 2\pi\left[\frac{x^4}{4}+\frac{x^2}{2}\right]\Big|_0^1$

$= 2\pi\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{3\pi}{2}$ cubic units.

2.



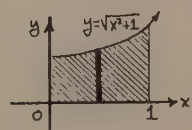
$V = 2\pi\int_1^4 x(\sqrt{x}+x^2)dx$

$= 2\pi\int_1^4\left(x^{\frac{3}{2}}+x^3\right)dx$

$= 2\pi\left[\frac{2}{5}x^{5/2}+\frac{x^4}{4}\right]\Big|_1^4$

$= 2\pi\left[\frac{2}{5}(32)+4^3-\frac{2}{5}-\frac{1}{4}\right] = \frac{1523\pi}{10}$ cubic unit

3.



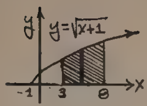
$V = 2\pi\int_0^1 x\sqrt{x^2+1}\,dx$

Let $u = x^2+1$, $du = 2xdx$.

When $x = 0$ $u=1$; $x=1$ $u=2$. S

$V = 2\pi\int_1^2\frac{1}{2}\sqrt{u}\,du = \pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2 = \pi\left[\frac{2}{3}\cdot 2^{\frac{3}{2}}-\frac{2}{3}\right]$

$= \frac{2\pi}{3}\left[2\sqrt{2}-1\right]$ cubic units.



$V = 2\pi \int_3^8 x\sqrt{x+1}\, dx$

Let $u = x+1$, so $du = dx$ and $x = u-1$. When $x=3$, $u=4$; when $x = 8$, $u = 9$.

$$V = 2\pi \int_4^9 (u-1)\sqrt{u}\, du = 2\pi \int_4^9 (u^{\frac{3}{2}} - u^{\frac{1}{2}})du$$

$$= 2\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_4^9$$

$$= 2\pi\left[\tfrac{2}{5}(3^5) - \tfrac{2}{3}(3^3) - (\tfrac{2}{5}\cdot 2^5 - \tfrac{2}{3}\cdot 2^3)\right]$$

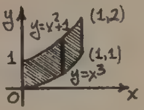
$$= 2\pi\left[\tfrac{422}{5} - \tfrac{38}{3}\right] = 2\pi(\tfrac{1076}{15}) = \tfrac{2152\pi}{15} \text{ cubic units.}$$

$$V = \int_0^1 2\pi\left[(x^2+1)-x^3\right]x\,dx$$

$$= 2\pi \int_0^1 (x^3+x-x^4)dx$$

$$= 2\pi(\tfrac{x^4}{4} + \tfrac{x^2}{2} - \tfrac{x^5}{5}) \Big|_0^1$$

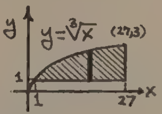
$$= \tfrac{11}{10}\pi \text{ cubic units.}$$



$$V = \int_0^{27} 2\pi(\sqrt[3]{x}-1)\, x\,dx$$

$$= 2\pi \int_0^{27} (x^{\frac{4}{3}} - x)dx$$

$$= 2\pi(\tfrac{3}{7}x^{7/3} - \tfrac{x^2}{2}) \Big|_0^{27} = \tfrac{8019}{7}\pi \text{ cubic units.}$$

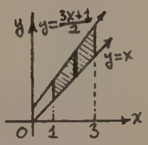


$$V = \int_1^3 2\pi(\tfrac{3x+1}{2} - x)x\,dx$$

$$= \pi\int_1^3 (x^2 + x)dx$$

$$= \pi(\tfrac{x^3}{3} + \tfrac{x^2}{2}) \Big|_1^3$$

$$= \pi(9 + \tfrac{9}{2}) - \pi(\tfrac{1}{3} + \tfrac{1}{2}) = \tfrac{38}{3}\pi \text{ cubic units.}$$

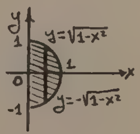


$$V = \int_0^1 2\pi(\sqrt{1-x^2} + \sqrt{1-x^2})x\,dx$$

$$= 4\pi \int_0^1 x\sqrt{1-x^2}\,dx.$$

Putting $u = 1-x^2$, we have $du = -2x\,dx$; hence, $V = -2\pi \int_1^0 \sqrt{u}\,du$

$$= 2\pi(\tfrac{2}{3}u^{3/2}) \Big|_0^1 = \tfrac{4}{3}\pi \text{ cubic units.}$$



9. $V = 2\pi \int_{\frac{\sqrt{\pi}}{2}}^{\sqrt{\pi}} x\sin x^2 dx$ Let $u=x^2$, $du=2x\,dx$.

When $x=\frac{\sqrt{\pi}}{2}$, $u=\frac{\pi}{4}$; when $x=\sqrt{\pi}$, $u=\pi$.

$$V = \tfrac{2\pi}{2}\int_{\frac{\pi}{4}}^{\pi} \sin u\, du = \pi\left[-\cos u\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \pi\left[-\cos\pi + \cos\tfrac{\pi}{4}\right] = \pi\left[1 + \tfrac{\sqrt{2}}{2}\right]$$

$$= (\tfrac{2+\sqrt{2}}{2})\pi \text{ cubic units.}$$

10. $V = 2\pi \int_0^{\frac{\sqrt{\pi}}{2}} x(\cos x^2 - \sin x^2)dx$

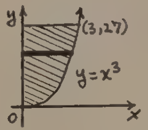
Let $u = x^2$, $du = 2x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\sqrt{\pi}}{2}$, $u = \frac{\pi}{4}$.

$$V = \tfrac{2\pi}{2}\int_0^{\frac{\pi}{4}} (\cos u - \sin u)\, du$$

$$= \pi\left[\sin u + \cos u\right]_0^{\frac{\pi}{x}}$$

$$= \pi\left[\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} - 0 - 1\right] = \pi(\sqrt{2}-1) \text{ cubic units.}$$

11.

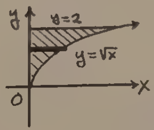


$$V = \int_0^{27} 2\pi y^{\frac{1}{3}} y\,dy$$

$$= \tfrac{6\pi}{7}y^{7/3} \Big|_0^{27}$$

$$= \tfrac{13,122}{7}\pi \text{ cubic units.}$$

12.

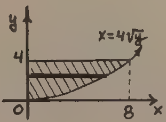


$$V = 2\pi\int_0^2 y(y^2)dy$$

$$= 2\pi\int_0^2 y^3 dy = 2\pi(\tfrac{y^4}{4}) \Big|_0^2$$

$$= 2\pi(4) = 8\pi \text{ cubic units.}$$

13.

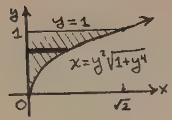


$$V = 2\pi\int_0^4 y(4\sqrt{y})dy$$

$$= 8\pi\int_0^4 y^{3/2}dy$$

$$= 8\pi\left[\tfrac{2}{5}y^{5/2}\right]_0^4 = 8\pi(\tfrac{2}{5}\cdot 32)$$

$$= \tfrac{512\pi}{5} \text{ cubic units.}$$

14. 

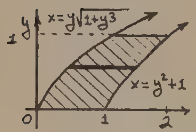
$V = 2\pi\int_0^1 y\cdot y^2\sqrt{1+y^4}\,dy$

$= 2\pi\int_0^1 y^3\sqrt{1+y^4}\,dy$

Let $u = 1+y^4$, $du = 4y^3dy$. When $y = 0$, $u = 1$; $y = 1$, $u = 2$.

$V = \frac{2\pi}{4}\int_1^2\sqrt{u}\,du = \frac{\pi}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2 = \frac{\pi}{3}(2^{\frac{3}{2}}-1)$

$= \frac{\pi}{3}(2\sqrt{2}-1)$ cubic units.

15. 

$V = 2\pi\int_0^1 y\left[y^2+1-(y\sqrt{1+y^3})\right]dy$

$= 2\pi\int_0^1(y^3+y-y^2\sqrt{1+y^3})\,dy$

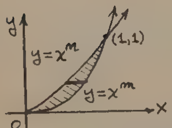
Let $u = 1+y^3$, $du = 3y^2dy$; so that $\int y^2\sqrt{1+y^3}\,dy = \int\sqrt{u}\,\frac{1}{3}\,du$

$= \frac{1}{3}\,\frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{9}(1+y^3)^{\frac{3}{2}}+c$

$V = 2\pi\left[\frac{y^4}{4}+\frac{y^2}{2}-\frac{2}{9}(1+y^3)^{\frac{3}{2}}\right]\Big|_0^1$

$= 2\pi\left[\frac{1}{4}+\frac{1}{2}-\frac{2}{9}\cdot 2^{\frac{3}{2}}+\frac{2}{9}\right]$

$= 2\pi(\frac{35}{36}-\frac{4\sqrt{2}}{9}) = \pi(\frac{35-16\sqrt{2}}{18})$ cubic units.

16. 

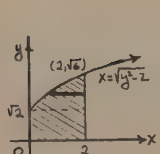
$V = 2\pi\int_0^1 y\left[\sqrt[m]{y}-\sqrt[n]{y}\right]dy$

$= 2\pi\int_0^1(y^{\frac{m+1}{m}} - y^{\frac{n+1}{n}})\,dy$

$= 2\pi\left[\frac{m}{2m+1}\,y^{\frac{2m+1}{m}} - \frac{n}{2n+1}\,y^{\frac{2n+1}{n}}\right]_0^1$

$= 2\pi\left[\frac{m}{2m+1}-\frac{n}{2n+1}\right] = 2\pi\left[\frac{2nm+m-2nm-n}{(2m+1)(2n+1)}\right]$

$= 2\pi\left[\frac{m-n}{(2m+1)(2n+1)}\right]$ cubic units.

17. 

$V = 2\pi\int_{\sqrt{2}}^{\sqrt{6}}(2y-y\sqrt{y^2-2})\,dy$

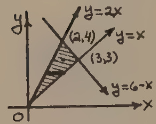
$+ 2\pi\int_0^{\sqrt{2}}2y\,dy$

$= 2\pi(y^2\Big|_{\sqrt{2}}^{\sqrt{6}})-2\pi\int_0^4\frac{1}{2}u^{\frac{1}{2}}du$

$+ 2\pi(2)$

$= 2\pi(4) - 2\pi(\frac{1}{3})u^{\frac{3}{2}}\Big|_0^4 + 4\pi$

$= 8\pi - \frac{16\pi}{3} + 4\pi = \frac{20\pi}{3}$ cubic units.

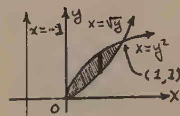
18. 

$V = 2\pi\int_0^3 y(y-\frac{y}{2})\,dy$

$+ 2\pi\int_3^4 y\left[(6-y)-(\frac{y}{2})\right]dy$

$= 2\pi(\frac{y^3}{6})\Big|_0^3 + 2\pi(3y^2-\frac{y^3}{2})\Big|_3^4$

$= 9\pi + 5\pi = 14\pi$ cubic units.

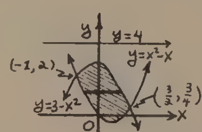
19. 

$V = 2\pi\int_0^1(x+1)(\sqrt{x}-x^2)\,dx$

$= 2\pi\int_0^1(x^{\frac{3}{2}}-x^3+x^{\frac{1}{2}}-x^2)\,dx$

$= 2\pi\left[\frac{2}{5}x^{\frac{5}{2}}-\frac{x^4}{4}+\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{3}\right]$

$= 2\pi(\frac{2}{5}-\frac{1}{4}+\frac{2}{3}-\frac{1}{3}) = 2\pi(\frac{29}{60}) = \frac{29\pi}{30}$ cubic units.

20. 

$V = 2\pi\int_{-1/4}^{3/4}(4-y)(\frac{1+\sqrt{1+4y}}{2}$

$- \frac{1-\sqrt{1+4y}}{2})\,dy + 2\pi\int_{3/4}^{2}$

$(4-y)(\sqrt{3-y}) - \frac{1-\sqrt{1+4y}}{2})\,dy + 2\pi\int_2^3(4-y)$

$\left[\sqrt{3-y}-(-\sqrt{3-y})\right]dy = 2\pi\int_{-1/4}^{3/4}(4-y)\sqrt{1+4y}\,dy$

$+ 2\pi\int_{3/4}^2(4-y)\sqrt{3-y}\,dy - \pi\int_{3/4}^2(4-y)\,dy$

$+ \pi\int_{3/4}^2(4-y)\sqrt{1+4y}\,dy + 4\pi\int_2^3(4-y)\sqrt{3-y}\,dy$

In the first and fourth integrals, make the change of variable $u = 1+4y$ and in the second and fifth integrals, make the change of variable $v = 3-y$ to obtain

$V = 2\pi\int_0^4\frac{17-u}{4}\sqrt{u}\,\frac{du}{4} + 2\pi\int_{9/4}^1(1+v)\sqrt{v}(-1)d$

$-\pi\int_{3/4}^2(4-y)\,dy + \pi\int_4^9\frac{17-u}{4}\sqrt{u}\,\frac{du}{4}$

$+ 4\pi\int_1^0 (1+v)\sqrt{v}(-1)dv.$ Therefore,

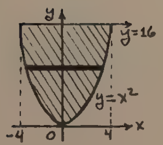
$$V = \frac{17\pi}{8}\int_0^4 u^{\frac12}du - \frac{\pi}{8}\int_0^4 u^{\frac32}du + 2\pi\int_1^{9/4} v^{\frac12}dv$$
$$+ 2\pi\int_1^{9/4} v^{\frac32}dv - \pi(4y - \frac{y^2}{2})\Big|_{3/4}^{2}$$
$$+ \frac{17\pi}{16}\int_4^9 u^{\frac12}du - \frac{\pi}{16}\int_4^9 u^{\frac32}du + 4\pi\int_0^1 v^{\frac12}dv$$

$+ 4\pi\int_0^1 v^{\frac32}dv.$ Therefore, $V = \frac{17\pi}{8}(\frac23 u^{\frac32})\Big|_0^4$

$$- \frac{\pi}{8}(\frac25 u^{\frac52})\Big|_0^4 + 2\pi(\frac23 v^{\frac32})\Big|_1^{9/4} + 2\pi(\frac25 v^{\frac52})\Big|_1^{9/4}$$
$$+ \pi(\frac{y^2}{2} - 4y)\Big|_{3/4}^{2} + \frac{17\pi}{16}(\frac23 u^{\frac32})\Big|_4^9 - \frac{\pi}{16}(\frac25 u^{\frac52})\Big|_4^9$$

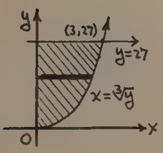
$+ 4\pi(\frac23 u^{\frac32})\Big|_0^1 + 4\pi(\frac25 u^{\frac52})\Big|_0^1.$ Consequently,

$$V = \frac{17\pi}{8}(\frac{16}{3}) - \frac{\pi}{8}(\frac{64}{5}) + 2\pi(\frac94 - \frac23)$$
$$+ 2\pi(\frac{243}{80} - \frac25) + \pi(-6 + \frac{87}{32}) + \frac{17\pi}{16}(18 - \frac{16}{3})$$
$$- \frac{\pi}{16}(\frac{486}{5} - \frac{64}{5}) + 4\pi(\frac23) + 4\pi(\frac25)$$
$$= \frac{875}{32}\pi \text{ cubic units.}$$

1. 

$V = \int_0^{16} 2\pi(2\sqrt{y})y\,dy = \frac{8\pi}{5}y^{\frac52}\Big|_0^{16}$

$= \frac{8192}{5}\pi$ cubic units.

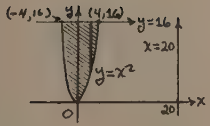
2. 

$V = \int_0^{27} 2\pi y^{\frac13}(27-y)dy$

$= 54\pi\int_0^{27} y^{\frac13}dy - 2\pi\int_0^{27} y^{\frac43}dy$

$= 54\pi(\frac34 y^{\frac43})\Big|_0^{27} - \frac{6\pi}{7}y^{\frac73}\Big|_0^{27}$

$= \frac{19,683}{14}\pi$ cubic units.

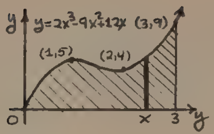
23. 

$V = \int_{-4}^{4} 2\pi(16-x^2)(20-x)dx$

$= 2\pi\int_{-4}^{4}(x^3-20x^2-16x+320)dx$

$= 2\pi(\frac{x^4}{4}-(20)\frac{x^3}{3}-8x^2+320x)\Big|_{-4}^{4}$

$= \frac{10,240}{3}\pi$ cubic units.

24. 

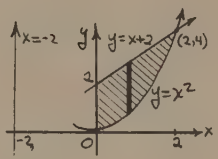
$V = \int_0^3 \pi(2x^3-9x^2+12x)^2dx$

$= \pi(\frac47 x^7 - 6x^6 + \frac{129}{5}x^5 - 54x^4$

$+ 48x^3)\Big|_0^3$

$= \pi\left[\frac{8748}{7} + \frac{31,347}{5} - 7452\right]$

$= \pi(\frac{263,169 - 260,820}{35}) = \pi(\frac{2349}{35})$ cu. units.

25. 

$V = 2\pi\int_0^2 (x+2)(x+2-x^2)dx$

$= 2\pi\int_0^2 (x^2+4x+4-x^3-2x^2)dx$

$= 2\pi\int_0^2 (4+4x-x^2-x^3)dx$

$= 2\pi\left[4x+2x^2-\frac{x^3}{3}-\frac{x^4}{4}\right]\Big|_0^2$

$= 2\pi\left[8 + 8 - \frac83 - 4\right] = \frac{56}{3}\pi$ cubic units.

26. $V = 2\pi\int_0^{27}(y+3)\left[-\sqrt{\frac{y}{3}} - \sqrt[3]{-y}\right]dy$

$= 2\pi\int_0^{27}(y+3)(y^{1/3} - \frac{y^{\frac12}}{\sqrt3})dy$

$= 2\pi\int_0^{27}(y^{\frac43} - \frac{\sqrt3}{3}y^{\frac32} + 3y^{\frac13} - \sqrt3 y^{\frac12})dy$

$= 2\pi(\frac37 y^{7/3} - \frac{2\sqrt3}{15}y^{5/2} + \frac94 y^{4/3} - \frac{2\sqrt3}{3}y^{3/2})\Big|_0^{27}$

$= 2\pi\left[\frac37(3)^7 - \frac{2\sqrt3}{15}(3^7\sqrt3) + \frac94(3)^4 - 2(3)^4\right]$

$= 162\pi\left[\frac{143}{140}\right] = \frac{11,583}{70}\pi$ cubic units.

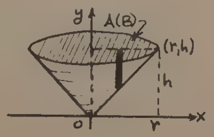
27. $V = \pi\int_1^4\left[(\sqrt{x}+2)^2 - 3^2\right]dx$

$= \pi\int_1^4(x+4\sqrt{x}-5)dx = \pi(\frac{x^2}{2} + \frac83 x^{\frac32} - 5x)\Big|_1^4$

$= \pi(8 + \frac83(8) - 20 - \frac12 - \frac83 + 5) = \pi(8 - \frac43 + \frac92)$

$= \frac{67\pi}{6}$ cubic units.

28. Let the cone have radius of base r and height h. The line through (0,0) and

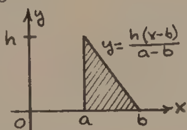


(r,h) has equation $y=\frac{h}{r}x$. Revolve the region bounded by the line $y=\frac{h}{r}x$ and the lines y=h and x=0 about the y axis.

$$V = 2\pi\int_0^r x(h - \frac{h}{r}x)dx = 2\pi(\frac{hx^2}{2} - \frac{hx^3}{3r})\Big|_0^r$$

$$= 2\pi(\frac{r^2h}{2} - \frac{r^3h}{3r}) = 2\pi(\frac{r^2h}{2} - \frac{r^2h}{3}) = 2\pi(\frac{r^2h}{6})$$

$= \frac{\pi r^2h}{3}$. Since the area of the base $A(B) = \pi r^2$, the volume is $\frac{1}{3}hA(B)$.

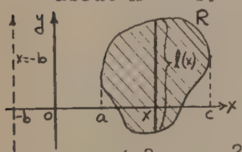


29. $V = 2\pi\int_a^b x\left[h\frac{x-b}{a-b}\right]dx = \frac{2\pi h}{a-b}\int_a^b(x^2-bx)dx$

$$= \frac{2\pi h}{a-b}(\frac{x^3}{3} - \frac{bx^2}{2})\Big|_a^b = \frac{2\pi h}{a-b}(\frac{b^3}{3} - \frac{b^3}{2}) - \frac{2\pi h}{a-b}(\frac{a^3}{3} - \frac{ba^2}{2})$$

$$= \frac{\pi h}{3}(b^2+ab-2a^2) = \frac{\pi h}{3}(b-a)(b+2a).$$

30. $V = 2\pi\int_a^c xl(x)dx$. Let V_b be the volume of the solid generated by revolving R about x = -b. Then $V_b = 2\pi\int_a^c(x+b)l(x)dx$



$$= 2\pi\int_a^c xl(x)dx + 2\pi b\int_a^c l(x)dx$$

$$= V + 2\pi bA.$$

31. $V = \pi\int_{-a}^a b^2(1-\frac{x^2}{a^2})dx = 2\pi b^2\int_0^a(1-\frac{x^2}{a^2})dx$

$$= 2\pi b^2(x-\frac{x^3}{3a^2})\Big|_0^a = 2\pi b^2(a - \frac{a^3}{3a^2})$$

$$= 2\pi b^2(\frac{2}{3}a) = \frac{4}{3}\pi ab^2.$$

32. Using same diagram in Problem 40, Section 6.1, we have,

$$V = 2\pi\int_{b-a}^{b+a} y\left[\sqrt{a^2-(y-b)^2}-(-\sqrt{a^2-(y-b)^2}\right]dy$$

$= 4\pi\int_{b-a}^{b+a} y\sqrt{a^2-(y-b)^2}dy$ Let u=y-b,

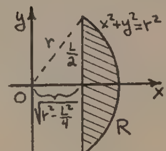
du=dy; y=b-a, u=-a; y=b+a, u=a

$$= 4\pi\int_{-a}^a(u+b)\sqrt{a^2-u^2}\,du = 4\pi\int_{-a}^a u\sqrt{a^2-u^2}du$$

$+ 4\pi b\int_{-a}^a\sqrt{a^2-u^2}du$. Since $f(u)=u\sqrt{a^2-u^2}$ is odd, $\int_{-a}^a f(u)du = 0$. Thus,

$V = 4\pi b\int_{-a}^a\sqrt{a^2-u^2}\,du$. But the integral is area of a semicircle with radius a.

So $V = 4\pi b(\frac{\pi a^2}{2}) = 2\pi^2a^2b$ cubic units.

33. 

The volume V remaining is generated by revolving the region R in the adjacent figure about the y axis; hence, $V = 2\pi\int_{\sqrt{r^2-\frac{L^2}{4}}}^{r} x(2\sqrt{r^2-x^2})dx$.

Making the substitution $u = r^2 - x^2$, we obtain $V = -2\pi\int_{\frac{L^2}{4}}^{0} u^{\frac{1}{2}}du = 2\pi(\frac{2}{3}u^{\frac{3}{2}})\Big|_0^{\frac{L^2}{4}}$

$= \frac{4\pi}{3}(\frac{L}{2})^3$ cubic units, which is the same as the volume of a sphere of diameter L.

Problem Set 6.3, page 373

1. Now $V = \int_{-3}^3 A(s)ds$ where $A(s) = x^2$. Now $\frac{x^2}{4} + s^2 = 9$, so $x^2 = 36-4s^2$.

$$V = \int_{-3}^3(36-4s^2)ds = (36s- \frac{4}{3}s^3)\Big|_{-3}^3$$

$$= (108-36)-(-108+36) = 144 \text{ cubic units.}$$

2. $V = \int_{-5}^5 A(s)ds$, where $A(s) = \frac{1}{2}xh$. Now $(\frac{x}{2})^2+s^2 = 25$, so $x^2 = 100 - 4s^2$. Also, $\frac{x^2}{4} + h^2= x^2$, so $h^2 = \frac{3x^2}{4}$ and $h = \frac{\sqrt{3x}}{2}$. So

$A(s) = \frac{1}{2}x\frac{\sqrt{3x}}{2} = \frac{\sqrt{3x}^2}{4} = \sqrt{3}\,(25 - s^2)$. Now

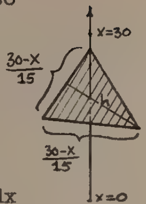
$$V = \int_{-5}^5\sqrt{3}(25-s^2)ds = \sqrt{3}(25s - \frac{s^3}{3})\Big|_{-5}^5$$

$$= \sqrt{3}\,(125-\frac{125}{3})-\sqrt{3}(-125+\frac{125}{3})$$

$$= \frac{500\sqrt{3}}{3} \text{ cubic centimeters.}$$

3. $V = \int_0^{30} A(x)dx$, where $A(x) = \frac{1}{2}\frac{(30-x)}{15}\cdot h$

and $h^2+\frac{(30-x)^2}{900}=\frac{(30-x)^2}{15^2}$, so



$h=\frac{30-x}{\sqrt{300}}$.

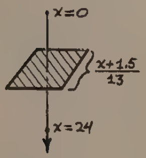
$$V=\int_0^{30}\frac{1}{2}\,\frac{(30-x)}{15}\cdot\frac{30-x}{\sqrt{300}}dx$$

$$=\frac{1}{300\sqrt{3}}\int_0^{30}(900-60x+x^2)dx$$

$$=\frac{1}{300\sqrt{3}}(900x-30x^2+\frac{x^3}{3})\Big|_0^{30}$$

$$=\frac{1}{300\sqrt{3}}(27,000-27,000+\frac{27,000}{3})$$

$=10\sqrt{3}$ cubic meters.

4. 

$$V=\int_0^{24}A(x)dx$$

$$=\int_0^{24}\frac{1}{13^2}(x+1.5)^2dx.$$

Let $u=x+1.5$, $du=dx$.

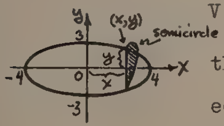
$$V=\int_0^{24}\frac{1}{13^2}(x+1.5)^2dx=\frac{1}{169}\int_{1.5}^{25.5}u^2du$$

$$=\frac{1}{507}u^3\Big|_{1.5}^{25.5}=\frac{1}{507}\left[(25.5)^3-(1.5)^3\right]$$

$=\frac{5526}{169}$ cubic meters.

5. $V=\frac{4}{3}\pi r_2^3-\frac{4}{3}\pi r_1^3=\frac{4\pi}{3}\left[(\frac{y_2}{2})^3-(\frac{y_1}{2})^3\right]$

$=\frac{\pi}{6}(y_2^3-y_1^3)$ cubic units.

6. 

$V=\int_{-4}^{4}\frac{1}{2}\pi y^2dx$. (x,y) on the ellipse satisfies the equation $\frac{x^2}{16}+\frac{y^2}{9}=1$, so $y^2=9-\frac{9x^2}{16}$ and

$$V=\int_{-4}^{4}\frac{1}{2}\pi(9-\frac{9x^2}{16})dx=\frac{\pi}{2}(9x-\frac{3x^3}{16})\Big|_{-4}^{4}$$

$$=\frac{\pi}{2}\left[(36-12)-(-36+12)\right]=\frac{\pi}{2}(48)$$

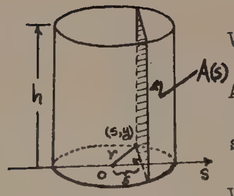
$=24\pi$ cubic units.

7. (a) $V=\int_0^5A(s)ds=\int_0^5(3s^2+2)ds$

$=(s^3+2s)\Big|_0^5=125+10=135$ cu. meters.

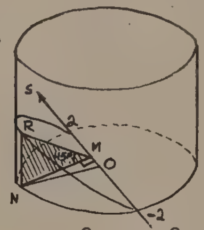
(b) $V=\int_0^5(s^2+s)ds=(\frac{s^3}{3}+\frac{s^2}{2})\Big|_0^5$

$=\frac{125}{3}+\frac{25}{2}=\frac{325}{6}$ cubic meters.

8. 

$V=\int_{-r}^{r}A(s)ds$, where $A(s)=2yh$ and $s^2+y^2=r^2$, so $y=\sqrt{r^2-s^2}$. Now $V=\int_{-r}^{r}2\sqrt{r^2-s^2}\cdot h\,ds$

$=2h\int_{-r}^{r}\sqrt{r^2-s^2}ds=2h\cdot\frac{1}{2}\pi r^2=\pi r^2h$ cubic units, since $\int_{-r}^{r}\sqrt{r^2-s^2}ds$ is the area of a semicircle of radius r.

9. 

$V=\int_{-2}^{2}A(s)ds$,

$V=\int_{-2}^{2}\frac{1}{2}|\overline{NR}|\cdot|\overline{NM}|\,ds.$

Now $|\overline{OM}|=|s|$.

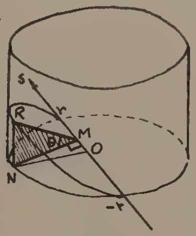
$|\overline{NM}|^2+|\overline{OM}|^2=|\overline{NO}|^2.$

$|\overline{NM}|^2=4-s^2$. In $\triangle NRM$, $\tan 45^\circ=1$

$=\frac{|\overline{NR}|}{|\overline{NM}|}=\frac{|\overline{NR}|}{\sqrt{4-s^2}}$. So $|\overline{NR}|=\sqrt{4-s^2}$.

$$V=\int_{-2}^{2}\frac{1}{2}(4-s^2)ds=\frac{1}{2}(4s-\frac{s^3}{3})\Big|_{-2}^{2}$$

$$=\frac{1}{2}\left[(8-\frac{8}{3})-(-8+\frac{8}{3})\right]=\frac{1}{2}(\frac{32}{3})=\frac{16}{3}\text{cu.ft.}$$

10. 

$V=\int_{-r}^{r}\frac{1}{2}|\overline{NR}||\overline{NM}|\,ds.$

$|\overline{OM}|=s$, $|\overline{NM}|^2+s^2=r^2$,

$|\overline{NM}|^2=r^2-s^2$. Now,

$\tan\theta=\frac{|\overline{NR}|}{|\overline{NM}|}$, so

$(\tan\theta)|\overline{NM}|=|\overline{NR}|$. So $V=\int_{-r}^{r}\frac{1}{2}\tan\theta\,|\overline{NM}||\overline{NM}|\,ds$

$$=\int_{-r}^{r}\frac{1}{2}\tan\theta(r^2-s^2)ds=\frac{1}{2}\tan\theta(r^2s-\frac{s^3}{3})\Big|_{-r}^{r}$$

$$=\frac{1}{2}\tan\theta\left[(r^3-\frac{r^3}{3})-(-r^3+\frac{r^3}{3})\right]$$

$=(\tan\theta)(\frac{2}{3}r^3)$ cubic units.

11. $V=\int_0^{1.5}A(x)dx$, where $A(x)=\pi\cdot y\cdot\frac{y}{2}$

$=\pi\frac{y^2}{2}$ and $y=\frac{1}{12}x^2+1$. So

$$V=\int_0^{1.5}\frac{\pi}{2}(\frac{1}{12}x^2+1)^2dx$$

$$=\frac{\pi}{2}\int_0^{1.5}(\frac{x^4}{144}+\frac{x^2}{6}+1)dx=\frac{\pi}{2}(\frac{x^5}{720}+\frac{x^3}{18}+x)\Big|_0^{1.5}$$

$=\frac{\pi}{2}\left[\frac{(1.5)^5}{720}+\frac{(1.5)^3}{18}+(1.5)\right] = \frac{4347}{5120}\pi$ cubic meters.

12. V_2(of small cone)$=(\frac{1}{3})(H-h)a$.

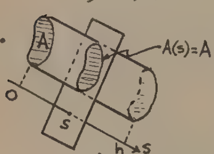
V_1(of large cone)$=(\frac{1}{3})HA$. The desired volume is $V = V_1-V_2$. So $V=(\frac{1}{3})$ $[HA-(H-h)a] = (\frac{1}{3})[H(A-a)+ha]$. Now $a = k(H-h)^2$, $A = kH^2$, so $k = \frac{A}{H^2}$. Hence, $a = \frac{A}{H^2}(H-h)^2$; so solving for H, we get $\frac{H-h}{H} = \sqrt{\frac{a}{A}}$, or $H = \frac{h}{1-\sqrt{\frac{a}{A}}}$. Now we have $V = (\frac{1}{3})\left[\frac{h}{1-\sqrt{\frac{a}{A}}}(A-a)+ha\right] =$

$\frac{h}{3}\left[\frac{A-a+a-a\sqrt{\frac{a}{A}}}{1-\sqrt{\frac{a}{A}}}\right] = \frac{h}{3}\left[\frac{A-a\sqrt{\frac{a}{A}}}{1-\sqrt{\frac{a}{A}}}\right]$. So

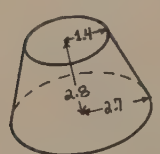
$V = \frac{h}{3}\left[\frac{A^{3/2}-a^{3/2}}{\sqrt{A}-\sqrt{a}}\right] = \frac{h}{3}\left[\frac{\sqrt{A}^3-\sqrt{a}^3}{\sqrt{A}-\sqrt{a}}\right]$

$= \frac{h}{3}\left[\sqrt{A^2}+\sqrt{aA}+\sqrt{a^2}\right]$. Therefore,

$V = \frac{h}{3}(A+\sqrt{aA}+a)$ cubic units.

13. 

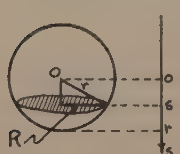
If A is the area of the base, then $A(s) = A$ and $V = \int_0^h A(s)ds = \int_0^h A ds$ $= A\int_0^h ds = Ah$ cubic units.

14. 

$a = \pi(1.4)^2$, $A = \pi(2.7)^2$, $h = 2.8$. $V = \frac{h}{3}(A+\sqrt{Aa}+a)$

$= \frac{2.8}{3}\left[(2.7)^2\pi+\pi(1.4)(2.7)+(1.4)^2\pi\right]$

$= \frac{2.8\pi}{3}(13.03) = 12.16\pi \approx 38.21$ cu.meters.

15. 

$R^2+s^2 = r^2$, so that $R^2 = r^2-s^2$ and $A(s) = \pi R^2 = \pi(r^2-s^2)$. Thus,

$V = \int_{r-h}^{r} A(s)ds = \pi\int_{r-h}^{r}(r^2-s^2)ds$

$= \pi(r^2s - \frac{s^3}{3})\Big|_{r-h}^{r} = \pi\left[r^3-\frac{r^3}{3}\right]-\pi\left[r^2(r-h) - \frac{(r-h)^3}{3}\right] = \pi h^2(r - \frac{h}{3})$ cubic units.

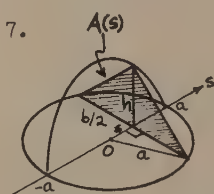
16. $V = \int_0^h A(s)ds = \int_0^h(as^2+bs+c)ds$

$= (\frac{as^3}{3}+\frac{bs^2}{2}+cs)\Big|_0^h = \frac{ah^3}{3}+\frac{bh^2}{2}+ch$

$= \frac{h}{6}(2ah^2+3bh+6c)$. By the prismoidal formula, since A(s) is a polynomial of degree ≤ 3, $\int_0^h A(s)ds$

$= \frac{h-0}{6}\left[A(0)+4A(\frac{0+h}{2})+A(h)\right] = \frac{h}{6}(A_0+4A_1+A_2$

So $V = \frac{h}{6}(A_0 + 4A_1 + A_2)$.

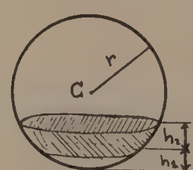
17. 

$V = \int_{-a}^{a}A(s)ds$, where $A(s) = \frac{1}{2}bh$. Now $a^2=s^2+h^2$ and $a^2=s^2+(\frac{b}{2})^2$, $h^2=a^2-s^2$ and $b^2=4(a^2-s^2)$. So

$\frac{1}{2}bh = \frac{1}{2}\cdot2\sqrt{a^2-s^2}\cdot\sqrt{a^2-s^2} = a^2-s^2$.

$V = \int_{-a}^{a}(a^2-s^2)ds = (a^2s-\frac{s^3}{3})\Big|_{-a}^{a}$

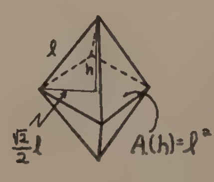
$= (a^3-\frac{a^3}{3})-(-a^3+\frac{a^3}{3}) = 2a^3-\frac{2a^3}{3} = \frac{4a^3}{3}$ cubic units.

18. 

We will subtract the volu of two spherical segments Using Problem 15 above, t desired volume V

$= \pi(h_1+h_2)^2(r-\frac{(h_1+h_2)}{3})-\pi(h_1)^2(r-\frac{h_1}{3})$.

So $V = \pi\left[(h_1+h_2)^2(r-\frac{(h_1+h_2)}{3})-(h_1)^2(r-\frac{h_1}{3})\right.$

19. 

By elementary geometry, the base is a square. Th diameter of the base, say d, satisfies $d^2 = 2(\ell)^2$, so $d = \sqrt{2}\cdot\ell$. Now the volume of a solid cone with base area

A(h) is given by $(\frac{1}{3})hA(h)$. So $V = \frac{2}{3}h[A(h)]$ where $A(h) = \ell^2$, and h satisfies $h^2 + (\frac{\sqrt{2}\cdot\ell}{2})^2 = \ell^2$, as seen in the diagram. Hence, $h = \frac{\ell}{\sqrt{2}}$.

$V = \frac{2}{3}\cdot\frac{\ell}{\sqrt{2}}\cdot\ell^2 = \frac{\sqrt{2}}{3}\cdot\ell^3$ cubic units.

0. The desired volume is the volume of the segment of two bases plus the volume of lower cone minus the volume of inner cone.

V(segment): $\pi[r-(h_2-h_1)]^2(r-\frac{r-(h_2-h_1)}{3}) -$

$\pi(r-h_2)^2(r-\frac{r-h_2}{3})$ (by Problem 18).

+V(lower cone): $\frac{\pi}{3}(h_2-h_1)(r^2-(h_2-h_1)^2)$

-V(inner cone): $\frac{\pi}{3}h_2(r^2-h_2^2)$. Let $k=h_2-h_1$.

Now V(lower cone)-V(inner cone) =

$\frac{\pi}{3}[h_2r^2-h_1r^2-(h_2-h_1)^3-h_2r^2+h_2^3]$

$= \frac{\pi}{3}(-h_1r^2-(h_2-h_1)^3+h_2^3) = \frac{\pi}{3}(-h_1r^2-k^3+h_2^3)$

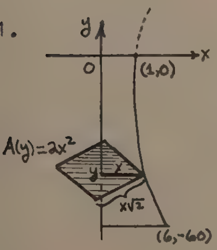
$= -\frac{\pi}{3}(h_1r^2+k^3-h_2^3)$.

$V = \pi(r-k)^2(r-\frac{r-k}{3})-\pi(r-h_2)^2(r-\frac{r-h_2}{3}) - \frac{\pi}{3}(h_1r^2+k^3-h_2^3)$

$V = \frac{\pi}{3}[(r^2-2rk+k^2)(2r+k)-(r^2-2rh_2+h_2^2)(2r+h_2)-h_1r^2-k^3+h_2^3]$

Write $h_1 = h_2-k$, so that $V = \frac{\pi}{3}[2r^3+r^2k-4r^2k-2rk^2+2rk^2+k^3-2r^3-r^2h_2+4r^2h_2+2rh_2^2-2h_2^2r-h_2^3-h_2r^3+kr^2-k^3+h_2^3] = \frac{\pi}{3}[-2kr^2+2r^2h_2]$

$= \frac{2\pi}{3}r^2(h_2-k)$. So $V = \frac{2}{3}\pi r^2h_1$.

1. 

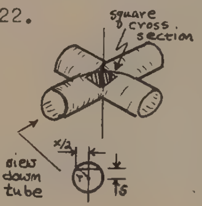
The parabola whose vertex is (1,0) has the equation $y^2 = 4p(x-1)$. Since (6,-60) belongs to the parabola, $(-60)^2 = 4p(6-1)$, so that $4p = 720$ and $y^2=720(x-1)$ is the equation of the parabola. Thus, $x = 1 + \frac{y^2}{720}$ and $A(y) = 2x^2 = 2(1+\frac{y^2}{720})^2$.

It follows that $V\int_{-60}^{0}2(1+\frac{y^2}{720})^2dy$

$= 2\int_{-60}^{0}(1+\frac{y^2}{360} + \frac{y^4}{720^2})dy$

$= 2(y + \frac{y^3}{1080} + \frac{y^5}{5(720)^2}\Big|_{-60}^{0}$

$= -2(-60 - \frac{60^3}{1080} - \frac{60^5}{5(720)^2}) = 1120$ cubic meters.

22. 

Take the reference axis perpendicular to the central axes of both cylinders as in the adjacent figure. Notice that a cross section of the solid region common to both cylinders will be square, say with side length x. If the origin of the reference axis is taken at the point of intersection of the central axes of the two cylinders, then a view down one of the cylinders will show an edge of the square cross section as shown. By the Pythagorean theorem, $s^2+(\frac{x}{2})^2 = r^2$; hence, $x^2 = 4(r^2-s^2)$. Since the area of the cross section is given by $A(s) = x^2 = 4(r^2-s^2)$, then

$V = \int_{-r}^{r}4(r^2-s^2)ds = [4(r^2s-\frac{s^3}{3})]\Big|_{-r}^{r}$

$= \frac{16r^3}{3}$ cubic units.

Problem Set 6.4, page 381

1. $\frac{dy}{dx} = 4$, so $s = \int_0^2\sqrt{1+4^2}dx = \int_0^2\sqrt{17}\,dx$

$= \sqrt{17}(x)\Big|_0^2 = 2\sqrt{17}$ units.

2. $\frac{dy}{dx} = -2$, so $s = \int_{-1}^{2}\sqrt{1+(-2)^2}dx = \int_{-1}^{2}\sqrt{5}\,dx$

$= \sqrt{5}x\Big|_{-1}^{2} = \sqrt{5}\,(2+1) = 3\sqrt{5}$ units.

3. $\frac{dy}{dx} = m$, so $s = \int_0^a \sqrt{1+m^2}\,dx = (\sqrt{1+m^2})x \Big|_0^a$

$= a\sqrt{1+m^2}$ units.

4. $y' = \frac{x^2}{2} - \frac{1}{2x^2}$, so $s = \int_1^3 \sqrt{1+(\frac{x^2}{2} - \frac{1}{2x^2})^2}\,dx$

$= \int_1^3 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\,dx$

$= \int_1^3 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\,dx = \int_1^3 \sqrt{(\frac{x^2}{2} + \frac{1}{2x^2})^2}\,dx$

$= \int_1^3 (\frac{x^2}{2} + \frac{1}{2x^2})\,dx = \left[\frac{x^3}{6} + (-\frac{1}{2x})\right]\Big|_1^3$

$= (\frac{27}{6} - \frac{1}{6}) - (\frac{1}{6} - \frac{1}{2}) = \frac{28}{6} = \frac{14}{3}$ units.

5. $s = \int_0^4 \sqrt{1+(\frac{3}{2}x^{\frac{1}{2}})^2}\,dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\,dx$

$= \frac{1}{2}\int_0^4 \sqrt{4+9x}\,dx$. Let $u=4+9x$, $du=9dx$,

$dx = (\frac{1}{9})du$. So $\int \sqrt{4+9x}\,dx = \int (\frac{1}{9})u^{\frac{1}{2}}du$

$= \frac{1}{9} \cdot \frac{2}{3}u^{\frac{3}{2}} + C$. So $\frac{1}{2}\int_0^4 (4+9x)dx$

$= \frac{1}{2} \cdot \frac{2}{27}(4+9x)^{\frac{3}{2}} \Big|_0^4 = \frac{1}{27}\left[(40)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$

$= \frac{8}{27}(10\sqrt{10}-1) \approx 9.0734$ units.

6. $s = \int_0^8 \sqrt{1+(\frac{1}{6}y^{-\frac{1}{3}})^2}\,dy = \int_0^8 \sqrt{1+\frac{1}{36y^{2/3}}}\,dy$

$= \frac{1}{6}\int_0^8 \frac{\sqrt{36y^{2/3}+1}}{y^{1/3}}\,dy$. Let $u=36y^{\frac{2}{3}}+1$, so

$du=24y^{-\frac{1}{3}}dy$. Hence, $\int \frac{\sqrt{36y^{\frac{2}{3}}+1}}{y^{1/3}}\,dy = \int \frac{u^{\frac{1}{2}}}{24}du$

$= \frac{2}{3} \cdot \frac{1}{24}u^{\frac{3}{2}} + C$. Now $s = \frac{1}{6}\int_0^8 \frac{\sqrt{36y^{2/3}+1}}{y^{1/3}}\,dy$

$= \frac{1}{6} \cdot \frac{1}{36}(36y^{\frac{2}{3}}+1)^{\frac{3}{2}} \Big|_0^8 = \frac{1}{216}(\left[36(8)^{2/3}+1\right]^{\frac{3}{2}}-1)$

$= \frac{1}{216}(145^{\frac{3}{2}}-1) \approx 8.08$ units.

7. $\frac{dx}{dy} = \frac{3}{2}y^{\frac{1}{2}}$, so $s = \int_1^4 \sqrt{1+\frac{9}{4}y}\,dy$. Let $u=1+\frac{9}{4}y$,

so $du=\frac{9}{4}dy$. When $y=1$, $u=\frac{13}{4}$; $y=4$, $u=10$. Thus,

$s = \int_{\frac{13}{4}}^{10} u^{\frac{1}{2}}\frac{4}{9}du = \frac{4}{9} \cdot \frac{2}{3}u^{\frac{3}{2}} \Big|_{\frac{13}{4}}^{10} = \frac{8}{27}(10^{\frac{3}{2}} - (\frac{13}{4})^{3/2})$

≈ 7.6337 units.

8. $y' = \frac{3}{2}(1-x^{\frac{2}{3}})^{\frac{1}{2}}(-\frac{2}{3}x^{-\frac{1}{3}})$.

Thus, $s = \int_{\frac{1}{8}}^1 \sqrt{1+(1-x^{\frac{2}{3}})(x^{-\frac{2}{3}})}\,dx$

$= \int_{\frac{1}{8}}^1 \sqrt{x^{-\frac{2}{3}}}\,dx = \int_{\frac{1}{8}}^1 x^{-\frac{1}{3}}dx = \frac{3}{2}x^{\frac{2}{3}} \Big|_{\frac{1}{8}}^1$

$= \frac{3}{2}(1 - \frac{1}{4}) = \frac{9}{8}$ units.

9. $\frac{dx}{dy} = (y-5)^{\frac{1}{2}}$, so $s = \int_5^6 \sqrt{1+y-5}\,dy$

$= \int_5^6 \sqrt{y-4}\,dy$. Let $u=y-4$, $du=dy$. When $y=5$, $u=1$; $y=6$, $u=2$,

$s = \int_1^2 u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} \Big|_1^2 = \frac{2}{3}(2^{\frac{3}{2}}-1)$

$= 1.21895$ units.

10. $\frac{dx}{dy} = \frac{y^3}{2} - \frac{y^{-3}}{2}$, so $s = \int_1^2 \sqrt{1+(\frac{y^3}{2} - \frac{y^{-3}}{2})}\,dy$

$= \int_1^2 \sqrt{\frac{y^6}{4} + \frac{1}{2} + \frac{y^{-6}}{4}}\,dy$

$= \int_1^2 \sqrt{(\frac{y^3}{2} + \frac{y^{-3}}{2})^2}\,dy = \int_1^2 \frac{y^3+y^{-3}}{2}\,dy$

$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^{-2}}{2}\right]\Big|_1^2 = \frac{1}{2}\left[4 - \frac{1}{8} - (\frac{1}{4} - \frac{1}{2})\right]$

$= \frac{33}{16}$ units.

11. $y = \frac{x^3}{3} + \frac{1}{4x}$, $y' = x^2 - \frac{1}{4x^2}$.

$s = \int_1^3 \sqrt{1+(x^2 - \frac{1}{4x^2})^2}\,dx = \int_1^3 \sqrt{1+x^4-\frac{1}{2}+\frac{1}{16x^4}}$

$= \int_1^3 \sqrt{x^4+\frac{1}{2}+\frac{1}{16x^4}}\,dx = \int_1^3 (x^2+\frac{1}{4x^2})\,dx$

$= (\frac{x^3}{3} - \frac{1}{4x}) \Big|_1^3 = (9-\frac{1}{12})-(\frac{1}{3}-\frac{1}{4}) = \frac{53}{6}$ units.

12. $s = \int_1^2 \sqrt{1+(\frac{dx}{dy})^2}\,dy = \int_1^2 \sqrt{1+(y^4 - \frac{1}{4y^4})^2}\,dy$

$= \int_1^2 \sqrt{(y^4 + \frac{1}{4y^4})^2}\,dy = \int_1^2 (y^4+\frac{1}{4y^4})\,dy$

$= (\frac{y^5}{5} - \frac{1}{12y^3}) \Big|_1^2 = (\frac{32}{5} - \frac{1}{96}) - (\frac{1}{5} - \frac{1}{12})$

$= \frac{3011}{480}$ units.

13. $s = \int_1^2 \sqrt{1+(\frac{dx}{dy})^2}\,dy = \int_1^2 \sqrt{1+(y^2 - \frac{1}{4y^2})^2}\,dy$

$= \int_1^2 \sqrt{(y^2 + \frac{1}{4y^2})^2}\,dy = \int_1^2 (y^2 + \frac{1}{4y^2})\,dy$

$= (\frac{1}{3}y^3 - \frac{1}{4y}) \Big|_1^2 = \frac{59}{24}$ units.

14. $s = \int_1^3 \sqrt{1+(\sqrt{x^4+x^2-1})^2}\,dx = \int_1^3 \sqrt{x^4+x^2}\,dx$

$= \int_1^3 x\sqrt{x^2+1}\ dx.$ Let $u = x^2+1$, $du = 2xdx$, $xdx = \frac{1}{2}du$. So $\int x\sqrt{x^2+1}\ dx = \int \frac{1}{2}u^{\frac{1}{2}}du = \frac{1}{3}u^{\frac{3}{2}}+C.$

Now $\int_1^3 x\sqrt{x^2+1}\ dx = \frac{1}{3}(x^2+1)^{\frac{3}{2}}\Big|_1^3$

$= \frac{1}{3}(10^{\frac{3}{2}}-2^{\frac{3}{2}}).$ So $s \approx 9.598$ units.

5. $S = \int_1^2\sqrt{1+(-\frac{1}{x^2})^2}dx = \int_1^2\sqrt{1+\frac{1}{x^4}}\ dx.$ Using Simpson's rule, we have

$S_4 = \frac{\frac{2-1}{4}}{3}(y_0+4y_1+2y_2+4y_3+y_4)$, where

$y_k = \sqrt{1+\frac{1}{(1+\frac{k}{4})^4}}$, $k = 0, 1, 2, 3, 4.$

$S_4 \approx \frac{1}{12}[1.41+4(1.19)+2(1.09)+4(1.05)+1.03]$

$\approx 1.13.$ Hence $s \approx 1.13$ units.

6. $s = \int_1^2\sqrt{1+(3x^2)^2}dx = \int_1^2\sqrt{1+9x^4}\ dx.$ Using Simpson's rule, $S_4 = \frac{\frac{2-1}{4}}{3}(y_0+4y_1+2y_2+4y_3+y_4)$

where $y_k = \sqrt{1+9(1+\frac{k}{4})^4}$, $k = 0, 1, 2, 3, 4.$

$S_4 = \frac{1}{12}[\sqrt{10}+4(\sqrt{1+9(\frac{5}{4})^4})+2(\sqrt{1+9(\frac{3}{2})^4})+4(\sqrt{1+9(\frac{7}{4})^4})$

$+\sqrt{145}] \approx 7.08.$ Hence, $s \approx 7.08$ units.

7. $s = \int_1^2\sqrt{1+(\frac{1}{x})^2}\ dx.$ Using Simpson's rule,

$S_4 = \frac{\frac{2-1}{4}}{3}(y_0+4y_1+2y_2+4y_3+y_4)$, where

$y_k = \sqrt{1+(\frac{1}{1+\frac{k}{4}})^2}$, $k = 0, 1, 2, 3, 4.$

$S_4 \approx \frac{1}{12}(1.41+4(1.28)+2(1.20)+4(1.15)+1.12)$

$\approx 1.22.$ $s \approx 1.22$ units.

8. $s = \int_0^\pi\sqrt{1+\cos^2 x}dx.$ Using Simpson's rule,

$S_4 = \frac{\frac{\pi-0}{4}}{3}(y_0+4y_1+2y_2+4y_3+y_4)$, where

$y_k = \sqrt{1+\cos^2\frac{k\pi}{4}}$, $k = 0, 1, 2, 3, 4.$

$S_4 \approx \frac{\pi}{12}(\sqrt{2}+4(\sqrt{1.5})+2(1)+4(\sqrt{1.5})+\sqrt{2}) \approx 3.829$

$s \approx 3.829$ units.

9. $(ds)^2 = (dx)^2+(dy)^2;$ $ds = \sqrt{(dx)^2+(dy)^2};$

$\frac{ds}{dt} = \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2} = \sqrt{[f'(t)]^2+[g'(t)]^2};$

$ds = \sqrt{[f'(t)]^2+[g'(t)]^2}dt.$

$s = \int_a^b\sqrt{[f'(t)]^2+[g'(t)]^2}dt.$

20. $\Delta s = \int_a^{a+\Delta x}\sqrt{1+[f'(x)]^2}\ dx$

$= \sqrt{1+[f'(c)]^2}\cdot\Delta x,\ a \le c \le a+\Delta x,$

by the mean value theorem for integrals.

$\Delta \ell = \sqrt{1+(\frac{\Delta y}{\Delta x})^2}\Delta x.$ So $\frac{\Delta \ell}{\Delta s}$

$= \frac{\sqrt{1+(\frac{\Delta y}{\Delta x})^2}\cdot\Delta x}{\sqrt{1+[f'(c)]^2}\cdot\Delta x} = \frac{\sqrt{1+(\frac{\Delta y}{\Delta x})^2}}{\sqrt{1+[f'(c)]^2}}$

$\lim_{\Delta x\to 0}\frac{\Delta \ell}{\Delta s} = \frac{\sqrt{1+(\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x})^2}}{\sqrt{1+\lim_{\Delta x\to 0}[f'(c)]^2}} = \frac{\sqrt{1+[f'(a)]^2}}{\sqrt{1+[f'(a)]^2}},$

since $f'(c)\to f'(a)$ as $\Delta x\to 0$, since f' is continuous. Therefore, $\lim_{\Delta x\to 0}\frac{\Delta \ell}{\Delta s} = 1.$

21. $A = \int_0^3 2\pi(3x+2)\sqrt{1+(3)^2}\ dx$

$= 2\pi\sqrt{10}\int_0^3(3x+2)dx = 2\pi\sqrt{10}(\frac{3}{2}x^2+2x)\Big|_0^3$

$= 2\pi\sqrt{10}(\frac{27}{2}+6) = 2\pi\sqrt{10}(\frac{39}{2})$

$= 39\pi\sqrt{10}$ square units.

22. $A = \int_a^b 2\pi\sqrt{kx}\sqrt{1+[\sqrt{k}\cdot\frac{1}{2}x^{-\frac{1}{2}}]^2}dx$

$A = \int_a^b 2\pi\sqrt{k}\sqrt{x}\sqrt{\frac{4x+k}{4x}}dx = \int_a^b\sqrt{4x+k}\ dx.$

Let $u = 4x+k$, $du = 4dx$, $dx = \frac{1}{4}du.$

So $\int\pi\sqrt{rx+k}\ dx = \int\frac{\pi}{4}u^{\frac{1}{2}}du = \frac{\pi}{6}u^{\frac{3}{2}}+C$, and now we have $A = \frac{\pi}{6}(4x+k)^{\frac{3}{2}}\Big|_a^b = \frac{\pi}{6}[(4b+k)^{\frac{3}{2}} - (4a+k)^{\frac{3}{2}}]$ square units.

23. $A = \int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}dx = \int_0^2 2\pi x^3\sqrt{9x^4+1}\ dx.$

Let $u = 9x^4+1$, $du = 36x^3dx$, so $2x^3dx = \frac{1}{18}du.$

So $\int_0^2 2\pi x^3\sqrt{9x^4+1}\ dx = \int_1^{145}\frac{\pi}{18}u^{\frac{1}{2}}du$

$= \frac{\pi}{27}u^{\frac{3}{2}}\Big|_1^{145} = \frac{\pi}{27}[(145)^{\frac{3}{2}}-1] \approx 203.0436$ sq. units.

24. $y' = \frac{1}{2}(2x-x^2)^{-\frac{1}{2}}(2-2x) = (2x-x^2)^{-\frac{1}{2}}(1-x).$

$$A = 2\pi\int_{1/2}^{3/2}\sqrt{2x-x^2}\sqrt{1+\left[\frac{1-x}{(2x-x^2)^{\frac{1}{2}}}\right]^2}\,dx$$

$$= 2\pi\int_{1/2}^{3/2}\frac{\sqrt{2x-x^2}}{\sqrt{2x-x^2}}\sqrt{2x-x^2+(1-x)^2}\,dx$$

$$= 2\pi\int_{1/2}^{3/2}1\cdot dx = 2\pi x\Big|_{1/2}^{3/2} = 2\pi(\tfrac{3}{2}-\tfrac{1}{2})$$

$= 2\pi$ square units.

25. $y' = \frac{x^3}{2} - \frac{x^{-3}}{2}$.

$$A = 2\pi\int_1^2(\frac{x^4}{8}+\frac{1}{4x^2})\sqrt{1+(\frac{x^3}{2}-\frac{x^{-3}}{2})^2}\,dx$$

$$= \pi\int_1^2(\frac{x^4}{4}+\frac{1}{2x^2})\sqrt{(\frac{x^3}{2}+\frac{x^{-3}}{2})^2}\,dx$$

$$= \pi\int_1^2(\frac{x^4}{4}+\frac{1}{2x^2})(\frac{x^3}{2}+\frac{x^{-3}}{2})dx$$

$$= \pi\int_1^2(\frac{x^7}{8}+\frac{x}{8}+\frac{x}{4}+\frac{x^{-5}}{4})dx$$

$$= \pi\left[\frac{x^8}{64}+\frac{x^2}{16}+\frac{x^2}{8}+\frac{x^{-4}}{-16}\right]\Big|_1^2$$

$$= \pi\left[4+\frac{1}{4}+\frac{1}{2}-\frac{1}{256}-\frac{1}{64}-\frac{1}{16}-\frac{1}{8}+\frac{1}{16}\right]$$

$= \pi\left[\frac{1179}{256}\right] \approx 14.4685$ square units.

26. $y' = \frac{\sqrt{2}}{4}\sqrt{1-x^2} + \frac{\sqrt{2}}{4}\frac{x(-2x)}{2\sqrt{1-x^2}} = \frac{\sqrt{2}}{4}(\frac{1-2x^2}{\sqrt{1-x^2}})$.

$$A = 2\pi\int_0^{\frac{1}{2}}\frac{\sqrt{2}}{4}x\sqrt{1-x^2}\sqrt{1+\frac{2}{16}\cdot\frac{(1-2x^2)^2}{1-x^2}}\,dx$$

$$= \frac{\pi\sqrt{2}}{2}\int_0^{\frac{1}{2}}\frac{x\sqrt{1-x^2}}{4\sqrt{1-x^2}}\sqrt{16-16x^2-8x^2+8x^4+2}\,dx$$

$$= \frac{\pi\sqrt{2}}{8}\int_0^{\frac{1}{2}}x\sqrt{2(2x^2-3)^2}dx = \frac{\pi}{4}\int_0^{\frac{1}{2}}x|2x^2-3|dx$$

$$= \frac{\pi}{4}\int_0^{\frac{1}{2}}(3x-2x^3)dx = \frac{\pi}{4}(\frac{3}{2}x^2-\frac{x^4}{2})\Big|_0^{\frac{1}{2}}$$

$= \frac{\pi}{4}(\frac{3}{8}-\frac{1}{32}) = \frac{11\pi}{128}$ square unit.

27. $A = 2\pi\int_0^{36}\frac{\sqrt{y}}{2}\sqrt{1+(\frac{dx}{dy})^2}dy = \pi\int_0^{36}\sqrt{y}\sqrt{1+(\frac{1}{4\sqrt{y}})^2}dy$

$$= \pi\int_0^{36}\sqrt{y}\sqrt{1+\frac{1}{16y}}\,dy = \pi\int_0^{36}\sqrt{y+\frac{1}{16}}\,dy.$$

Now, let $u = y + \frac{1}{16}$, so that

$$A = \pi\int_{\frac{1}{16}}^{\frac{577}{16}}u^{\frac{1}{2}}du = \frac{2\pi}{3}u^{\frac{3}{2}}\Big|_{\frac{1}{16}}^{\frac{577}{16}}$$

$= \frac{\pi}{96}(577^{\frac{3}{2}}-1) \approx 453.5352$ square units.

28. $A = 2\pi\int_0^8 y^{\frac{2}{3}}\sqrt{1+(\frac{dx}{dy})^2}dy$

$$= 2\pi\int_0^8 y^{\frac{2}{3}}\sqrt{1+(\tfrac{2}{3}y^{-1/3})^2}\,dy.$$

Making the change of variable $y = x^{3/2}$, so that $dy = \frac{3}{2}x^{\frac{1}{2}}dx$, we obtain A

$$= 2\pi\int_0^4 x\sqrt{1+(\tfrac{2}{3}x^{-\frac{1}{2}})^2}\,\tfrac{3}{2}x^{\frac{1}{2}}dx = 3\pi\int_0^4 x\sqrt{x+\tfrac{4}{9}}\,dx.$$

Now let $u = x + \frac{4}{9}$, so that $du = dx$ and

$$A = 3\pi\int_{4/9}^{40/9}(u-\tfrac{4}{9})u^{\frac{1}{2}}du = 3\pi(\tfrac{2}{5}u^{\frac{5}{2}}-\tfrac{8}{27}u^{\frac{3}{2}})\Big|_{4/9}^{40/9}$$

$= 2\pi\,\frac{8000\sqrt{10}+64}{1215} \approx 131.1568.$

29. $A = 2\pi\int_0^2\frac{y^3}{9}\sqrt{1+(\frac{dx}{dy})^2}dy = \frac{2\pi}{9}\int_0^2 y^3\sqrt{1+(\frac{y^2}{3})^2}$

$= \frac{2\pi}{9}\int_0^2 y^3\sqrt{1+\frac{y^4}{9}}\,dy$. Putting $u = 1 + \frac{y^4}{9}$

we have $du = \frac{4}{9}y^3dy$,

so that $A = \frac{2\pi}{9}\int_1^{25/9}\frac{9}{4}\sqrt{u}\,du$

$= \frac{\pi}{2}(\frac{2}{3}u^{\frac{3}{2}})\Big|_1^{25/9} = \frac{98\pi}{81}$ square units.

30. $A = 2\pi\int_1^2(\frac{y^3}{12}+\frac{1}{y})\sqrt{1+(\frac{y^2}{4}-\frac{1}{y^2})^2}dy$

$$= 2\pi\int_1^2(\frac{y^3}{12}+\frac{1}{y})\sqrt{(\frac{y^2}{4}+\frac{1}{y^2})^2}\,dy$$

$$= 2\pi\int_1^2(\frac{y^3}{12}+\frac{1}{y})(\frac{y^2}{4}+\frac{1}{y^2})dy$$

$$= 2\pi\int_1^2(\frac{y^5}{48}+\frac{y}{12}+\frac{y}{4}+y^{-3})dy$$

$$= (\frac{y^6}{288}+\frac{y^2}{6}+\frac{y^{-2}}{-2})\Big|_1^2$$

$$= (\frac{2}{9}+\frac{2}{3}-\frac{1}{8}-\frac{1}{288}-\frac{1}{6}+\frac{1}{2})$$

$= \frac{315}{288} \approx 1.0938$ square units.

31. $A = 2\pi\int_0^{\pi}\sin x\sqrt{1+\cos^2 x}\,dx$. Now

$$A \approx 2\pi S_4 = 2\pi\frac{\frac{\pi-0}{4}}{3}(y_0+4y_1+2y_2+4y_3+y_4),$$

where $y_k = \sin\frac{k\pi}{4}\sqrt{1+\cos^2\frac{k\pi}{4}}$. So

$A = \frac{\pi^2}{6}(0+4(\sqrt{\tfrac{3}{2}})+2(1)+4(\sqrt{\tfrac{3}{2}})+0) \approx 14.69$ squ[are] uni[ts]

32. $y' = \frac{-2x}{3\sqrt{9-x^2}}$, $A = 2\pi\int_0^{2\sqrt{2}}\frac{2}{3}\sqrt{9-x^2}\sqrt{1+\frac{4x^2}{9(9-x^2)}}\,dx$

$= \frac{4\pi}{9}\int_0^2 \sqrt{81-5x^2}dx$. We estimate $\int_0^2\sqrt{81-5x^2}dx$

by Simpson's parabolic rule with n = 2.

Thus, $S_4 = \frac{(\frac{2-0}{4})}{3}(y_0+4y_1+2y_2+4y_3+y_4)$, where

$y_k = \sqrt{81-5(\frac{k}{2})^2}$ for k = 0,1,2,3,4. Therefore,

$S_4 = \frac{1}{6}(9+4\sqrt{\frac{319}{2}} + 2\sqrt{\frac{304}{2}} + 4\sqrt{\frac{279}{2}} + \sqrt{\frac{244}{2}})$

≈ 17.23; so that $A \approx \frac{4\pi}{9}(17.23) \approx 24.06$ sq. units.

3. 

Total surface area

$A = \pi ar + \pi r^2$; $a^2=h^2+r^2$,

$a = \sqrt{h^2+r^2}$. So the total

surface area is

$\pi r\sqrt{h^2+r^2}+\pi r^2 = \pi r(r+\sqrt{h^2+r^2})$ square units.

4. (a) The total surface area is multiplied by k^2, since $\pi kr(kr+\sqrt{k^2h^2+k^2r^2})$
$= \pi k^2r(r+\sqrt{h^2+r^2}) = k^2\cdot$ (total A).

(b) Increased by a factor of k^2.

5. (a) Find arc length in first quadrant and multiply by 4. Using implicit differentiation we find that $y' = \frac{x^{-1/3}}{y^{-1/3}}$

so $(y')^2 = (-\frac{x^{-1/3}}{y^{-1/3}})^2 = \frac{y^{2/3}}{x^{2/3}} = \frac{1-x^{2/3}}{x^{2/3}}$

$= x^{-2/3}-1$. $s = 4\int_0^1\sqrt{1+(x^{-2/3}-1)}\,dx$

$= 4\int_0^1 x^{-1/3}dx = 4(\frac{3}{2}x^{2/3})\Big|_0^1 = 6(1-0)$

$= 6$ units.

(b) Find area from first quadrant rotation and multiply by 2.

$\frac{A}{2} = 2\pi\int_0^1(1-x^{\frac{2}{3}})^{\frac{3}{2}}\sqrt{1+(x^{-\frac{2}{3}} - 1)}dx$

$= 2\pi\int_0^1 x^{-1/3}(1-x^{2/3})^{3/2}dx$. Let $u=1-x^{\frac{2}{3}}$,

so $du = -\frac{2}{3}x^{-1/3}dx$. $\frac{A}{2} = 2\pi\int_1^0 u^{\frac{3}{2}}(-\frac{3}{2})du$

$= 3\pi[\frac{2}{5}u^{\frac{5}{2}}]_0^1 = 3\pi(\frac{2}{5}) = \frac{6\pi}{5}$.

Therefore, $A = \frac{12\pi}{5}$ square units.

36. $A = 2\pi\int_a^b [f(x)+k]\sqrt{1+[f'(x)]^2}\,dx$

$= 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}dx$

$+ 2\pi k\int_a^b\sqrt{1+[f'(x)]^2}dx = A_0 + 2\pi kS_0$.

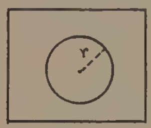
37. (a) $A = 4n\pi r^2$, where $\frac{4}{3}\pi r^3 n=1$, so $r^3 = \frac{3}{4\pi n}$,

$r = \sqrt[3]{\frac{3}{4\pi n}}$. Hence, $A = 4(n\pi)(\frac{3}{4\pi n})^{2/3}$

$= \sqrt[3]{36n\pi}$ square centimeters.

(b) As n grows larger and larger, the surface area increases without bound; therefore, its limit is infinity.

(c) As the substance is divided into finer and finer pieces, the total surface area increases according to the formula in part (a); hence, the rate at which it will dissolve increases.

38. 

Mass of the organism

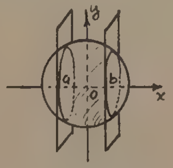
$= \frac{4}{3}\pi r^3 d$. Required intake

of nutrients is at least

$(\frac{4}{3}\pi r^3 d)b$ grams per second.

Intake of nutrients per second from surrounding fluid is $(4\pi r^2)a$ grams per second. Therefore, we require that

$\frac{4}{3}\pi r^3 db \le 4\pi r^2 k$ or $r \le \frac{3k}{db}$ centimeters.

39. 

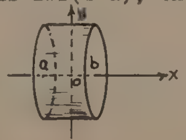
The spherical zone has

surface area A =

$\int_a^b 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$

$= \int_a^b 2\pi\sqrt{r^2}\,dx = 2\pi r\int_a^b dx = 2\pi rx\Big|_a^b$

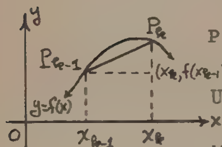
$= 2\pi r(b-a)$. The surface area cut off on the cylinder is $2\pi r(b-a)$, the same as above.



40. In Problem 39, take $b = (6.371 \times 10^3)\sin 23.45°$. Here $a = -b$, $b-a = 2b$, and so $A = 2\pi(6.371 \times 10^3)(b-a)$

$= 4\pi(6.371 \times 10^3)^2 \sin 23.45°$

$= 2.030 \times 10^8$ square kilometers.

41. 

$P_k = (x_k, f(x_k))$ $k=0,\ldots,n$. Using the distance formula, we have $L(\mathcal{P}) = \sum_{k=1}^{n} |\overline{P_{k-1}P_k}|$

$$= \sum_{k=1}^{n} \sqrt{(x_{k-1}-x_k)^2 + (f(x_{k-1})-f(x_k))^2}$$

$$= \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

42. Assuming f has a continuous first derivative, then by the mean value theorem, there is a c_k in $[x_{k-1}, x_k]$ such that

$\dfrac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k)$ or equivalently

$f(x_k)-f(x_{k-1}) = f'(c_k)\,\Delta x_k$ where $\Delta x_k = x_k - x_{k-1}$ for $k = 1,2,\ldots,n$.

Thus $L(\mathcal{P}) = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(c_k)\Delta x_k]^2}$

$= \sum_{k=1}^{n} \sqrt{1+(f'(c_k))^2}\,\Delta x_k$ Hence,

$$S = \lim_{\|\mathcal{P}\| \to 0} L(\mathcal{P}) = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

Problem Set 6.5, page 391

1. $F = ks$ and 6 inches $= \frac{1}{2}$ foot, so

$2s = k(\frac{1}{2})$ or $k = 50$

$W = \int_{1/4}^{1/2} 50\,s\,ds = 25s^2 \Big|_{1/4}^{1/2} = 25(\frac{1}{4} - \frac{1}{16})$

$= 25(\frac{3}{16}) = \frac{75}{16}$ foot - lbs.

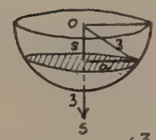
2. $F = ks$, so $200 = k(3)$ or $k = \frac{200}{3}$

$W = \int_0^5 \frac{200}{3}\,s\,ds = \frac{200}{3}\frac{s^2}{2}\Big|_0^5 = \frac{200}{3}(\frac{25}{2})$

$= \frac{2500}{3}$ newtons per centimeter. Thus,

$W = \frac{2500}{3} \cdot \frac{1}{100}$ newtons per meter

$= \frac{25}{3}$ joules.

3. 

$9 = s^2+a^2$; $9-s^2=a^2$.

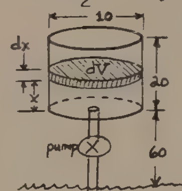
$A(s) = \pi(9-s^2)$.

$W = \int_0^3 sw\pi(9-s^2)\,ds$

$= 10{,}110\pi \int_0^3 (9s-s^3)\,ds$

$= 10{,}110\pi\left[\frac{9s^2}{2} - \frac{s^4}{4}\right]_0^3 = 10{,}110\pi\left[\frac{81}{2} - \frac{81}{4}\right]$

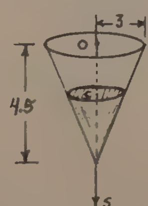
$= \frac{409{,}455}{2}\pi$ joules.

4. 

Let x be the distance from the bottom of the tank to the surface of the wate The slab of water of heigh dx in the adjacent figure has volume $dV = 25\pi dx$ and weighs $62.4dV = 1560\pi dx$ pounds. To lift it from the surface of the lake requires an amount of work given by $dW = (1560\pi dx)(x+60)$ foot-pounds. To fill the tank requires $\int_0^{20} 1560\pi(x+60)\,dx$

$= 1560\pi\left[\frac{x^2}{2} + 60x\right]\Big|_0^{20} = 1560\pi[200+1200]$

$= 2{,}184{,}000\pi$ foot-pounds of work. This will require $\frac{2{,}184{,}000\pi}{33{,}000} = \frac{728}{11}\pi$ horsepower minutes of work. Since we have available 1.5 horsepower, it will require $\frac{728\pi}{(11)(1.5)}$ minutes to fill the tank. This is approximately 2.31 hours.

5. 

The radius a of the surfac of the water satisfies $\frac{3}{4.5} = \frac{a}{4.5-s}$ by similar triangles; hence,

$a = \frac{3}{4.5}(4.5-s)$. The required work is given by $W = 9800\int_{1.5}^{4.5}\pi a^2 s\,ds$

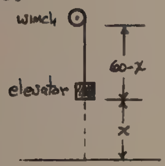
$= 9800\pi\int_{1.5}^{4.5}\left[\frac{3}{4.5}(4.5-s)\right]^2 s\,ds$

$= 9800\pi\left(\frac{3}{4.5}\right)^2\int_{1.5}^{4.5}\left[(4.5)^2 s-9s^2+s^3\right]ds$

$= 9800\pi\left(\frac{3}{4.5}\right)^2\left[\frac{(4.5)^2 s^2}{2} - 3s^3 + \frac{s^4}{4}\right]_{1.5}^{4.5}$

$= 9800\pi\left(\frac{3}{4.5}\right)^2\left[\frac{(4.5)^4}{2} - 3(4.5)^3 + \frac{(4.5)^4}{4} - \frac{(4.5)^2(1.5)^2}{2} + 3(1.5)^3 - \frac{(1.5)^4}{4}\right]$

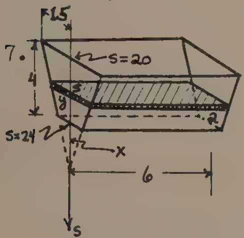
$= 9800\pi(9) = 88{,}200\pi$ joules.

6. 

The required work is given by $W\int_0^{60}\left[(60-x)150 + 10{,}000\right]dx$

$= \int_0^{60}(19{,}000 - 150x)\,dx$

$= 19{,}000x - 75x^2 \Big|_0^{60} = 870{,}000$ joules.

7. 

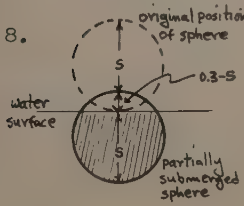
The work done is given by $w\int_{20}^{24} sA(s)\,ds =$ $62.4\int_{20}^{24}sA(s)\,ds$, where $A(s) = 6(2y) = 12y$ sq.ft.

We use similar triangles from the diagram to find y; $\frac{y}{1} = \frac{(24-s)+x}{x}$. Also $\frac{1.5}{1}=\frac{4+x}{x}$, so that $1.5x = 4+x$ and $x=8$. So, $y = \frac{24-s+8}{8} = \frac{32-s}{8}$. So the work done is

$\frac{62.4}{8}\int_{20}^{24} 12s(32-s)\,ds = 12\left(\frac{62.4}{8}\right)\left(16s^2-\frac{s^3}{3}\right)\Big|_{20}^{24}$

$= (12)7.8\left[16(24)^2 - \frac{(24)^3}{3} - 16(20)^2 + \frac{(20)^3}{3}\right]$

$= 81{,}868.80$ foot-pounds.

8. 

The volume of the displaced fluid is equal to the volume of the spherical segment (of one base) in the adjacent figure. By the method of slicing (Section 6.3), this volume can be seen to be $= \frac{1}{3}\pi s^2(0.45-s)$ cubic meters. So the weight w of the displaced fluid is $w = \frac{9800}{3}\pi s^2(0.45-s)ds$. The work is given by

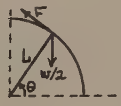
$W = \int_0^{0.3}\frac{9800\pi}{3}s^2(0.45-s)\,ds = \frac{9800\pi}{3}\left[0.15s^3-\frac{s^4}{4}\right]\Big|_0^{0.3}$

$= \frac{9800\pi}{3}\left[s^3\left(0.15-\frac{s}{4}\right)\right]\Big|_0^{0.3} = \frac{9800\pi}{3}(0.027)(0.075)$

$= 6.615\,\pi$ joules ≈ 20.782 joules.

9. $W = \int_0^{25}(25-x)20\,dx = 20\left(25x - \frac{x^2}{2}\right)\Big|_0^{25}$

$= 20\left(625 - \frac{625}{2}\right) = 6250$ joules.

10. 

$F = \frac{w}{2}\cos\theta$

$ds = L\,d\theta$

Hence, $W = \frac{wL}{2}\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$

$= \frac{wL}{2}(1)$ newton-meters $= \frac{wL}{2}$ joules.

11. $W = \int_{25}^{50}100V^{-1.3}dV = 100\,\frac{V^{-0.3}}{-0.3}\Big|_{25}^{50}$

$= -\frac{1000}{3}\left(\frac{1}{50^{0.3}} - \frac{1}{25^{0.3}}\right)$

$= \frac{1000}{3}\left(\frac{1}{25^{0.3}} - \frac{1}{50^{0.3}}\right) \approx 23.83$ inch - pounds ≈ 1.99 foot-pounds.

12. For adiabatic compression, we have $P = \frac{K}{V^\gamma}$, where $K = P_0V_0^\gamma$. For isothermal compression, we have $P = \frac{C}{V}$, where $C = P_0V_0$. Hence, the work required for isothermal compression is given by $\int_{V_1}^{V_0}\frac{P_0V_0}{V}\,dV$. The work required for adiabatic compression is given by $\int_{V_1}^{V_0}\frac{P_0V_0^\gamma}{V^\gamma}\,dV$. Now, since $V_0 \geq V$, $\frac{V_0}{V} \geq 1$, then $\left(\frac{V_0}{V}\right)^\gamma \geq \frac{V_0}{V}$. Since $V_1 \leq V_0$ and since $\left(\frac{V_0}{V}\right)^\gamma \geq \frac{V_0}{V_1}$, then we have

$\int_{V_1}^{V_0} (\frac{V_0}{V})^{\gamma} dV \geq \int_{V_1}^{V_0} \frac{V_0}{V_1} dV$ by the monotonicity theorem. Hence, adiabatic compression requires more work than isothermal compression.

13. Since $PV^{1.4} = k$, then $2 \times 10^4(0.5)^{1.4} = k$, so $k \approx 7578.58$. The work done is given by $W = \int_{0.5}^{0.8} 7578.58V^{-1.4}dV$

$= 7578.58(\frac{V^{-0.4}}{-0.4}) \Big|_{0.5}^{0.8} = \frac{-75785.8}{4}(\frac{1}{(0.8)^{0.4}}$

$- \frac{1}{(0.5)^{0.4}}) \approx 4284.7$ joules.

14. Since $PV^{1.4} = k$, then $1.013 \times 10^5(0.02)^{1.4} = k$, so $k \approx 423.69$. The volume at pressure 5×10^5 N/m^2 is $V = (\frac{k}{P})^{\frac{1}{1.4}} \approx (\frac{423.69}{5 \times 10^5})^{\frac{1}{1.4}}$

≈ 0.0064 m^3. The work done is given by

$W = \int_{0.0064}^{0.02} 423.69V^{-1.4}dV$

$= -\frac{423.69}{0.4} V^{-0.4} \Big|_{0.0064}^{0.02}$

≈ 2924.50 joules

$T = \frac{W}{1500} \approx 1.95$ seconds.

15. $F = \int_0^6 ws\ell(s)ds = 9800 \int_0^6 s(30)ds$

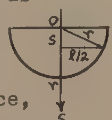
$= (9800)(30)\frac{s^2}{2} \Big|_0^6 = (9800)(30)(18)$

$= 5.292 \times 10^6$ newtons.

16. $\frac{\ell^2}{4} + s^2 = r^2$, so that $\ell = 2\sqrt{r^2-s^2}$ and

$F = \int_0^r ws\ell(s)ds = w\int_0^r 2s\sqrt{r^2-s^2}\, ds$

$= w\left[-\frac{2}{3}(r^2-s^2)^{3/2}\right] \Big|_0^r = \frac{2wr^3}{3}$.



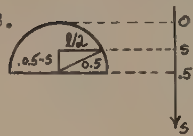
Here $w = 9800$ and $r = 1.9$m; hence,

$F = \frac{2(9800)(1.9)^3}{3} = 4.4812 \times 10^4$ newtons.

17. $F = \int_0^{0.4} ws\ell(s)ds = 9114\int_0^{0.4} s(0.2)ds$

$= 9114(0.2)\frac{s^2}{2}\Big|_0^{0.4} = 9114(0.2)\frac{(0.4)^2}{2}$

$= 145.824$ newtons.

18. 

We begin by finding the force F_1 on the upper semicircle exerted by the oil. We have

$(\frac{\ell}{2})^2+(0.5-s)^2 = (0.5)^2$ or

$\ell = 2\sqrt{0.25-(0.5-s)^2}$; hence,

$F_1 = 9,114\int_0^{0.5} 2\sqrt{0.25-(0.5-s)^2}\, sds$.

Letting $u = 0.5-s$, we have F_1

$= -9,114\int_{0.5}^{0} 2(0.5-u)\sqrt{0.25-u^2}\, du$

$= 9,114\int_0^{0.5} (1-2u)\sqrt{0.25-u^2}\, du$

$= 9,114\int_0^{0.5} \sqrt{0.25-u^2}\, du -$

$9,114\int_0^{0.5} 2u\sqrt{0.25-u^2}\, du$.

The integral $\int_0^{0.5}\sqrt{0.25-u^2}\, du$ represents the area of the quadrant of a circle of radius 0.5; hence, $\int_0^{0.5}\sqrt{0.25-u^2}\, du$

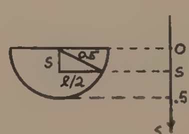
$= \frac{0.25\pi}{4}$. Making the change of variable $V = 0.25-u^2$, so $dV = -2udu$, we have

$\int_0^{0.5} 2u\sqrt{0.25-u^2}\, du = -\int_{0.25}^{0}\sqrt{v}\, dv$

$= \frac{2}{3}v^{\frac{3}{2}}\Big|_0^{0.25} = \frac{0.250}{3}$. Hence, F_1

$= 9,114(\frac{0.25\pi}{4})-9,114(\frac{0.250}{3})$

$= 189.875(3\pi-4)$.



To find the force F_2 on the lower semicircle exerted by the water, we begin by noticing that th pressure s centimeters below the dividin line between the oil and the water is $9114\ (0.5)+1000s = 4557+1000s$. Hence, since $s^2 + (\frac{\ell}{2})^2 = (0.5)^2$, we have

$\ell = 2\sqrt{0.25-s^2}$ and

$F_2 = \int_0^{0.5} (4557+1000s)\ell ds$

$= 4557\int_0^{0.5} \ell ds + 1000\int_0^{0.5} s\ell ds$

$= 4557(2)\int_0^{0.5}\sqrt{0.25-s^2}\, ds$

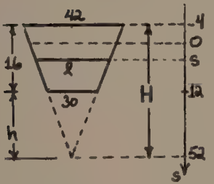
$+ 1000\int_0^{0.5} s\sqrt{0.25-s^2}ds$

$= 9114(\frac{0.25\pi}{4}) + (1000)(\frac{0.250}{3})$.

Thus the total force is given by $F_1 + F_2$

$= 189.875(3\pi-4)+\frac{2278.5\pi}{4} +(\frac{0.250}{3})(1000)$

≈ 2902.89 newtons.

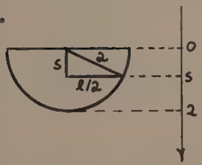
19. 

In the adjacent figure, $H = h+16$ and, by similar triangles, $\frac{h}{30} = \frac{H}{42} = \frac{h+16}{42}$; hence, $h = 40$ feet.

Again, by similar triangles, $\frac{\ell}{52-s} = \frac{30}{40}$; hence, $\ell = 39 - \frac{3}{4}s$. Therefore,

$F = w\int_0^{12} s\ell ds = 62.4\int_0^{12} (39s-\frac{3}{4}s^2)ds$

$= 62.4(\frac{39s^2}{2} - \frac{s^3}{4})\Big|_0^{12} = 148,262.4$ pounds.

20. 

The force F_1 on the upper semicircle is given by

$F_1 = 1.013 \times 10^5(\frac{1}{2}\pi)(2)^2$

$= 2.026 \times 10^5 \times \pi$.

The pressure s meters below the surface of the water is $1.013 \times 10^5 + 1000s$. The force F_2 on the lower semicircle is given by $F_2 = \int_0^2 (1.013 \times 10^5+1000s)\cdot 2\sqrt{4-s^2}\, ds$

$= 1.013 \times 10^5\int_0^2 2\sqrt{4-s^2}ds+1000\int_0^2 2s\sqrt{4-s^2}ds$

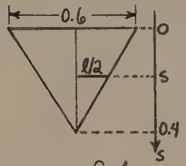
The first integral is the area of a semicircle of radius 2 and the second integral is evaluated by letting $u = 4-s^2$. Thus,

$F_2 = 1.013 \times 10^5(\frac{4\pi}{2})+1000\int_0^4\sqrt{u}\, du$

$= 2.026 \times 10^5 \times\pi + 1000(\frac{2}{3}u^{\frac{3}{2}})\Big|_0^4$

$= 2.026 \times 10^5 \times \pi + \frac{16,000}{3}$. The total force is $F_1+F_2 = 4.052 \times 10^5 \times\pi + \frac{16,000}{3}$

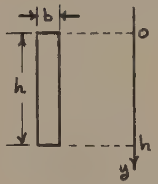
$\approx 1,278,306.68$ newtons.

21. 

By similar triangles: $\frac{\ell/2}{0.4-s} = \frac{0.3}{0.4}$. Hence, $\ell = \frac{3}{2}(0.4-s)$. Therefore,

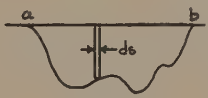
$F = w\int_0^{0.4} s\ell ds = 9,800\int_0^{0.4} (\frac{3}{2})(0.4s-s^2)ds$

$= 14,700(0.2s^2- \frac{1}{3}s^2)\Big|_0^{0.4} = 156.8$ newtons

22. 

The force on a rectangle of width b and height h whose upper edge is at the surface of the fluid is given by $\int_0^h wbydy = \frac{wh^2}{2}\cdot b$.

Therefore, the force dF on an "infinitesimal" vertical rectangle of



width ds and height h(s) is given by $dF= \frac{w[h(s)]^2}{2}\, ds$.

Integrating to obtain the total force, we obtain

$F = \int_a^b \frac{w[h(s)]^2}{2}ds = \frac{w}{2}\int_a^b[h(s)]^2\, ds$.

23. (a) The weight is $9.8(5.2 \times 10^{-3})$

$= 5.096 \times 10^{-2}$ newtons.

(b) 5.096×10^{-2} (1) $= 5.096 \times 10^{-2}$ joules.

(c) $\frac{5.096 \times 10^{-2}}{1.8 \times 10^6} = 35,321,821$.

24. The potential energy is numerically equal to the amount of work done in stretching the spring from its relaxed position; that is, $\int_0^L ksds = k\,\frac{s^2}{2}\Big|_0^L = \frac{kL^2}{2}$ joules.

25. (a) No, since there is more force on the spring at 22 centimeters, so there is more work done in stretching it from 22 to 23 centimeters.

(b) Yes, since the force is constant.

26. Let h be the height in meters. Here $m = 1,200$ kg and $v = \frac{88,000}{3,600}$ m/s, and $k = \frac{1}{2}mv^2$, so $k \approx 358,518.5$ joules. Thus, $w \cdot h = 358,518.5$ and $w = 1,000$, so $h \approx 358.52$ meters.

27. If the spring is stretched by s meters, the work done will be $\int_0^s F ds = \int_0^s 500 s ds = 500 \cdot \frac{s^2}{2} = 250s^2$. Therefore, we require $200 = 250s^2$ or $s = \sqrt{\frac{20}{25}} = \frac{2\sqrt{5}}{25}$

$\approx$ 0.8944 meter.

28. $\frac{d}{dt}\left[m_1v_1+m_2v_2\right] = m_1\frac{dv_1}{dt} + m_2\frac{dv_2}{dt}$

$= F_1+F_2 = F_1+(-F_1) = 0$. Hence, $m_1v_1+m_2v_2$ remains constant during motion.

29. (a) $\frac{mv^2}{2} - \frac{1}{s} = 0$. Hence, $v = \pm\sqrt{\frac{2}{ms}}$ (m=mass).

(b) $\frac{dt}{ds} = -\sqrt{\frac{m}{2}} \cdot s^{\frac{1}{2}}$. Hence, $t = -\frac{2}{3}\sqrt{\frac{m}{2}}(s^{\frac{3}{2}}-s_0^{3/2})$, and so when $s_0 = 25$, $\frac{3}{2}\sqrt{\frac{2}{m}}t = 125-s^{\frac{3}{2}}$. Thus, solving for s, we have $s = (125 - \frac{3}{2}\sqrt{\frac{2}{m}}t)^{\frac{2}{3}}$.

(c) $\frac{dt}{ds} = \sqrt{\frac{m}{2}}\, s^{\frac{1}{2}}$. Hence, $t = \frac{2}{3}\sqrt{\frac{m}{2}}(s^{\frac{3}{2}}-s_0^{\frac{3}{2}})$ and so for $s_0 = 25$, $s = (125+\frac{3}{2}\sqrt{\frac{2}{m}}t)^{2/3}$.

(d) $v = \frac{ds}{dt} = \sqrt{\frac{2}{m}}(125+\frac{3}{2}\sqrt{\frac{2}{m}}\,t)^{-1/3}$. Hence, $\lim_{t\to\infty} v = 0$.

30. (a) $F = -ky$, so $V = -\int_0^s F dy = -\int_0^s (-ky)dy = -\frac{1}{2}ky^2$.

(b) $E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$.

(c) When $y = A_0$, $v = 0$ and $E = \frac{1}{2}kA_0^2$. Hence, since E is constant, $\frac{1}{2}kA_0^2 = \frac{1}{2}mv^2+\frac{1}{2}ky^2$ holds for all values of y. When $y = 0$, we have $\frac{1}{2}kA_0^2 = \frac{1}{2}mv^2$, so that $|v| = A_0\sqrt{\frac{k}{m}}$. On the first passage throug the origin, $v < 0$, so $v = -A_0\sqrt{\frac{k}{m}}$.

(d) When $y = \pm A_0$, $v = 0$ and the energy is entirely potential. When $y = 0$, $V = 0$ and the energy is entirely kinetic

31. $\frac{dV}{ds} = F = \frac{k}{s^2}$. Hence, $\int_s^\infty dV = \int_s^\infty \frac{k}{r^2}dr = -\frac{k}{s}$.

But $\lim_{s\to\infty} V = 0$. Therefore, $V = \frac{k}{s} = \frac{8.99 \text{ X } 10^9}{s}$.

32. (a) Take $s_0 > r$. Then $V = \int_s^{s_0} F ds$.

If $0 \leq s \leq r$, then $V = \int_s^r F ds + \int_r^{s_0} F ds$

$= \int_s^r (-\frac{4}{3}\gamma m\pi w s)ds + \int_r^{s_0} \frac{-4\gamma m\pi r^3 w}{3s^2} ds$

$= \frac{-2\gamma m\pi w s^2}{3}\Big|_s^r + \frac{4\gamma m\pi r^3 w}{3s}\Big|_r^{s_0}$

$= \frac{2\gamma m\pi w s^2}{3} - \frac{2\gamma m\pi w r^2}{3} + \frac{4\gamma m\pi r^3 w}{3s_0} - \frac{4\gamma m\pi r^2 w}{3}$

$= \frac{2\gamma m\pi w s^2}{3} + \frac{4\gamma m\pi r^3 w}{3s_0} - \frac{6\gamma m\pi r^2 w}{3}$. If $r < s$ then $V = \int_s^{s_0} F ds = \int_s^{s_0} \frac{-4\gamma m\pi r^3 w}{3s^2} ds$

$= \frac{4\gamma m\pi r^3 w}{3s}\Big|_s^{s_0} = \frac{4\gamma m\pi r^3 w}{3s_0} - \frac{4\gamma m\pi r^3 w}{3s}$

$$V = \begin{cases} \frac{2\gamma m\pi w s^2}{3} + \frac{4\gamma m\pi r^3 w}{3s_0} - 2\gamma m\pi r^2 w & 0 \leq s \leq r \\ \frac{4\gamma m\pi r^3 w}{3s_0} - \frac{4\gamma m\pi r^3 w}{3s} & r < s \end{cases}$$

If $\lim_{s\to+\infty} V = 0$, then $\lim_{s\to+\infty}\left[\frac{4\gamma m\pi r^3 w}{3s_0} - \frac{4\gamma m\pi r}{3s}\right.$

$= 0$ when s_0 is infinite. So,

$$V = \begin{cases} \frac{2\gamma m\pi w s^2}{3} - 2\gamma m\pi r^2 w & 0 \leq s \leq r \\ \frac{-4\gamma m\pi r^3 w}{3s} & r < s. \end{cases}$$

(b) Let v_0 be the required "escape velocity". The total energy E is given by $E = V + \frac{1}{2}mv^2$. "At $+\infty$", we have $V=0$ and $v = 0$; hence $E = 0$. Since E is a constant, then $0 = V_0+\frac{1}{2}mv_0^2$, where V_0 is

the value of the potential energy when s=r. From part (a), $V_0 = \frac{-4\gamma m\pi r^2 w}{3}$. So $\frac{1}{2}mv_0^2$

$= \frac{4\gamma m\pi r^2 w}{3}$ and $v_0 = \sqrt{\frac{8}{3}\pi\gamma r^2 w} = 2r\sqrt{\frac{2}{3}\pi w\gamma}$.

33. By Example 8, $v^2 = v_0^2 - 2gs$; hence

$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - mgs$.

34. From Newton's law of gravitation, we have $F = -G\frac{m_1 m}{(r+s)^2}$, where m_1 is the earth's mass, m is the projectile's mass, and s is the height of the projectile above the surface of the earth. Hence,

$$V = Gm_1 m\int_0^s \frac{du}{(6.371 \times 10^6 + u)^2}$$

$$= Gm_1 m\left[\frac{1}{6.37 \times 10^6} - \frac{1}{6.37 \times 10^6 + s}\right]$$

Since $G = 6.63 \times 10^{-11} N\cdot m^2/kg^2$ and $m_1 = 5.983 \times 10^{24}kg$, then

$$V = 3.967 \times 10^{14} m\left[\frac{1}{6.371 \times 10^6} - \frac{1}{6.371\times10^6 + s}\right]$$

$= \frac{6.226 \times 10^7 ms}{6.371 \times 10^6 + s}$. Now $E = \frac{1}{2}mv^2 + V$, and when $s = 0$, $E = \frac{1}{2}mv_0^2$. Thus, $\frac{1}{2}mv^2 = E - V$

$= \frac{1}{2}mv_0^2 - \frac{6.226 \times 10^7 ms}{6.371 \times 10^6 + s}$.

Problem Set 6.6, page 397

1. (a) $q = 100(4-\sqrt{4p})$, so that $\sqrt{4p} = 4 - \frac{q}{100}$,

$4p = (4 - \frac{q}{100})^2$, $p = \frac{1}{4}(4 - \frac{q}{100})^2$

$= (2 - \frac{q}{200})^2$.

(b) Consumer's surplus

$= \int_1^c 100(4-\sqrt{4p})dp = 100(4p - 2\cdot\frac{2}{3}\cdot p^{\frac{3}{2}})\Big|_1^c$

$= 100(4c - \frac{4}{3}c^{\frac{3}{2}} - 4 + \frac{4}{3}) = \frac{100}{3}(12c - 4c^{\frac{3}{2}} - 8)$.

Now when $q = 0$, $c = 4$. Hence consumer's surplus $= \frac{100}{3}(48-32-8) = \frac{800}{3} = \$266.67$

(c) $\int_0^{q_0} f(q)dq - f(q_0)q_0 = \int_0^{200}(2-\frac{q}{200})^2 dq - 200(1)$

Let $u = 2 - \frac{q}{200}$ so $\int_0^{200}(2-\frac{q}{200})^2 dq - 200$

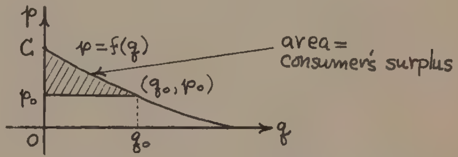
$= \int_2^1 u^2(-200)du - 200$

$= 200\,\frac{u^3}{3}\Big|_1^2 - 200$

$= 200(\frac{8}{3} - \frac{1}{3}) - 200 = 200\left[\frac{7}{3} - 1\right]$

$= 200(\frac{4}{3}) = \frac{800}{3} = \$266.67$.

2.



3. Let p be the price in cents of each souvenir, and let q be the number of souvenirs demanded. Then,

$q = 5000 - (\frac{p-100}{5})500 = 15,000 - 100p$

so $p = \frac{15,000-q}{100}$

Consumer's surplus $= \int_0^{q_0} f(q)dq - f(q_0)q_0$

and $p = 125$.

Now $125 = \frac{15,000-q_0}{100}$ so $q_0 = 2,500$.

Thus, consumer's surplus

$= \int_0^{2500}\frac{15,000-q}{100}dq - (2,500)(125)$

$= (150q - \frac{q^2}{200})\Big|_0^{2,500} - (2,500)(125)$

$= 2,500(150 - \frac{2500}{200} - 125) = 2,500(\frac{25}{2})$

$= 1,250(25) = 31,250¢$

Hence, in dollars, consumer's surplus

$= \$312.50$.

4. Assuming that the manufacturers supply what is demanded, we have the revenue function $R = pq$. Thus, $dR = pdq + qdp$

$= g(q)dp + qdp$. The producer's surplus

$= \int_c^{p_0} q\,dp = \int_{q=c}^{q=p_0} dR - \int_{q=0}^{q=q_0} g(q)dq$

$= R\Big|_{q=c}^{q=p_0} - \int_0^{q_0} g(q)dq = p_0 q_0 - \int_0^{q_0} g(q)dq$.

5. $\frac{dx}{dt} = 30\sqrt{t}$, so that during the first 36 weeks, $x = \int_0^{36} 30\sqrt{t}\,dt = 20t^{3/2}\Big|_0^{36}$

$= 20(36^{3/2}) = 4{,}320$ automobiles.

6. $\frac{dx}{dt} = A\left[1-(\frac{k}{t+k})^p\right]$, so that during the nth week of production, $x = \int_{n-1}^{n} A\left[1-(\frac{k}{t+k})^p\right]dt.$

Now put $u = t+k$, so $du = dt$, $u = n+k$ when $t=n$, and $u = n-1+k$ when $t = n-1$. Thus,

$x = \int_{n-1+k}^{n+k} A(1-k^p u^{-p})du = A(u+\frac{k^p u^{1-p}}{p-1})\Big|_{n-1+k}^{n+k}$

$= A(n+k+ \frac{k^p(n+k)^{1-p}}{p-1} - n+1-k-\frac{k^p(n-1+k)^{1-p}}{p-1}$

$= A\left[1+\frac{k^p}{p-1}((n+k)^{1-p}-(n+k-1)^{1-p})\right]$

$= \frac{A}{p-1}(k^p\left[(n+k)^{1-p}-(n+k-1)^{1-p}\right]+p-1).$

7. $\frac{dx}{dt} = \frac{t^{2/3}}{600}$, so that after 10 years = 520 weeks of operation, $x = \int_0^{520} \frac{t^{2/3}}{600}\,dt$

$= \frac{1}{600} \cdot \frac{3}{5}(t^{5/3})\Big|_0^{520} = \frac{1}{1{,}000}(520)^{5/3}$

$= \frac{32}{1{,}000}(65)^{5/3} = \frac{4}{125}(65^{5/3})$ tons $\approx$ 33.63tons.

8. $\frac{t^{2/3}}{600} = 0.015$, or $t^{2/3} = 9$ so that $t = 9^{3/2}$ = 27 weeks. Thus, at the end of 27 weeks, pollutants are being removed from the lake at the same rate they are being added. Subsequently, pollutants begin to accumulate in the lake. The amount of pollutant in the lake after ten years is, therefore, given by

$\int_{27}^{520}(\frac{t^{2/3}}{600} -0.015)dt = (\frac{t^{5/3}}{1{,}000} - 0.015t)\Big|_{27}^{520}$

$= (\frac{520^{5/3}}{1{,}000} - 7.80)-(\frac{27^{5/3}}{1{,}000} - 0.405)$

$= \frac{520^{5/3}}{1{,}000} - 0.243 - 7.395$

$\approx 33.626 - 7.638 \approx 25.99$ tons.

9. The rate of flow, measured by the total volume of blood passing a cross section of the vessel in unit time, is given by

$V = 2\pi\int_0^{0.1}(0.30-30r^2)r\,dr$

$= 2\pi\int_0^{0.1}(0.30r-30r^3)dr$

$= 2\pi(\frac{0.30r^2}{2} - \frac{30r^4}{4})\Big|_0^{0.1}$

$= 2\pi\left[0.15(0.1)^2 - \frac{15}{2}(0.1)^4\right]$

$= \frac{2\pi(15)}{(2)(10^4)} = 0.004712\ \text{cm}^3/\text{second}.$

10. $V = \int_{r=0}^{r=R} dV = 2\pi\int_0^R K(R^2-r^2)r\,dr$

$= 2\pi K\int_0^R(R^2r-r^3)dr = 2\pi K(\frac{R^2r^2}{2} - \frac{r^4}{4})\Big|_0^R$

$= 2\pi K(\frac{R^4}{2} - \frac{R^4}{4}) = \frac{\pi K R^4}{2}.$

Review Problem Set, Chapter 6, page 397

1. $V = \int_0^8 \pi(\sqrt[3]{x})^2dx = \pi\int_0^8 x^{2/3}dx = \pi(\frac{3}{5}x^{\frac{5}{3}})\Big|_0^8$

$= \frac{96\pi}{5}$ cubic units.

2. $V = \pi\int_0^1\left[1^2-(\sqrt{y})^2\right]dy = \pi\int_0^1(1-y)dy$

$= \pi(y - \frac{y^2}{2})\Big|_0^1 = \frac{\pi}{2}$ cubic units.

3. $V = \pi\int_0^2\left[(\frac{x^2+4}{4})^2-x^2\right]dx = \frac{\pi}{16}\int_0^2(x^4-8x^2+16)dx$

$= \frac{\pi}{16}(\frac{x^5}{5} - \frac{8x^3}{3} + 16x)\Big|_0^2 = \frac{16}{15}\pi$ cubic units.

4. $V = \pi\int_0^1\left[3^2-(2+\sqrt[3]{y})^2\right]dy = \pi\int_0^1(5-4y^{\frac{1}{3}}-y^{\frac{2}{3}})dy$

$= \pi(5y-3y^{\frac{4}{3}}-\frac{3}{5}y^{\frac{5}{3}})\Big|_0^1 = \frac{7}{5}\pi$ cubic units.

5. $V = \pi\int_0^4\left[(\sqrt{y})^2-(\frac{y}{2})^2\right]dy = \pi\int_0^4(y-\frac{y^2}{4})dy$

$= \pi(\frac{y^2}{2} - \frac{y^3}{12})\Big|_0^4 = \pi(\frac{4^2}{2} - \frac{4^3}{12}) = \frac{8\pi}{3}$ cubic units.

6. $V = \pi\int_{-2}^2\left[3^2-(3-(1-\frac{x^2}{4}))^2\right]dx = \pi\int_{-2}^2(5-x^2-\frac{x^4}{16})dx$

$= 2\pi\int_0^2(5-x^2 - \frac{x^4}{16})dx = 2\pi(5x-\frac{x^3}{3} - \frac{x^5}{80})\Big|_0^2$

$= \frac{208}{15}\pi.$

7. $V = \pi\int_1^2\left[(3-y)^2-(\frac{2}{y})^2\right]dy = \pi\int_1^2(9-6y+y^2-\frac{4}{y^2})dy$

$= \pi(9y-3y^2+\frac{y^3}{3}+\frac{4}{y})\Big|_1^2 = \pi(18-12+\frac{8}{3}+2-9+3-\frac{1}{3}-4$

$= \frac{\pi}{3}$ cubic units.

8. $V = \pi\int_{-4}^4\left[6^2-(2+\frac{y^2}{4})^2\right]dy = 2\pi\int_0^4(32-y^2-\frac{y^4}{16})dy$

$= 2\pi(32y - \frac{y^3}{3} - \frac{y^5}{80})\Big|_0^4 = \frac{2816}{15}\pi$ cubic units.

9. $V = \pi\int_1^4 (y^{3/2})^2 dy = \pi\int_1^4 y^3 dy = \frac{\pi y^4}{4}\Big|_1^4$

$= \frac{\pi}{4}(4^4-1) = \frac{255\pi}{4}$ cubic units.

10. Using cylindrical shells, we have

$V = 2\pi\int_{-4}^0 (-x)\cdot 2x^2\sqrt{x+4}\,dx$. Changing the variable according to $u = x+4$, we have $x = u-4$ and $V = 2\pi\int_0^4(-2)(u-4)^3u^{\frac12}du$

$= 2\pi\int_0^4(-2u^{\frac72}+24u^{\frac52}-96u^{\frac32}+128u^{\frac12})du$

$= 2\pi(-\frac49 u^{\frac92}+\frac{48}{7}u^{\frac72}-\frac{192}{5}u^{\frac52}+\frac{256}{3}u^{\frac32})\Big|_0^4$

$= \frac{65,536}{315}\pi$ cubic units.

11. $V = \pi\int_0^4[(x+2)^2-2^2]dx = \pi\int_0^4(x^2+4x)dx$

$= \pi(\frac{x^3}{3}+2x^2)\Big|_0^4 = \frac{160}{3}\pi$ cubic units.

12. $V = \pi\int_1^2[(y^2)^2-1^2]dy + \pi\int_2^{17}[(\sqrt{18-y})^2-1^2]dy$

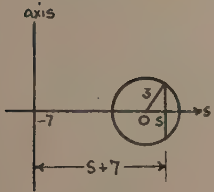
$= \pi\int_1^2(y^4-1)dy + \pi\int_2^{17}(17-y)dy$

$= \pi[\frac{y^5}{5}-y]\Big|_1^2 + \pi(17y-\frac{y^2}{2})\Big|_2^{17}$

$= \pi[\frac{32}{5}-2-\frac15+1] + \pi(289-\frac{289}{2}-34+2)$

$= \pi[\frac{31}{5}-1+\frac{289}{2}-32] = \frac{1177\pi}{10}$ cubic units.

13.

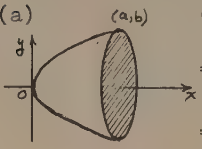


Using cylindrical shells, we have $V = 2\pi\int_{-3}^3(s+7)\cdot2\sqrt{9-s^2}ds =$ $2\pi\int_{-3}^3 2s\sqrt{9-s^2}ds + 28\pi\int_{-3}^3\sqrt{9-s^2}ds$

Since $2s\sqrt{9-s^2}$ is an odd function of s, it follows that the first integral is zero. Since $\int_{-3}^3\sqrt{9-s^2}ds$ is the area of a semicircle of radius 3, we have $V = 28\pi(\frac12\pi\cdot3^2) = 126\pi^2$ cubic centimeters.

14. $V = a^2+7a = \pi\int_0^a[f(x)]^2dx$. Differentiating with respect to a and using the fundamental theorem of calculus, we obtain $2a+7 = \pi[f(a)]^2$; hence, $f(a) = \sqrt{\frac{2a+7}{\pi}}$. Replacing a by x, we find that $f(x) = \sqrt{\frac{2x+7}{\pi}}$.

15. (a)



(b) $V = \pi\int_0^a[f(x)]^2dx$

$= \pi\int_0^a \frac{b^2x}{a}dx$

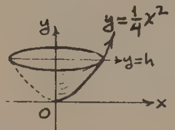
$= \frac{\pi b^2}{a}\cdot\frac{x^2}{2}\Big|_0^a = \frac{\pi b^2a}{2}$.

(c) $V = 2\pi\int_0^b y(a-x)dy = a(2\pi)\int_0^b y(1-\frac{y^2}{b^2})dy$

$= 2a\pi\int_0^b(y-\frac{y^3}{b^2})dy = 2a\pi(\frac{y^2}{2}-\frac{y^4}{4b^2})\Big|_0^b$

$= 2a\pi(\frac{b^2}{2}-\frac{b^2}{4}) = 2a\pi(\frac{b^2}{4}) = \frac{\pi b^2a}{2}$.

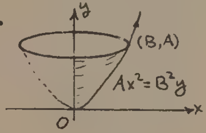
16.



$V = \pi\int_0^h x^2dy$; $V = \pi\int_0^h 4ydy$;

$\frac{dV}{dt} = \pi(4h)\frac{dh}{dt}$. When h = 4 and $\frac{dV}{dt} = -4$, then $-4 = 16\pi\frac{dh}{dt}$; hence $-\frac{1}{4\pi} = \frac{dh}{dt}$. The height is decreasing at the rate of $\frac{1}{4\pi}$ unit per minute.

17.



$V = \pi\int_0^A x^2dy = \pi\int_0^A\frac{B^2y}{A}dy$

$= \frac{\pi B^2y^2}{2A}\Big|_0^A = \frac{\pi B^2A}{2}$. A cylinder of height A and base radius B has volume πB^2A, so its volume is twice the volume of the paraboloid just discussed.

18. $U_n = 2\pi\int_0^1 x(x^n)dx = \frac{2\pi x^{n+2}}{n+2}\Big|_0^1 = \frac{2\pi}{n+2}$.

$V_n = \pi\int_0^1(x^n)^2dx = \pi\frac{x^{2n+1}}{2n+1}\Big|_0^1 = \frac{\pi}{2n+1}$.

(a) $\lim_{n\to+\infty} U_n = \lim_{n\to+\infty}\frac{2\pi}{n+2} = 0$.

(b) $\lim_{n\to+\infty} V_n = \lim_{n\to+\infty}\frac{\pi}{2n+1} = 0$.

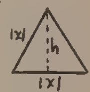
(c) $\lim_{n\to+\infty}\frac{U_n}{V_n} = \lim_{n\to+\infty}\frac{2\pi/(n+2)}{\pi/(2n+1)}$

$= \lim_{n\to+\infty}\frac{4n+2}{n+2} = 4$.

19. $V = \int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$ cubic units.

20. $A = \frac{1}{2}(|x|)(h) = \frac{|x|}{2}(\frac{\sqrt{3}|x|}{2}) = \frac{\sqrt{3x^2}}{4}$.

$V = \int_{-2}^{3} \frac{\sqrt{3x^2}}{4} dx = \frac{\sqrt{3}}{4}(\frac{x^3}{3})\Big|_{-2}^{3}$



$= \frac{9\sqrt{3}}{4} - (\frac{-8\sqrt{3}}{12}) = \frac{35\sqrt{3}}{12}$ cubic units.

21. $V = \int_0^{26} (\frac{26-x}{13})(\frac{26-x}{20})\, dx$

$= \frac{1}{260}\int_0^{26}(676-52x+x^2)dx$

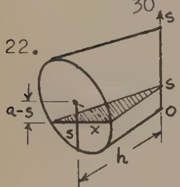
$= \frac{1}{260}(676x - 26x^2 + \frac{x^3}{3})\Big|_0^{26}$

$= \frac{1}{260}(676(26) - 26^3 + \frac{26^3}{3})$

$= \frac{1}{260}(26)(676 - \frac{2(26)^2}{3}) = \frac{1}{10}(\frac{676}{3})$

$= \frac{676}{30} = \frac{338}{15}$ cubic feet.

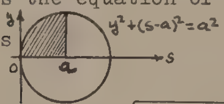
22.



$V = 2\int_0^a \frac{1}{2}(2x)h\,ds$ where $x^2+(a-s)^2=a^2$ (see diagram), so $x = \sqrt{a^2-(a-s)^2}$.

$V = 2\int_0^a \sqrt{a^2-(a-s)^2}\,h\,ds$

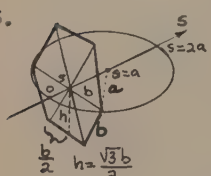
$= 2h\int_0^a \sqrt{a^2-(s-a)^2}ds$. Now $y^2+(s-a)^2=a^2$ is the equation of a circle whose graph is



Hence, $\int_0^a \sqrt{a^2-(s-a)^2}ds = \frac{\pi}{4}a^2$.

Hence, $V = 2h(\frac{\pi}{4}a^2) = \frac{\pi h a^2}{2}$ cubic units.

23.



$b^2 = a^2-(a-s)^2 = 2as-s^2$.

Area of hexagon $= 6(\frac{1}{2})b\cdot h$

$= 3b(\frac{\sqrt{3}b}{2}) = \frac{3\sqrt{3}}{2}b^2$

$= \frac{3\sqrt{3}}{2}(2as - s^2)$.

So $V = 2\int_0^a \frac{3\sqrt{3}}{2}(2as-s^2)ds = 3\sqrt{3}(as^2-\frac{s^3}{3})\Big|_0^a$

$= 3\sqrt{3}(a^3-\frac{a^3}{3}) = 3\sqrt{3}(\frac{2a^3}{3}) = 2\sqrt{3}a^3$ cubic units.

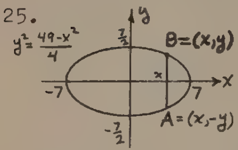
24.

We want to show that $\frac{dx}{dt}$ is a constant. $V = \int_0^x A(s)ds$.

$\frac{dV}{dt} = \frac{d}{dx}\left[\int_0^x A(s)ds\right]\frac{dx}{dt}$

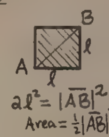
$K\cdot A(x) = A(x)\cdot\frac{dx}{dt}$. Therefore, $\frac{dx}{dt} = K$.

25.



In the figure, $\overline{AB}$ is the diagonal of a square cross section of area $A(x)$ $= \frac{|\overline{AB}|^2}{2} = \frac{|2y|^2}{2} = \frac{4}{2}y^2$

$= 2(\frac{49-x^2}{4}) = \frac{49-x^2}{2}$. Hence,

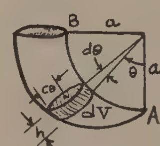


$V = \int_{-7}^{7}\frac{49-x^2}{2}dx$

$= 2\int_0^7 \frac{49-x^2}{2}dx = (49x - \frac{x^3}{3})\Big|_0^7$

$= \frac{686}{3}$ cubic units.

26.



$h=(a+c\theta)d\theta$

$dV = \pi(c\theta)^2(a+c\theta)d\theta$

$= (\pi c^2 a\theta^2+\pi c^3\theta^3)d\theta$.

$V = \int_0^{\pi/2}(\pi c^2 a\theta+\pi c^3\theta^3)d\theta$

$= \pi c^2\int_0^{\pi/2}(a\theta^2+c\theta^3)d\theta$

$= \pi c^2(\frac{a\theta^3}{3}+\frac{c\theta^4}{4})\Big|_0^{\pi/2}$

$= \frac{\pi c^2}{12}\left[4a(\frac{\pi}{2})^3+3c(\frac{\pi}{2})^4\right] = \frac{\pi c^2}{12}(\frac{a\pi^3}{2}+\frac{3c\pi^4}{16})$

$= \frac{\pi^4 c^2}{192}(8a+3\pi c)$ cubic units.

27. $y = x^{3/2}+8$, $y' = \frac{3}{2}x^{\frac{1}{2}}$, $(y')^2 = \frac{9}{4}x$,

$s = \int_0^1\sqrt{1+\frac{9}{4}x}\,dx$. Let $u = 1+\frac{9}{4}x$,

so that $du = \frac{9}{4}dx$ and $s = \int_1^{13/4}\sqrt{u}\cdot\frac{4}{9}du$

$= \frac{4}{9}(\frac{2}{3}u^{3/2})\Big|_1^{13/4} = \frac{8}{27}\left[(\frac{13}{4})^{3/2}-1\right]$

$= \frac{8}{27}\left[\frac{13^{3/2}}{8}-\frac{8}{8}\right] = \frac{13\sqrt{13}-8}{27}$ units.

28. $y = \sqrt[3]{4}\,x^{2/3}$, $y' = \frac{2}{3}\sqrt[3]{4}x^{-1/3}$,

$(y')^2 = \frac{4}{9}\sqrt[3]{16}x^{-2/3} = \frac{8}{9}\sqrt[3]{2}x^{-2/3}$.

$s = \int_4^{32}\sqrt{1+\frac{8}{9}\sqrt[3]{2}x^{-2/3}}dx$

$= \int_4^{32}\frac{\sqrt{x^{2/3}+\frac{8}{9}\sqrt[3]{2}}}{x^{1/3}}dx$. Let $u=x^{2/3}+\frac{8}{9}\sqrt[3]{2}$

so that $du = \frac{2}{3}x^{-1/3}dx$ and $\frac{dx}{x^{1/3}} = \frac{3}{2}du$.

$s = \int_{\frac{26}{9}\sqrt[3]{2}}^{\frac{80}{9}\sqrt[3]{2}}\frac{3}{2}\sqrt{u}du = u^{\frac{3}{2}}\Big|_{\frac{26}{9}\sqrt[3]{2}}^{\frac{80}{9}\sqrt[3]{2}}$

$= (\frac{80}{9})^{3/2}\sqrt{2} - (\frac{26}{9})^{3/2}\sqrt{2} = \frac{320}{27}\sqrt{10} - \frac{26}{27}\sqrt{52}$

$= \frac{4}{27}(80\sqrt{10} - 13\sqrt{13})$ units.

29. $y' = x^{\frac{1}{2}}$, $(y')^2 = x$, $s = \int_0^4 \sqrt{1+x}\,dx$. Let $u = 1+x$, $du = dx$, $s = \int_1^5 \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_1^5$

$= \frac{2}{3}(5^{\frac{3}{2}}-1) = \frac{2}{3}(5\sqrt{5}-1)$ units.

30. $y' = \frac{1}{8}(4x^3 - \frac{4}{x^3}) = \frac{1}{2}(\frac{x^6-1}{x^3})$.

$s = \int_1^2 \sqrt{1+\frac{(x^6-1)^2}{4x^6}}\,dx = \int_1^2 \frac{\sqrt{4x^6+x^{12}-2x^6+1}}{2x^3}\,dx$

$= \int_1^2 \frac{\sqrt{(x^6+1)^2}}{2x^3}\,dx = \int_1^2 \frac{x^6+1}{2x^3}\,dx = \int_1^2 (\frac{x^3}{2}+\frac{x^{-3}}{2})\,dx$

$= \frac{1}{2}(\frac{x^4}{4}+\frac{x^{-2}}{-2})\Big|_1^2 = \frac{1}{2}[(\frac{16}{4}-\frac{1}{8})-(\frac{1}{4}-\frac{1}{2})]$

$= \frac{1}{2}(4 - \frac{1}{8} - \frac{1}{4} + \frac{1}{2}) = \frac{33}{16}$ units.

31. $y = \frac{2}{3}x^{\frac{3}{2}}$ from $(0,0)$ to $(1,\frac{2}{3})$, so $y' = \sqrt{x}$.

Hence, $s = \int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^1$

$= \frac{2}{3}(2\sqrt{2} - 1)$ units.

32. $x' = y^4 - \frac{1}{4y^4}$. $s = \int_{\frac{1}{2}}^1 \sqrt{1+\frac{(4y^8-1)^2}{16y^8}}\,dy$

$= \int_{\frac{1}{2}}^1 \frac{\sqrt{16y^8+16y^{16}-8y^8+1}}{4y^4}\,dy = \int_{\frac{1}{2}}^1 \frac{4y^8+1}{4y^4}\,dy$

$= \int_{\frac{1}{2}}^1 (y^4 + \frac{1}{4}y^{-4})\,dy = (\frac{y^5}{5} - \frac{y^{-3}}{12})\Big|_{\frac{1}{2}}^1$

$= (\frac{1}{5} - \frac{1}{12}) - (\frac{1}{160} - \frac{2}{3}) = \frac{373}{480}$ units.

33. $y' = \frac{3}{2}(x+1)^{\frac{1}{2}}$. $s = \int_3^8 \sqrt{1+\frac{9}{4}(x+1)}\,dx$

$= \frac{1}{2}\int_3^8 \sqrt{4+9x+9}\,dx = \frac{1}{2}\int_3^8 \sqrt{9x+13}\,dx$. Let

$u = 9x+13$, $du = 9dx$, $dx = \frac{1}{9}du$.

$s = \frac{1}{2}\int_3^8 \sqrt{9x+13}\,dx = \frac{1}{2}\int_{40}^{85} u^{\frac{1}{2}}(\frac{1}{9})\,du$

$= \frac{1}{18}(\frac{2}{3})u^{\frac{3}{2}}\Big|_{40}^{85} = \frac{1}{27}[(85)^{3/2} - 40^{3/2}]$

$= \frac{1}{27}(85\sqrt{85} - 40\sqrt{40}) = \frac{5}{27}(17\sqrt{85} - 16\sqrt{10})$ units.

34. $y' = \sqrt{x^2+2x}$. $s = \int_0^1 \sqrt{1+x^2+2x}\,dx$

$= \int_0^1 \sqrt{(x+1)^2}\,dx = \int_0^1 (x+1)\,dx$

$= (\frac{x^2}{2} + x)\Big|_0^1 = \frac{3}{2}$ units.

35. $y = \sqrt[3]{\frac{1}{4}}(x+1)^{2/3}$. $\frac{dy}{dx} = \frac{2}{3}\sqrt[3]{\frac{1}{4}}(x+1)^{-1/3}$.

$s = \int_{-1}^1 \sqrt{1+\frac{4}{9}\sqrt[3]{\frac{1}{16}}(x+1)^{-2/3}}\,dx$

$= \int_{-1}^1 \frac{\sqrt{(x+1)^{2/3} + \frac{1}{9}\sqrt[3]{4}}}{(x+1)^{1/3}}\,dx$.

Let $u = (x+1)^{2/3} + \frac{1}{9}\sqrt[3]{4}$, $du = \frac{2}{3}(x+1)^{-\frac{1}{3}}dx$.

$s = \int_{\frac{1}{9}\sqrt[3]{4}}^{2^{2/3}+\frac{1}{9}\sqrt[3]{4}} \sqrt{u}\,\frac{3}{2}\,du$

$= u^{3/2}\Big|_{\frac{1}{9}\sqrt[3]{4}}^{\frac{10}{9}\sqrt[3]{4}} = \sqrt{(\frac{10}{9})^3\cdot 4} - \sqrt{(\frac{1}{9})^3\cdot 4}$

$= \frac{20}{27}\sqrt{10} - \frac{2}{27} = \frac{2}{27}(10\sqrt{10}-1)$ units.

36. $\frac{dx}{dy} = (y+1)^{\frac{1}{2}}$. $s = \int_3^8 \sqrt{1+y+1}\,dy = \int_3^8 \sqrt{y+2}\,dy$

$= \frac{2}{3}(y+2)^{3/2}\Big|_3^8 = \frac{2}{3}(10^{3/2}-5^{3/2})$

$= \frac{2}{3}(10\sqrt{10} - 5\sqrt{5})$ units.

37. $s = \int_1^2 \sqrt{1+(\sqrt{2x^4+x^7-1})^2}\,dx = \int_1^2 \sqrt{x^7+2x^4}\,dx$

$= \int_1^2 x^2\sqrt{x^3+2}\,dx$. Let $u = x^3+2$, $du = 3x^2dx$,

$x^2dx = \frac{1}{3}du$. $s = \int_1^2 x^2\sqrt{x^3+2}\,dx$

$= \frac{1}{3}\int_3^{10} u^{\frac{1}{2}}\,du = \frac{1}{3}(\frac{2}{3})u^{3/2}\Big|_3^{10}$

$= \frac{2}{9}(10^{3/2}-3^{3/2}) = \frac{2}{9}(10\sqrt{10}-3\sqrt{3})$ units.

38. $y = \sqrt{4-x^2}$ from $(-2,0)$ to $(2,0)$ is a semicircle of radius 2. Hence, $s = \frac{1}{2}(2\pi r) = \pi(2) = 2\pi$ units.

39. $y' = \frac{1}{\sqrt{x}}$. $s = \int_1^4 \sqrt{1+\frac{1}{x}}\,dx$. Now $S_4 = \frac{4-1}{12}[y_0+4y_1+2y_2+4y_3+y_4]$, where $\Delta x = \frac{4-1}{4} = \frac{3}{4}$ and $y_k = \sqrt{1+\frac{1}{(1+\frac{3}{4}k)}}$, $k = 0,1,2,3,4$. $S_4 = \frac{3}{12}[\sqrt{2}+4\sqrt{1+(\frac{4}{7})} + 2\sqrt{1+(\frac{2}{5})} + 4\sqrt{1+(\frac{4}{13})} + \sqrt{1+\frac{1}{4}}] \approx 3.62$. Hence, $s \approx 3.62$ units.

40. $y' = 2(\frac{1}{2})(1+x^2)^{-\frac{1}{2}}(2x) = \frac{2x}{\sqrt{1+x^2}}.$

$s = \int_0^1 \sqrt{1+\frac{4x^2}{1+x^2}}\,dx = \int_0^1 \sqrt{\frac{1+5x^2}{1+x^2}}\,dx.$

$S_4 = \frac{1-0}{12}[y_0+4y_1+2y_2+4y_3+y_4]$, where $\Delta x = \frac{1}{4}$ and $y_k = \sqrt{\frac{1+5(\frac{k}{4})^2}{1+(\frac{k}{4})^2}}\,dx$, $k=0,1,2,3,4$.

$S_4 = \frac{1}{12}\left[\sqrt{1}+4\sqrt{\frac{21}{17}}+2\sqrt{\frac{9}{5}}+4\sqrt{\frac{61}{25}}+\sqrt{3}\right] \approx 1.34.$

So $s \approx 1.34$ units.

41. $s = \int_0^1 \sqrt{1+4x^2}\,dx$. $S_4 = \frac{1-0}{12}[y_0+4y_1+2y_2+4y_3+y_4]$ where $\Delta x_k = \frac{1-0}{4} = \frac{1}{4}$ and $y_k = \sqrt{1+4(\frac{k}{4})^2}$, $k = 0, 1, 2, 3, 4$. So $S_4 = \frac{1}{12}\left[1+4\sqrt{1+\frac{1}{4}} + 2\sqrt{1+1} + 4\sqrt{1+\frac{9}{4}} + \sqrt{5}\,\right] \approx 1.48$. So $s \approx 1.48$ units.

42. $s = \int_0^2 \sqrt{1+(\frac{1}{1+x^2})^2}\,dx$. $S_4 = \frac{2-0}{12}[y_0+4y_1+2y_2+4y_3+y_4]$, where $\Delta x_k = \frac{2}{4} = \frac{1}{2}$ and

$y_k = \sqrt{1+\left[\frac{1}{1+(\frac{k}{2})^2}\right]^2}$, $k = 0, 1, 2, 3, 4.$

$S_4 = \frac{1}{6}\left[\sqrt{1+1}+4\sqrt{1+\frac{16}{25}}+2\sqrt{\frac{5}{4}}+4\sqrt{1+\frac{16}{169}}+\sqrt{1+\frac{1}{25}}\right] \approx 2.33.$

So $s \approx 2.33$ units.

43. Equation of line through P_1 and P_2 is $y-y_1 = m(x-x_1)$, where $m = \frac{y_2-y_1}{x_2-x_1}$. Without loss of generality, $s = \int_{x_1}^{x_2} \sqrt{1+m^2}\,dx$

$= \sqrt{1+m^2}(x_2-x_1) = \sqrt{1+(\frac{y_2-y_1}{x_2-x_1})^2}(x_2-x_1)$

$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$

44. Let s denote the arc length of the curve $y=f(x)$ between $(0,1)$ and $(x,f(x))$. The potential energy of the mass at the beginning, when the particle is at $(0,1)$, is $mgh=mg(1)=mg$ and its kinetic energy at this instant is 0; hence, the total energy is $E=mg$. When the particle reaches the point $(x,f(x))$, its potential energy will have decreased to $mgf(x)$ and its kinetic energy will have increased to $\frac{1}{2}mv^2=\frac{1}{2}m(\frac{ds}{dt})^2$. Hence, the total energy when the particle is at $(x,f(x))$ is $mgf(x)+\frac{1}{2}m(\frac{ds}{dt})^2$. By the law of conservation of energy, $mg=mgf(x)+\frac{1}{2}m(\frac{ds}{dt})^2$; hence, $\frac{ds}{dt} = \sqrt{2g[1-f(x)]}$. Since $ds=\sqrt{1+[f'(x)]^2}\,dx$, then $dt = \sqrt{\frac{1+[f'(x)]^2}{2g[1-f(x)]}}\,dx$; hence,

$t = \int_0^1 \sqrt{\frac{1+[f'(x)]^2}{2g[1-f(x)]}}\,dx.$

45. $y' = \frac{3}{2}x^{-\frac{1}{2}}$. $A = 2\pi\int_1^4 3\sqrt{x}\sqrt{1+\frac{9}{4x}}\,dx$

$= 2\pi(\frac{3}{2}) \int_1^4 \sqrt{4x+9}\,dx = 3\pi\int_1^4 \sqrt{4x+9}\,dx.$

Let $u = 4x+9$, $du = 4dx$. $\int\sqrt{4x+9}\,dx$

$= \int u^{\frac{1}{2}}(\frac{1}{4})du = \frac{2}{3}(\frac{1}{4})u^{3/2}+C$. So $3\pi\int_1^4\sqrt{4x+9}\,dx$

$= 3\pi(\frac{1}{6})(4x+9)^{3/2}\Big|_1^4 = \frac{\pi}{2}(25^{3/2}-13^{3/2})$

$= \frac{\pi}{2}(125 - 13\sqrt{13})$ square units.

46. $x = \sqrt{\frac{y}{3}}$, $x' = \frac{1}{2}(\frac{y}{3})^{-\frac{1}{2}}(\frac{1}{3})$.

$A = 2\pi\int_0^{12}\sqrt{\frac{y}{3}}\sqrt{1+\frac{1}{36}\cdot\frac{3}{y}}\,dy$

$= 2\pi\int_0^{12}\frac{\sqrt{y/3}}{\sqrt{12}\sqrt{y}}\sqrt{12y+1}\,dy = \frac{\pi}{3}\int_0^{12}\sqrt{12y+1}\,d$

Let $u = 12y+1$, $du = 12dy$. So $\int\sqrt{12y+1}\,dy$

$= \int\frac{1}{12}u^{\frac{1}{2}}du = \frac{2}{3}(\frac{1}{12})u^{3/2}+C$. So

$A = \frac{\pi}{3}\left[\frac{1}{18}(12y+1)^{3/2}\right]\Big|_0^{12} = \frac{\pi}{3}\left[\frac{(145)^{3/2}}{18} - \frac{1}{18}\right.$

$= \frac{\pi}{54}(145\sqrt{145} - 1)$ square units.

47. $y' = 3x^2$. $A = 2\pi\int_1^3 x^3\sqrt{1+9x^4}\,dx$. Let $u = 1+9x^4$, $du = 36x^3dx$. So $\int x^3\sqrt{1+9x^4}\,dx$

$= \frac{1}{36}\int u^{\frac{1}{2}}du = \frac{1}{54}u^{3/2}+C$. So,

$A = 2\pi\left[\frac{1}{54}(1+9x^4)^{3/2}\right]\Big|_1^3$

$= \frac{\pi}{27}\left[(730)^{3/2}-10^{3/2}\right]$

$= \frac{10\pi}{27}(73\sqrt{730} - \sqrt{10})$ square units.

48. $A = 2\pi\int_0^4 \sqrt{\frac{y}{2}}\sqrt{1+(\frac{1}{4\sqrt{y}})^2}\,dy = \frac{\pi}{4}\int_0^4\sqrt{16y+1}\,dy.$

Let $u = 16y+1$; $du = 16dy$. So,

$\int\sqrt{16y+1}\,dy = \int u^{\frac12}(\frac{1}{16})du = \frac23(\frac{1}{16})u^{3/2}+C.$

So $A = \frac{\pi}{4}\int_0^4\sqrt{16y+1}\,dy = \frac{\pi}{4}(\frac{1}{24})(16y+1)^{3/2}\Big|_0^4$

$= \frac{\pi}{96}(65^{3/2}-1)$ square units.

49. $A = 2\pi\int_0^3 \frac13x^3\sqrt{1+x^4}\,dx.$ Let $u = 1+x^4$, $du = 4x^3dx.$ $\int\frac13x^3\sqrt{1+4x}\,dx = \int\frac14(\frac13)u^{\frac12}du$ $= \frac{1}{18}u^{3/2}+C.$ $A = 2\pi(\frac{1}{18})(1+x^4)^{3/2}\Big|_0^3$ $= \frac{\pi}{9}[82\sqrt{82}-1]$ square units.

50. $y' = x^3 - \frac{1}{4x^3}.$ So $A = 2\pi\int_1^3(\frac{x^4}{4}+\frac{1}{8x^2})\sqrt{1+(\frac{4x^6-1}{4x^3})^2}\,dx = 2\pi\int_1^3\frac{(2x^6+1)}{8x^2}\sqrt{\frac{(4x^6+1)^2}{16x^6}}\,dx$

$= 2\pi\int_1^3\frac{(2x^6+1)(4x^6+1)}{32x^5}dx$

$= \frac{\pi}{16}\int_1^3(8x^7+6x+x^{-5})dx = \frac{\pi}{16}(x^8+3x^2-\frac{x^{-4}}{4})\Big|_1^3$

$= \frac{\pi}{16}(3^8+27-\frac{1}{4(3)^4}-1-3+\frac14)$

≈ 1292.81 square units.

51. $A = 2\pi\int_0^4 2\sqrt{y}\sqrt{1+\frac1y}\,dy = 2\pi\int_0^4 2\sqrt{y+1}\,dy.$

Let $u = y+1$, $du = dy$. $\int\sqrt{y+1}\,dy$ $= \int u^{\frac12}du = \frac23u^{3/2}+C.$ So $4\pi\int_0^4\sqrt{y+1}\,dy$ $= 4\pi(\frac23)(y+1)^{3/2}\Big|_0^4 = \frac{8\pi}{3}(5\sqrt5-1)$ square units.

52. $x = y^{2/3}$, $x' = \frac23y^{-1/3}.$

$A = 2\pi\int_1^8 y^{2/3}\sqrt{1+\frac{4}{9y^{2/3}}}\,dy$

$= 2\pi\int_1^8\frac{y^{2/3}}{3y^{1/3}}\sqrt{9y^{2/3}+4}\,dy.$ Let

$u = 9y^{2/3}+4$, $du = 6y^{-1/3}dy.$

So $\int\frac{y^{2/3}}{3y^{1/3}}\sqrt{9y^{2/3}+4}\,dy = \int\frac{(u-4)}{3\cdot9}(\frac16)u^{\frac12}du$

$= \frac{1}{162}\int(u^{3/2}-4u^{\frac12})du = \frac{1}{162}(\frac25u^{5/2}-\frac83u^{3/2})+C.$

So $2\pi\int_1^8\frac{y^{2/3}}{3y^{1/3}}\sqrt{9y^{2/3}+4}\,dy$

$= \frac{2\pi}{162}\left[\frac25(9y^{2/3}+4)^{5/2}-\frac83(9y^{2/3}+4)^{3/2}\right]\Big|_1^8$

$= \frac{2\pi}{162}\left[\frac25(40)^{5/2}-\frac83(40)^{3/2}-\frac25(13)^{5/2}+\frac83(13)^{3/2}\right] \approx 126.22$ square units.

53. $y = \sqrt{9-x}$, $y' = \frac12(9-x)^{-\frac12}(-1).$

$A = 2\pi\int_0^9\sqrt{9-x}\sqrt{1+\frac{1}{4(9-x)}}\,dx$

$= 2\pi\int_0^9\frac{\sqrt{9-x}}{2\sqrt{9-x}}\sqrt{36-4x+1}\,dx$

$= \pi\int_0^9\sqrt{37-4x}\,dx.$ Let $u = 37-4x$,

$du = -4dx$, $-\frac14du = dx$. So $\pi\int_0^9\sqrt{37-4x}\,dx$ $= \frac{\pi}{4}(-\frac23u^{3/2})\Big|_{37}^{1}.$ So $A = -\frac{\pi}{6}u^{3/2}\Big|_{37}^{1}$

$= -\frac{\pi}{6}(1-37^{3/2}) = \frac{\pi}{6}(37\sqrt{37}-1)$ square units.

54. $y' = x^4 - \frac{x^{-4}}{4}.$ $A = 2\pi\int_1^2(\frac{x^5}{5}+\frac{1}{12x^3})\sqrt{1+(\frac{4x^8-1}{4x^4})^2}\,dx$

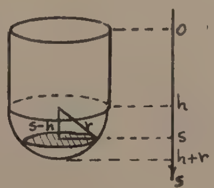
$= 2\pi\int_1^2\frac{(12x^8+5)}{60x^3}\sqrt{\frac{(4x^8+1)^2}{16x^8}}\,dx$

$= \frac{\pi}{30(4)}\int_1^2\frac{(12x^8+5)(4x^8+1)}{x^7}dx$

$= \frac{\pi}{120}\int_1^2(48x^9+32x+5x^{-7})dx$

$= \frac{\pi}{120}(\frac{48x^{10}}{10}+16x^2-\frac56x^{-6})\Big|_1^2$

$= \frac{\pi}{120}(4.8\times2^{10}+64-\frac{5}{6.2^6}-4.8-16+\frac56) \approx 129.83$ square units.

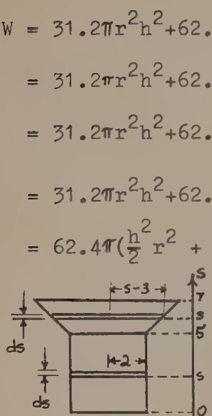
55. 

$W = 62.4\pi\int_0^h sr^2ds + 62.4\pi\int_h^{h+r}s\left[r^2-(s-h)^2\right]ds = 62.4\pi r^2\frac{s^2}{2}\Big|_0^h + 62.4\pi\int_h^{h+r}s\left[r^2-(s-h)^2\right]ds$

$= 31.2\pi r^2h^2+62.4\pi\int_h^{h+r}s\left[r^2-(s-h)^2\right]ds.$

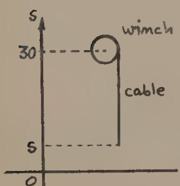
Changing the variable in the latter integral according to $u = s-h$, we have

$$W = 31.2\pi r^2h^2+62.4\pi\int_0^r(u+h)(r^2-u^2)du$$
$$= 31.2\pi r^2h^2+62.4\pi\int_0^r(hr^2+r^2u-hu^2-u^3)du$$
$$= 31.2\pi r^2h^2+62.4\pi(hr^2u+r^2\cdot\tfrac{u^2}{2}-h\tfrac{u^3}{3}-\tfrac{u^4}{4})\Big|_0^r$$
$$= 31.2\pi r^2h^2+62.4\pi(hr^3+\tfrac{r^4}{2}-h\tfrac{r^3}{3}-\tfrac{r^4}{4})$$
$$= 62.4\pi(\tfrac{h^2}{2}r^2+\tfrac{2}{3}hr^3+\tfrac{r^4}{4})\text{ foot-lbs.}$$

56. 

Weight of the water is 9800 newtons per cubic meter.

$$W = \int_0^5 9800(s\pi)(4)ds + \int_5^7 9800s\pi(s-3)^2ds$$
$$= 19{,}600\pi s^2\Big|_0^5 + 9800\pi(\tfrac{s^4}{4}-2s^3+\tfrac{9}{2}s^2)\Big|_5^7$$
$$= 19{,}600\pi(25)+9800\pi\left[\tfrac{7^4}{4}-2(7)^3+\tfrac{9}{2}(7)^2-\tfrac{5^4}{4}+2(5)^3-\tfrac{9}{2}(25)\right]$$
$$= 9800\pi\left[50+116\right] = 1{,}626{,}800\pi\approx 5{,}110{,}742.93\text{ joules.}$$

57. 

$$W = \int_0^{15}4(30-s)ds$$
$$= 40\,(30s-\tfrac{s^2}{2})\Big|_0^{15}$$
$$= 4(450-\tfrac{15^2}{2})$$
$$= 1350\text{ joules.}$$

58. $\text{Work} = W\cdot k+\int_0^k w(\ell-s)ds = Wk+w(\ell s-\tfrac{s^2}{2})\Big|_0^k$

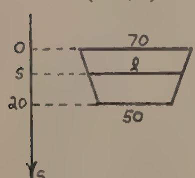
$$= Wk+w(\ell k-\tfrac{k^2}{2}) = (W+w\ell)k-\tfrac{wk^2}{2}\text{ units of work.}$$

59. $W = -\int_{400}^{30} kV^{-1.4}dV$, where $PV^{1.4} = k$,

$k = 15(144)400^{1.4} = 9{,}491{,}563.094$

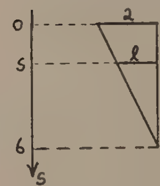
$$W = -\frac{kV^{-0.4}}{-0.4}\Big|_{400}^{30} = \frac{k}{0.4}\left(\frac{1}{30^{0.4}}-\frac{1}{400^{0.4}}\right)$$

$k(0.41)\approx 3{,}927{,}363.64$ foot-pounds.

60. 

Evidently ℓ varies linearly with s, so that $\ell = ms+b$. When $s=0$, $\ell=70$, so $b=70$ and $\ell=ms+70$. When $s=20$, $\ell=50$, so $50=m(20)+70$, and it follows that $m = -1$. Hence, $\ell = -s+70 = 70-s$ and

$$F = 9800\int_0^{20}s(70-s)\,ds$$
$$= 9800(35s^2-\tfrac{s^3}{3})\Big|_0^{20}$$
$$= 9800\left[35(400)-\tfrac{8000}{3}\right]$$
$$= \frac{333{,}200{,}000}{3}\text{ newtons.}$$

61. 

$\frac{\ell}{2} = \frac{6-s}{6}$, so that $\ell = \frac{6-s}{3}$.

$dF = 9800s\ell ds = 9800(\frac{6s-s^2}{3})\,ds$ and

$$F = \frac{9800}{3}\int_0^6(6s-s^2)ds$$
$$= \frac{9800}{3}\left[3s^2-\frac{s^3}{3}\right]\Big|_0^6 = \frac{9800}{3}\left[3(36)-\frac{216}{3}\right]$$
$$= 9800(36-24) = 9800(12) = 117{,}600\text{ newto}$$

62. (a) $s = \frac{-g}{2}t^2+v_0t+s_0$ so that $v = \frac{ds}{dt} = -gt$

Solving the equation $0 = \frac{-g}{2}t^2+v_0t+(s_0-s)$

for t, we have $t = \dfrac{-v_0\pm\sqrt{v^2+2g(s_0-s)}}{-g}$, but

we know that t is positive; consequently

$t = \dfrac{-\left[v_0+\sqrt{v_0^2+2g(s_0-s)}\right]}{-g}$. Substituting:

$$v = -g\left[\frac{v_0+\sqrt{v_0^2+2g(s_0-s)}}{g}\right]+v_0,$$
$$v = -\sqrt{v_0^2 2g(s_0-s)}\,.$$

(b) $K = \frac{1}{2}mv^2 = \frac{1}{2}m\left[v_0^2+2g(s_0-s)\right]$. At $s=s_0$

$K = \frac{1}{2}mv_0^2$ and at $s=s_1$, $K = \frac{1}{2}m\left[v_0^2+2g(s_0-s_1\right.$

So the increase in K is given by

$\frac{1}{2}m\left[v_0^2+2g(s_0-s_1)\right]-\frac{1}{2}mv_0^2 = mg(s_0-s_1)$.

(c) $V = \int_{s_0}^s(-F)ds = \int_{s_0}^s\left[-(-mg)\right]ds$

$= mgs-mgs_0$.

(d) Using (c), at $s=s_0$, $V = 0$; at $s=s_1$,

$V = mgs_1-mgs_0$. The decrease in V is given by mgs_1-mgs_0.

(e) $K = \frac{1}{2}mv^2$, and $v=v_0-gt$, so $K=\frac{1}{2}mv(v_0-g$

Now $V = mgs-mgs_0$, so that $V =$

$= mg(\frac{-g}{2}t + v_0t + s_0) - mgs_0$. So,

$V = -\frac{mg^2t^2}{2} + mgv_0t$ or $mg(-\frac{g}{2}t^2 + v_0t)$.

63. $v = \frac{ds}{dt} = 20-2t$, $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m(20-2t)^2$, so

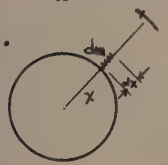
$k = \frac{1}{2}(100)(20-2t)^2 = 50(20-2t)^2$

When $t = 0$, $E_k = 50(20)^2 = 20{,}000$;

When $t = 10$, $E_k = 50(0)^2 = 0$.

The change in kinetic energy E_k is $-20{,}000$ joules.

Since the change in potential energy is $-E_k$, its value is 20,000 joules.

64. 

The mass $dm = f(x)dx$ is moving on a circle of radius x and has a (linear) velocity of $v = (2\pi x)n$ units per second; hence, its kinetic energy is $d = \frac{1}{2}[f(x)dx][2\pi xn]^2$.

The total kinetic energy of the rod is therefore given by

$K = \int_0^b \frac{1}{2}f(x)(2\pi xn)^2dx = 2\pi^2n^2\int_0^b x^2f(x)dx$.

65. Consumer's surplus $= \int_0^{q_0} f(q)dq - q_0p_0$ and $p_0 = 15$. Now $15 = \frac{-q_0^2}{100} - \frac{7q_0}{20} + 30$ so that

$q_0^2 + 35q_0 - 1500 = 0$, and since x_0 cannot be negative, $q_0 = 25$. Thus, consumer's

surplus $= \int_0^{25}(-\frac{q^2}{100} - \frac{7q}{20} + 30)dq - 25(15)$

$= (-\frac{q^3}{300} - \frac{7q^2}{40} + 30q)\Big|_0^{25} - 375$

$= -\frac{625}{12} - \frac{875}{8} + 750 - 375$

$= \frac{-3875}{24} + 375 = \frac{5125}{24} \approx 213.5¢$

In dollars, consumer's surplus is \$2.14.

66. Let p be the price of each pumpkin, and let q be the number of pumpkins demanded.

Then $q = 100 - (\frac{p-75}{25})(20)$ and so

$p = f(q) = 200 - \frac{5}{4}q$. Now the revenue

$R = pq = q(200 - \frac{5}{4}q) = 200q - \frac{5}{4}q^2$, so that

R is maximum when $R' = 200 - \frac{5}{2}q = 0$, or

$q = 80$; that is, when the price per pumpkin is $200 - \frac{5}{4}(80) = 100¢$. Hence,

when $q_0 = 80$ and $p_0 = 100$, consumer's

surplus $= \int_0^{80}(200 - \frac{5}{4}q)dq - 80(100)$

$= (200q - \frac{5q^2}{8})\Big|_0^{80} - 8000$

$= 16{,}000 - 4000 - 8000 = 4000¢ = \$40.00$.

67. The net profit is given by $\int_0^{49} 100\sqrt{t}\,dt -$

$(200)(49) = \frac{2}{3}(100)t^{3/2}\Big|_0^{49} - 9800$

$= \frac{200}{3}(49)^{3/2} - 9800 = \frac{68{,}600}{3} - 9800$

$= \$13{,}066.67$.

68. $\int_0^8(50 - \frac{50}{\sqrt{t+1}})\,dt = 50t\Big|_0^8 - \int_1^9 50u^{-\frac{1}{2}}du$

$= 400 - (100u^{\frac{1}{2}})\Big|_1^9 = 400 - (300 - 100)$

$= 200$ connections. (Here we took $u = t+1$.)

69. The rate of flow of blood, when R denotes the radius of the blood vessel, is given

by $V = 2\pi\int_0^R vrdr = 2\pi\int_0^R K(R^2 - r^2)rdr$

$= 2\pi K(\frac{R^2r^2}{2} - \frac{r^4}{4})\Big|_0^R = 2\pi K(\frac{R^4}{2} - \frac{R^4}{4}) = \frac{\pi KR^4}{2}$.

Now, when the radius is increased by 5%,

we have $V = \frac{\pi K}{2}(R + 0.05R)^4 = \frac{\pi K}{2}(1.05R)^4$

$= \frac{\pi K}{2}(1.05)^4R^4$. Hence, the increase in the rate of flow is given by

$\frac{\pi K}{2}\left[(1.05)^4R^4 - R^4\right] = \frac{\pi KR^4}{2}\left[(1.05)^4 - 1\right]$, and

the percentage increase in the rate of

flow $= \frac{\pi KR^4[(1.05)^4 - 1]100\%}{2\cdot\pi KR^4/2} \approx 21.55\%$.

7 TRANSCENDENTAL FUNCTIONS

Problem Set 7.1, page 407

1. $f(g(x)) = f(\frac{x+3}{2}) = 2(\frac{x+3}{2})-3 = (x+3)-3 = x$ and $g(f(x)) = g(2x-3) = \frac{(2x-3)+3}{2} = \frac{2x}{2} = x$. Therefore, f and g are inverses of each other.
2. $f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$ and $g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$. Therefore, f and g are inverses of each other.
3. $f(g(x)) = f(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x,\ x \neq 0$, and

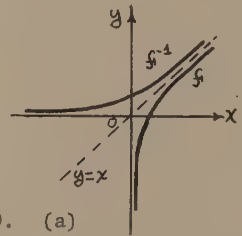
   $g(f(x)) = g(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x,\ x \neq 0$.

   Therefore, f and g are inverses of each other.
4. $f(g(x)) = f(\frac{2x-3}{3x-2}) = \frac{2(\frac{2x-3}{3x-3})-3}{3(\frac{2x-3}{3x-2})-2}$

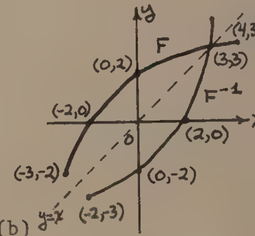
   $= \frac{2(2x-3)-3(3x-2)}{3(2x-3)-2(3x-2)} = \frac{4x-6-9x+6}{6x-9-6x+4} = \frac{-5x}{-5} = x$

   and $g(f(x)) = g(\frac{2x-3}{3x-2}) = \frac{2(\frac{2x-3}{3x-2})-3}{3(\frac{2x-3}{3x-2})-2} = x$.

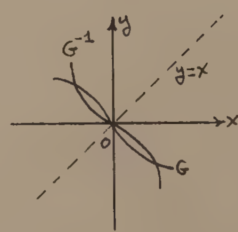
   Therefore, f and g are inverses of each other.
5. $f(g(x)) = f(x^3-8) = \sqrt[3]{(x^3-8)+8} = \sqrt[3]{x^3} = x$ and $g(f(x)) = g(\sqrt[3]{x+8}) = (\sqrt[3]{x+8})^3-8 = x+8-8 = x$. Therefore, f and g are inverses of each other.
6. $f(g(x)) = f(\sqrt{x-1}) = (\sqrt{x-1})^2+1 = (x-1)+1 = x$ and $g(f(x)) = g(x^2+1) = \sqrt{(x^2+1)-1} = \sqrt{x^2} = |x| = x$ since $x \geq 1$. Therefore, f and g are inverses of each other.
7. (a) Invertible (b) Not invertible (c) Not invertible
8. Linear function is invertible if it is not parallel to the x axis, that is, if and only if $m \neq 0$.
9. (a)



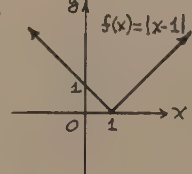
(b)



(c)



10.



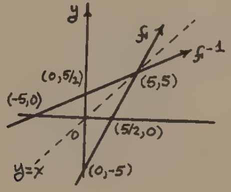
Graph of $f(x)=|x-1|$ not invertible since graph fails horizontal-line test.

11. (a) $f(x) = 2x-5$, or $y=2x-5$; $x=2y-5$ or $2y=x+5$ or $y = \frac{x+5}{2}$, so $f^{-1}(x) = \frac{x+5}{2}$.

(b) $f^{-1}(f(x)) = f^{-1}(2x-5) = \frac{(2x-5)+5}{2}$
$= \frac{2x}{2} = x.$

(c) $f(f^{-1}(x)) = f(\frac{x+5}{2}) = 2(\frac{x+5}{2})-5$
$= (x+5)-5 = x.$

(d)

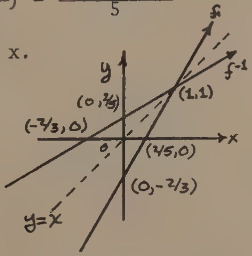


12. (a) $f(x) = \frac{3x+2}{5}$ or $y = \frac{3x+2}{5}$; $x = \frac{3y+2}{5}$ or $5x = 3y+2$ or $3y = 5x-2$ or $y = \frac{5x-2}{3}$, so $f^{-1}(x) = \frac{5x-2}{3}$.

(b) $f^{-1}(f(x)) = f^{-1}(\frac{3x+2}{5}) = \frac{5(\frac{3x+2}{5})-2}{3} =$
$\frac{(3x+2)-2}{3} = \frac{3x}{3} = x.$

(c) $f(f^{-1}(x)) = f(\frac{5x-2}{3}) = \frac{3(\frac{5x-2}{3})+2}{5}$
$= \frac{(5x-2)+2}{5} = \frac{5x}{5} = x.$

(d)

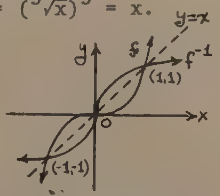


13. (a) $f(x) = x^3$ or $y = x^3$; $x = y^3$ or $y = \sqrt[3]{x}$, so $f^{-1}(x) = \sqrt[3]{x}$.

(b) $f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x.$

(c) $f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x.$

(d)

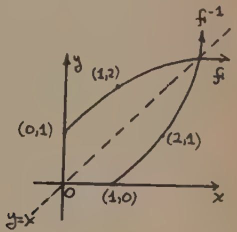


14. (a) $f(x) = 1+\sqrt{x}$ or $y = 1+\sqrt{x}$; $x = 1+\sqrt{y}$ or $\sqrt{y} = x-1$ or $y = (x-1)^2$, so $f^{-1}(x)=(x-1)^2$.

(b) $f^{-1}(f(x)) = f^{-1}(1+\sqrt{x}) = (1+\sqrt{x}-1)^2$
$= (\sqrt{x})^2 = x.$

(c) $f(f^{-1}(x)) = f[(x-1)^2] = 1+\sqrt{(x-1)^3}$
$= 1+|x-1| + 1+x-1 = x$, since $x \geq 1$.

(d)

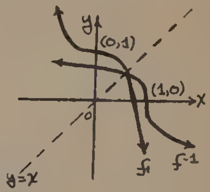


15. (a) $f(x) = 1-2x^3$ or $y = 1-2x^3$; $x = 1-2y^3$ or $2y^3 = 1-x$ or $y^3 = \frac{1-x}{2}$ or $y = \sqrt[3]{\frac{1-x}{2}}$, so $f^{-1}(x) = \sqrt[3]{\frac{1-x}{2}}$.

(b) $f^{-1}(f(x)) = f^{-1}(1-2x^3) = \sqrt[3]{\frac{1-(1-2x^3)}{2}}$
$= \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x.$

(c) $f(f^{-1}(x)) = f(\sqrt[3]{\frac{1-x}{2}}) = 1-2(\sqrt[3]{\frac{1-x}{2}})^3$
$= 1-2(\frac{1-x}{2}) = 1-1+x = x.$

(d)

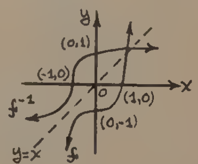


16. (a) $f(x) = x^5-1$ or $y = x^5-1$; $x = y^5-1$ or $y^5 = x+1$ or $y = \sqrt[5]{x+1}$, so $f^{-1}(x)=\sqrt[5]{x+1}$.

(b) $f^{-1}(f(x)) = f^{-1}(x^5-1) = \sqrt[5]{x^5-1+1}=\sqrt[5]{x^5}=x.$

(c) $f(f^{-1}(x)) = f(\sqrt[5]{x+1}) = (\sqrt[5]{x+1})^5-1=(x+1)-1=x.$

(d)

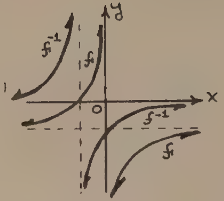


17. (a) $f(x) = -\frac{1}{x} - 1$ or $y = -\frac{1}{x} - 1$ $(x \neq 0, y \neq -1)$; $x = -\frac{1}{y} - 1$ or $\frac{1}{y} = -x-1$ or $y = \frac{-1}{x+1}$, so $f^{-1}(x) = \frac{-1}{x+1}$.

(b) $f^{-1}(f(x)) = f^{-1}(-\frac{1}{x} - 1) = \frac{-1}{-\frac{1}{x} - 1 + 1}$
$= \frac{-1}{-\frac{1}{x}} = x.$

(c) $f(f^{-1}(x))=f(\frac{-1}{x+1})=-\frac{1}{-\frac{1}{x+1}} - 1=(x+1)-1 = x.$

(d)



18. (a) $f(x) = \frac{3}{x+2}$ or $y = \frac{3}{x+2}$ $(x \neq -2,\ y \neq 0)$;

$x = \frac{3}{y+2}$ or $xy+2x = 3$ or $y = \frac{3-2x}{x}$,

so $f^{-1}(x) = \frac{3-2x}{x}$.

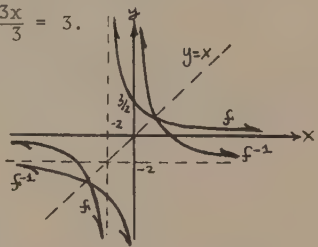
(b) $f^{-1}(f(x)) = f^{-1}(\frac{3}{x+2}) = \dfrac{3-2(\frac{3}{x+2})}{\frac{3}{x+2}}$

$= \frac{3(x+2)-6}{3} = \frac{3x}{3} = x.$

(c) $f(f^{-1}(x)) = f(\frac{3-2x}{x}) = \dfrac{3}{\frac{3-2x}{x}+2}$

$= \frac{3}{3-2x+2x} = \frac{3x}{3} = 3.$

(d)



19. (a) $f(x) = \frac{3x-7}{x+1}$ or $y(x+1) = 3x-7$ $(x \neq -1)$

or $3x-xy = y+7$, so $x = \frac{y+7}{3-y}$. Thus,

$f^{-1}(x) = \frac{x+7}{3-x}$.

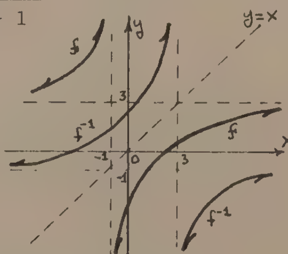
(b) $f^{-1}(\frac{3x-7}{x+1}) = \dfrac{(\frac{3x-7}{x+1})+7}{3-(\frac{3x-7}{x+1})} = \frac{3x-7+7x+7}{3x+3-3x+7}$

$= \frac{10x}{10} = x.$

(c) $f(\frac{x+7}{3-x}) = \dfrac{3(\frac{x+7}{3-x})-7}{\frac{x+7}{3-x}+1} = \frac{3x+21-21+7x}{x+7\ +\ 3-x}$

$= \frac{10x}{10} = x.$

(d)



20. (a) $f(x) = (x+3)^5-2$ or $y = (x+3)^5-2$;

$x = (y+3)^5-2$ or $(y+3)^5 = x+2$ or

$y+3 = \sqrt[5]{x+2}$, so $f^{-1}(x) = \sqrt[5]{x+2}-3$.

(b) $f^{-1}(f(x)) = f^{-1}[(x+3)^5-2]$

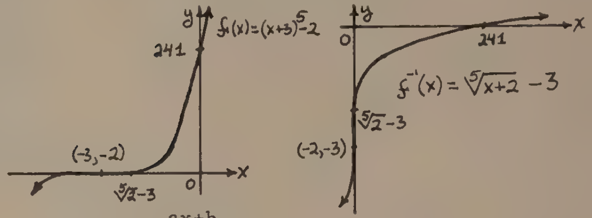
$= \sqrt[5]{[(x+3)^5-2]+2} - 3 = \sqrt[5]{(x+3)^5}-3$

$= (x+3)-3 = x.$

(c) $f(f^{-1}(x)) = f(\sqrt[5]{x+2}-3) = (\sqrt[5]{x+2}-3+3)^5-$

$= (\sqrt[5]{x+2})^5-2 = (x+2)-2 = x.$

(d) Because of the difference in the scale on the x and y axes, we have drawn two separate graphs for f and f^{-1}.



21. Let $y = \frac{ax+b}{cx+d}$, $cyx + yd = ax + b$,

$cyx - ax = b - yd$, $x(cy - a) = b - yd$,

$x = \frac{b - yd}{cy - a}$, so $f^{-1}(x) = \frac{-dx + b}{cx - a}$.

22. Let $y = \sqrt{4-x^2}$, $0 \leq x \leq 2$. Then $x^2 = 4-y^2$ and $x = \sqrt{4-y^2}$. Hence, $f^{-1}(x) = \sqrt{4-x^2} = f(x)$

23. When $y=1$, $x=1$ and $\frac{dx}{dy} = \frac{1}{(\frac{dy}{dx})} = \frac{1}{5x^4} = \frac{1}{5(1)^4} = \frac{1}{5}$.

24. When $y=64$, $x=2$ and $\frac{dx}{dy} = \frac{1}{(\frac{dy}{dx})} = \frac{1}{6x^5} = \frac{1}{6(2)^5} = \frac{1}{192}$.

25. When $y=4$, $x=1$ and $\frac{dx}{dy} = \frac{1}{(\frac{dy}{dx})} = \frac{1}{2x+2} = \frac{1}{2(1)+2} = \frac{1}{4}$.

26. When $y = -3$, $x = 0$, and $\frac{dx}{dy} = \frac{1}{(\frac{dy}{dx})}$

$= \dfrac{1}{\frac{(x-1)(2)-(2x+3)}{(x-1)^2}} = \frac{(x-1)^2}{(-5)} = \frac{(0-1)^2}{-5} = -\frac{1}{5}$.

27. When $y = -1$, $x = 1$ and $\frac{dx}{dy} = \frac{1}{(\frac{dy}{dx})}$

$= \dfrac{1}{\frac{(2x-7)(7)-(7x-2)(2)}{(2x-7)^2}} = \frac{(2x-7)^2}{(-45)}$

$= \frac{(2(1)-7)^2}{-45} = -\frac{5}{9}$.

28. When $y = \frac{1}{2}$, $x = 30° = \frac{\pi}{6}$ radians, $\frac{dx}{dy} =$

$\frac{1}{(\frac{dy}{dx})} = \frac{1}{\cos x} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$.

29. When $y = \frac{2}{3}\sqrt{3}$, $x = 2$, $\frac{dx}{dy} = \frac{1}{(\frac{dy}{dx})} =$

$\dfrac{1}{\frac{\sqrt{x^2-1} - x\cdot\frac{2x}{2\sqrt{x^2-1}}}{x^2-1}} = -(x^2-1)^{3/2} = -3(3^{3/2})$.

30. By either the algebraic method or by reflecting the graph of f across the line $y = x$, we find that

$$f^{-1}(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \leq x \leq 81 \\ \frac{x^2}{729} & \text{if } x > 81 \end{cases}$$

Hence, $(f^{-1})'(x) = \begin{cases} 1 & \text{if } x < 1 \\ \frac{1}{2\sqrt{x}} & \text{if } 1 < x < 81 \\ \frac{2x}{729} & \text{if } x > 81. \end{cases}$

Notice that $(f^{-1})'(x)$ does not exist for $x = 1$ or for $x = 81$.

31. $(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(3)} = \frac{1}{2}$.

32. $(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(5)} = \frac{1}{7}$.

33. $(f^{-1})'(-1) = \frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(4)} = 7$.

34. $(f^{-1})'(\frac{1}{3}) = \frac{1}{f'(f^{-1}(-\frac{2}{3}))} = \frac{1}{f'(1)} = \frac{3}{2}$.

35. $(f^{-1})'(\frac{\sqrt{2}}{2}) = \frac{1}{f'(f^{-1}(\frac{\sqrt{2}}{2}))} = \frac{1}{f'(\frac{\pi}{4})} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

36. $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = 1$.

37. (a) Let $g(x) = \frac{3x+1}{2x-1}$. We must show that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. Now, $(f \circ g)(x) = f(g(x)) = \frac{g(x) + 1}{2g(x) - 3} =$

$$\frac{\frac{3x+1}{2x-1} + 1}{\frac{6x+2}{2x-1} - 3} = \frac{(3x+1) + (2x-1)}{(6x+2) - (2x-1)} = \frac{5x}{5} = x$$ for

$x \neq \frac{1}{2}$. Similarly, $(g \circ f)(x) = x$ for $x \neq \frac{3}{2}$. Hence, $g(x) = \frac{3x+1}{2x-1} = f^{-1}(x)$.

(b) $(f^{-1})'(x) = \frac{(2x-1)(3) - (3x+1)(2)}{(2x-1)^2} =$

$\frac{-5}{(2x-1)^2}$, so $(f^{-1})'(0) = \frac{-5}{(2(0)-1)^2} = -5$.

(c) $f^{-1}(0) = -1$, so $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))}$

$= \frac{1}{f'(-1)}$. Now $f'(x) = \frac{(2x-3)-(x+1)(2)}{(2x-3)^2} =$

$\frac{-5}{(2x-3)^2}$, so $f'(-1) = \frac{-5}{(-5)^2} = -\frac{1}{5}$ and

$(f^{-1})'(0) = \frac{1}{f'(-1)} = \frac{1}{(-\frac{1}{5})} = -5$.

38. (a) To prove $f^{-1} = f$, we must show that $(f \circ f)(x) = x$ for $x \neq \frac{2}{3}$. We have

$(f \circ f)(x) = f(f(x)) = \frac{2f(x)-1}{3f(x)-2}$

$$= \frac{2(\frac{2x-1}{3x-2})-1}{3(\frac{2x-1}{3x-2})-2} = \frac{2(2x-1) - (3x-2)}{3(2x-1) - 2(3x-2)} = \frac{x}{1} = x.$$

(b) Since $f^{-1} = f$, we have $(f^{-1})'(x) =$

$f'(x) = \frac{(3x-2)(2) - (2x-1)(3)}{(3x-2)^2} = \frac{-1}{(3x-2)^2}$

for $x \neq \frac{2}{3}$.

(c) By (b), $f'(x) = -\frac{1}{(3x-2)^2}$. By the inverse function rule, and the fact that $f^{-1} = f$, $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$$= \frac{1}{f'(f(x))} = \frac{1}{\left[\frac{-1}{(3f(x)-2)^2}\right]} = -(3f(x)-2)^2$$

$$= -(3(\frac{2x-1}{3x-2})-2)^2 = -(\frac{6x-3}{3x-2} - \frac{2(3x-2)}{3x-2})^2$$

$$= -(\frac{1}{3x-2})^2 = \frac{-1}{(3x-2)^2}$$ for $x \neq \frac{2}{3}$.

39. (a) $2x^2-x+(1-y) = 0$, $x = \frac{1 \pm \sqrt{1-4(1-y)2}}{4}$

$= \frac{1 \pm \sqrt{8y-7}}{4}$. Since $x > \frac{1}{4}$, we must use the plus sign, so $x = \frac{1+\sqrt{8y-7}}{4}$, $y > \frac{7}{8}$.

Thus $f^{-1}(x) = \frac{1+\sqrt{8x-7}}{4}$, $x > \frac{7}{8}$.

(b) $(f^{-1})'(x) = \frac{\frac{1}{2}(8x-7)^{-\frac{1}{2}}(8)}{4} = \frac{1}{\sqrt{8x-7}}$,

so $(f^{-1})'(2) = \frac{1}{\sqrt{16-7}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$.

(c) $(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)}$

$= \frac{1}{4(1)-1} = \frac{1}{3}$.

40. $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(2)}$

$= \frac{1}{3(2)^2-2(2)} = \frac{1}{8}$,

where $f'(x) = 3x^2 - 2x$.

41. $f'(x) = \dfrac{(x^2+1)(3x^2) - (x^3-1)(2x)}{(x^2+1)^2}$

$= \dfrac{x^4+3x^2+2x}{(x^2+1)^2}$. Therefore,

$(f^{-1})(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(1)}$

$= \dfrac{1}{\dfrac{1+3(1)^2+2(1)}{(1^2+1)^2}} = \dfrac{2}{3}$.

42. f and g are inverses of each other. If (b,a) is on the graph of g, then a = g(b). Thus, f(a) = f(g(b)) = b. Thus, (a,b) is on the graph of f.

43. If no horizontal line intersects the graph of f more than one time, then to each b in the range of f there corresponds a unique a in the domain of f. Therefore, the relation g defined by "(b,a) is in g if and only if (a,b) is in the graph of f" defines a function, and

$(f\circ g)(x) = f(g(x)) = x,$

$(g\circ f)\ x = g(f(x)) = x.$

Hence, f is invertible.

44. Assume f is invertible and that for a and b in the domain of f, f(a) = f(b). Then $a = f^{-1}(f(a)) = f^{-1}(f(b)) = b$. Hence, a = b and f is one-to-one.

Now, if f is one-to-one, then to each value f(a) in the range of f corresponds a unique value a in the domain of f, and then each horizontal line intersects the graph of f once; hence, f is invertible.

45. $C = f^{-1}(t)$.

46. By definition, the range of f is contained in the domain of f^{-1}. Now, if u is in the domain of f^{-1}, then, by definition, $f^{-1}(u) = x$ is in the domain of f. Hence, $f(x) = f(f^{-1}(u)) = u$. Since f(x) is in the range of f, we conclude that u is in the range of f. This shows that the range of f is the same as the domain of f^{-1}. A similar argument will show that the domain of f is the same as the range of f^{-1}.

47. Let T_1 and T_2 be the transformations of the plane onto itself obtained by turning the plane over and by a clockwise 90° rotation around the origin, respectively. Then $T_1(x,y) = (-x,y)$ and

$T_2(x,y) = (y,-x).$

Now, if (a,b) is in the graph of f, then $(T_2\circ T_1)(a,b) = T_2(-a,b) = (b,-(-a)) = (b,a)$ is in f^{-1}.

48. If $f^{-1}(a) = f^{-1}(b)$, then $a = f(f^{-1}(a)) = f(f^{-1}(b)) = b$; hence, f^{-1} is one-to-one. Problem 44 tells us that f^{-1} is invertibl[e]. That $(f^{-1})^{-1} = f$ is an immediate consequence of the equations:

$f(f^{-1}(x)) = x,$

$f^{-1}(f(y)) = y.$

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1. $\sin^{-1}1 = x$ then $\sin x = 1$ where $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$; so $x = \frac{\pi}{2}$.

2. $\arcsin\frac{\sqrt{3}}{2} = x$ then $\sin x = \frac{\sqrt{3}}{2}$ where $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$; so $x = \frac{\pi}{3}$.

3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = x$ then $\sin x = -\frac{\sqrt{2}}{2}$ where $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$; so $x = -\frac{\pi}{4}$.

4. $\cos^{-1}\left(-\frac{1}{2}\right) = x$ then $\cos x = -\frac{1}{2}$ where $0\le x\le\pi$; so $x = \frac{2\pi}{3}$.

5. $\arccos 1 = x$ then $\cos x = 1$ where $0 \le x \le \pi$; so $x = 0$.

6. $\cos^{-1}\frac{\sqrt{3}}{2} = x$ then $\cos x = \frac{\sqrt{3}}{2}$ where $0 \le x \le \pi$; so $x = \frac{\pi}{6}$.

7. $\sin^{-1} \frac{\sqrt{2}}{2} = x$ then $\sin x = \frac{\sqrt{2}}{2}$ where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$; so $x = \frac{\pi}{4}$.

8. $\cos^{-1}0 = x$ then $\cos x = 0$ where $0 \le x \le \pi$; so $x = \frac{\pi}{2}$.

9. $\arccos \frac{1}{2} = x$ then $\cos x = \frac{1}{2}$ where $0 \le x \le \pi$; so $x = \frac{\pi}{3}$.

10. $\arctan 1 = x$ then $\tan x = 1$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$; so $x = \frac{\pi}{4}$.

11. $\tan^{-1}(-1) = x$ then $\tan x = -1$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$; so $x = -\frac{\pi}{4}$.

12. $\tan^{-1} \frac{\sqrt{3}}{3} = x$ then $\tan x = \frac{\sqrt{3}}{3}$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$; so $x = \frac{\pi}{6}$.

13. $\arctan (-\frac{\sqrt{3}}{3}) = x$ then $\tan x = -\frac{\sqrt{3}}{3}$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$; so $x = -\frac{\pi}{6}$.

14. $\tan^{-1}\sqrt{3} = x$ then $\tan x = \sqrt{3}$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$; so $x = \frac{\pi}{3}$.

15. $\cot^{-1}(-1) = x$ then $\cot x = -1$ where $0 < x < \pi$; so $x = \frac{3\pi}{4}$.

16. $\cot^{-1} \sqrt{3} = x$ then $\cot x = \sqrt{3}$ where $0 < x < \pi$; so $x = \frac{\pi}{6}$.

17. $\operatorname{arccot}(-\frac{\sqrt{3}}{3}) = x$ then $\cot x = -\frac{\sqrt{3}}{3}$ where $0 < x < \pi$; so $x = \frac{2\pi}{3}$.

18. $\sec^{-1}\sqrt{2} = x$ then $\sec x = \sqrt{2}$ where $0 \le x \le \pi$, $x \ne \frac{\pi}{2}$; so $x = \frac{\pi}{4}$.

19. $\operatorname{arcsec}(-2) = x$ then $\sec x = -2$ where $0 \le x \le \pi$, $x \ne \frac{\pi}{2}$; so $x = \frac{2\pi}{3}$.

20. $\csc^{-1}\sqrt{2} = x$ then $\csc x = \sqrt{2}$ where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $x \ne 0$; so $x = \frac{\pi}{4}$.

21. $\csc^{-1}2 = x$ then $\csc x = 2$ where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $x \ne 0$; so $x = \frac{\pi}{6}$.

22. $\operatorname{arccsc}(-\sqrt{2})$ then $\csc x = -\sqrt{2}$ where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $x \ne 0$; so $x = -\frac{\pi}{4}$.

23. $\arcsin 0.6442 = 0.6999768749$.

24. $\arccos 0.6675 = 0.8399500768$.

25. $\cos^{-1} 0.9051 = 0.4391814802$.

26. $\tan^{-1} 0.2500 = 0.2449786631$.

27. $\arctan 2 = 1.107148718$.

28. $\sin^{-1}(-0.5495) = -0.5817656715$.

29. $\tan^{-1}(-3.224) = -1.270032196$.

30. $\cos^{-1}(-\frac{1}{8}) = 1.445468496$.

31. $\arcsin (-0.5505) = -0.5829630403$.

32. $\arccos (-\frac{5}{11}) = 2.042658164$.

33. $\cos^{-1} \frac{\sqrt{5}}{4} = 0.9775965507$.

34. $\tan^{-1}(-\frac{\sqrt{7}}{3}) = -0.7227342478$.

35. $\cot^{-1}(3.217) = \frac{\pi}{2} - \tan^{-1}(3.217)$
$= 0.3013796990$.

36. $\operatorname{arcsec} 1.732 = \arccos (\frac{1}{1.732})$
$= 0.9552958753$.

37. $\sec^{-1} 2.718 = \cos^{-1}(\frac{1}{2.718}) = 1.194027796$.

38. $\csc^{-1}(-3.709) = \sin^{-1}(\frac{1}{-3.709})$
$= -0.2729926338$.

39. $\csc^{-1}(-1.747) = \sin^{-1}(\frac{1}{-1.747})$
$= -0.6094418025$.

40. $\operatorname{arccsc} (-5.432) = \arcsin (\frac{1}{-5.432})$
$= -0.1851502893$.

41. (a) By definition $y = \cos^{-1}x$ if and only if $x = \cos y$ and $0 \le y \le \pi$. Hence, with $x = \cos y$, $\cos^{-1}(\cos y) = y$ for $0 \le y \le \pi$, or $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$.

(b) $x = \cos y = \cos(\cos^{-1}x)$ for $-1 \le x \le 1$.

42. (a) $\tan^{-1}x = y$ if and only if $-\frac{\pi}{2} < y < \frac{\pi}{2}$

and $\tan y = x$. Since $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan y = \tan y$ it follows that $\tan^{-1}(\tan y) = y$.

(b) Let $y = \tan^{-1}x$. Then $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan y = x$; that is, $\tan(\tan^{-1}x) = x$ for all x.

(c) $\cot^{-1}x = y$ if and only if $0 < y < \pi$ and $\cot y = x$. Since $0 < y < \pi$ and $\cot y = \cot y$, it follows that $\cot^{-1}(\cot y) = y$.

(d) Let $y = \cot^{-1}x$. Then $0 < y < \pi$ and $\cot y = x$; that is, $\cot(\cot^{-1}x) = x$ for all x.

(e) $\sec^{-1}x = y$ if and only if $y \neq \frac{\pi}{2}$, $0 \leq y \leq \pi$, and $x = \sec y$. Since $y \neq \frac{\pi}{2}$, $0 \leq y \leq \pi$ and $\sec y = \sec y$, and it follows that $\sec^{-1}(\sec y) = y$.

(f) Suppose $|x| \geq 1$ and let $y = \sec^{-1}x$. Then $y \neq \frac{\pi}{2}$, $0 \leq y \leq \pi$, and $x = \sec y$; that is, $x = \sec(\sec^{-1}x)$.

(g) $\csc^{-1}x = y$ if and only if $x = \csc y$, $y \neq 0$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Since $y \neq 0$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, and $\csc y = \csc y$, it follows that $\csc^{-1}(\csc y) = y$.

(h) Suppose $|x| \geq 1$ and let $y = \csc^{-1}x$. Then $y \neq 0$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, and $x = \csc y$; that is, $x = \csc(\csc^{-1}x)$.

43. $\sin(\sin^{-1}\frac{3}{5}) = \frac{3}{5}$ since $\sin(\sin^{-1}x) = x$ for $-1 \leq x \leq 1$.

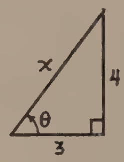
44. $\cos^{-1}(\cos\frac{5\pi}{4}) = \cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$ since $\cos^{-1}(\cos x) = x$ for $0 \leq x \leq \pi$.

45. $\sin^{-1}(\sin\frac{\pi}{6}) = \frac{\pi}{6}$ since $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

46. $\tan(\arctan 3) = 3$ since $\tan(\tan^{-1}x) = x$ for all x.

47. $\tan^{-1}(\tan\frac{3\pi}{4}) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

48. $\cos^{-1}(\cos(-\frac{\pi}{3})) = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$.

49. 

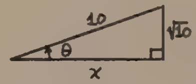
Let $\theta = \arctan\frac{4}{3}$, so that $\tan\theta = \frac{4}{3}$; want to find $\sin\theta$.

By the Pythagorean theorem, $4^2+3^2 = x^2$ or $x = 5$. Thus, $\sin\theta = \frac{4}{5}$.

50. Let $y = \arctan(-2)$; want to find $\cos y$. $\tan y = -2$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Now $1 + \tan^2 y = \sec^2 y$ or $1+4 = \sec^2 y$ or $\sec^2 y = 5$; so $\sec y = \pm\sqrt{5}$ since $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $\sec y > 0$. Thus, $\sec y = \sqrt{5}$ or $\cos y = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

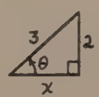
51. Let $\theta = \sin^{-1}\frac{\sqrt{10}}{10}$, so that $\sin\theta = \frac{\sqrt{10}}{10}$; want to find $\cos\theta$.



By the Pythagorean theorem, $100 = 10+x^2$ or $x^2 = 90$, so $x = \sqrt{90} = 3\sqrt{10}$.

Thus, $\cos\theta = \frac{3\sqrt{10}}{10}$.

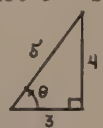
52. Let $\theta = \arcsin\frac{2}{3}$, so that $\sin\theta = \frac{2}{3}$; want to find $\sin 2\theta = 2\sin\theta\cos\theta$.



By the Pythagorean theorem, $9 = 4+x^2$ or $x^2 = \sqrt{5}$ or $x = \sqrt{5}$.

$\sin 2\theta = 2\sin\theta\cos\theta = 2(\frac{2}{3})\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$.

53. Let $\theta = \sin^{-1}\frac{4}{5}$, so that $\sin\theta = \frac{4}{5}$;



want to find $\tan\theta$. $\tan\theta = \frac{4}{3}$.

54. Let $\theta = \operatorname{arcsec}(-5)$; want to find $\tan\theta$. $\sec\theta = -5$, $0 \leq \theta \leq \pi$, $0 \neq \frac{\pi}{2}$. Since $\sec\theta < 0$, $\frac{\pi}{2} < \theta \leq \pi$. But $1 + \tan^2\theta = \sec^2\theta$ or $1+\tan^2\theta = 25$ or $\tan^2\theta = 24$;

so $\tan\theta = -\sqrt{24} = -2\sqrt{6}$ since
$\frac{\pi}{2} < \theta \le \pi$.

55. Let $\theta = \cos^{-1}\frac{7}{10}$; want to find $\sec\theta$.
$\cos\theta = \frac{7}{10}$, so $\sec\theta = \frac{10}{7}$.

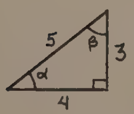
56. Let $\theta = \cot^{-1}(-2)$; want to find $\csc\theta$.
$\cot\theta = -2$, $0 < \theta < \pi$; but $\cot\theta < 0$,
so $\frac{\pi}{2} < \theta < \pi$. $1 + \cot^2\theta = \csc^2\theta$, so $1+4-\csc^2\theta$
or $\csc^2\theta = 5$; so $\cos\theta = \sqrt{5}$ since
$\frac{\pi}{2} < \theta < \pi$.

57. Let $\theta = \csc^{-1}\sqrt{2}$; want to find $\sec\theta$.
$\csc\theta = \sqrt{2}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; since $\csc\theta > 0$,
$0 < \theta < \frac{\pi}{2}$. Hence, $\theta = \frac{\pi}{4}$, and so
$\sec\theta = \sqrt{2}$.

58. Let $\theta = \sin^{-1}\frac{1}{8}$; want to find $\sec 2\theta$.
$\sin\theta = \frac{1}{8}$, $\cos^2\theta = \cos^2\theta - \sin^2\theta =$
$1 - 2\sin^2\theta = 1-2(\frac{1}{64}) = 1 - \frac{1}{32} = \frac{31}{32}$;
so $\sec^2\theta = \frac{32}{31}$.

59. Let $\theta = \operatorname{arccsc} 7$; want to find $\cot\theta$.
$\csc\theta = 7$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\theta \ne 0$. Since
$\csc\theta > 0$, $0 < \theta < \frac{\pi}{2}$.
$1+\cot^2\theta = \csc^2\theta$ or $1+\cot^2\theta = 49$ or
$\cot^2\theta = 48$ or $\cot\theta = \sqrt{48} = 4\sqrt{3}$, since
$0 < \theta < \frac{\pi}{2}$.

60.

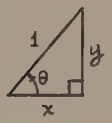


$\tan\alpha = \frac{3}{4}$, so $\alpha = \tan^{-1}\frac{3}{4}$
and $\alpha \approx 0.644$.
$\beta = \frac{\pi}{2} - \alpha \approx 0.927$.

61. Let $\sin^{-1}x = \theta$, $-1 \le x \le 1$. Now
$\cos(\frac{\pi}{2} - \sin^{-1}x) = \cos(\frac{\pi}{2} - \theta) =$
$\cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta = \sin\theta = x$.
Hence, for $-1 \le x \le 1$, $\frac{\pi}{2} - \sin^{-1}x = \cos^{-1}x$.

62. Let $\cos^{-1}x = \theta$, $-1 \le x \le 1$. Now
$\cos(\pi - \cos^{-1}x) = \cos(\pi - \theta)$
$= \cos\pi\cos\theta + \sin\pi\sin\theta$
$= -\cos\theta = -\cos(\cos^{-1}x) = -x$; hence,
$\pi - \cos^{-1}x = \cos^{-1}(-x)$, for $-1 \le x \le 1$.

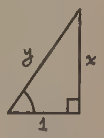
63. Let $\theta = \cos^{-1}x$; show $\sin\theta = \sqrt{1-x^2}$.
Now $\cos\theta = x$. Consider $0 < \theta < \frac{\pi}{2}$;
that is, $0 < x \le 1$. We can draw the
following triangle. Then by the



Pythagorean theorem,
$x^2+y^2 = 1$ or $y = \sqrt{1-x^2}$.
But $\sin\theta = y$, so for
$0 \le x \le 1$, $\sin(\cos^{-1}x) = \sqrt{1-x^2}$. When
$-1 \le x \le 0$, then $0 < -x \le 1$, and so
$\sin(\cos^{-1}(-x)) = \sin(\pi - \cos^{-1}(x))$ (Problem 62)
$= \sin(\cos^{-1}x) = \sqrt{1-(-x)^2} = \sqrt{1-x^2}$.
Result holds for all x in $[-1,1]$.

64. $\tan(\sin^{-1}x) = \dfrac{\sin(\sin^{-1}x)}{\cos(\sin^{-1}x)}$. Now $\sin(\sin^{-1}x)$
$= x$ for $-1 \le x \le 1$, and $\cos(\sin^{-1}x)$
$= \sqrt{1-x^2}$ for $-1 \le x \le 1$. Hence,
$\tan(\sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$, $-1 < x < 1$.

65. Let $\theta = \arctan x$; show $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$.
For $0 \le \theta < \frac{\pi}{2}$, $\tan\theta = x = \frac{x}{1}$ and $x > 0$.
The right triangle is:



Then by the Pythagorean
theorem, $1+x^2 = y^2$ or
$y = \sqrt{1+x^2}$. Thus,
$\sin\theta = \frac{x}{y} = \dfrac{x}{\sqrt{1+x^2}}$. If $-\frac{\pi}{2} < \theta < 0$, then
$x < 0$, $0 < -\theta < \frac{\pi}{2}$, and $-x > 0$; thus,
$\sin(-\theta) = -\sin\theta = \dfrac{-x}{\sqrt{1+x^2}}$ and $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$.

66. Let $\theta = \arccos x$, so $\cos\theta = x$, $-1 \le x \le 1$.
Now, $\tan\frac{\theta}{2} = \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \dfrac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{\frac{1+\cos\theta}{2}}} = \dfrac{\sqrt{1-x}}{\sqrt{1+x}}$,

$-1 < x \le 1$. Hence, $\tan(\frac{1}{2}\arccos x)$
$= \sqrt{\frac{1-x}{1+x}}$, $-1 < x \le 1$.

67. $\tan(\tan^{-1}x + \tan^{-1}y) =$

$\dfrac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)}.$

But $\tan(\tan^{-1}z) = z$ for all z, so

$\tan(\tan^{-1}x + \tan^{-1}y) = \frac{x+y}{1-xy}$ where $xy \neq 1$.

68. $\cos\left[\sin^{-1}x + \sin^{-1}y\right] =$

$\cos\left[\sin^{-1}x\right]\cos\left[\sin^{-1}y\right] - \sin(\sin^{-1}x)\sin(\sin^{-1}y)$,

$-1 \le x \le 1$ and $-1 \le y \le 1$. Now $\sin(\sin^{-1}t)$

$= t$ for $-1 \le t \le 1$ and $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

by Example 4 in this section. So

$\cos\left[\sin^{-1}x + \sin^{-1}y\right] = \sqrt{1-x^2}\ \sqrt{1-y^2} - xy$

$= \sqrt{1-x^2-y+x^2y^2} - xy.$

69. Let $\cos^{-1}\frac{1}{x} = \alpha$, so $\cos\alpha = \frac{1}{x}$, $0 \le \alpha \le y$.

Now $\sec\alpha = x$, $|x| \ge 1$.

Thus, $\sec(\cos^{-1}\frac{1}{x}) = x$, so that $\cos^{-1}\frac{1}{x} =$

$\sec^{-1}x$ for $|x| \ge 1$.

70. Let $\sin^{-1}\frac{1}{x} = \alpha$, so $\sin\alpha = \frac{1}{x}$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

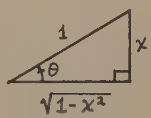
Now $\csc\alpha = x$ for $|x| \ge 1$, so

$\csc(\sin^{-1}\frac{1}{x}) = x$ or $\sin^{-1}\frac{1}{x} = \csc^{-1}x$, $|x| \ge 1$.

71. Let $\theta = \sin^{-1}x$; find $\sin 2\theta$.

For $0 < \theta < \frac{\pi}{2}$, $\sin\theta = x$,

and the right triangle is:



Thus, $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2x\sqrt{\frac{1-x^2}{1}} = 2x\sqrt{1-x^2}$, for $0 \le x \le 1$.

Continuing the argument as in Example 4,

we get $\sin 2\theta = 2x\sqrt{1-x^2}$ for $-1 \le x \le 1$.

72. Let $\theta = \csc^{-1}x$; find $\sin\theta$.

$\csc\theta = x$ for $|x| \ge 1$,

so $\sin\theta = \frac{1}{x}$; and therefore

$\sin(\csc^{-1}x) = \frac{1}{x}$, $|x| \ge 1$.

73. $\cos(\sin^{-1}x - \cos^{-1}x)$

$= \cos(\sin^{-1}x)\cos(\cos^{-1}x) + \sin(\sin^{-1}x)\sin(\cos^{-1}x)$

$\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$; $\sin(\sin^{-1}x)$

$= x$ for $-1 \le x \le 1$.

$\cos(\sin^{-1}x) = \sqrt{1-x^2}$ from Example 4 in

this section; $\sin(\cos^{-1}x) = \sqrt{1-x^2}$ from

Problem 63; hence, $\cos(\sin^{-1}x - \cos^{-1}x) =$

$\sqrt{1-x^2}(x) + x(\sqrt{1-x^2}) = 2x\sqrt{1-x^2}$, $-1 \le x \le 1$.

74. Let $\theta = \sec^{-1}x$; find $\cos^2\theta$.

$\sec\theta = x$ or $\cos\theta = \frac{1}{x}$, $|x| \ge 1$.

$\cos^2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 2(\frac{1}{x})^2 - 1 = \frac{2}{x^2} - 1.$

75. Let $\alpha = \tan^{-1}\frac{1}{3}$,

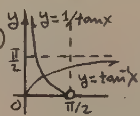
So $\tan\alpha = \frac{1}{3}$;

$\beta = \tan^{-1}\frac{1}{2}$, so $\tan\beta = \frac{1}{2}$.

$x = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{1}{\sqrt{10}}\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{5}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$

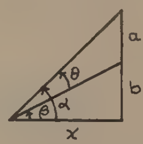
76. (a)



Yes, there is a value x for which $\tan^{-1} = \frac{1}{\tan x}$.

(b) $x = 0.928394860$.

77.



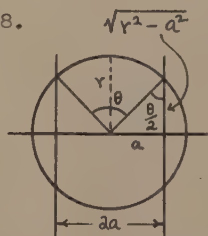
$\tan\alpha = \frac{a+b}{x}$, so $\alpha = \tan^{-1}\frac{a+}{}$

$\tan\beta = \frac{b}{x}$, so $\beta = \tan^{-1}\frac{b}{x}$

$\theta = \alpha - \beta$.

Hence, $\theta = \alpha - \beta = \tan^{-1}\frac{a+b}{x} - \tan^{-1}\frac{b}{x}$.

78.



The area of $\frac{A}{2} =$

$\frac{1}{2}r^2\theta + 2(\frac{1}{2})a(\sqrt{r^2-a^2})$. So

$A = r^2\theta + 2a\sqrt{r^2-a^2}$. Now

$\sin\frac{\theta}{2} = \frac{a}{r}$, so that $\frac{\theta}{2} =$

$\sin^{-1}\frac{a}{r}$ and $\theta = 2\sin^{-1}\frac{a}{r}$.

So $A = 2r^2\sin^{-1}\frac{a}{r} + 2a\sqrt{r^2-a^2}$ square units.

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1. $f'(x) = \dfrac{3}{\sqrt{1-9x^2}}$.

2. $g'(x) = \dfrac{-7}{\sqrt{1-49x^2}}$.

3. $h'(x) = \dfrac{1/5}{1 + \frac{x^2}{25}} = \dfrac{5}{25 + x^2}$.

4. $H'(x) = \dfrac{-2/3}{1 + \frac{4x^2}{9}} = \dfrac{-6}{9 + 4x^2}.$

5. $G'(t) = \dfrac{3t^2}{|t^3|\sqrt{t^6-1}} = \dfrac{3}{|t|\sqrt{t^6-1}}.$

6. $f'(x) = \dfrac{-2x}{x^2\sqrt{x^4-1}}.$

7. $f'(t) = \dfrac{-2t}{1+(t^2+3)^2} = \dfrac{-2t}{t^4+6t^2+10}.$

8. $F'(x) = \dfrac{1}{1+\frac{4x^2}{(1-x^2)^2}}\left[\dfrac{2-2x^2+4x^2}{(1-x^2)^2}\right]$

$= \dfrac{2(x^2+1)}{(1-x^2)^2+4x^2} = \dfrac{2}{x^2+1}.$

9. $g'(x) = \dfrac{-(-\frac{3}{2x^2})}{\left|\frac{3}{2x}\right|\sqrt{\frac{9}{4x^2}-1}} = \dfrac{1}{\left|\frac{x}{2x}\right|\sqrt{9-4x^2}} = \dfrac{2}{\sqrt{9-4x^2}}.$

10. $f'(r) = \dfrac{1}{1+\left[\frac{r+2}{1-2r}\right]^2}\cdot\dfrac{(1-2r)-(r+2)(-2)}{(1-2r)^2}$

$= \dfrac{1}{r^2+1}.$

11. $h'(u) = \dfrac{1}{\sqrt{\frac{1}{1-u^2}}\sqrt{\frac{1}{1-u^2}-1}}\cdot\dfrac{u}{(1-u^2)^{3/2}}$

$= \dfrac{u}{(1-u^2)\sqrt{\frac{u^2}{1-u^2}}} = \dfrac{u}{|u|\sqrt{1-u^2}}.$

12. $f'(x) = x\left(-\dfrac{x}{\sqrt{4-x^2}}\right)+\sqrt{4-x^2}+\dfrac{4(\frac{1}{2})}{\sqrt{1-\frac{x^2}{4}}}$

$= \dfrac{-x^2+4-x^2+4}{\sqrt{4-x^2}} = \dfrac{8-2x^2}{\sqrt{4-x^2}}.$

13. $f'(s) = \dfrac{-2/s^2}{\sqrt{1-\frac{4}{s^2}}} + \dfrac{-\frac{1}{2}}{1+\frac{s^2}{4}} = -\dfrac{2}{\frac{s^2}{|s|}\sqrt{s^2-4}} - \dfrac{2}{4+s^2}$

$= \dfrac{-2}{|s|\sqrt{s^2-4}} - \dfrac{2}{4+s^2}.$

14. $g'(t) = t\left(\dfrac{-2}{\sqrt{1-4t^2}}\right)+\cos^{-1}(2t)-\frac{1}{2}(\frac{1}{2})\left(\dfrac{-8t}{\sqrt{1-4t^2}}\right)$

$= \cos^{-1}(2t).$

15. $G'(r) = \dfrac{1}{|r|\sqrt{r^2-1}} - \dfrac{1}{|r|\sqrt{r^2-1}} = 0.$

16. $F'(x) = \dfrac{1}{\sqrt{x^2+9}\,(\sqrt{x^2+9-1})}\cdot\dfrac{2x}{2\sqrt{x^2+9}}$

$= \dfrac{x}{(x^2+9)\sqrt{x^2+8}}.$

17. $g'(x) = x^2\left(\dfrac{-3}{\sqrt{1-9x^2}}\right) + 2x\cos^{-1}(3x)$

$= \dfrac{-3x^2}{\sqrt{1-9x^2}} + 2x\cos^{-1}(3x).$

18. $h'(t) = t(3)(\sin^{-1}t)^2\left(\dfrac{1}{\sqrt{1-t^2}}\right)+(\sin^{-1}t)^3-3$

$= (\sin^{-1}t)^2\left[\dfrac{3t}{\sqrt{1-t^2}} + \sin^{-1}t\right] - 3.$

19. $H'(x) = \dfrac{1}{x^2}\left(\dfrac{\frac{5}{x^2}}{1+\frac{25}{x^2}}\right) - \dfrac{2}{x^3}\left(\tan^{-1}\dfrac{5}{x}\right)$

$= \dfrac{-5}{x^2(x^2+25)} - \dfrac{2}{x^3}\tan^{-1}\dfrac{5}{x}.$

20. $F'(x) = \dfrac{(x^2+1)\dfrac{\frac{1}{2\sqrt{x}}}{\sqrt{x}\sqrt{x-1}}-(\sec^{-1}\sqrt{x})(2x)}{(x^2+1)^2}$

$= \dfrac{x^2+1-4x^2\sqrt{x-1}\,\sec^{-1}\sqrt{x}}{2x\sqrt{x-1}\,(x^2+1)^2}.$

21. $g'(x) =$

$\dfrac{\sqrt{x^2+1}\left(\dfrac{-2x}{(x^2+1)\sqrt{(x^2+1)^2-1}}\right)-\csc^{-1}(x^2+1)\left(\dfrac{2x}{2\sqrt{x^2+1}}\right)}{x^2+1}$

$= \dfrac{-2x-x\sqrt{x^4+2x^2}\,\csc^{-1}(x^2+1)}{(x^2+1)^{3/2}\sqrt{x^4+2x^2}}.$

22. (a) Let $y = \cos^{-1}u$, $-1 \le u \le 1$, and assume u is a differentiable function of x; then $\cos y = u$ or $-\sin y\,\dfrac{dy}{dx} = \dfrac{du}{dx}$.

Now by Problem 63 in Problem Set 7.2,

$\dfrac{dy}{dx} = \dfrac{\frac{du}{dx}}{-\sin y} = \dfrac{-\frac{du}{dx}}{\sqrt{1-u^2}},\ -1 < u < 1.$

(b) Let $y = \cot^{-1}u$, so $\cot y = u$.

Now $D_x \cot y = -\csc^2 y\, D_x y = D_x u$,

so $D_x y = \dfrac{D_x u}{y} = \dfrac{-D_x u}{1+\cot^2 y} = \dfrac{-D_x u}{1+u^2}.$

(c) Let $y = \csc^{-1}u$, so $\csc y = u$, $|u| \ge 1$.

$D_x \csc y = -\csc y \cot y\, D_x y = D_x u$,

so $D_x y = \dfrac{-D_x u}{\csc y \cot y} = \dfrac{-D_x u}{|u|\sqrt{u^2-1}}$,

where $|u| > 1$ and where $1+\cot^2 y = \csc^2 y = u^2$.

23. $\sin^{-1} y + \dfrac{x}{\sqrt{1-y^2}} D_x y = 1 + D_x y$, so

$$D_x y = \frac{1-\sin^{-1} y}{\frac{x}{\sqrt{1-y^2}} - 1} = \frac{\sqrt{1-y^2}(1-\sin^{-1} y)}{x-\sqrt{1-y^2}}.$$

24. $-\dfrac{1}{\sqrt{1-x^2y^2}} \cdot (y+xD_x y) = \dfrac{1}{\sqrt{1-(x+y)^2}} \cdot (1+D_x y)$,

$\left(-\dfrac{x}{\sqrt{1-x^2y^2}} - \dfrac{1}{\sqrt{1-(x+y)^2}}\right) D_x y =$

$\dfrac{1}{\sqrt{1-(x+y)^2}} + \dfrac{y}{\sqrt{1-x^2y^2}}$, $D_x y =$

$$\frac{\sqrt{1-x^2y^2}+y\sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^2}\sqrt{1-x^2y^2}} \cdot \frac{\sqrt{1-x^2y^2}\sqrt{1-(x+y)^2}}{-(x\sqrt{1-(x+y)^2}+\sqrt{1-x^2y^2})}.$$

So $D_x y = -\dfrac{y\sqrt{1-(x+y)^2}+\sqrt{1-x^2y^2}}{x\sqrt{1-(x+y)^2}+\sqrt{1-x^2y^2}}$.

25. $\dfrac{1}{1+x^2} - \dfrac{1}{1-y^2} D_x y = 0$. $D_x y = \dfrac{1+y^2}{1+x^2}$.

26. $\dfrac{1}{|x|\sqrt{x^2-1}} - \dfrac{1}{|y|\sqrt{y^2-1}} D_x y = 0$.

$$D_x y = \frac{|y|\sqrt{y^2-1}}{|x|\sqrt{x^2-1}}.$$

27. $\dfrac{dy}{dx} = \sec^2(2\tan^{-1}\frac{x}{2}) \cdot \dfrac{d(2\tan^{-1}\frac{x}{2})}{dx}$

$$= \sec^2(2\tan^{-1}\tfrac{x}{2})\left(2 \cdot \frac{1}{1+\frac{x^2}{4}} \cdot \frac{1}{2}\right)$$

$$= \left[\tan^2(2\tan^{-1}\tfrac{x}{2})+1\right] \frac{4}{4+x^2} = \frac{4(y^2+1)}{4+x^2}.$$

28. $D_x y = \left[5+(\tan^{-1} 2x)^2\right]^{20} \cdot D_x(\tan^{-1} 2x)$

$$= \left[5+(\tan^{-1} 2x)^2\right]^{20} \left(\frac{2}{1+4x^2}\right).$$

29. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + C.$

30. Let $u = 3t$, $du = 3dt$, $dt = \frac{1}{3}du$.

$$\int \frac{dt}{\sqrt{16-9t^2}} = \int \frac{1}{3}\frac{du}{\sqrt{16-u^2}} = \frac{1}{3}\sin^{-1}\frac{u}{4} + C$$

$$= \frac{1}{3}\sin^{-1}\frac{3t}{4} + C.$$

31. Let $u = 2x$, $du = 2dx$, $dx = \frac{1}{2}du$.

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{1}{2}\frac{du}{\sqrt{9-u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{3} + C$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + C.$$

32. Let $u = \sqrt{11}\, y$, $du = \sqrt{11}\, dy$, $dy = \frac{1}{\sqrt{11}}du$.

$$\int \frac{dy}{25-11y^2} = \int \frac{1}{\sqrt{11}}\frac{du}{\sqrt{25-u^2}} = \frac{1}{\sqrt{11}}\sin^{-1}\frac{u}{5} + C$$

$$= \frac{1}{\sqrt{11}}\sin^{-1}\frac{\sqrt{11}}{5}y + C.$$

33. Let $u = 3t$, $du = 3dt$, $dt = \frac{1}{3}du$.

$$\int \frac{dt}{\sqrt{1-9t^2}} = \int \frac{1}{3}\frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1} u + C$$

$$= \frac{1}{3}\sin^{-1} 3t + C.$$

34. $\displaystyle\int \frac{dx}{x^2+9} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$

35. Let $u = 3y$, $du = 3dy$, $dy = \frac{1}{3}du$.

$$\int \frac{dy}{4+9y^2} = \int \frac{1}{3}\frac{du}{4+u^2} = \frac{1}{3}\left(\frac{1}{2}\right)\tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{6}\tan^{-1}\frac{3y}{2} + C.$$

36. Let $V = 3u$, $dV = 3du$, $du = \frac{1}{3}dV$.

$$\int \frac{du}{9u^2+1} = \int \frac{1}{3}\frac{dV}{V^2+1} = \frac{1}{3}\tan^{-1}(V) + C$$

$$= \frac{1}{3}\tan^{-1}(3u) + C.$$

37. Let $u = 2x$, $du = 2dx$, $dx = \frac{1}{2}du$.

$$\int \frac{dx}{4x^2+9} = \int \frac{1}{2}\cdot\frac{du}{u^2+9} = \frac{1}{2}\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{u}{3}\right)+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2}{3}x\right) + C.$$

38. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\sec^{-1}\left|\frac{x}{2}\right| + C.$

39. Let $u = 4t$, $du = 4dt$, $dt = \frac{1}{4}du$.

$$\int \frac{\frac{1}{4}du}{\frac{u}{4}\cdot\sqrt{u^2-25}} = \frac{1}{5}\sec^{-1}\left|\frac{u}{5}\right| + C$$

$$= \frac{1}{5}\sec^{-1}\left|\frac{4t}{5}\right| + C.$$

40. Let $V = 3u$, $dV = 3du$, $du = \frac{1}{3}dV$.

$$\int \frac{du}{u\sqrt{9u^2-100}} = \int \frac{\frac{1}{3}\,dV}{\frac{1}{3}V\sqrt{V^2-100}} = \frac{1}{10}\sec^{-1}\left|\frac{V}{10}\right| + C$$

$$= \frac{1}{10}\sec^{-1}\left|\frac{3u}{10}\right| + C.$$

41. $\int \frac{4dx}{x\sqrt{x^2-16}} = 4(\frac{1}{4})\sec^{-1}\left|\frac{x}{4}\right| + C = \sec^{-1}\left|\frac{x}{4}\right| + C.$

42. $\int_0^{\frac{1}{2}} \frac{dt}{\sqrt{1-t^2}} = (\sin^{-1}t)\Big|_0^{\frac{1}{2}} = \sin^{-1}(\frac{1}{2}) - \sin^{-1}(0)$

$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

43. $\int_{-3}^{3} \frac{dx}{\sqrt{12-x^2}} = \sin^{-1}(\frac{x}{\sqrt{12}})\Big|_{-3}^{3}$

$$= \sin^{-1}(\frac{3\sqrt{12}}{12}) - \sin^{-1}(\frac{-3\sqrt{12}}{12})$$

$$= \sin^{-1}(\frac{\sqrt{3}}{2}) - \sin^{-1}(-\frac{\sqrt{3}}{2})$$

$$= \frac{\pi}{3} - (-\frac{\pi}{3}) = \frac{2\pi}{3}.$$

44. $\int_0^2 \frac{2du}{\sqrt{8-u^2}} = 2\left[\sin^{-1}\frac{u}{\sqrt{8}}\right]\Big|_0^2$

$$= 2(\sin^{-1}\frac{2}{\sqrt{8}} - \sin^{-1}0)$$

$$= 2(\sin^{-1}\frac{\sqrt{2}}{2} - 0) = 2(\frac{\pi}{4}) = \frac{\pi}{2}.$$

45. $\int_0^3 \frac{dt}{3+t^2} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\Big|_0^3 = \frac{1}{\sqrt{3}}(\tan^{-1}\frac{3}{\sqrt{3}} - \tan^{-1}0)$

$$= \frac{1}{\sqrt{3}}(\frac{\pi}{3} - 0) = \frac{\pi\sqrt{3}}{9}.$$

46. $\int_{-1}^{1} \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_{-1}^{1} = \frac{1}{2}\left[\tan^{-1}\frac{1}{2} - \tan^{-1}(-\frac{1}{2})\right]$

$$= \frac{1}{2}(2\tan^{-1}\frac{1}{2}) = \tan^{-1}\frac{1}{2}.$$

47. $\int_{-2}^{-\sqrt{2}} \frac{dt}{t\sqrt{t^2-1}} = \sec^{-1}|t|\Big|_{-2}^{-\sqrt{2}}$

$$= \sec^{-1}\sqrt{2} - \sec^{-1}(2) = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}.$$

48. Let $V = 2u$, $dV = 2du$, $du = \frac{1}{2}dV$.

$$\int \frac{du}{u\sqrt{4u^2-1}} = \int \frac{\frac{1}{2}dV}{\frac{1}{2}V\sqrt{V^2-1}} = \sec^{-1}|V| + C.$$

So $\int_{\frac{\sqrt{2}}{2}}^{1} \frac{du}{u\sqrt{4u^2-1}} = \sec^{-1}|2u|\Big|_{\frac{\sqrt{2}}{2}}^{1}$

$$= \sec^{-1}2 - \sec^{-1}\sqrt{2} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

49. $u = \sin x$, $du = \cos x\,dx$.

$$\int \frac{\cos x}{\sqrt{36-\sin^2 x}}dx = \int \frac{du}{\sqrt{36-u^2}} = \sin^{-1}(\frac{u}{6}) + C$$

$$= \sin^{-1}(\frac{\sin x}{6}) + C.$$

50. Let $u = \tan t$, $du = \sec^2 t\,dt$.

$$\int \frac{\sec^2 t}{1+\tan^2 t}dt = \int \frac{du}{1+u^2} = \tan^{-1}(u) + C$$

$$= \tan^{-1}(\tan t) + C = t + C.$$

(Note that $\sec^2 t = 1 + \tan^2 t$.)

51. $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{4+x^4}dx = \int \frac{\frac{1}{2}du}{4+u^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}(\frac{u}{2})\right] + C$$

$$= \frac{1}{4}\tan^{-1}(\frac{x^2}{2}) + C.$$

52. Let $u = 3x-1$, $du = 3dx$, $dx = \frac{1}{3}du$.

$$\int \frac{dx}{7+(3x-1)^2} = \int \frac{(\frac{1}{3})du}{7+u^2}$$

$$= \frac{1}{3}\left[\frac{1}{\sqrt{7}}\tan^{-1}(\frac{u}{\sqrt{7}})\right] + C = \frac{1}{3\sqrt{7}}\tan^{-1}(\frac{3x-1}{\sqrt{7}}) + C.$$

53. $u = \sin\frac{x}{2}$, $du = \frac{1}{2}\cos\frac{x}{2}dx$.

$$\int \frac{\cos\frac{x}{2}}{1+\sin^2\frac{x}{2}}dx = \int \frac{2du}{1+u^2} = 2\tan^{-1}u + C$$

$$= 2\tan^{-1}(\sin\frac{x}{2}) + C.$$

54. Let $u = 3\tan t$, $du = 3\sec^2 t\,dt$.

$$\int \frac{\sec^2 t\,dt}{\sqrt{1-9\tan^2 t}} = \int \frac{\frac{1}{3}du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}u + C$$

$$= \frac{1}{3}\sin^{-1}(3\tan t) + C.$$

55. Let $u = 3\csc t$, $du = -3\csc t\cot t\,dt$.

$$\int_{\pi/6}^{\pi/3} \frac{\csc t\cot t}{1+9\csc^2 t}dt = -\frac{1}{3}\int_6^{2/\sqrt{3}} \frac{du}{1+u^2}$$

$$= -\frac{1}{3}\tan^{-1}(u)\Big|_6^{2\sqrt{3}} = -\frac{1}{3}\tan^{-1}(2\sqrt{3}) + \frac{1}{3}\tan^{-1}6$$

$$= \frac{1}{3}(\tan^{-1}6 - \tan^{-1}2\sqrt{3}).$$

56. Let $u = \cos^{-1}x$, $du = \frac{-1}{\sqrt{1-x^2}}$.

$$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{\cos^{-1}x\,dx}{\sqrt{1-x^2}} = -\int_{\pi/4}^{\pi/6} u\,du =$$

$$-\frac{u^2}{2}\Big|_{\pi/4}^{\pi/6} = -\frac{(\frac{\pi^2}{36})}{2} + \frac{(\frac{\pi^2}{16})}{2} = \frac{5\pi^2}{288}.$$

57. Let $u = t^{\frac{1}{3}}$, $du = \frac{1}{3t^{2/3}}\,dt$, $3du = \frac{dt}{t^{2/3}}$.

So $\int_1^8 \frac{dt}{t^{2/3}(1+t^{2/3})} = \int_1^2 \frac{3du}{1+u^2} = 3\tan^{-1}(u)\Big|_1^2$

$= 3(\tan^{-1}2) - \frac{3\pi}{4}$.

58. Let $u = \cot^{-1}\frac{x}{2}$, $du = -\frac{\frac{1}{2}}{1+\frac{x^2}{4}}\,dx = \frac{-2dx}{4+x^2}$.

$\int_0^2 \frac{\cot^{-1}\frac{x}{2}}{4+x^2}dx = \int_{\pi/2}^{\pi/4}(-\tfrac{1}{2}u)\,du = -\frac{u^2}{4}\Big|_{\pi/2}^{\pi/4}$

$= \frac{3\pi^2}{64}$.

59. Let $u = bx$, $du = b\,dx$, $dx = \frac{1}{b}du$.

$\int \frac{dx}{\sqrt{a^2-b^2x^2}} = \int \frac{\frac{1}{b}\,du}{\sqrt{a^2-u^2}} = \frac{1}{b}\sin^{-1}(\frac{u}{a}) + C$

$= \frac{1}{b}\sin^{-1}\frac{bx}{a} + C$.

60. $\int \frac{du}{4u^2-4u+5} = \int \frac{du}{4(u^2-u+\frac{1}{4})+5-1} = \int \frac{du}{4(u-\frac{1}{2})^2+4}$

Let $V = u - \frac{1}{2}$, $dV = du$.

So $\int \frac{du}{4(u-\frac{1}{2})^2+4} = \int \frac{dV}{4(V^2+1)} =$

$\frac{1}{4}\tan^{-1}V + C = \frac{1}{4}\tan^{-1}(u-\frac{1}{2})+C$.

61. $y = \sin^{-1}(\cos x)+\cos^{-1}(\sin x)$

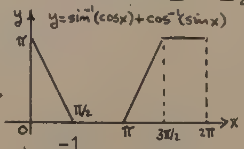
$= \frac{\pi}{2} - x + \frac{\pi}{2} - x = \pi - 2x$ for $0 \le x \le \frac{\pi}{2}$.

$y = (\frac{\pi}{2} - x) + (x - \frac{\pi}{2}) = 0$ for $\frac{\pi}{2} \le x \le \pi$.

$y = (x - \frac{3\pi}{2})+(x - \frac{\pi}{2}) = 2x-2\pi$ for $\pi \le x \le \frac{3\pi}{2}$.

$y = (x - \frac{3\pi}{2})+(\frac{5\pi}{2}-x)$

$= \pi$ for $\frac{3\pi}{2} \le x \le 2\pi$.



62. $m_1 = \frac{dy}{dx} = \frac{1}{1+x^2}$ for $y = \tan^{-1}x$;

$m_2 = \frac{dy}{dx} = -\frac{1}{1+x^2}$ for $y = \cot^{-1}x$.

We want to find θ. Now $\theta = \beta - \alpha$, so that

$\tan\theta = \tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha}$.

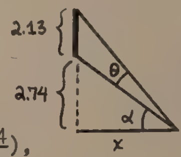
$m_1 = \tan\alpha$ and $m_2 = \tan\beta$, so that

$\tan\theta = \frac{-\frac{1}{1+x^2} - \frac{1}{1+x^2}}{1+(-\frac{1}{1+x^2})(\frac{1}{1+x^2})} = \frac{2(1+x^2)}{(1+x^2)^2-1}$

$= -\frac{2(1+x^2)}{x^4+2x^2}$. The point of intersection of the two curves is the point where $x = 1$. So $\tan\theta = \frac{-2(2)}{1+2} = -\frac{4}{3}$.

We want $\theta = \tan^{-1}(-\frac{4}{3})$, so $\theta \approx -53.13°$ or -0.93 radian.

63. $\theta + \alpha = \tan^{-1}\frac{4.87}{x}$,



$\theta = \tan^{-1}(\frac{4.87}{x})-\tan^{-1}(\frac{2.74}{x})$,

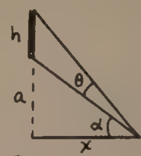
$\frac{d\theta}{dx} = \frac{1}{1+\frac{(4.87)^2}{x^2}}(-\frac{4.87}{x^2}) - \frac{(\frac{-2.74}{x^2})}{1+\frac{(2.74)^2}{x^2}}$.

We want $\frac{d\theta}{dx} = \frac{-4.87}{x^2+(4.87)^2} + \frac{2.74}{x^2+(2.74)^2} = 0$

or $-4.87\left[x^2+(2.74)^2\right]+2.74\left[x^2+(4.87)^2\right]=0$

and $x = \sqrt{13.3438} \approx 3.65$ meters.

64. $\theta + \alpha = \tan^{-1}\frac{a+h}{x}$,



$\theta = \tan^{-1}\frac{a+h}{x} =$

$\tan^{-1}\frac{a+h}{x} - \tan^{-1}\frac{a}{x}$,

$\frac{d\theta}{dt} = \frac{1}{1+(\frac{a+h}{x})^2}\cdot\left[\frac{-(a+h)}{x^2}\right] - \frac{(-\frac{a}{x^2})}{1+(\frac{a}{x})^2} =$

$\frac{-a+h}{x^2+(a+h)^2} + \frac{a}{x^2+a^2}$. So $\frac{d\theta}{dt} = 0$ provided

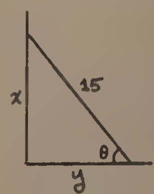
$-(a+h)(x^2+a^2)+a\left[x^2+(a+h)^2\right] = 0$ or

$-ax^2-a^3-hx^2-ha^2+ax^2+a(a+h)^2 = 0$.

$-hx^2 = -a^2h-ah^2$, $x^2 = a^2+ah$. Hence,

$x = \sqrt{a^2+ah}$ units.

65. $\frac{dx}{dt} = -2$, and when $y = 6$,



$x = \sqrt{225-36} = \sqrt{189}$.

Now $\theta = \sin^{-1}\frac{x}{15}$.

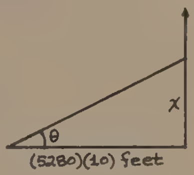
Hence, $\frac{d\theta}{dt} = \frac{(1/15)\frac{dx}{dt}}{\sqrt{1-\frac{x^2}{225}}} =$

$$\frac{\frac{dx}{dt}}{\sqrt{225-x^2}} = \frac{-2}{\sqrt{225-189}} = \frac{-2}{\sqrt{36}} = \frac{1}{3} \text{ radian}$$

per minute.

66. $\frac{dx}{dt} = 4{,}000$ feet/minute when $x = 2{,}000$.

Now $\theta = \tan^{-1} \frac{x}{52{,}800}$,



so that $\frac{d\theta}{dt} = \frac{\frac{1}{52{,}800}\frac{dx}{dt}}{1+(\frac{x}{52{,}800})^2}$

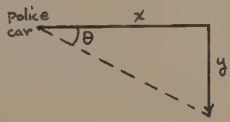
$$= \frac{\frac{4{,}000}{52{,}800}}{1+(\frac{2{,}000}{52{,}800})^2} = \frac{\frac{5}{66}}{1+(\frac{5}{132})^2}$$

$= \frac{1{,}320}{17{,}449} \approx 0.08$ radian/minute.

67. $\frac{dx}{dt} = -80$ feet/second, $\frac{dy}{dt} = 60$ feet/second.

$\left.\begin{array}{l} y = 120 \text{ feet} \\ x = 210-160=50 \text{ feet} \end{array}\right\}$ 2 seconds later.

$\theta=\tan^{-1} \frac{y}{x}$, so that

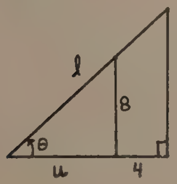


$\frac{d\theta}{dt} = \frac{1}{1 + (\frac{y}{x})^2}$.

$\frac{x\frac{dy}{dt}-y\frac{dx}{dt}}{x^2}$, and so $\frac{d\theta}{dt} = \frac{(50)(60)+120(80)}{\left[1+(\frac{120}{50})^2\right](50)^2} =$

$\frac{126}{169} \approx 0.75$ radian/second.

68.



$\tan\theta = \frac{8}{u}$, so $u = \frac{8}{\tan\theta} = 8\cot\theta$. Let ℓ be length of the ladder; then $\cos\theta = \frac{u+4}{\ell} = \frac{8\cot\theta + 4}{\ell}$,

so $\ell = \frac{8\cot\theta + 4}{\cos\theta} = 8\csc\theta + 4\sec\theta$.

$\frac{d\ell}{d\theta} = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta = 0$.

Thus, $\frac{-8\cos\theta}{\sin^2\theta} + \frac{4\sin\theta}{\cos^2\theta} = 0$ or

$4\sin^3\theta = 8\cos^3\theta$ or $\tan^3\theta = 2$ or

$\tan\theta = \sqrt[3]{2}$ so $\theta = \tan^{-1}\sqrt[3]{2}$ hus

$\ell = 8\left(\frac{\sqrt{1+\sqrt[3]{4}}}{\sqrt[3]{2}}\right) + 4\left(\sqrt{1+\sqrt[3]{4}}\right) = 4\sqrt{1+\sqrt[3]{4}}(\sqrt[3]{4}+1)$

$= 4(1 + \sqrt[3]{4})^{3/2} \approx 16.65$ feet.

69. $\int_{\cos x}^{\sin x} \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u \Big|_{\cos x}^{\sin x} =$

$\sin^{-1}(\sin x)-\sin^{-1}(\cos x) = x - \frac{\pi}{2} + x$

$= 2x - \frac{\pi}{2}$.

70. $A = \int_0^{\sqrt{3}} \frac{3}{9+x^2}dx = \frac{3}{3}\tan^{-1}\frac{x}{3}\Big|_0^{\sqrt{3}}$

$= \tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}0 = \frac{\pi}{6}$ square unit.

71. $V = \pi\int_0^1 \left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = \pi\int_0^1 \frac{1}{1+x^2}dx$

$= \pi\tan^{-1}x\Big|_0^1 = \pi(\tan^{-1}1-\tan^{-1}0)$

$= \pi(\frac{\pi}{4}) = \frac{\pi^2}{4}$ cubic units.

72. $\frac{\pi}{6} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} \approx S_4 = \frac{\frac{1}{2}-0}{6(2)}(y_0+4y_1+2y_2+4y_3+$

$y_4)$, where $\Delta x = \frac{\frac{1}{2}-0}{4} = \frac{1}{8}$ and $y_k = \frac{1}{\sqrt{1-(\frac{k}{8})^2}}$,

$k = 0, 1, 2, 3, 4$. So

$S_4 = \frac{1}{24}\left[1+4(\frac{8}{\sqrt{63}})+2(\frac{4}{\sqrt{15}})+4(\frac{8}{\sqrt{55}})+\frac{2}{\sqrt{3}}\right]$

≈ 0.52362. Hence, $\frac{\pi}{6} \approx 0.52362$ and π

≈ 3.1417. Note that the correct value of π rounded to five places is 3.14159.

73. Put $x = au$, so that $dx = adu$.

So $\int\frac{dx}{a^2+x^2} = \int\frac{adu}{a^2+a^2u^2} = \frac{1}{a}\int\frac{du}{1+u^2}$

$= \frac{1}{a}\tan^{-1}u+C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$.

74. Put $x = au$, so that $dx = adu$.

So $\int\frac{dx}{x\sqrt{x^2-a^2}} = \int\frac{adu}{au\sqrt{a^2u^2-a^2}} =$

$\int\frac{1}{|a|}\frac{du}{u\sqrt{u^2-1}} = \frac{1}{|a|}\sec^{-1}|u|+C =$

$\frac{1}{|a|}\sec^{-1}\left|\frac{x}{a}\right| + C$.

75. Let f be the function defined by $f(x) = \tan x$; $f'(x) = \sec^2x \neq 0$ for all x in $(-\frac{\pi}{2}, \frac{\pi}{2})$. By the inverse function theorem,

f is invertible, f^{-1} is differentiable, and $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$. But $f^{-1} = \tan^{-1}$, so $D_x \tan^{-1} x = \dfrac{1}{\sec^2(\tan^{-1} x)}$. But $1+\tan^2 t = \sec^2 t$, so $\sec^2(\tan^{-1} x) = 1+\tan^2(\tan^{-1} x) = 1+\left[\tan(\tan^{-1} x)\right]^2 = 1+x^2$. Thus, $D_x \tan^{-1} x = \dfrac{1}{1+x^2}$.

76. $D_x \sec^{-1} x = \dfrac{1}{(\sec)'(\sec^{-1} x)} = \dfrac{1}{\sec(\sec^{-1} x)\tan(\sec^{-1} x)} = \pm \dfrac{1}{x\sqrt{\sec^2(\sec^{-1} x)-1}}$

$= \dfrac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1.$

Problem Set 7.4, page 430

1. $f'(x) = \dfrac{8x}{4x^2+1}$.

2. $g'(x) = \dfrac{1}{\cos x^2}(-\sin x^2 \cdot (2x)) = -2x \tan x^2$.

3. $f'(x) = \cos(\ln x) \cdot \frac{1}{x} = \frac{1}{x}\cos(\ln x)$.

4. $f'(t) = \dfrac{1}{1+(\ln t)^2}\left(\frac{1}{t}\right) = \dfrac{1}{t\left[1+(\ln t)^2\right]}$.

5. $g'(x) = 1 - \dfrac{1}{\sin 6x} \cdot (6\cos 6x) = 1-6\cot 6x$.

6. $H'(x) = \dfrac{1}{x+\cos x}(1-\sin x) - \dfrac{1}{1+x^2}$

$= \dfrac{1-\sin x}{x+\cos x} - \dfrac{1}{1+x^2}$.

7. $f'(t) = (\sin t)\left(\dfrac{2t}{t^2+7}\right) + (\cos t)\ln(t^2+7)$.

8. $G'(x) = \dfrac{1}{4x+x^2+5}(4+2x) = \dfrac{2x+4}{x^2+4x+5}$.

9. $F'(u) = \dfrac{1}{\ln(u)} \cdot \dfrac{1}{u} = \dfrac{1}{u \ln u}$.

10. $f'(x) = \dfrac{1}{\csc x - \cot x} \cdot \left[-\csc x \cot x + \csc^2 x\right]$

$= \dfrac{\csc x(\csc x - \cot x)}{\csc x - \cot x} = \csc x$.

11. $h'(x) = \ln\left(\dfrac{\sec x}{5}\right) + x \cdot \dfrac{5}{\sec x} \cdot \left(\frac{1}{5}\sec x \tan x\right)$

$= \ln\left(\dfrac{\sec x}{5}\right) + x\tan x$.

12. $f'(r) = \dfrac{1}{5\sqrt[5]{1+5r^3}} \cdot \dfrac{\left(\frac{1}{5}\right)15r^2}{(1+5r^3)^{4/5}} = \dfrac{3r^2}{1+5r^3}$.

13. $g'(v) = \dfrac{1}{v^2\sqrt{v+1}} \cdot \left[2v\sqrt{v+1} + v^2 \cdot \dfrac{1}{2\sqrt{v+1}}\right]$

$= \dfrac{5v^2+4v}{2v^2(v+1)} = \dfrac{5v+4}{2v(v+1)}$.

14. $g'(t) = \dfrac{1}{t^3 \ln t^2} \cdot \left[t^3\dfrac{(2t)}{t^2} + 3t^2 \ln t^2\right]$

$= \dfrac{2+3\ln t^2}{t \ln t^2}$.

15. $h'(x) = \dfrac{-2\cos x \sin x}{\cos^2 x} = -2\tan x$.

16. $g(r) = r^2\left[-\csc(\ln r^2)\cot(\ln r^2) \cdot \dfrac{2r}{r^2}\right] + 2r\csc(\ln r^2) = 2r\csc\ln(r^2)\left[1-\cot(\ln r^2\right.$

17. $f'(x) = \frac{1}{6}\left(\dfrac{4x^3+1}{8x^2}\right)\left[\dfrac{(4x^3+1)(16x)-8x^2(12x^2)}{(4x^3+1)^2}\right]$

$= \dfrac{1-2x^3}{3x(4x^3+1)}$.

18. $g'(x) = \sqrt[3]{\dfrac{x^2+1}{x}} \cdot \frac{1}{3}\left(\dfrac{x}{x^2+1}\right)^{-2/3} \cdot \dfrac{1-x^2}{(x^2+1)^2}$

$= \dfrac{1-x^2}{3x(x^2+1)}$.

19. $h'(t) = \dfrac{(t^3+5)\left(\frac{1}{t}\right) - (\ln t)(3t^2)}{(t^3+5)^2}$

$= \dfrac{t^3+5-3t^3(\ln t)}{t(t^3+5)^2}$.

20. $H'(x) = \dfrac{1}{2\sqrt{\ln \frac{x}{x+2}}} \cdot \dfrac{x+2}{x} \cdot \dfrac{2}{(x+2)^2}$

$= \dfrac{1}{x(x+2)\sqrt{\ln \frac{x}{x+2}}}$.

21. $g'(x)$

$= \dfrac{x^2 \cdot \dfrac{1}{\tan^2 x}(2\tan x)(\sec^2 x) - \left[\ln(\tan^2 x)\right](2x)}{x^4}$

$= \dfrac{\dfrac{2x\sec^2 x}{\tan x} - 2\ln(\tan^2 x)}{x^3}$

$= \dfrac{2x\sec^2 x - 2\tan x \ln(\tan^2 x)}{x^3 \tan x}$.

22. $f'(x) = \dfrac{x\cdot\dfrac{\frac{1}{3}(-\csc^2\frac{x}{3})}{\cot\frac{x}{3}} - \ln(\cot\frac{x}{3})}{x^2}$

$= \dfrac{-x\csc^2\frac{x}{3} - 3\cot\frac{x}{3}\ln(\cot\frac{x}{3})}{3x^2\cot\frac{x}{3}}.$

23. $\dfrac{\frac{y}{x}(y-x\frac{dy}{dx})}{y^2} + \dfrac{x\frac{dy}{dx} - y}{x^2} = 0,$

$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0,$

$\frac{dy}{dx}(\frac{1}{x} - \frac{1}{y}) = \frac{y}{x^2} - \frac{1}{x}, \quad \frac{dy}{dx} = \frac{y}{x}.$

24. $\frac{dy}{dx} = \frac{1}{\sec x+\tan x}\cdot\left[\sec x\tan x+\sec^2 x\right] +$

$\csc y\cot y\,\frac{dy}{dx}, \quad \frac{dy}{dx}(1-\csc y\cot y) = \sec x,$

$\frac{dy}{dx} = \frac{\sec x}{1-\csc y\cot y}.$

25. $\frac{y}{\sin x}\cdot\cos x+\ln(\sin x)\frac{dy}{dx}-y^2-2xy\frac{dy}{dx} = 0,$

$\frac{dy}{dx}\left[\ln(\sin x)-2xy\right] = y^2-y\cot x,$

$\frac{dy}{dx} = \frac{y^2-y\cot x}{\ln(\sin x)-2xy}.$

26. $\frac{1}{y}\frac{dy}{dx} + \sin(x+y)\left[1+\frac{dy}{dx}\right] = 0,$

$\frac{dy}{dx}(\frac{1}{y} + \sin(x+y)) = -\sin(x+y),$

$\frac{dy}{dx} = \frac{-y\sin(x+y)}{1+y\sin(x+y)}.$

27. (a) $D_x\int_1^{\ln x}\cos t^2\,dt = \cos(\ln x)^2\cdot\frac{1}{x}$

$= \frac{\cos(\ln x)^2}{x}.$

(b) $\frac{dy}{dx} = \ln\left[\tan(\cos x)^4\right]\cdot(-\sin x)$

$= -\sin x\ln\left[\tan(\cos^4 x)\right].$

28. $D_x\left[\frac{1}{2\sqrt{ab}}\ln\frac{x\sqrt{a}-\sqrt{b}}{x\sqrt{a}+\sqrt{b}}\right] =$

$\frac{1}{2\sqrt{ab}}\,\frac{x\sqrt{a}+\sqrt{b}}{x\sqrt{a}-\sqrt{b}}\left[\frac{(x\sqrt{a}+\sqrt{b})\sqrt{a}-(x\sqrt{a}-\sqrt{b})\sqrt{a}}{(x\sqrt{a}+\sqrt{b})^2}\right] =$

$\frac{1}{2\sqrt{ab}}\left(\frac{ax+\sqrt{ab}-ax+\sqrt{ab}}{(x\sqrt{a}-\sqrt{b})(x\sqrt{a}+\sqrt{b})}\right) = \frac{1}{ax^2-b}.$

29. $u = 7+5x$, so that $du = 5dx$. So $\int\frac{dx}{7+5x} =$

$\frac{1}{5}\int\frac{du}{u} = \frac{1}{5}\ln|u|+C = \frac{1}{5}\ln|7+5x|+C.$

30. Let $u = 9+\cos x$, $du = -\sin x\,dx$. So

$\int\frac{\sin x}{9+\cos x}dx = -\int\frac{du}{u} = -\ln|u|+C$

$= -\ln|9+\cos x|+C.$

31. $u = \sin x$, $du = \cos x\,dx$. So $\int\cot x\,dx$

$= \int\frac{\cos x}{\sin x}dx = \int\frac{du}{u} = \ln|u|+C$

$= \ln|\sin x|+C.$

32. Let $u = \ln(x+2)$, $du = \frac{1}{x+2}dx$. Thus,

$\int\frac{dx}{(x+2)\ln(x+2)} = \int\frac{du}{u} = \ln|u|+C$

$= \ln|\ln(x+2)|+C.$

33. $u = \ln 4x$, $du = \frac{4}{4x}dx = \frac{1}{x}dx$. So

$\int\frac{\sec^2(\ln 4x)dx}{x} = \int\sec^2 u\,du = \tan u+C$

$= \tan(\ln 4x)+C.$

34. Let $u = 1+\sqrt[3]{x}$, $du = \frac{1}{3x^{2/3}}dx$. Thus,

$\int\frac{dx}{3\sqrt[3]{x^2}(1+\sqrt[3]{x})} = \int\frac{du}{u} = \ln|u|+C$

$= \ln|1+\sqrt[3]{x}|+C.$

35. Let $u = x^2+7$, $du = 2x\,dx$. Hence,

$\int\frac{4x\,dx}{x^2+7} = \int\frac{2\,du}{u} = 2\ln|u|+C$

$= 2\ln(x^2+7)+C.$

36. Let $u = \ln 5x$, $du = \frac{5}{5x}dx = \frac{1}{x}dx$. So

$\int\frac{(\ln 5x)^2}{x}dx = \int u^2du = \frac{u^3}{3}+C$

$= \frac{(\ln 5x)^3}{3}+C.$

37. Let $u = \ln x$, $du = \frac{1}{x}dx$. So $\int\frac{\cos(\ln x)}{x}dx$

$= \int\cos u\,du = \sin u+C = \sin(\ln x)+C.$

38. Let $u = \sec x+\tan x$, $du = (\sec x\tan x +$ $\sec^2 x)dx$. Thus, $\int\frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$

$= \int\frac{du}{u} = \ln|u|+C = \ln|\sec x+\tan|x+C.$

(Note that we have found $\int\sec x\,dx =$ $\ln|\sec x+\tan x|+C$.)

39. Let $u = \ln x$, $du = \frac{1}{x}dx$. Thus, $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$

$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u+C = \sin^{-1}(\ln x)+C.$

40. Let $u = \ln x$, $du = \frac{1}{x}dx$. So $\int \frac{dx}{x\left[1+(\ln x)^2\right]}$

$= \int \frac{du}{1+u^2} = \tan^{-1}u+C = \tan^{-1}(\ln x)+C.$

41. $\int_{1/8}^{1/5} \frac{dx}{x} = \ln|x| \Big|_{1/8}^{1/5} = \ln \frac{1}{5} - \ln \frac{1}{8}.$

42. $\int_{0.01}^{10} \frac{dt}{t} = \ln|t| \Big|_{0.01}^{10} = \ln 10-\ln 0.01.$

43. Let $u = x^2+3$, $du = 2x\,dx$. So

$\int_1^{\sqrt{6}} \frac{xdx}{x^2+3} = \frac{1}{2}\int_4^9 \frac{du}{u} = \frac{1}{2}\ln|u| \Big|_4^9$

$= \frac{1}{2}(\ln 9-\ln 4).$

44. Let $u = 1+\sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$, $2\,du = \frac{1}{\sqrt{x}}dx.$

So $\int_1^9 \frac{dx}{\sqrt{x}(1+\sqrt{x})} = \int_2^4 \frac{2\,du}{u} = 2\ln|u| \Big|_2^4$

$= 2(\ln 4-\ln 2).$

45. Let $u = \ln x$, $du = \frac{1}{x}dx$. $\int_1^4 \frac{\cos(\ln x)}{x}dx$

$= \int_{\ln 1}^{\ln 4} \cos u\,du = \sin u \Big|_{\ln 1}^{\ln 4}$

$= \sin(\ln 4)-\sin(\ln 1) = \sin(\ln 4)-\sin 0$

$= \sin(\ln 4).$

46. Let $u = 2-\cos x$, $du = \sin x\,dx$.

So $\int_0^{\pi/2} \frac{\sin x}{2-\cos x}dx = \int_1^2 \frac{du}{u} = \ln|u| \Big|_1^2$

$= \ln 2-\ln 1 = \ln 2.$

47. $s = \int_0^3 \frac{dt}{1+t} = \ln(1+t) \Big|_0^3$

$= \ln 4$ meters per second.

48. $y' = \frac{x}{2} - \frac{1}{2x}$, so that $s = \int_1^2\sqrt{1+(\frac{x}{2} - \frac{1}{2x})^2}dx$

$= \int_1^2 \frac{\sqrt{4x^2+x^4-2x^2+1}}{4x^2} dx = \int_1^2 \frac{x^2+1}{2x}dx$

$= \frac{1}{2}\int_1^2 (x+\frac{1}{x})dx = \frac{1}{2}(\frac{x^2}{2} + \ln|x|) \Big|_1^2$

$= \frac{1}{2}(2+\ln 2-\frac{1}{2}-0) = \frac{1}{2}(\frac{3}{2}+\ln 2)$ units

≈ 1.10 units.

49. $M = \frac{1}{3}\int_1^4 \frac{\ln x^2}{x}dx$. Let $u = \ln x^2$. Then

$du = \frac{1}{x^2} 2xdx = \frac{2dx}{x}$. Thus,

$M = \frac{1}{3}\int_0^{\ln 16} u \frac{du}{2} = \frac{1}{6}\int_0^{\ln 16} u\,du =$

$\frac{1}{6}\frac{u^2}{2} \Big|_0^{\ln 16} = \frac{1}{12}(\ln 16)^2.$

50. We use Problem 38 to evaluate $\int \sec x\,dx$.

$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x.$

$s = \int_{\pi/4}^{\pi/3}\sqrt{1+(-\tan x)^2}\,dx = \int_{\pi/4}^{\pi/3} \sqrt{1+\tan^3 x}\,dx$

$= \int_{\pi/4}^{\pi/3} \sqrt{\sec^2 x\,dx} = \int_{\pi/4}^{\pi/3} \sec x\,dx$

$= \ln|\sec x + \tan x| \Big|_{\pi/4}^{\pi/3}$

$= \ln\left|\sec \frac{\pi}{3} + \tan \frac{\pi}{3}\right|-\ln\left|\sec\frac{\pi}{4} + \tan \frac{\pi}{4}\right|$

$= \ln\left[2+\sqrt{3}\right]- \ln(\sqrt{2} + 1)$ units.

Problem Set 7.5, page 437

1. $\ln ac = \ln a+\ln c = 0.6931 + 1.6094$

$= 2.3025.$

2. $\ln(\frac{1}{b}) = \ln 1 - \ln b = 0 - \ln b = -1.0986.$

3. $\ln a^2c^2 = \ln(ac)^2 = 2\ln ac = 2(\ln a+\ln c)$

$= 2(2.3025) = 4.605.$

4. $\ln \frac{ab}{c} = \ln a + \ln b - \ln c$

$= 0.6931 + 1.0986 - 1.6094 = 0.1823.$

5. $\ln\sqrt{b} = \ln b^{\frac{1}{2}} = \frac{1}{2}\ln b = \frac{1}{2}(1.0986) = 0.5493$

6. $\ln c^{-\frac{1}{2}} = -\frac{1}{2}\ln c = -\frac{1}{2}(1.6094) = -0.8047.$

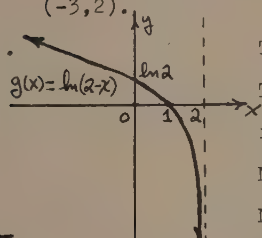
7. $\ln(x-1)+\ln(x-2) = \ln 6$, $\ln(x-1)(x-2)=\ln 6,$ so $x^2-3x+2 = 6$ and $x^2-3x-4 = 0$. Now $(x-4)(x+1) = 0$ implies $x=4$ or $x=-1$. But $x-1 < 0$ for $x=-1$. The solution is $x=4$.

8. $\ln(x^2-4)+\ln(x-2) = \ln 3$, $\ln(x^2-4)(x-2)$ $= \ln 3$, so that $x^3-2x^2-4x+8 = 3$ and so $x^3-2x^2-4x+5 = 0$, so that $(x-1)(x^2-x-5)=0$ a $x=1$ or $x = \frac{1-\sqrt{21}}{2}$ or $x = \frac{1+\sqrt{21}}{2}$. The

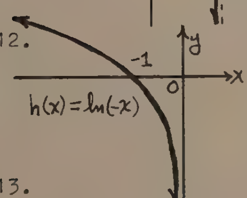
solution is $\frac{1+\sqrt{21}}{2}$ since $\frac{1-\sqrt{21}}{2} < 0$ and $x = 1$ makes $x-2 < 0$.

9. $2 \ln(x-2) = \ln x$, $\ln(x-2)^2 = \ln x$, $(x-2)^2 = x$, $x^2-4x+4 = x$, $x^2-5x+4 = 0$, $(x-4)(x-1) = 0$, $x = 4$ or $x = 1$. However, $x = 1$ does not satisfy the original equation since $1-2 = -1 < 0$. The solution is $x = 4$.

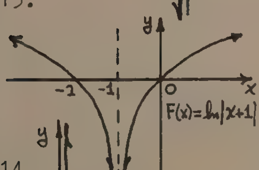
10. $\ln(6-x-x^2)-\ln(x+3) = \ln(2-x)$, $\ln(6-x-x^2) = \ln(x+3)+\ln(2-x)$, $\ln(6-x-x^2) = \ln\left[(x+3)(2-x)\right]$, $\ln(6-x-x^2) = \ln(6-x-x^2)$. Therefore, the original equation holds provided that $x+3 > 0$ and $2-x > 0$; that is, provided $-3 < x < 2$. The solution consists of all values of x in the open interval $(-3,2)$.

11. 

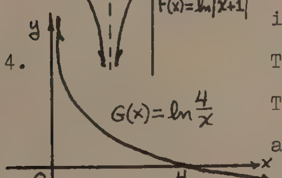
The domain is $(-\infty,2)$. The range is $\mathbb{R}$. $x=2$ is a vertical asymptote. No maximum or minimum. No inflection points.

12. 

The domain is $(-\infty,0)$. The range is $\mathbb{R}$. $x=0$ is a vertical asymptote.

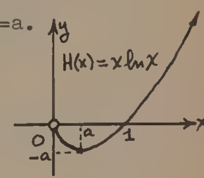
13. 

The domain is all real numbers except for -1. The range is $\mathbb{R}$. $x=-1$ is a vertical asymptote.

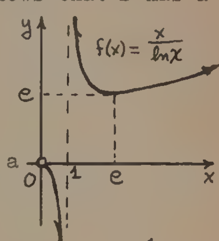
14. 

The domain is $(0,\infty)$. The range is $\mathbb{R}$. $x=0$ is a vertical asymptote.

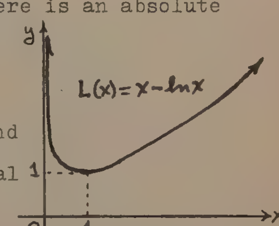
15. Let a be the positive real number whose natural logarithm is -1; that is, $\ln a = -1$. From tables of the logarithm, $a \approx 0.37$. We have $H'(x) = x(\frac{1}{x})+\ln(x) = 1+\ln(x)$, so that a is a critical number; that is, $H'(a)=0$. Also, $H''(x) = \frac{1}{x}$, so that $H''(x) > 0$ for all $x > 0$. It follows that H has a minimum value of $H(a) = a \ln(a) = -a$ at $x=a$. The domain is $(0,\infty)$. The range is $\left[-a,\infty\right) = \left[-\frac{1}{e},\infty\right)$.



16. The domain is all positive real numbers except for 1. Here, $f'(x) = \frac{\ln x-1}{(\ln x)^2}$. Let e be the positive real number whose natural logarithm is 1; that is, $\ln e = 1$. From tables of the logarithm, $e \approx 2.7$. Thus, e is a critical number for f, $f'(x) > 0$ for $x > e$ and $f'(x) < 0$ for $0 < x < e$, $x \neq 1$. It follows that f has a relative minimum at e. The range of f is $(-\infty,0)$ together with $\left[e,\infty\right)$. Also, $x = 1$ is a vertical asymptote.

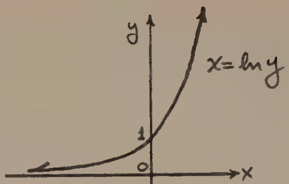


17. The domain is $(0,\infty)$. $L'(x) = 1 - \frac{1}{x}$, so that 1 is a critical number. $L''(x) = \frac{1}{x^2}$, so that $L''(x) > 0$ for all $x > 0$. Hence, there is an absolute minimum value of 1 at $x = 1$. The range is $\left[1,\infty\right)$ and $x = 0$ is a vertical asymptote.



18. From the text, the points $(0,1)$, $(0.69,2)$, $(1.38,4)$, $(2.07,8)$, $(-0.69,\frac{1}{2})$, and $(-1.38,\frac{1}{4})$ are on the graph. By implicit differentiation, $1 = \frac{1}{y}\frac{dy}{dx}$, so $\frac{dy}{dx} = y$.

So the slope at $(\ln 2, 2)$ is 2.



19. $\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + 2x \ln x = x + 2x \ln x.$

When $x = 2$, $\frac{dy}{dx} = 2+4 \ln 2$. The tangent line has equation $y-4 \ln 2 = (2+4 \ln 2)(x-2)$. The normal line has equation $y-4 \ln 2 = -\frac{1}{(2+4 \ln 2)}(x-2)$.

20. $\frac{ds}{dt} = \frac{4t^2+5}{8t}\left[\frac{(4t^2+5)8-8t(8t)}{(4t^2+5)^2}\right] = \frac{-32t^2+40}{8t(4t^2+5)} = \frac{5-4t^2}{t(4t^2+5)}.$

$$\frac{d^2s}{dt^2} = \frac{t(4t^2+5)(-8t)-(5-4t^2)(12t^2+5)}{t^2(4t^2+5)^2} = \frac{16t^4-80t^2-25}{t^2(4t^2+5)^2}.$$

21. $A = \int_5^7 \frac{1}{x}\,dx = \ln|x|\Big|_5^7 = \ln 7 - \ln 5 = \ln \frac{7}{5}$ square unit.

22. $A = \int_2^3 \frac{4}{x-1}\,dx$. Let $u = x-1$, $du = dx$. Then $A = \int_2^3 \frac{4}{x-1}\,dx \doteq \int_1^2 \frac{4\,du}{u} = 4 \ln|u|\Big|_1^2 = 4(\ln 2 - \ln 1) = 4 \ln 2$ square units.

23. $A = \int_3^4 \frac{3}{x-2}\,dx$. Let $u = x-2$, $du = dx$. Thus, $A = \int_1^2 \frac{3}{u}\,du = 3 \ln|u|\Big|_1^2 = 3(\ln 2 - \ln 1) = 3 \ln 2$ square units.

24. $A = \int_2^3 \frac{1}{2x-1}\,dx$. Let $u = 2x-1$, $du = 2\,dx$. So $A = \int_2^3 \frac{1}{2x-1}dx = \frac{1}{2}\int_3^5 \frac{1}{u}du = \frac{1}{2} \ln|u|\Big|_3^5 = \frac{1}{2}(\ln 5 - \ln 3) = \frac{1}{2} \ln \frac{5}{3}$ square unit.

25. $V = \int_1^4 \pi\left(\frac{x}{1+x^2}\right)dx$. Let $u = 1+x^2$, $du = 2x\,dx$. $V = \pi\int_1^4 \frac{x}{1+x^2}dx = \frac{\pi}{2}\int_2^{17} \frac{du}{u} = \frac{\pi}{2} \ln|u|\Big|_2^{17} = \frac{\pi}{2}(\ln 17 - \ln 2) = \frac{\pi}{2} \ln \frac{17}{2}$ cubic units.

26. $V = \pi\int_7^{10} \frac{x}{x-6}\,dx$. Let $u = x-6$, $du = dx$.

$V = \pi\int_7^{10} \frac{x}{x-6}\,dx = \pi\int_1^4 \frac{u+6}{u}du = \pi\int_1^4 \left(1+\frac{6}{u}\right)du = \pi(u+6 \ln|u|)\Big|_1^4 = \pi\left[4+6 \ln 4 - 1 - 6 \ln 1\right] = \pi(3+12 \ln 2)$ cubic units.

27. $V = \pi\int_1^4 \frac{x+1}{x^2+2x}\,dx$. Let $u = x^2+2x$, $du = (2x+2)dx$, $\frac{1}{2}du = (x+1)dx$.

$V = \pi\int_1^4 \frac{x+1}{x^2+2x}\,dx = \frac{\pi}{2}\int_3^{24} \frac{du}{u} = \frac{\pi}{2} \ln|u|\Big|_3^{24} = \frac{\pi}{2}(\ln 24 - \ln 3) = \frac{\pi}{2} \ln 8 = \frac{\pi}{2}(3) \ln 2 = \frac{3\pi}{2} \ln 2$ cubic units.

28. Let $y = \ln x$, $dy = \frac{1}{x}\,dx$. Choose $x = 10$ and $dx = 0.007$. So $\ln 10.007 = \ln 10 + \Delta y = \ln 10 + dy \approx \ln 10 + \frac{1}{x}\,dx = \ln 10 + \frac{0.007}{10} \approx 2.3025851 + 0.0007 = 2.3032851.$

29. $\ln 4126 = 8.325063694.$

30. $\ln 2.704 = 0.994732158.$

31. $\ln 0.040404 = -3.208826489.$

32. $\ln(7.321 \times 10^8) \doteq \ln 7.321 + 8 \ln 10 = 20.41142767.$

33. $\ln(1.732 \times 10^{-7}) = \ln 1.732 - 7 \ln 10 = -15.56881884.$

34. $\ln \pi = 1.144729886.$

35. (a) $A = \ln \frac{7}{5} \approx 0.3365$ square unit.

(b) $V = \frac{\pi}{2} \ln \frac{17}{2} \approx 3.3616$ cubic units.

36. Let $f(x) = \ln(1+x)$. By the mean value theorem, there is a c between 0 and x such that $f(x)-f(0) = f'(c)(x-0)$ or $\ln(1+x) = \frac{1}{1+c}(x) \approx x$ if $|x|$ is small (for then c is small). As $|x|$ gets smaller and smaller, then so does c,

and the approximation becomes more accurate.

37. $f'(x) = \frac{x(\frac{1}{x})-\ln x-1}{x^2} = \frac{-\ln x}{x^2} = 0$ for $\ln x = 0$, so $x = 1$. Now $f'(x) > 0$ for $0 < x < 1$; $f'(x) < 0$ for $x > 1$. Hence there is a maximum at 1; it is an absolute maximum since f is negative for x close to zero, and f approaches 0 as x gets large. The absolute maximum occurs at (1,1).

38. $W = \int_{V_1}^{V_0} P\,dV = \int_{V_1}^{V_0} \frac{C}{V}\,dV = C \ln V \Big|_{V_1}^{V_0}$

$= C(\ln V_0 - \ln V_1) = C \ln (\frac{V_0}{V_1})$.

Since $PV = C$, it follows that $C = P_0V_0$ and $V_1 = \frac{C}{P_1} = \frac{P_0V_0}{P_1}$. Hence,

$W = P_0V_0 \ln\left[\frac{V_0}{(\frac{P_0V_0}{P_1})}\right] = P_0V_0 \ln(\frac{P_1}{P_0})$.

39. Let $f(x) = x^2 \ln \frac{1}{x}$. $f'(x) = x^2(x)(-\frac{1}{x^2}) + 2x \ln \frac{1}{x} = 0$ provided $x(2 \ln \frac{1}{x} - 1) = 0$ or when $\ln \frac{1}{x} = \frac{1}{2}$; that is, $\ln x = -\frac{1}{2}$. The speed will be maximum when $x = a$, where $\ln a = -\frac{1}{2}$.

40. $P = 25x - \left[250 + x(6+2 \ln x)\right]$.
$P' = 25 - x(\frac{2}{x}) - 6 - 2 \ln x = 0$ when $17 - 2 \ln x = 0$; that is, when $\ln x = \frac{17}{2}$. The output level is maximum when $x = a$, where $\ln a = \frac{17}{2}$.

41. $N(t) = 100(\frac{t}{10} - \ln \frac{t}{10}) - 30$, $1 \le t \le 12$.
$N'(t) = 100[\frac{1}{10} - \frac{10}{t}(\frac{1}{10})] = 100(\frac{1}{10} - \frac{1}{t}) = 0$ for $t = 10$. The pollution reaches a minimum at $t = 10$ days after treatment.

Problem Set 7.6, page 443

1. (a) $e^{\ln 5} = 5$.
(b) $e^{-3 \ln 2} = e^{\ln 2^{-3}} = 2^{-3} = \frac{1}{8}$.
(c) $e^{3+4 \ln 2} = e^3 e^{\ln 2^4} = e^3(2^4) = 16e^3$.
(d) $\ln e^{\frac{1}{x}} = \frac{1}{x}$.
(e) $\ln e^{x-x^2} = x - x^2$.
(f) $e^{-\ln \frac{1}{x}} = e^{\ln(\frac{1}{x})^{-1}} = (\frac{1}{x})^{-1} = x$.
(g) $\ln e^{x^2-4} = x^2 - 4$.
(h) $e^{\ln x^2 - 4} = e^{\ln x^2} e^{-4} = \frac{x^2}{e^4}$.
(i) $\frac{e^{\ln(x^2-4)}}{x+2} = \frac{x^2-4}{x+2} = x - 2$.
(j) $e^{\ln x - 3 \ln y} = e^{\ln x} e^{\ln y^{-3}} = \frac{x}{y^3}$.

2. (a) Let $y = e^x$. So $2y+1 = \frac{1}{y}$, $2y^2+y-1=0$ and $(2y-1)(y+1) = 0$; $y=\frac{1}{2}$ or $y=-1$. Hence, $e^x=\frac{1}{2}$ or $e^x=-1$. Hence, $x=\ln \frac{1}{2}$; that is, $x = -\ln 2$. (We cannot have $e^x = -1$.)
(b) Let $y = e^x$. So $y + \frac{20}{y} = 21$, and $y^2-21y+20 = 0$; and so $(y-20)(y-1)=0$ and $y=20$ or $y=1$. Now $e^x = 20$, so $x=\ln 20$; and $e^x= 1$, so that $x = 0$. Hence, $x = \ln 20$ or $x = 0$.

3. (a) 0.3678794412 (b) 0.1353352832
(c) 20.08553692 (d) 1.648721271
(e) 9.356469012 (f) 15.15426223
(g) 0.0446009553 (h) 0.06598803588
(i) 23.14069264 (j) 0.6608598017

4. (a) $e^{\sqrt{2}} e^{\sqrt{3}} = e^{\sqrt{2}+\sqrt{3}}$ since (4.113250379)(5.652233674) ≈ 23.24905230 and

$e^{\sqrt{2}+\sqrt{3}} \approx 23.24905230.$

(b) $e^{\sqrt{5}-\pi} = e^{\sqrt{5}} \cdot e^{-\pi}$ since $0.4043296870 =$

$9.356469012(0.0432139183).$

(c) $(e^{\pi/2})^{(1-\sqrt{3})} = e^{\pi/2(1-\sqrt{3})}$ since

$(4.810477382)^{1-\sqrt{3}} = 0.3166675732$

and $e^{-1.149902720} = 0.316675733$

(d) $e^{3.9-2.5} = \dfrac{e^{3.9}}{e^{2.5}}$ since 4.055199967

$= \dfrac{49.40244911}{12.18249396}.$

5. $f'(x) = 7e^{7x}.$

6. $g'(t) = 3(e^{4t})^2(4e^{4t}) = 12e^{12t}.$

7. $g(x) = x^3$, so that $g'(x) = 3x^2.$

8. $f'(u) = [\exp(\sin u)][\cos u]$

$= \cos u \exp(\sin u).$

9. $f'(x) = [-\sin(\exp x)] \cdot [\exp x]$

$= (-\exp x)\sin(\exp x).$

10. $g'(x) = -e^{-x}\cos 2x - 2(\sin 2x)e^{-x}$

$= -e^{-x}(\cos 2x + 2 \sin 2x).$

11. $f'(t) = -2e^{-2t}\sin t + e^{-2t}\cos t$

$= e^{-2t}(\cos t - 2 \sin t).$

12. $g'(r) = \dfrac{1}{1+(\exp r)^2}(\exp r) = \dfrac{\exp r}{1+\exp(2r)}.$

13. $h'(x) = e^{x^2+5 \ln x}(2x+\frac{5}{x}).$

14. $F'(x) = \exp\sqrt{4-x^2}\left(\dfrac{-2x}{2\sqrt{4-x^2}}\right)$

$= -\dfrac{x}{\sqrt{4-x^2}} \exp\sqrt{4-x^2}.$

15. $f'(t) = e^{t \ln t}(t \cdot \frac{1}{t} + \ln t)$

$= (1+\ln t)e^{t \ln t}.$

16. $g'(x) = e^{x^2}(-\csc^2 4x)(4) + 2xe^{x^2} \cot 4x$

$= 2e^{x^2}[x \cot(4x) - 2 \cdot \csc^2 4x].$

17. $h'(x) = \dfrac{1}{e^{3x}\sqrt{(e^{3x})^2-1}} \cdot 3e^{3x} = \dfrac{3}{\sqrt{e^{6x}-1}}.$

18. $f'(x) = e^{\sqrt{x}} \cdot \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}} + \ln\sqrt{x}\, e^{\sqrt{x}} \dfrac{1}{2\sqrt{x}}$

$= \dfrac{e^{\sqrt{x}}}{2\sqrt{x}} \left(\dfrac{1}{\sqrt{x}} + \ln\sqrt{x}\right).$

19. $G'(s) = 2(1-e^{3s})(-3e^{3s}) = -6e^{3s}(1-e^{3s})$

$= 6e^{3s}(e^{3s}-1).$

20. $f'(t) = \dfrac{(e^t+1)(2e^{2t}) - e^{2t}(e^t)}{(e^t+1)^2}$

$= \dfrac{e^{2t}(2e^t+2-e^t)}{(e^t+1)^2} = \dfrac{e^{2t}(e^t+2)}{(e^t+1)^2}.$

21. $f'(x) = \dfrac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) [-x] = -\dfrac{x}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}).$

22. $f'(t) = \dfrac{t+1}{e^t}\left[\dfrac{(t+1)e^t - e^t}{(t+1)^2}\right] = \dfrac{t}{t+1}.$

23. $f'(x) = (8x^2-3x+1)(-e^{-x}) + e^{-x}(16x-3)$

$= e^{-x}(16x-3-8x^2+3x-1)$

$= e^{-x}(-8x^2+19x-4).$

24. $H'(r) = \sqrt{1+(e^r)^2}\, e^r = \sqrt{1+e^{2r}} \cdot e^r.$

25. $(1+e^x)\frac{dy}{dx} + ye^x - y^2 - 2xy\frac{dy}{dx} = 0,$

$\dfrac{dy}{dx} = \dfrac{y^2-ye^x}{1+e^x-2xy} = \dfrac{y(y-e^x)}{1+e^x-2xy}.$

26. $e^y \frac{dy}{dx} - \cos(x+y)\left[1+\frac{dy}{dx}\right] = 0,$

$\dfrac{dy}{dx} = \dfrac{\cos(x+y)}{e^y - \cos(x+y)}.$

27. $x(\cos y)\frac{dy}{dx} + \sin y = e^{x+y}(1+\frac{dy}{dx}),$

$\dfrac{dy}{dx} = \dfrac{e^{x+y} - \sin y}{x \cos y - e^{x+y}}.$

28. $-e^{-x}\ln y + \frac{e^{-x}}{y} \frac{dy}{dx} + \frac{e^y}{x} + (\ln x)(e^y)\frac{dy}{dx} = 0,$

$\dfrac{dy}{dx} = \dfrac{e^{-x}\ln y - \frac{e^y}{x}}{\frac{e^{-x}}{y} + e^y \ln x} = \dfrac{xye^{-x}\ln y - ye^y}{xe^{-x} + xye^y \ln x}.$

29. $y' = -3e^{-3x}$, $y'' = 9e^{-3x}$; $y''+2y'-3y =$

$9e^{-3x} - 6e^{-3x} - 3e^{-3x} = 0.$

30. $\dfrac{dy}{dx} = -20xe^{-4x} + 5e^{-4x}$; $\dfrac{d^2y}{dx^2} =$

$-20e^{-4x}+80xe^{-4x}-20e^{-4x}=80xe^{-4x}-40e^{-4x}.$

$\frac{d^2y}{dx^2}+8\,\frac{dy}{dx}+16y=80xe^{-4x}-40e^{-4x}-160xe^{-4x}+$

$40e^{-4x}+80xe^{-4x}=0.$

31. $\int e^{3x}dx=\frac{e^{3x}}{3}+C.$

32. $\int e^{-7x}dx=\frac{e^{-7x}}{-7}+C.$

33. $u=5x+3$, so $du=5\,dx$. $\int e^{5x+3}\,dx=\frac{1}{5}\int e^u du$

$=\frac{e^u}{5}+C=\frac{e^{5x+3}}{5}+C.$

34. Let $u=-4x+5$, $du=-4\,dx$. So $\int e^{-4x+5}dx$

$=-\frac{1}{4}\int e^u du=-\frac{1}{4}e^u+C=\frac{-e^{-4x+5}}{4}+C.$

35. $u=5x^2$, $du=10x\,dx$. So $\int xe^{5x^2}\,dx=$

$\frac{1}{10}\int e^u du=\frac{e^u}{10}+C=\frac{e^{5x^2}}{10}+C.$

36. Let $u=\sin x$, $du=\cos x\,dx$. So

$\int e^{\sin x}\cos x\,dx=\int e^u du=e^u+C=e^{\sin x}+C.$

37. $u=e^x$, so $du=e^x dx$. So $\int\frac{e^x dx}{1+e^{2x}}=\int\frac{du}{1+u^2}=$

$\tan^{-1}u+C=\tan^{-1}e^x+C.$

38. $u=x^{1/3}$, so $du=\frac{1}{3}x^{-2/3}dx$. So $\int\frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx=$

$\int 3e^u du=3e^u+C=3e^{\sqrt[3]{x}}+C.$

39. Let $u=e^x+4$, $du=e^x dx$. So $\int\frac{3e^x dx}{\sqrt{e^x+4}}=\int\frac{3\,du}{\sqrt{u}}=$

$6u^{\frac{1}{2}}+C=6\sqrt{e^x+4}+C.$

40. Let $u=e^{-3x}+7$, $du=-3e^{-3x}dx$. So $\int\frac{5e^{-3x}}{(e^{-3x}+7)^8}dx$

$=-\frac{5}{3}\int\frac{du}{u^8}=-\frac{5}{3(-7)}u^{-7}+C=\frac{5}{21(e^{-3x}+7)^7}+C.$

41. $u=\cot x$, $du=-\csc^2 x\,dx$. So $\int e^{\cot x}\csc^2 x\,dx$

$=-\int e^u du=-e^u+C=-e^{\cot x}+C.$

42. Let $u=\sec x$, $du=\sec x\tan x\,dx$. So

$\int e^{\sec x}\sec x\tan x\,dx=\int e^u du=e^u+C$

$=e^{\sec x}+C.$

43. $\int_0^1 e^{2x}dx=\frac{e^{2x}}{2}\Big|_0^1=\frac{e^2}{2}-\frac{1}{2}=\frac{e^2-1}{2}.$

44. $\int_0^{\ln 5}e^{-3x}dx=\frac{e^{-3x}}{-3}\Big|_0^{\ln 5}=\frac{e^{-3\ln 5}}{-3}-\frac{e^0}{-3}$

$=-\frac{1}{375}+\frac{1}{3}=\frac{125-1}{375}=\frac{124}{375}.$

45. Let $u=x^3$, so $du=3x^2dx$. So $\int_0^1 2x^2(e^{x^3}+1)dx$

$=\frac{2}{3}\int_0^1(e^u+1)du=\frac{2}{3}(e^u+u)\Big|_0^1=\frac{2}{3}\left[(e+1)-(1+0)\right]$

$=\frac{2}{3}e.$

46. $\int_1^2(1+e^{-x})^2dx=\int_1^2(1+2e^{-x}+e^{-2x})dx=$

$(x-2e^{-x}-\frac{1}{2}e^{-2x})\Big|_1^2=2-2e^{-2}-\frac{1}{2}e^{-4}-1+2e^{-1}+\frac{1}{2}e^{-2}$

$=\frac{e^{-4}}{2}(4e^4-4e^2-1-2e^4+4e^3+e^2)$

$=\frac{e^{-4}}{2}(2e^4+4e^3-3e^2-1)=\frac{2e^4+4e^3-3e^2-1}{2e^4}.$

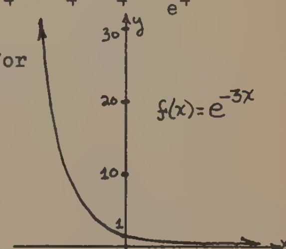
47. Let $u=\sin 2x$, $du=2\cos 2x\,dx$.

So $\int_0^{\pi/2}e^{\sin 2x}\cos 2x\,dx=\frac{1}{2}\int_0^0 e^u du=0.$

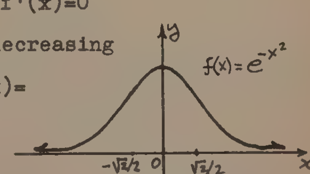
48. $\int_0^1\frac{3+e^{4x}}{e^{4x}}dx=\int_0^1(3e^{-4x}+1)dx=(\frac{3e^{-4x}}{-4}+x)\Big|_0^1$

$=-\frac{3}{4}e^{-4}+1+\frac{3}{4}=-\frac{3}{4}e^{-4}+\frac{7}{4}=\frac{1}{4}(7-\frac{3}{e^4}).$

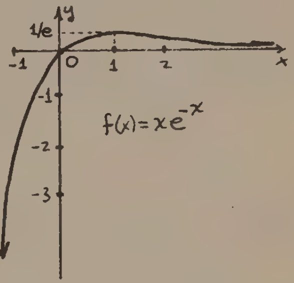
49. $f'(x)=-3e^{-3x}<0$ for all x, so f is decreasing on R. $f''(x)=9e^{-3x}>0$ for all x, so f is concave upward on R. $y=0$ is a horizontal asymptote.



50. $f'(x)=-2xe^{-x^2}$. $f'(x)=0$ for $x=0$. f is decreasing on $[0,\infty)$. $f''(x)=-2e^{-x^2}+4x^2e^{-x^2}$; $f''(0)<0$. So $(0,1)$ is a (relative) maximum. $f''(x)>0$ for $x\geq\frac{1}{\sqrt{2}}$ or $x\leq\frac{-1}{\sqrt{2}}$, so f is concave



upward on $\left[\frac{\sqrt{2}}{2},\infty\right)$ and $\left(-\infty,\frac{-\sqrt{2}}{2}\right]$; f is concave downward on $\left[\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]$. Inflection points are $\left(\frac{-\sqrt{2}}{2},0.61\right)$ and $\left(\frac{\sqrt{2}}{2},0.61\right)$ (approximate). $y=0$ is a horizontal asymptote.



51. $f'(x) = e^{-x}-xe^{-x}=0$ for $1-x=0$ or when $x=1$. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$. f is increasing on $(-\infty,1]$ and decreasing on $[1,\infty)$. Hence, f has an absolute maximum value of $\frac{1}{e}$ at 1. $f''(x)=-e^{-x}+xe^{-x}-e^{-x} = e^{-x}(x-2) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$. f is concave upward on $(2,\infty)$ and concave downward on $(-\infty,2)$. So $\left(2,\frac{2}{e^2}\right)\approx(2,0.27)$ is an inflection point. $y=0$ is an asymptote.

52. $A = \int_1^5(e^{2x}-x)dx = \left(\frac{e^{2x}}{2} - \frac{x^2}{2}\right)\Big|_1^5 = \frac{e^{10}}{2} - \frac{25}{2} - \frac{e^2}{2} + \frac{1}{2} = \frac{e^{10}-e^2-24}{2}$ square units.

53. $e^{2x}=e^{3x}$ for $x=0$. $A = \int_0^1(e^{3x}-e^{2x})dx = \left(\frac{e^{3x}}{3} - \frac{e^{2x}}{2}\right)\Big|_0^1 = \frac{e^3}{3} - \frac{e^2}{2} - \frac{1}{3} + \frac{1}{2} = \frac{e^3}{3} - \frac{e^2}{2} + \frac{1}{6} = \frac{2e^3-3e^2+1}{6}$ square units.

54. $V = \pi\int_0^2(e^{2x})^2dx = \pi\int_0^2 e^{4x}dx = \frac{\pi e^{4x}}{4}\Big|_0^2 = \frac{\pi}{4}(e^8-1)$ cubic units.

55. We want to show that $\exp(x-y)=\frac{\exp x}{\exp y}$. Let $A = \exp x$, $B = \exp y$. Now $\ln A = x$ and $\ln B = y$. So $\ln A-\ln B = x-y$; that is, $\ln(\frac{A}{B})=x-y$. So it follows that $\exp(x-y)= \exp(\ln \frac{A}{B}) = \frac{A}{B} = \frac{\exp x}{\exp y}$.

56. $S = \int_0^1 \sqrt{1+(y')^2}\,dx$ where $y' = \frac{e^x-e^{-x}}{2}$.

So $S = \int_0^1 \sqrt{1+\frac{e^{2x}-2+e^{-2x}}{4}}\,dx$

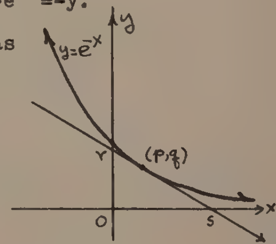
$= \int_0^1 \frac{1}{2}\sqrt{e^{2x}+2+e^{-2x}}\,dx$

$= \int_0^1\frac{1}{2}(e^x+e^{-x})dx = \frac{1}{2}(e^x-e^{-x})\Big|_0^1$

$= \frac{1}{2}\left[(e-\frac{1}{e})-(1-1)\right] = \frac{1}{2}\left(\frac{e^2-1}{e}\right)$ units.

57. $f'(x)=e^x-1 \geq 0$ for $e^x \geq 1$; that is, $\ln e^x \geq \ln 1$ for $x \ln e \geq 0$ or for $x \geq 0$. Also $f'(x) = e^x-1 \leq 0$ when $e^x \leq 1$ or when $\ln e^x \leq \ln 1$, that is, for $x \leq 0$. So $f'(x) \geq 0$ if $x \geq 0$; $f'(x) \leq 0$ if $x \leq 0$. Now, take $x \geq 0$. Since f is increasing for $x \geq 0$, $f(x) \geq f(0)$, that is, $e^x-1-x \geq e^0-1-0$, that is, $e^x-1-x \geq 0$ or $e^x \geq 1+x$; since $x \geq 0$, $-x \leq 0$, so that, since f is decreasing, $f(-x) \geq f(0)$, that is, $e^{-x}-1+x \geq e^0-1-0$, or $e^{-x}-1+x \geq 0$, or $e^{-x} \geq 1-x$.

58. $y=e^{-x}$, so that $y'=-e^{-x}=-y$. The tangent line has equation $y-q = (-q)(x-p)$, so $y= -qx+q(p+1)$. Hence,



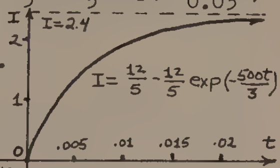
$$\begin{cases} r=q(p+1) \\ s=p+1 \end{cases}$$

So $r=qs$. Now $q=e^{-p}$ and $p=s-1$, so that $q=e^{1-s}$ and so $r=e^{1-s}(s)$. Now $\frac{dr}{dt} = (e^{1-s}-se^{1-s})\frac{ds}{dt}$, so that $\frac{dr}{dt}=(e^{-9}-10e^{-9})(5) = -9e^{-9}(5) = \frac{-45}{e^9}$ units per second $\approx$ 0.0056 unit per second.

59. $\frac{dV}{dt} = 10{,}000(0.1)e^{-0.1t}=1000e^{-0.1t}$. When $t=5$, $\frac{dV}{dt} = \frac{1000}{e^{0.5}} \approx \$606.53$/year.

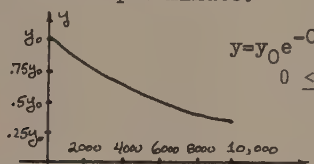
60. $I = \frac{E}{R} - \frac{E}{R}\exp(-\frac{Rt}{L})$. For $E=12$, $R=5$, and

$L=0.03$, we have $I = \frac{12}{5} - \frac{12}{5}\exp(-\frac{5}{0.03}t) = \frac{12}{5} - \frac{12}{5}\exp(-\frac{500}{3}t)$.



61. $\frac{dP}{dt} = -0.0060e^{-0.0004h}(\frac{dh}{dt})$. When $h=10{,}000$ and $\frac{dh}{dt} = 1000$, then $\frac{dP}{dt} = -0.0060e^{-4}(1000)$ so $\frac{dP}{dt} = -6e^{-4} = -\frac{6}{e^4}$ pound per square inch per minute ≈ -0.1098938 pound per square inch per minute.

62. (a)



$y=y_0e^{-0.0001212t}$, $0 \le t \le 10{,}000$

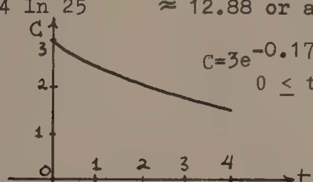
(b) $y = 10e^{-0.0001212(10{,}000)}$

$= 2.976014809$ grams.

63. $P = 3(10{,}000)(1-e^{-0.25t})-300t-500$.

$\frac{dP}{dt} = 30{,}000(0.25)(e^{-0.25t})-300 = 0$ provided $25e^{-0.25t}=1$ or when $e^{-0.25t}=\frac{1}{25}$; that is, $-0.25t = \ln\frac{1}{25}$, so $-0.25t=-\ln 25$ or $t=4\ln 25 \approx 12.88$ or about 13 days.

64. (a)



$C=3e^{-0.173t}$, $0 \le t \le 4$

(b) $C = 3e^{-0.173(4)}=1.501721758$ milligrams per liter.

Problem Set 7.7, page 450

1. (a) $2^{\sqrt{2}} = 2.665144143$.

(b) $2^{-\sqrt{2}} = 0.37521442272$.

(c) $2^{\pi} = 8.824977827$.

(d) $2^{-\pi} = 0.1133147323$.

(e) $\sqrt{2}^{\sqrt{2}} = 1.63256919$.

(f) $\pi^{\pi} = 36.46215961$.

(g) $\sqrt{3}^{-\sqrt{5}} = 0.2927940321$.

(h) $3.0157^{2.7566} = 20.96434682$.

2. (Intermediate steps are given for check.)

(a) $3.074^{2.183} \times 3.074^{1.075} = 11.60533104 \times 3.344118458 = 38.80960174$, whereas $3.074^{2.183+1.075} = 38.80960175$.

(b) $2.471^{5.507} \times 2.471^{0.012} = 145.7304174 \times 1.010914610 = 147.3210081$, whereas $2.471^{5.507+0.012} = 147.3210080$.

(c) $(1.777^{-2.058})^{3.333} = (0.30629733741)^{3.333} = 0.0193782936$;

$1.777^{(-2.058)(3.333)}=1.777^{-6859314}$

$= 0.0193782936$.

(d) $\sqrt{2}^{\sqrt{5}-\sqrt{3}}=\sqrt{2}^{0.5040171690}=1.190863935$;

$\frac{\sqrt{2}^{\sqrt{5}}}{\sqrt{2}^{\sqrt{3}}} = \frac{2.170509877}{1.822634654} = 1.190863935$.

(e) $(\sqrt{7}\pi)^{\sqrt{\pi}} = 42.67011803$;

$(\sqrt{7})^{\sqrt{\pi}}\pi^{\sqrt{\pi}} =(5.609816271)(7.606330756)$

$= 42.67011804$.

(f) $(\frac{2+\pi}{\sqrt{2}-1})^{\sqrt{5}-\sqrt{3}} = (12.41290273)^{0.5040171690}$

$= 3.559024275$;

$\frac{(2+\pi)^{\sqrt{5}-\sqrt{3}}}{(\sqrt{2}-1)^{\sqrt{5}-\sqrt{3}}} = \frac{2.282471864}{0.6413195550}$

$= 3.559024275$.

3. $f(x) = x^{-3\pi} = e^{-3\pi\ln x}$.

$f'(x) = e^{-3\pi\ln x}(-\frac{3\pi}{x})=(-\frac{3\pi}{x})x^{-3\pi}=3\pi x^{-3\pi-1}$.

4. $g(t) = t^{\pi-2}=e^{(\pi-2)\ln t}$.

$g'(t)=e^{(\pi-2)\ln t}(\frac{\pi-2}{t})=\frac{\pi-2}{t}(t^{\pi-2})=(\pi-2)t^{-3}$.

5. $h'(x)=6^{-5x}\ln 6(-5)=-5(\ln 6)6^{-5x}$.

6. $f'(x) = 2^{7x^2}\ln 2(14x) = 14x(\ln 2)2^{7x^2}$.

7. $g'(x) = 3^{2x+1}\ln 3(2) = 2(\ln 3)3^{2x+1}$.

8. $h'(t) = 3e(t^2+1)^{3e-1}(2t) = 6et(t^2+1)^{3e-1}$.

9. $G'(t) = 5^{\sin t}\ln 5(\cos t) = \cos t(\ln 5)5^{\sin t}$.

10. $H'(x) = 3^{\cos x^2}(\ln 3)(-\sin x^2)(2x)$

$= -2x\ln 3\sin x^2(3^{\cos x^2})$.

11. $h'(x) = (x^2+5)2^{-7x^2}(\ln 2)(-14x)+(2x)(2^{-7x^2})$

$= 2^{-7x^2}(x)\left[2-14\ln 2(x^2+5)\right]$.

12. $g'(t) = \sin t\, 3^{5t^2}(\ln 3)(10t)+3^{5t^2}(\cos t)$

$= 3^{5t^2}\left[(\ln 3)10t\sin + \cos t\right]$.

13. $f'(x) = \dfrac{(x^2+5)\ln 2(2^{x+1})-2^{x+1}(2x)}{(x^2+5)^2}$

$= \dfrac{2^{x+1}[(x^2+5)(\ln 2)-2x]}{(x^2+5)^2}$

14. $h'(x) = 2^{5x}(-\csc^2 x)+\ln 2(2^{5x})(5)\cot x$

$= 2^{5x}\left[5\ln 2(\cot x)-\csc^2 x\right]$.

15. $g'(x) = \dfrac{1}{5^x+5^{-x}}(5^x\ln 5-5^{-x}\ln 5)$

$= \dfrac{\ln 5(5^x-5^{-x})}{5^x+5^{-x}}$.

16. $h(t) = 7^{t/3}$

$h'(t) = 7^{t/3}\ln 7(\frac{1}{3}) = \frac{1}{3}(\ln t)7^{t/3}$.

17. $f'(x) = \dfrac{(3^x+1)3^x\ln 3-(3^x-1)3^x\ln 3}{(3^x+1)^2}$

$= \dfrac{3^x\ln 3(3^x+1-3^x+1)}{(3^x+1)^2} = \dfrac{2(3^x\ln 3)}{(3^x+1)^2}$.

18. $F'(x) = 4^{-3x}(\frac{1}{x^2+8})(2x)+\ln(x^2+8)\left[4^{-3x}\ln 4(-3)\right]$

$= 4^{-3x}\left[\dfrac{2x}{x^2+8} - 3\ln 4\ln(x^2+8)\right]$.

19. $h'(t) = \dfrac{1+t}{2t(\ln 10)}\cdot\dfrac{2}{(1+t)^2} = \dfrac{1}{t(\ln 10)(1+t)}$.

20. $f'(t) = \dfrac{1}{(\ln\cos t)(\ln 5)}\cdot\dfrac{1}{\cos t}(-\sin t)$

$= -\dfrac{\tan t}{(\ln\cos t)(\ln 5)}$.

21. $F'(u) = 3^{\tan u}\cdot\dfrac{1}{u\ln 8} +$

$3^{\tan u}(\ln 3)(\sec^2 u)\cdot\log_8 u =$

$3^{\tan u}[(\ln 3)(\sec^2 u)\log_8 u + \dfrac{1}{u\ln 8}]$.

22. $g'(t) = \dfrac{1}{2\sqrt{\log_5 t}}\cdot\dfrac{1}{(\ln 5)(t)}$.

23. $f'(x) = \dfrac{(x+2)\dfrac{2x}{(\ln 3)(x^2+5)} - \log_3(x^2+5)}{(x+2)^2}$

$\dfrac{2x(x+2)-\log_3(x^2+5)(\ln 3)(x^2+5)}{(\ln 3)(x^2+5)(x+2)^2}$.

24. $F'(x) = \csc x\cdot\dfrac{4x^3}{(\ln 3)(x^4+1)} +$

$\log_3(x^4+1)(-\csc x\cot x) =$

$\csc x\left[\dfrac{4x^3}{(\ln 3)(x^4+1)}-\cot x\log_3(x^4\right.$

25. $x2^y\ln 2\dfrac{dy}{dx} + 2^y + 2y\dfrac{dy}{dx} = 0$, so

$\dfrac{dy}{dx} = \dfrac{-2^y}{4(2^y)\ln 2 + 2y}$.

26. $3^{xy}\ln 3(x\frac{dy}{dx} + y) = 2x$,

$x\dfrac{dy}{dx} = \dfrac{2x}{3^{xy}\ln 3} - y$, so $\dfrac{dy}{dx} = \dfrac{2}{3^{xy}\ln 3} - \dfrac{y}{x}$.

27. $\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$, $\dfrac{1}{y}\dfrac{dy}{dx} = \sqrt{x}(\frac{1}{x})+\dfrac{\ln}{2\sqrt{x}}$

$\dfrac{dy}{dx} = x^{\sqrt{x}}(\dfrac{2+\ln x}{2\sqrt{x}})$.

28. $\ln y = x\ln(\cos x)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x(-\sin x)}{\cos x} +$

$\ln(\cos x)$, $\dfrac{dy}{dx} = (\cos x)^x\left[\ln(\cos x) -\right.$

$\left.x\tan x\right]$.

29. $\ln y = \ln(\sin x^2)^{3x} = 3x\ln(\sin x^2)$,

$\dfrac{1}{y}\dfrac{dy}{dx} = 3\ln(\sin x^2)+\dfrac{3x(\cos x^2)(2x)}{\sin x^2}$,

$\dfrac{dy}{dx} = (\sin x^2)^{3x}\left[3\ln(\sin x^2)+6x^2\cot x^2\right]$.

30. $\ln y = \ln(x+1)^x = x\ln(x+1)$,

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x}{x+1} + \ln(x+1)$,

$\dfrac{dy}{dx} = (x+1)^x\left[\dfrac{x}{x+1} + \ln(x+1)\right]$.

31. $\ln y = \ln(x^2+4)^{\ln x} = (\ln x)\ln(x^2+4)$,

$\dfrac{1}{y}\dfrac{dy}{dx} = (\ln x)(\dfrac{2x}{x^2+4})+ \dfrac{\ln(x^2+4)}{x}$,

$\frac{dy}{dx} = (x^2+4)^{\ln x}\left[(\ln x)(\frac{2x}{x^2+4})+\frac{\ln(x^2+4)}{x}\right].$

32. $\ln y = \cos x \ln(\sin x),$

$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{(\cos x)}{\sin x} - \sin x \ln(\sin x),$

$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\cos x \cot x - \sin x\ln(\sin x)\right].$

33. $\ln y = \ln\left[(x^2+7)^2(6x^3+1)^4\right]$

$= 2\ln(x^2+7)+4\ln(6x^3+1),$

$\frac{1}{y}\frac{dy}{dx} = \frac{2(2x)}{x^2+7} + \frac{4(18x^2)}{6x^3+1},$

$\frac{dy}{dx} = (x^2+7)^2(6x^3+1)^4\left[\frac{4x}{x^2+7} + \frac{72x^2}{6x^3+1}\right]=$

$(6x^3+1)^3(x^2+7)\left[4x(6x^3+1)+72x^2(x^2+7)\right] =$

$(6x^3+1)^3(x^2+7)(4x)(24x^3+126x+1).$

34. $\ln y = \ln\left[x^2\sin x^3\cos(3x+7)\right] =$

$2\ln x+\ln(\sin x^3)+\ln\cos(3x+7),$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{(\cos x^3)(3x^2)}{\sin x^3} + \frac{-\sin(3x+7)(3)}{\cos(3x+7)},$

$\frac{dy}{dx} = x^2\sin x^3\cos(3x+7)\left[\frac{2}{x} + \frac{3x^2\cos x^3}{\sin x^3} - \frac{3\sin(3x+7)}{\cos(3x+7)}\right]$

$= 2x\sin x^3\cos(3x+7)+3x^4\cos(3x+7)\cos x^3-$
$3x^2\sin x^3\sin(3x+7).$

35. $\ln y = \ln\left[\frac{(\sin x)\sqrt[3]{1+\cos x}}{\sqrt{\cos x}}\right]=$

$\ln(\sin x) + \frac{1}{3}\ln(1+\cos x)-\frac{1}{2}\ln\cos x,$

$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{-\sin x}{3(1+\cos x)} + \frac{\sin x}{2\cos x}.$

$\frac{dy}{dx} = \frac{(\sin x)\sqrt[3]{1+\cos x}}{\sqrt{\cos x}}\left[\cot x-\frac{\sin x}{3(1+\cos x)} + \frac{1}{2}\tan x\right].$

36. $\ln y = \ln\frac{\tan^2 x}{\sqrt{1-4\sec x}}=\ln(\tan^2 x)-\frac{1}{2}\ln(1-4\sec x),$

$\frac{1}{y}\frac{dy}{dx} = \frac{2\tan x\sec^2 x}{\tan^2 x} - \frac{(-4\sec x\tan x)}{2(1-4\sec x)},$

$\frac{dy}{dx} = \frac{\tan^2 x}{\sqrt{1-4\sec x}}(\frac{2\sec^2 x}{\tan x} + \frac{2\sec x\tan x}{1-4\sec x}).$

37. $\ln y = \ln(\frac{x^2\sqrt[5]{x^2+7}}{\sqrt[4]{11x+8}})$

$= 2\ln x+\frac{1}{5}\ln(x^2+7)-\frac{1}{4}\ln(11x+8),$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{2x}{5(x^2+7)} - \frac{11}{4(11x+8)},$

$\frac{dy}{dx} = \frac{x^2\cdot\sqrt[5]{x^2+7}}{\sqrt[4]{11x+8}}\left[\frac{2}{x} + \frac{2x}{5(x^2+7)} - \frac{11}{4(11x+8)}\right].$

38. $\ln y = \ln\sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}} =$

$\frac{1}{2}\left[\ln(\sec x + \tan x)-\ln(\sec x-\tan x)\right],$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x} - \frac{\sec x\tan x - \sec^2 x}{\sec x-\tan x}\right],$

$\frac{dy}{dx} = \sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}}(\frac{1}{2})(\sec x+\sec x)$

$= \sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}}(\sec x).$

39. Let $u = 5x$, $du = 5dx$. So $\int 3^{5x}dx =$

$\frac{1}{5}\int 3^u du = \frac{1}{5}\frac{3^u}{\ln 3} + C = \frac{3^{5x}}{5\ln 3} + C.$

40. Let $u = \ln x^2$, $du = \frac{2x\,dx}{x^2} = \frac{2}{x}dx.$

So $\int \frac{5^{\ln x^2}}{x}dx = \frac{1}{2}\int 5^u du = \frac{1}{2}(\frac{5^u}{\ln 5})+C$

$= \frac{5^{\ln x^2}}{2\ln 5} + C.$

41. $u=x^4+4x^3$, $du=(4x^3+12x^2)dx=4(x^3+3x^2)dx.$

So $\int 7^{x^4+4x^3}(x^3+3x^2)dx = \frac{1}{4}\int 7^u du =$

$\frac{1}{4\ln 7}\cdot 7^u+C = \frac{7^{x^4+4x^3}}{4\ln 7} + C.$

42. Let $u=\tan x$, $du=\sec^2 x\,dx$.

So $\int 3^{\tan x}\sec^2 x\,dx = \int 3^u du=\frac{3^u}{\ln 3} + C$

$= \frac{3^{\tan x}}{\ln 3} + C.$

43. $u=\ln x$, $du=\frac{1}{x}dx$. So $\int \frac{2^{\ln(\frac{1}{x})}}{x}dx =$

$\int 2^{-u}du = \frac{-2^{-u}}{\ln 2} + C = \frac{-2^{-\ln x}}{\ln 2} + C.$

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44. Let u=sec x, du=sec x tan x dx.
So $\int 8^{\sec x}\sec x \tan x\, dx = \int 8^u du = \frac{8^u}{\ln 8} + C = \frac{8^{\sec x}}{\ln 8} + C.$

45. Let u=cot x, du=-$\csc^2 x$ dx.
So $\int 4^{\cot x}\mathrm{cxc}^2 x\, dx = -\int 4^u du = \frac{-4^u}{\ln 4} + C = \frac{-4^{\cot x}}{\ln 4} + C.$

46. Let u=x ln x, du=(ln x+1)dx.
So $\int 2^{x \ln x}(1+\ln x)dx = \int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{x \ln x}}{\ln 2} + C.$

47. Let u=-2x, du=-2dx. So $\int_0^1 5^{-2x}dx = -\frac{1}{2}\int_0^{-2} 5^u du = \frac{-1}{2 \ln 5} 5^u \Big|_0^{-2} = -\frac{1}{2(\ln 5)(25)} + \frac{1}{2 \ln 5} = \frac{24}{50 \ln 5} = \frac{12}{25 \ln 5}.$

48. Let u=sin x, du=cos x dx.
So $\int 3^{\sin x}\cos x\, dx = \int 3^u du = \frac{3^u}{\ln 3} + C = \frac{3^{\sin x}}{\ln 3} + C.$ Hence, $\int_0^{\pi/2} 3^{\sin x}\mathrm{cox}\, x\, dx = \frac{3^{\sin x}}{\ln 3}\Big|_0^{\pi/2} = \frac{3^{\sin\frac{\pi}{2}}}{\ln 3} - \frac{3^0}{\ln 3} = \frac{2}{\ln 3}.$

49. (a) $\log_2 25 = \frac{\ln 25}{\ln 2} = 4.64385619.$

(b) $\log_3 2 = \frac{\ln 2}{\ln 3} = 0.6309297536.$

(c) $\log_8 e = \frac{\ln e}{\ln 8} = \frac{1}{\ln 8} = 0.480898347.$

(d) $\log_\pi 5 = \frac{\ln 5}{\ln \pi} = 1.405954306.$

(e) $\log_{\sqrt{2}} 0.07301 = \frac{\ln 0.07301}{\ln \sqrt{2}} = -7.55152422.$

50. $\log x = \log_{10} x = \frac{\ln x}{\ln 10}$, so ln x = (ln 10)(log x). Let M = ln 10. Hence, ln x = M log x, x > 0 .

51. $e^\pi \approx 23.14$ and $\pi^e \approx 22.46$. Hence, $e^\pi > \pi^e$.

52. Let $f(x) = \frac{\ln x}{x}$. $f'(x) = \frac{1-\ln x}{x^2} = 0$ provided ln x = 1; that is, f'(x) = 0 for x = e. Now f"(x) =
$$\frac{x^2(-\frac{1}{x})-(1-\ln x)(2x)}{x^4} = \frac{-3+2\ln x}{x^3}.$$
So $f''(e) = \frac{-3+2}{e^3} = \frac{1}{e^3} < 0$. Hence, f(e) = $\frac{1}{e}$ is a maximum value. Hence, $\frac{\ln x}{x} \le \frac{1}{e}$, so that $\frac{\ln \pi}{\pi} < \frac{1}{e}$ and $e \ln \pi < \pi$; so we have $e^{e \ln \pi} < e^\pi$, that is, $\pi^e < e^\pi$.

53. $\frac{b^x}{b^y} = \frac{e^{x \ln b}}{e^{y \ln b}} = e^{x\cdot\ln b - y\cdot\ln b} = e^{(x-y)\ln b} = b^{x-y}.$

54. $b^{-x} = e^{-x \ln b} = (e^{x \ln b})^{-1} = \frac{1}{e^{x \ln b}} = \frac{1}{b^x}.$

55. $\ln b^x = \ln e^{x\cdot\ln b} = x \ln b$ if $b > 0$.

56. $\ln[f(x)] = \frac{1}{x}\ln x$, so that $\frac{1}{f(x)}f'(x) = \frac{1-\ln x}{x^2}$ and $f'(x) = x^{\frac{1}{x}}(\frac{1-\ln x}{x^2}) = 0$ provided ln x = 1 or x = e. Now
$$f''(x) = x^{\frac{1}{x}}\left[\frac{-x-(1-\ln x)(2x)}{x^4}\right] + \left[\frac{(1-\ln x)}{x^2}\right]^2 x^{\frac{1}{x}},$$
so that $f''(e) = \frac{e^{1/e}(-e)}{e^4} + 0 = \frac{e^{1/e}}{e^3} < 0,$ and so f(e) is a maximum. Hence, f(x) = $x^{1/x}$, x > 0, has a maximum for x = e.

57. $\log_a b = \frac{\ln b}{\ln a}$ and $\log_b a = \frac{\ln a}{\ln b}$. So $(\log_a b)(\log_b a) = 1$. Hence, $\log_a b = \frac{1}{\log_b a}$.

58. By Theorem 6, $D_x \log_a u = \frac{1}{u \ln a} D_x u$ and by Problem 57, $\ln a = \log_e a =$

$\frac{1}{\log_a e}$. Hence, $D_x\left[\log_a u\right] = \frac{1}{u\left(\frac{1}{\log_a e}\right)} D_x u$

$= \frac{\log_a e}{u} D_x u.$

59. (a) $\log_a xy = \frac{\ln xy}{\ln a} = \frac{1}{\ln a}(\ln x + \ln y) =$

$\frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} = \log_a x + \log_a y.$

(b) $\log_a \frac{x}{y} = \frac{\ln \frac{x}{y}}{\ln a}$

$= \frac{\ln x}{\ln a} - \frac{\ln y}{\ln a} = \log_a x - \log_a y.$

(c) $\log_a x^u = \frac{\ln x^u}{\ln a} = \frac{u \ln x}{\ln a} = u \log_a x.$

60. $D_x(\log_x a) = D_x\left(\frac{1}{\log_a x}\right)$ by Problem 57.

But $D_x\left(\frac{1}{\log_a x}\right) = \frac{-\frac{1}{x \ln a}}{(\log_a x)^2}$. Hence,

$D_x(\log_x a) = \frac{-1}{x \ln a (\log_a x)^2}.$

61. $V = \int_0^2 \pi(3^x)^2 dx = \pi \int_0^2 3^{2x} dx$

$= \pi \int_0^2 e^{2x \ln 3} dx = \pi \left.\frac{e^{2x(\ln 3)}}{2 \ln 3}\right|_0^2$

$= \left.\frac{\pi}{2 \ln 3} 3^{2x}\right|_0^2 = \frac{\pi}{2 \ln 3}(3^4 - 3^0)$

$= \frac{80\pi}{2(\ln 3)} = \frac{40\pi}{\ln 3}$ cubic units.

62. Let $Y = \log_b y$ and $X = \log_b x$.

$\lim_{\Delta x \to 0} \frac{\left(\frac{\Delta Y}{y}\right)}{\left(\frac{\Delta X}{x}\right)} = \lim_{\Delta x \to 0} \frac{x}{y} \cdot \frac{\Delta y}{\Delta x} = \frac{x}{y}\left(\frac{dy}{dx}\right).$

Also, $\frac{dY}{dx} = \frac{d}{dx}(\log_b y) = \frac{1}{y \ln b}\left(\frac{dy}{dx}\right)$ and

$\frac{dX}{dx} = \frac{1}{x \ln b}$; hence, $\frac{dY}{dX} = \frac{\left(\frac{dY}{dx}\right)}{\left(\frac{dX}{dx}\right)} = \frac{\frac{1}{y \ln b}\left(\frac{dy}{dx}\right)}{\frac{1}{x \ln b}}$

$= \frac{x}{y}\left(\frac{dy}{dx}\right).$

Problem Set 7.8, page 456

1. $\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -\frac{e^x - e^{-x}}{2}$

$= -\sinh x.$

2. $\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x.$

3. $\sinh x \cosh y + \sinh y \cosh x =$

$\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^y - e^{-y}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) =$

$\frac{1}{4}(e^{x+y} - e^{-x+y} + e^{x-y} - e^{-x-y} + e^{y+x} - e^{-y+x} + e^{y-x} - e^{-y-x}) =$

$\frac{1}{4}(2e^{x+y} - 2e^{-(x+y)}) = \frac{e^{x+y} - e^{-(x+y)}}{2}$

$= \sinh(x+y).$

4. $\cosh x \cosh y + \sinh x \sinh y =$

$\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) =$

$\frac{1}{4}(e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y})$

$= \frac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \frac{e^{x+y} + e^{-(x+y)}}{2}$

$= \cosh(x+y).$

5. We use the identity $\cosh^2 x - \sinh^2 x = 1$ (proved in text). Divide both sides by $\cosh^2 x$; so $1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$ or

$1 - \left(\frac{\sinh x}{\cosh x}\right)^2 = \left(\frac{1}{\cosh x}\right)^2.$

Thus, $1 - \tanh^2 x = \operatorname{sech}^2 x.$

6. Divide the identity $\cosh^2 x - \sinh^2 x = 1$ on both sides by $\sinh^2 x$ to obtain

$\left(\frac{\cosh x}{\sinh x}\right)^2 - 1 = \frac{1}{\sinh^2 x}$; that is,

$\coth^2 x - 1 = \operatorname{csch}^2 x.$

7. $\sinh x + \cosh x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}$

$= \frac{2e^x}{2} = e^x.$

8. $\sinh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{x - x^{-1}}{2}$

$= \frac{x - \frac{1}{x}}{2} = \frac{x^2 - 1}{2x}.$

9. (a) $\sinh 1.2 = \frac{e^{1.2} - e^{-1.2}}{2} = 1.509461355.$

(b) $\cosh(-1.4) = \frac{e^{-1.4} + e^{1.4}}{2} = 2.150898465.$

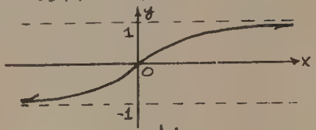
(c) $\tanh 0.7 = \frac{e^{0.7} - e^{-0.7}}{e^{0.7} + e^{-0.7}} = 0.6043677771.$

(d) $\coth 1.3 = 1.160465504$.

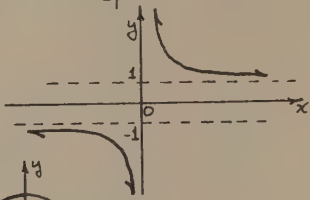
(e) $\operatorname{sech} 0.6 = 0.8435506876$.

(f) $\operatorname{csch}(-0.9) = 0.9741682480$.

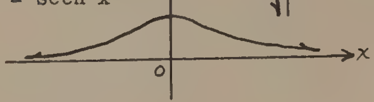
10. (a) $y = \tanh x$



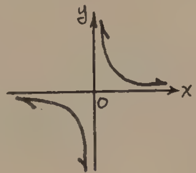
(b) $y = \coth x$



(c) $y = \operatorname{sech} x$



(d) $y = \operatorname{csch} x$



11. $f'(x) = 6x\cosh(3x^2+5)$.

12. $g'(x) = \dfrac{\sinh(\ln x)}{x}$.

13. $f'(t) = \dfrac{1}{\sinh t^3}(\cosh t^3)(3t^2)$

$= 3t^2\coth t^3$.

14. $f'(u) = e^{2u}\operatorname{sech}^2 u + 2e^{2u}\tanh u$.

15. $h'(t) = \operatorname{sech}^2 e^{3t}(e^{3t})(3) = 3e^{3t}\operatorname{sech}^2 e^{3t}$.

16. $F'(r) = \sin^{-1} r \operatorname{sech}^2(3r+5)(3) + \dfrac{\tanh(3r+5)}{\sqrt{1-r^2}}$.

17. $G'(s) = \dfrac{1}{\sqrt{1-\tanh^2 s}}\operatorname{sech}^2 s = \dfrac{\operatorname{sech}^2 s}{\sqrt{\operatorname{sech}^2 s}}$

$= \dfrac{\operatorname{sech}^2 s}{|\operatorname{sech} s|} = \operatorname{sech} s$.

18. $f'(x) = \dfrac{1}{1+\tanh^2 x}(\operatorname{sech}^2 x) = \dfrac{\operatorname{sech}^2 x}{1+\tanh^2 x}$.

19. $2x = \cosh y \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = 2x \operatorname{sech} y$.

20. $\cosh x = \sinh y \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \cosh x \operatorname{csch} y$.

21. $\cos x = \cosh y \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{\cos x}{\cosh y}$.

22. $2\tanh x \operatorname{sech}^2 x - 2\cosh y \dfrac{dy}{dx} = \operatorname{sech}^2 y \dfrac{dy}{dx}$,

$\dfrac{dy}{dx} = \dfrac{2\tanh x \operatorname{sech}^2 x}{\operatorname{sech}^2 y + 2\cosh y}$.

23. $u = 7x$, $du = 7dx$. $\int \cosh 7x\, dx$

$= \frac{1}{7}\int \cosh u\, du = \frac{1}{7}\sinh u + C = \dfrac{\sinh 7x}{7} + C$.

24. Let $u = \frac{5x}{3}$, $du = \frac{5}{3}dx$. So $\int \sinh \frac{5x}{3}dx$

$= \frac{3}{5}\int \sinh u\, du = \frac{3}{5}\cosh u + C = \frac{3}{5}\cosh \frac{5x}{3} + C$.

25. Let $u = 3x$, $du = 3dx$, Then

$\int \operatorname{sech}^2 3x\,dx = \frac{1}{3}\int \operatorname{sech}^2 u\, du = \frac{1}{3}\tanh u + C$

$= \frac{1}{3}\tanh 3x + C$.

26. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}dx$, so

$\int \dfrac{\operatorname{sech}^2\sqrt{x}}{\sqrt{x}}dx = 2\int \operatorname{sech}^2 u\, du = 2\tanh u + C$

$= 2\tanh\sqrt{x} + C$.

27. Let $u = \cosh 5x$, $du = 5\sinh 5x\, dx$.

So $\frac{1}{5}\int \dfrac{du}{u^3} = \frac{1}{5}\cdot\dfrac{u^{-2}}{-2} + C = -\dfrac{1}{10\cosh^2 5x} + C$

$= -\frac{1}{10}\operatorname{sech}^2 5x + C$.

28. $u = \sinh 3x$, $du = 3\cosh 3x\, dx$.

$\int \sinh^{10} 3x \cosh 3x\, dx = \frac{1}{3}\int u^{10} du$

$= \dfrac{u^{11}}{33} + C = \dfrac{\sinh^{11} 3x}{33} + C$.

29. Let $u = \cosh x$, $du = \sinh x\, dx$.

$\int_0^1 \cosh^3 x \sinh x\, dx = \int_{\cosh 0}^{\cosh 1} u^3 du =$

$\dfrac{u^4}{4}\Big|_{\cosh 0}^{\cosh 1} = \dfrac{\cosh^4 1 - \cosh^4 0}{4}$

$= \dfrac{\cosh^4 1 - 1}{4}$.

30. Let $u = \sinh x$, $du = \cosh x\, dx$.

$\int \sinh^4 \cosh x\, dx = \int u^4 du = \dfrac{u^5}{5} + C =$

$\dfrac{\sinh^5 x}{5} + C$. So $\int_0^2 \sinh^4 x \cosh x\, dx =$

$\dfrac{\sinh^5 x}{5}\Big|_0^2 = \dfrac{\sinh^5 2 - \sinh^5 0}{5} = \dfrac{\sinh^5 2}{5}$.

31. $\int \cosh(\ln x)dx = \int \frac{e^{\ln x}+e^{-\ln x}}{2}dx =$
$\int \frac{x+\frac{1}{x}}{2}\,dx = \frac{1}{2}(\frac{x^2}{2} + \ln|x|)+C = \frac{x^2}{4} + \frac{1}{2}\ln|x| + C.$

32. (a) $D_x\cosh u = D_x \frac{e^u+e^{-u}}{2} = \frac{D_x e^u+D_x e^{-u}}{2}$

$= \frac{e^u-e^{-u}}{2}D_x u = \sinh u\, D_x u.$

(b) $D_x \coth u = D_x \frac{1}{\tanh u}$

$= \frac{\tanh u(0)-1(\operatorname{sech}^2 u)D_x u}{\tanh^2 u} = \frac{\operatorname{sech}^2 u}{\tanh^2 u}D_x u$

$= -\frac{1}{\sinh^2 u}D_x u = -\operatorname{csch}^2 u\, D_x u.$

(c) $D_x\operatorname{sech} u = D_x \frac{1}{\cosh u}$

$= \frac{\cosh u(0)-1(\sinh u)}{\cosh^2 u}D_x u = -\frac{\sinh u}{\cosh^2 u}D_x u$

$= -\frac{\sinh u}{\cosh u}\cdot\frac{1}{\cosh u}D_x u$

$= -\tanh u\,\operatorname{sech} u\, D_x u.$

(d) $D_x\operatorname{csch} u = D_x \frac{1}{\sinh u}$

$= \frac{\sinh u(0)-1(\cosh u)D_x u}{\sinh^2 u} = -\frac{\cosh u}{\sinh^2 u}D_x u$

$= -\coth u\,\operatorname{csch} u\, D_x u.$

33. $\frac{dy}{dx} = Ak\cdot\cosh kx+Bk\cdot\sinh kx.$

$\frac{d^2y}{dx^2} = Ak^2\sinh kx+Bk^2\cosh kx = k^2y.$

Thus, $\frac{d^2y}{dx^2} - k^2y = 0.$

34. $A = \int_0^1 \sinh x\,dx = \cosh x\,\Big|_0^1$

$= \cosh 1-\cosh 0 = \frac{e+\frac{1}{e}}{2} - 1$

$= \frac{e^2+1-2e}{2e} = \frac{(e-1)^2}{2e}$ square units.

35. $f'(x) = \frac{1}{\sqrt{1+x^6}}D_x\, x^3 = \frac{3x^2}{\sqrt{1+x^6}}.$

36. $g'(x) = \frac{1}{\sqrt{\sec^2 x-1}}\,D_x\sec x = \frac{\sec x\tan x}{\sqrt{\tan^2 x}}$

$= \frac{\sec x\tan x}{|\tan x|}.$

37. $g'(x) = \frac{1}{\sqrt{(\frac{x}{3})^2-1}}\cdot(\frac{1}{3}) = \frac{1}{\sqrt{x^2-9}}.$

38. $F'(t) = \frac{1}{1-\sin^2 t}D_t\sin t = \frac{\cos t}{\cos^2 t} = \frac{1}{\cos t}$

$= \sec t.$

39. $G'(t) = \frac{1}{1-(5t)^2}\cdot(5) = \frac{5}{1-25t^2}.$

40. $H'(t) = \frac{1}{\sqrt{t^2-1}}\cdot\frac{(2t)}{2\sqrt{t^2-1}} - \frac{t}{1-t^2} - \tanh^{-1}t$

$= \frac{2t-(t^2-1)\tanh^{-1}t}{t^2-1}.$

41. $f'(x) = \frac{xe^x}{\sqrt{e^{2x}-1}} + \cosh^{-1}e^x.$

42. $g'(x) = x\frac{1}{\sqrt{1+x^2}} + \sinh^{-1}x - \frac{1(2x)}{2\sqrt{1+x^2}}$

$= \sinh^{-1}x.$

43. $h'(u) = u\cdot\frac{1}{1-(\ln u)^2}\cdot(\frac{1}{u})+\tanh^{-1}(\ln u)$

$= \frac{1}{1-(\ln u)^2} + \tanh^{-1}(\ln u).$

44. $F'(w) = \frac{(w^2+4)\frac{1}{\sqrt{w^2-1}} - \cosh^{-1}w(2w)}{(w^2+4)^2}$

$= \frac{w^2+4-2w\sqrt{w^2-1}\cosh^{-1}w}{(w^2+4)^2\sqrt{w^2-1}}.$

45. (a) $x = \frac{1}{2}(e^y-e^{-y})$

or $2x = e^y-e^{-y}$ or $2xe^y = e^{2y}-1$

or $e^{2y}-2xe^y-1 = 0.$

So $(e^y)^2-2xe^y-1 = 0.$

(b) Now $e^y = \frac{2x\pm\sqrt{4x^2-4(-1)}}{2} = x\pm\sqrt{x^2+1}.$

Since $e^y > 0$, then $e^y = x+\sqrt{x^2+1}.$

(c) From (b), $y = \ln(x+\sqrt{x^2+1})$. But $y = \sinh^{-1}x$, so $\sinh^{-1}x=\ln(x+\sqrt{x^2+1}).$

46. $y=\tanh^{-1}x$, so $x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y-e^{-y}}{e^y+e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$ by dividing each term by e^{-y}. Thus, $xe^{2y}+x = e^{2y}-1$ or

$e^{2y}(1-x) = 1+x$, so $e^{2y} = \frac{1+x}{1-x}$, $x \neq 1$.

So $2y = \ln(\frac{1+x}{1-x})$ provided $\frac{1+x}{1-x} > 0$ or $|x|<1$,

or $y = \frac{1}{2}\ln(\frac{1+x}{1-x})$ or $\tanh^{-1}x = \frac{1}{2}\ln(\frac{1+x}{1-x})$.

47. $D_x(\sinh^{-1}\frac{x}{a} + C) = \frac{1}{\sqrt{1+(\frac{x}{a})^2}}(\frac{1}{a})+0 = \frac{1}{\sqrt{a^2+x^2}}$

provided $a > 0$.

48. $D_x(\frac{1}{a}\tanh^{-1}\frac{x}{a} + C) = \frac{1}{a}\,\frac{1}{1-(\frac{x}{a})^2}(\frac{1}{a})+0$

$= \frac{1}{a^2-x^2}$ provided $|\frac{x}{a}|<1$, that is, $|x|<a$.

49. $D_x(\cosh^{-1}\frac{x}{a} + C) = \frac{1}{\sqrt{(\frac{x}{a})^2-1}}(\frac{1}{a})+0 = \frac{1}{\sqrt{x^2-a^2}}$

provided $\frac{x}{a} > 1$, that is, $x > a > 0$.

50. (a) $y = \cosh^{-1}u$, so $\cosh y = u$, where $u > 1$, $y > 0$. Since $\cosh u$ is differentiable and $\neq 0$, then $\cosh^{-1}u$ exists and is differentiable. Now $\sinh y\, D_xy = D_xu$. Hence, $D_xy = \frac{D_xu}{\sinh y}$, where $\sinh y > 0$. Now $\cosh^2 - \sinh^2 y = 1$, so $\sinh y = \sqrt{\cosh^2 y-1} = \sqrt{u^2-1}$, since $\sinh y > 0$ and $u > 1$. It follows that $D_x(\cosh^{-1}u) = D_xy = \frac{D_xu}{\sqrt{u^2-1}}$, $u > 1$.

(b) $y = \tanh^{-1}u$; then $\tanh y = u$, $|u|<1$. The derivative of $\tanh u$ exists and $\neq 0$, so $\tanh^{-1}u$ is differentiable. Now, $\operatorname{sech}^2 y\, D_xy = D_xu$ or $D_xy = \frac{D_xu}{\operatorname{sech}^2 y}$

$= \frac{D_xu}{1-\tanh^2 y} = \frac{D_xu}{1-u^2}$, $|u| < 1$.

51. Let $a = 3$. Then $\int\frac{dx}{\sqrt{9+x^2}} = \sinh^{-1}\frac{x}{3} + C$.

52. $A \int_0^{3\sqrt{2}} \frac{16}{\sqrt{16+x^2}}dx = 16\sinh^{-1}\frac{x}{4}\Big|_0^{3\sqrt{2}}$

$= 16\sinh^{-1}\frac{3\sqrt{2}}{4}$ square units.

(By Problem 45, $A = 16\ln(\frac{3\sqrt{2}}{4} + \sqrt{\frac{18}{16}+1})$

$= 16\ln(\frac{3\sqrt{2}+\sqrt{34}}{4})$ square units.)

53. $s = \int_{-b}^{b}\sqrt{1+(y')^2}dx = \int_{-b}^{b}\sqrt{1+[a(\frac{1}{a})\sinh\frac{x}{a}]^2}dx =$

$\int_{-b}^{b}\sqrt{1+\sinh^2\frac{x}{a}}dx = \int_{-b}^{b}\sqrt{\cosh^2\frac{x}{a}}dx =$

$\int_{-b}^{b}\cosh\frac{x}{a}\,dx$. Let $u=\frac{x}{a}$, $du=\frac{1}{a}dx$, so that

$\int\cosh\frac{x}{a}dx = a\int\cosh u\,du = a\sinh u+C =$

$a\sinh\frac{x}{a} + C$. Hence, $s = \int_{-b}^{b}\cosh\frac{x}{a}\,dx =$

$a\sinh\frac{x}{a}\Big|_{-b}^{b} = a[\sinh\frac{b}{a} - \sinh(-\frac{b}{a})] =$

$a[\sinh\frac{b}{a} + \sinh\frac{b}{a}] = 2a\sinh\frac{b}{a}$ units.

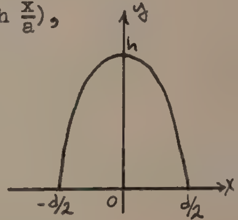
54 When $y=0$, $0=h+a(1-\cosh\frac{x}{a})$,

so $1-\cosh\frac{x}{a} = -\frac{h}{a}$

or $\cosh\frac{x}{a} = 1+\frac{h}{a}$.

So $\frac{x}{a} = \cosh^{-1}(1+\frac{h}{a})$

or $x=a\cosh^{-1}(1+\frac{h}{a})$.



But distance is twice this x intercept, so $d=2a\cosh^{-1}(1+\frac{h}{a})$ units.

Problem Set 7.9, page 465

1. $\lim_{h\to+\infty}(1+\frac{6}{h})^h = e^6$.

2. $\lim_{n\to+\infty}(1-\frac{2}{n})^n = e^{-2}$.

3. $\lim_{x\to 0^+}(1-5x)^{1/x} = \lim_{y\to+\infty}(1-\frac{5}{y})^y = e^{-5}$

where $y = \frac{1}{x}$.

4. $\lim_{n\to-\infty}(1-\frac{3}{n})^n = e^{-3}$.

5. $\lim_{n\to+\infty}(1+\frac{1}{n})^{4n} = \left[\lim_{n\to\infty}(1+\frac{1}{n})^n\right]^4$

$= (e^1)^4 = e^4$.

6. $\lim_{h\to+\infty}(1+\frac{3}{h})^{h/5} = \left[\lim_{h\to+\infty}(1+\frac{3}{h})^h\right]^{\frac{1}{5}}$

$= (e^3)^{1/5} = e^{3/5}$.

7. $\lim_{u\to 0^-} (1-4u)^{1/u} = \lim_{y\to -\infty} (1 - \frac{4}{y})^y$
$= e^{-4}$, where $y = \frac{1}{u}$.

8. $\lim_{u\to 0^+} (1 + \frac{u}{7})^{1/u} = \lim_{y\to +\infty} (1+y)^{1/y} = e$,
where $y = \frac{1}{u}$.

9. Let $f(h) = (1 + \frac{6}{h})^h$.
$f(10) \approx 109.95$. $f(100) \approx 339.302$.
$f(1000) \approx 396.2604$. $f(10,000) \approx 402.7036$.
$f(100,000) \approx 403.356$.
$f(1,000,000) \approx 403.421532$.
$f(10,000,000) \approx 403.428067$.
$f(100,000,000) \approx 403.428721$.
$e^6 \approx 403.42879$.

10. $P(k) = \lim_{n\to +\infty} \left[\frac{n(n-1)(n-2)\ \dots\ (n-k+1)}{k!}\right](\frac{c}{n})^k \cdot$
$(1-\frac{c}{n})^{n-k} =$
$\frac{c^k}{k!} \lim_{n\to +\infty} \left[\frac{n}{n}\ \frac{(n-1)}{n}\ \frac{(n-2)}{n} \dots \frac{(n-k+1)}{n}\right] \frac{(1-\frac{c}{n})^n}{(1-\frac{c}{n})^k} =$
$\frac{c^k}{k!} \lim_{n\to +\infty} (1)(1-\frac{1}{n})(1-\frac{2}{n})\dots(1- \frac{(k-1)}{n}) \cdot$
$\frac{\lim_{n\to+\infty}(1-\frac{c}{n})^n}{\lim_{n\to+\infty}(1-\frac{c}{n})^k} = \frac{c^k}{k!}\ \frac{e^{-c}}{1} = \frac{c^k \cdot e^{-c}}{k!}$.

11. (a) $R = (1+\frac{r}{n})^n-1 = (1+\frac{0.07}{1})^1-1 = 0.07 = 7\%$.
(b) $S = P(1+\frac{r}{n})^{nt} = 1000(1+\frac{0.07}{1})^{13}$
$\approx \$2,409.85$.

12. (a) $R = (1+\frac{r}{n})^n-1 = (1+\frac{0.07}{12})^{12}- 1$
$\approx 0.07229 = 7.23\%$.
(b) $S = P(1+\frac{r}{n})^{nt} = 1000(1+\frac{0.07}{12})^{156}$
$\approx \$2,477.76$.

13. (a) $R = (1+\frac{r}{n})^n-1 = (1+\frac{0.07}{52})^{52} - 1$
$\approx 0.072458 = 7.25\%$.
(b) $S = P(1+\frac{r}{n})^{nt} = 1000(1+\frac{0.07}{52})^{676}$
$= \$2,482.80$.

14. (a) $R = (1+\frac{r}{n})^n-1 = (1 + \frac{0.12}{12})^{12} - 1$
$\approx 0.1268 = 12.68\%$.
(b) $S = P(1 + \frac{r}{n})^{nt} = 1000(1 + \frac{0.12}{12})^{156}$
$\approx \$4,722.09$.

15. (a) $R = (1 + \frac{r}{n})^n-1 = (1 + \frac{0.135}{12})^{12}-1$
$\approx 0.143674 = 14.37\%$.
(b) $S = P(1 + \frac{r}{n})^{nt} = 50,000(1+\frac{0.135}{12})^6$
$= \$53,471.36$.

16. (a) $R = (1+\frac{r}{n})^n-1 = (1+\frac{0.155}{52})^{52}-1$
$\approx 0.167389 = 16.74\%$.
(b) $S = P(1+\frac{r}{n})^{nt} = 25,000(1+ \frac{0.155}{52})^{13}$
$\approx \$25,986.27$.

17. $S = P(1+\frac{r}{n})^{nt}$
$= Pe^{rt}$. $P = 1000$, $r = 0.08$, $t = 2$.
(a) $S = 1000(1+\frac{0.08}{1})^2 = \$1,166.40$.
(b) $S = 1000(1+\frac{0.08}{2})^4 \approx \$1,169.86$.
(c) $S = 1000(1+\frac{0.08}{4})^8 \approx \$1,171.66$.
(d) $S = 1000(1+\frac{0.08}{12})^{24} \approx \$1,172.89$.
(e) $S = 1000(1+\frac{0.08}{52})^{104} \approx \$1,173.37$.
(f) $S = 1000(1+\frac{0.08}{365})^{730} \approx \$1,173.49$.
(g) $S = 1000(1+\frac{0.08}{8760})^{17520} \approx \$1,173.51$.
(h) $S = 1000\ e^{0.08(2)} \approx \$1,173.51$.

18. $r = 7\%$, $n = 12$.
$R = (1+\frac{r}{n})^n-1 = (1+\frac{0.07}{12})^{12}-1$
$= 0.0723 = 7.23\%$.

19. $r = 0.08$, $S = 100$, $t = \frac{1}{2}$.
(a) $n = 4$, $P = S(1+\frac{r}{n})^{-nt} = 100(1+\frac{0.08}{4})^{-2}$
$\approx \$96.12$.

(b) $n=12, P=100(1+\frac{0.08}{12})^{-6} \approx \$96.09$.

(c) $n=52$, $P=100(1+\frac{0.08}{52})^{-26} \approx \$96.08$.

(d) $P=Se^{-rt}=100e^{-0.04} \approx \$96.08$.

20. $r = 0.07$

(a) $2P = P(1+\frac{0.07}{4})^{4t}$ or $2=(1+\frac{0.07}{4})^{4t}$ or

$4t \ln(1.0175)=\ln 2$, so

$t = \frac{\ln 2}{4 \ln(1.0175)} \approx 9.99$ years.

(b) $2P = P(1+\frac{0.07}{12})^{12t}$ or $2=(1+\frac{0.07}{12})^{12t}$

or $12t \ln(1+\frac{0.07}{12}) = \ln 2$,

so $t = \frac{\ln 2}{12 \ln(1+\frac{0.07}{12})} \approx 9.93$ years.

(c) $2P = Pe^{0.07t}$ or $2 = e^{0.07t}$

or $0.07t = \ln 2$,

so $t = \frac{\ln 2}{0.07} \approx 9.902$ years.

21. (a) $R = e^r-1 = e^{0.055}-1 \approx 0.05654062$

$= 5.65406\%$.

(b) When the bank compounds interest quarterly at 5.5% interest, the accounts will be worth $\$(28,000,000)(1+\frac{5.5}{100(4)})^4 \approx \$29,572,054.65$ at the end of the year, so that the interest paid out is \$1,572,054.65. If the bank compounds continuously at 5.5%, then, at the end of the year, the accounts will be worth $\$28,000,000e^{\frac{(5.5)}{100}}$ $\approx \$29,583,137.22$, so that the interest paid out is \$1,583,137.22. The bank would have to pay out \$11,082.57 more in interest with the new plan.

22. $r=0.07$, $S=50,000$, $t=5$.

$P=50,000e^{-(0.07)(5)} \approx \$35,234.40$.

23. (a) $2P=Pe^{r(9.9)}$ or $2=e^{9.9r}$ or $9.9r=\ln 2$,

so $r = \frac{\ln 2}{9.9} \approx 0.07 = 7\%$.

(b) $R=e^r-1 = e^{0.07}-1 \approx 0.0725 = 7.25\%$.

24. If x is the cost of a loaf of bread five years ago, then the present cost is $1.00 = (1.11)^5x$; so $x = (1.11)^{-5} \approx \$0.59$.

25. $S = 40,000$, $r = 0.08$, $t = 15$.

$P = 40,000e^{-(0.08)(15)} \approx \$12,047.77$.

26. (a) At the end of two months, the article would cost $(P+Pr)+(P+Pr)r = P(1+r)^2$; at the end of three months, it would cost $P(1+r)^2+P(1+r)^2r = P(1+r)^3$, and so forth. At the end of one year, it would cost $P(1+r)^{12}$. Thus, $P(1+r)^{12} = P(1+r)$, so $R = (1+r)^{12}-1$.

(b) If $r = 1\% = 0.01$, then $R = (1.01)^{12}- 1 \approx 0.126825$, or approximately 12.7%.

27. $\frac{dq}{dt} = 5q$, so $\frac{dq}{q} = 5dt$.

So $\ln q = 5t+c$ or $q = e^ce^{5t}$.

When $t=0$, $q=2$, so $e^c=2$.

Hence, $q=2e^{5t}$.

28. $\frac{dy}{y} = 2\,dx$ so $\ln y = 2x+C$ or $y = e^ce^{2x}$.

When $x=0$, $y=10$, so $e^c=10$. Hence, $y=10e^{2x}$

29. $\frac{dN}{N} = -4dt$ or $\ln N = -4t+C$ or $N=e^{-4t}e^c$.

When $t=0$, $N=40$, so $e^c=40$. Hence, $N=40e^{-4}$

30. $\frac{dy}{y} = -2dx$, so $\ln y = -2x+C$ or $y=e^ce^{-2x}$.

When $x=0$, $y=-10$, so e^c-10.

Hence, $y = -10e^{-2x}$.

31. $\frac{dq}{10-q} = dt$, so $-\ln(10-q) = t+C$

$\ln(10-q) = -t-C$

$10-q = e^{-t-c} = e^{-c}e^{-t}$

When $t=0$, $q=3$, so $e^{-c} = 7$. Thus, $10-q = 7e^{-t}$, and so $q=10-7e^{-t}$.

32. $dx = -0.2(80-t)dt$.

$x = -0.2(80t-\frac{t^2}{2})+C$

$= -16t + \frac{1}{10}t^2 + C.$

When $t=0$, $x=0$, so $C=0$.

Hence, $x = -16t + \frac{1}{10}t^2$.

3. $q = 1000e^{kt}$, so $2000 = 1000e^{k(\frac{1}{4})}$ or $2 = e^{k/4}$, so $\frac{k}{4} = \ln 2$. Thus, $k = 4 \ln 2$.

$$2{,}000{,}000 = 1000e^{(4 \ln 2)t},$$
$$2000 = e^{4t \ln 2},$$
$$\ln 2000 = 4\, t \ln 2,$$
$$t = \frac{\ln 2000}{4 \ln 2} \approx 2.741446070 \text{ hours}$$
$$\approx 164.49 \text{ minutes.}$$

4. Rate of increase per hour $= 0.25 - 0.20 = 0.05$. Suppose the number of bacteria at time t hours is given by $q = q_0e^{kt}$. After one hour, we have $q_0 + 0.05q_0 = q_0e^k$, so that $k = \ln 1.05$. Let T be the doubling time, so that $2q_0 = q_0e^{kT}$, $kT = \ln 2$, $T = \ln 2/k = \ln 2/\ln 1.05 \approx 14.21$ hours.

5. $N = N_0e^{kt}$. $N = 239$, $N_0 = 225$, $t = 1980 - 1977 = 3$.

$239 = 225e^{k(3)}$, $\frac{239}{225} = e^{3k}$,

$3k = \ln \frac{239}{225}$, so $k = \frac{1}{3} \ln \frac{239}{225}$.

$N = 225e^{(\frac{1}{3}\ln \frac{239}{225})(13)} \approx 292$ bears.

6. $N = N_0e^{kt}$. At the beginning of the $(t+1)$st year, the population is N_0e^{kt}; at the end of this year, it is $N_0e^{k(t+1)}$. During the year, the population increase is $N_0e^{k(t+1)} - N_0e^{kt} = N_0e^{kt}(e^k - 1)$. The percent of the increase in population during the year is $\dfrac{N_0e^{kt}(e^k-1)}{N_0e^{kt}} \times 100\%$

$= (e^k - 1) \times 100\%.$

7. $N = N_0e^{kt}$. $3N_0 = N_0e^{k(2)}$ or $3 = e^{2k}$ or $2k = \ln 3$, so $k = \frac{1}{2} \ln 3$.

$50N_0 = N_0e^{kt}$ or $50 = e^{kt}$ or $kt = \ln 50$, so $t = \frac{\ln 50}{k}$. $t = \frac{\ln 50}{\frac{1}{2} \ln 3} \approx 7.12$ hours.

Time $= 7.12 - 2 = 5.12$ hours.

38. $q = q_0e^{kt}$. Thus, $q_1 = q_0e^{kt_1}$ and $q_2 = q_0e^{kt_2}$. $q_0 = \frac{q_1}{e^{kt_1}} = \frac{q_2}{e^{kt_2}}$ or $\frac{e^{kt_2}}{e^{kt_1}} = e^{k(t_2-t_1)} = \frac{q_2}{q_1}$.

Then $k(t_2-t_1) = \ln \frac{q_2}{q_1}$ and $k = \frac{1}{t_2-t_1} \ln \frac{q_2}{q_1}$ provided $t_1 \neq t_2$.

39. $q = 10e^{kt}$, $20 = 10e^{k(20)}$ or $2 = e^{20k}$ or $20k = \ln 2$, so $k = \frac{\ln 2}{20} \approx 0.034657359$.

(a) With calculator: $q = 10e^{0.034657359t}$.

Without calculator: $q = 10e^{(\ln 2/20)t} = 10e^{(\ln 2)(t/20)} = 10(2^{t/20})$.

(b) $q = 10(2^{\frac{60}{20}}) = 10(2^3) = 80$.

40. $D_t\left[f(t)e^{-kt}\right] = f(t)\left[-ke^{-kt}\right] + e^{-kt}(f'(t))$
$= e^{-kt}\left[f'(t) - kf(t)\right]$.

Now, if $q = f(t)$ is a solution, then $f(t) = q_0e^{kt}$ for all t and $f'(t) = q_0ke^{kt}$, so $D_t\left[f(t)e^{-kt}\right] = e^{-kt}\left[q_0ke^{kt} - k(q_0e^{kt})\right] = 0$. Hence, $f(t)e^{-kt} = C$. When $t = 0$, $f(0)\ e^0 = C = q_0$, so $C = q_0$.

$f(t)e^{-kt} = q_0$ or $f(t) = q_0e^{kt}$.

41. $q = q_0e^{-kt}$. $20 = 100e^{-4k}$, so $e^{-4k} = 0.2$ or $-4k = \ln 0.2$; $k = -\frac{1}{4} \ln 0.2$.

(a) $q = 100e^{-(-\frac{1}{4} \ln 0.2)} = 100e^{2 \ln 0.2} = 100e^{\ln 0.04} = 100(0.04) = 4$ grams.

(b) $50 = 100e^{-kt}$ or $\frac{1}{2} = e^{-kt}$, so $-kt = \ln 0.5$. Thus, $t = -\frac{\ln 0.5}{k} = \frac{4 \ln 0.5}{\ln 0.2} \approx 1.7227$ years.

42. After t years, $q = Kq_0$. Hence, $q_0e^{kt} = Kq_0$ or $kt = \ln K$; $k = \ln K/t$. If T is the half-life, then $\frac{q_0}{2} = q_0e^{kT}$, so $kT = \ln(\frac{1}{2})$ $= -\ln 2$ and $T = -\ln 2/k = t \ln 2/(-\ln K)$.

43. $0.1 = 2e^{-kt}$.

From Problem 41, $T = \frac{\ln 2}{k} = 140$, so $k = \frac{\ln 2}{140}$. Thus, $0.1 = 2e^{-t(\frac{\ln 2}{140})}$ or $\ln(\frac{0.1}{2}) = -t(\frac{\ln 2}{140})$. Thus,

$t = -\frac{140}{\ln 2}(\ln 0.05) \approx 605.07$ days.

44. $q = q_0e^{-kt}$.

$70 = 100e^{-k(8)}$, so $-8k = \ln \frac{7}{10}$ or $k = -\frac{1}{8}\ln\frac{7}{10}$.

$$q = 100e^{-(-\frac{1}{8}\ln\frac{7}{10})24} = 100e^{3 \ln \frac{7}{10}}$$

$$= 100e^{\ln(\frac{7}{10})^3} = 100(\frac{7}{10})^3 = 34.3 \text{ kilograms.}$$

45. $y = y_0e^{-kt}$. We know the half-life

$T = \frac{\ln 2}{k}$, so $k = \frac{\ln 2}{1656}$. When $y_0 = 1$, $t = 60$.

Thus, $y = (1)e^{-k(60)} = e^{-60(\frac{\ln 2}{1656})} =$

$e^{\frac{-\ln 2}{27.6}} \approx 0.9752$ gram.

46. $\frac{dq}{dt} = k(A-q)$.

$\frac{dq}{A-q} = kdt$ or $-\ln|A-q| = kt + C$ or

$|A-q| = e^{-(kt+C)} = C_1e^{-kt}$, so $A-q = \pm C_1e^{-kt}$

or $q = A-C_2e^{-kt}$, where $C_2 = \pm C_1$.

When $t = 0$, $q = q_0$, so $q_0 = A-C_2e^0 = A-C_2$.

Thus, $C_2 = A-q_0$. Hence, $q = A-(A-q_0)e^{-kt}$.

$$\lim_{t\to+\infty} q = \lim_{t\to+\infty} [A-(A-q_0)e^{-kt}] = A.$$

After a long period of time, the concentration of glucose in the blood stabilizes at A.

47. $q = q_0e^{-rt}$ where $q_0 = 28{,}000$, so $q = 28{,}000e^{-rt}$.

When $t = 2$, $q = 20{,}000$, so $20{,}000 = 28{,}000e^{-r(2)}$; $r = -\frac{1}{2} \ln \frac{5}{7}$.

When $t = 10$, $q = 28{,}000e^{-(-\frac{1}{2} \ln \frac{5}{7})(10)} =$

$(28{,}000)(\frac{5}{7})^5 \approx \$5206.16$.

48. $q_0 = V = 2(9) = 18$. When $t_1 = 12$, $q_1 = 9$. Find q_2 for $t_2 = 48$.

$q = q_0e^{-kt}$, so $9 = 18e^{-12k}$; $k = -\frac{1}{12} \ln \frac{1}{2}$.

$$q = 18e^{-(-\frac{1}{12} \ln \frac{1}{2})(48)} = 18e^{4 \ln \frac{1}{2}}$$

$$= 18(\frac{1}{2})^4 = \frac{18}{16} = 1.125 \text{ cubic meters.}$$

49. $y = y_0e^{-kt}$. $\frac{y}{y_0} = e^{kt}$, so $F = e^{-kt}$.

When $t = T$, $F = \frac{1}{2}$, so $\frac{1}{2} = e^{-kT}$ or

$\ln \frac{1}{2} = -kT$ or $-\ln 2 = -kT$. Thus, $k = \frac{\ln 2}{T}$

Now $F = e^{-kt}$ or $\ln F = -kt$, so $t = \frac{\ln F}{-k} =$

$\frac{\ln F}{-\frac{\ln 2}{T}} = -T \frac{\ln F}{\ln 2}$.

50. Let $a < 0$ and let $h \to +\infty$. Then, if $u = \frac{h}{a}$, $u \to -\infty$. Then $\lim_{h\to+\infty} (1+\frac{a}{h})^h = \lim_{h\to+\infty} (1+\frac{a}{h})^{(h/a)}$

$= \lim_{u\to-\infty} (1+\frac{1}{u})^{ua} = \lim_{u\to-\infty} [(1+\frac{1}{u})^u]^a$.

Put $V = (1+\frac{1}{u})^u$. By Theorem 1, $\lim_{u\to-\infty} V = e$.

So using Theorem 2, we have

$$\lim_{h\to+\infty} (1+\frac{a}{h})^h = \lim_{u\to-\infty} V^a = (\lim_{u\to-\infty} V)^a = e^a.$$

Let $a > 0$ and let $h \to -\infty$. Then, if $u = \frac{h}{a}$, $u \to -\infty$. So $\lim_{h\to-\infty} (1+\frac{a}{h})^h = \lim_{u\to-\infty} (1+\frac{1}{u})^{ua} =$

e^a by Theorem 1.

51. $t = -5580 \frac{\ln 0.76}{\ln 2} = 2209.282014$

≈ 2209 years.

Problem Set 7.10, page 473

1. $N = N_0 e^{kt}$.

(a) When $t=0$, $N=10$ million $= 10^7$.

Thus, $N = 10^7 e^{kt}$.

Now $k=\ln(1+K)$ where K is the yearly percent increase, so

$k=\ln(1+0.03) = \ln 1.03$

Thus, $N = 10^7 e^{t(\ln 1.03)}$

$\approx 10^7 e^{0.03t}$ (million).

(b) When $t = 20$, $N = 10^7 e^{(0.03)(20)} = 10^7 e^{0.6} =$

$18{,}221{,}188 \approx 18.22$ (million).

(c) Doubling time $= T = \frac{\ln 2}{k} = \frac{\ln 2}{\ln 1.03}$

$= 23.44977225 \approx 23$ years.

2. $N = N_0 e^{kt}$, $T = \frac{\ln 2}{k} = \frac{\ln 2}{\ln(1+K)}$.

3. $N = N_0 e^{kt}$. When $t=35$, $N=2N_0$, so

$2N_0 = N_0 e^{k(35)}$ or $\ln 2 = 35k$;

$k = \frac{\ln 2}{35}$. However, $k = \ln(1+K)$ where K is the yearly percent increase. Thus,

$\frac{\ln 2}{35} = \ln(1+K)$ or $1+K = e^{(\ln 2)/35}$,

so $K = e^{(\ln 2)/35} - 1 = 0.020001609$

$\approx 2\%$ per year.

4. In logistic model, $N = \frac{C}{1+C_0 e^{-kt}} = \frac{Ce^{kt}}{e^{kt}+C_0}$.

When t is small (close to 0), then $e^{kt} \approx 1$, so

$N \approx \frac{ce^{kt}}{1+C_0} \approx N_0 e^{kt}$, the Malthusian model.

5. $T = \frac{\ln 2}{k}$, so $k = \frac{\ln 2}{T} = \frac{\ln 2}{2}$.

Now $k = \ln(1+K) = \frac{\ln 2}{2} = \ln 2^{\frac{1}{2}}$,

so $1+K = \sqrt{2}$ or $K = \sqrt{2}-1 = 0.4142135624$

$\approx 41.42\%$.

6. $N = N_0 e^{k\bar{t}}$, $\bar{t}$=time,

$k = \frac{\ln 2}{T}$. So $N = N_0 e^{(\ln 2/T)\bar{t}}$;

t units later

$N = N_0 e^{(\ln 2/T)t} = N_0 e^{\ln 2(t/T)} = N_0 e^{\ln 2^{\frac{t}{T}}}$

Therefore, $N = N_0 2^{t/T}$.

7. $N_0 = 300$, $k = 0.1$, $t = 5$, $N = 387$.

(a) $N = \frac{C}{1+C_0 e^{-kt}}$

$N_0 = \frac{C}{1+C_0}$ so $300 = \frac{C}{1+C_0}$

or $C_0 = \frac{C-300}{300}$.

So $N = \frac{C}{1+\frac{C-300}{300}(e^{-0.1t})} = \frac{300C}{300+(C-300)e^{-0.1t}}$;

thus, $387 = \frac{300C}{300+(C-300)e^{-0.1(5)}}$,

or $300C = 387(300)+387(C-300)e^{-0.5}$

$300C = 387(300)+387e^{-0.5}C-387(300)e^{-0.5}$

$C(300-387e^{-0.5}) = 387(300)(1-e^{-0.5})$

$C = \frac{387(300)(1-e^{-0.5})}{300-387e^{-0.5}}$

$= 699.8612914 \approx 700$ deer.

(b) $N = \frac{300(700)}{300+(700-300)e^{-0.1(7)}}$

$= \frac{210000}{300+400e^{-0.7}} = 421.1504808$

≈ 421 deer.

(c) $t_I = \frac{1}{K} \ln C_0 = \frac{1}{K} \ln \frac{C-N_0}{N_0}$

$= \frac{1}{0.1} \ln \frac{700-300}{300} = 10 \ln \frac{4}{3}$

$= 2.876820724 \approx 2.88$ years

8. (a) $N(t) = \frac{C}{1+C_0 e^{-kt}}$, $N_0 = N(0) = \frac{C}{1+C_0}$.

Hence, $1+C_0 = \frac{C}{N_0} \leq 1$; then $C_0 \leq 0$.

(b) $N(t) = \frac{C}{1+C_0 e^{-kt}}$, $N_0 = N(0) = \frac{C}{1+C_0}$

$= \frac{N_0}{1+C_0}$. Hence, $N_0 + N_0 C_0 = N_0$

implies that $N_0C_0 = 0$. If $N_0 = 0$ then $N(t) = 0$; if $C_0 = 0$, then $N(t) = N_0$. In both instances, $N(t)$ is constant.

(c) $N(t) = \frac{C}{1+C_0e^{-kt}}$, $N'(t) = \frac{CC_0ke^{-kt}}{(1+C_0e^{-kt})^2}$

< 0 ($C_0 = \frac{C-N_0}{N_0} < 0$); and so $N(t)$ decreases as t increases.

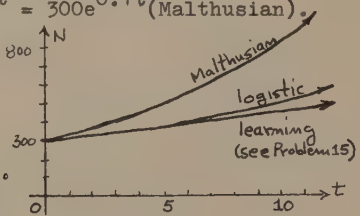
(d) $N'' = -\frac{k}{C}NN'+\frac{k}{C}(C-N)N'$. For $C < N_0$, $N' < 0$ and $C-N < 0$; hence, $N'' > 0$ and the graph of N is concave upward. Finally, $N = C$ is a horizontal asymptote because

$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \left[C/(1+C_0e^{-kt})\right] = C.$

9. (a) $N = \frac{300(700)}{300+400e^{-0.1t}} = \frac{2100}{3+4e^{-t/10}}$

(logistic).

(b) $N = N_0e^{kt} = 300e^{0.1t}$ (Malthusian).



10. $N = \frac{C}{1+C_0e^{-kt}}$.

$N + NC_0e^{-kt} = C$, $NC_0e^{-kt} = C-N$,

$e^{-kt} = \frac{C-N}{NC_0}$, $-kt = \ln \frac{C-N}{NC_0}$,

$t = -\frac{1}{k}\ln\frac{C-N}{NC_0} = \frac{1}{k}\ln\left(\frac{C-N}{NC_0}\right)^{-1} = \frac{1}{k}\ln\frac{NC_0}{C-N}.$

11. $N_0 = 2.35$ million, $k = 0.023$, $t = 100$, $N = 5.14$ million.

(a) $N = \frac{C}{1+C_0e^{-kt}}$. $C_0 = \frac{C-N_0}{N_0} = \frac{C-2.35}{2.35}$.

$N = \frac{C}{1+\frac{C-2.35}{2.35}e^{-kt}} = \frac{2.35C}{2.35+(C-2.35)e^{-kt}}$;

so $5.14 = \frac{2.35C}{2.35+(C-2.35)e^{-0.023(100)}}$

$= \frac{2.35C}{2.35+(C-2.35)e^{-2.3}}$,

$2.35C = 5.14(2.35)+5.14(C-2.35)e^{-2.3}$,

$C(2.35-5.14e^{-2.3}) = 5.14(2.35)(1-e^{-2.3})$,

$C = \frac{5.14(2.35)(1-e^{-2.3})}{2.35-5.14e^{-2.3}}$

$= 5.923668091 \approx 5.92$ million.

(b) $N = \frac{5.92}{1+\frac{6-(2.35)}{2.35}e^{-0.023(170)}}$

$= \frac{(5.92)(2.35)}{2.35+3.65e^{-3.91}} = 5.741292$

≈ 5.74 million.

12. If N is growing according to the logistic model, $kt = \ln\left(\frac{C_0N}{C-N}\right) = \ln C_0+\ln\left(\frac{N}{C-N}\right)$.

If we let $y = \ln\frac{N}{C-N}$, then $y = kt-\ln C_0$.

Thus, k can be obtained by computing the slope of the line that best fits the measurements.

13. $\frac{dN}{dt} = \left(\frac{C}{N}-1\right)k(N) = (C-N)k$.

(a) $\frac{dN}{C-N} = k\,dt$, $-\ln|C-N| = kt+C_1$,

$\ln|C-N| = -kt-C_1$, $|C-N| = e^{-kt-C_1} = C_2e^{-kt}$,

$C-N = \pm C_2e^{-kt} = C_3e^{-kt}$.

Thus, $N = C-C_3e^{-kt}$.

When $t=0$, $N=N_0$, so $N_0=C-C_3$ and $C_3=C-N_0$.

Hence, $N=C-(C-N_0)e^{-kt} = C-C_0Ce^{-kt}$

$= C(1-C_0e^{-kt})$, where $CC_0=C-N_0$; that is,

$C_0 = 1-\frac{N_0}{C}$.

(b) $\lim_{t\to+\infty} N = \lim_{t\to\infty} C(1-C_0e^{-kt})$

$= C\lim_{t\to+\infty}\left(1-\frac{C_0}{e^{kt}}\right) = C(1-0) = C.$

Hence, $N = C$ is a horizontal asymptote of the graph of N as a function of t.

14. $N = \frac{C}{1+C_0e^{-kt}}$

$\frac{dN}{dt} = \frac{-C\left[-kC_0e^{-kt}\right]}{\left[1+C_0e^{-kt}\right]^2} = \frac{kCC_0e^{-kt}}{\left[1+C_0e^{-kt}\right]^2}.$

$$\frac{d^2N}{dt^2} = \frac{\left[1+C_0e^{-kt}\right]^2\left[-k^2CC_0e^{-kt}\right]}{(1+C_0e^{-kt})^4} - \frac{kCC_0e^{-kt}(2)(1+C_0e^{-kt})(-kC_0e^{-kt})}{(1+C_0e^{-kt})^4}$$

$$= \frac{k^2CC_0e^{-kt}(C_0e^{-kt}-1)}{(1+C_0e^{-kt})^3} = 0 \text{ when}$$

$C_0e^{-kt} = 1$; that is, $-kt = \ln \frac{1}{C_0}$ or

$t = -\frac{1}{k}\ln\frac{1}{C_0} = -\frac{1}{k}(-\ln C_0) = \frac{1}{k}\ln C_0$.

Hence, $t_I = \frac{1}{k}\ln C_0$.

Therefore, $N_I = \dfrac{C}{1+C_0e^{-k(\frac{1}{k}\ln C_0)}}$

$$= \frac{C}{1+C_0e^{-\ln C_0}} = \frac{C}{1+C_0e^{\ln 1/C_0}} = \frac{C}{1+C_0(1/C_0)} = \frac{C}{2}.$$

Now, if $t < t_I$, that is, $t < \frac{1}{k}\ln C_0$, we have $-kt > -\ln C_0$, so $-kt > \ln(\frac{1}{C_0})$ or $e^{-kt} > \frac{1}{C_0}$ or $C_0e^{-kt} > 1$, so $C_0e^{-kt}-1 > 0$; and we have $\frac{d^2N}{dt^2} > 0$ and the graph is concave upward. If $t > t_I$, a similar argument shows the graph is concave downward. Thus, (t_I,N_I) is point of inflection.

. $N = 700\left[1-(1-\frac{300}{700})e^{-0.05t}\right]$

$= 700\left[1-\frac{400}{700}e^{-0.05t}\right] = 700-400e^{-0.05t}$.

For the graph of N and the comparisons, see Problem 9; also Problem 23.

. Solving for t in the equation $N = C(1-C_0e^{-kt})$, we get $kt = \ln C_0 - \ln(1-\frac{N}{C_0})$; that is, $y = kt - \ln C_0$, where $y = \ln(\frac{C}{C-N})$. Hence, k can be obtained as the slope of the line L that best fits the measurement (t,y) with $y = \ln(\frac{C}{C-N(t)})$.

. $N(t) = C(1-C_0e^{-kt})$, $N'(t) = kCC_0e^{-kt}$, and $N''(t) = -k^2CC_0e^{-kt}$. If $0 < N_0 < C$, then $C_0 = 1-\frac{N_0}{C} > 0$ and so $N''(t) < 0$ for all $t \geq 0$. Therefore, N does not have a point of inflection.

18. $\frac{dN}{dt} = \left[k\cos(\omega t-\phi)\right]N$,

$\frac{dN}{N} = k\cos(\omega t-\phi)dt$,

$\ln N = \frac{k\sin(\omega t-\phi)}{\omega} + C$,

$N = C_1e^{\frac{k}{\omega}\sin(\omega t-\phi)}$.

When $t=0$, $N=N_0$, so $N_0 = C_1e^{\frac{k}{\omega}\sin(-\phi)}$

$= C_1e^{-k/\omega\sin\phi}$. So $C_1 = N_0e^{k/\omega\sin\phi}$.

Hence, $N = N_0\exp(\frac{k}{\omega}\sin\phi)\exp(\frac{k}{\omega}\sin(\omega t-\phi))$

$= \exp[\frac{k}{\omega}(\sin\phi + \sin(\omega t-\phi))]$.

19. $N = C(1-C_0e^{-kt})$. $C_0 = 1-\frac{N_0}{C}$.

When $t = 0$, $N_0 = 10$. When $C = 100$, $k = 0.025$. $C_0 = 1-\frac{10}{100} = 1-\frac{1}{10} = \frac{9}{10}$.

$N = 100(1-\frac{9}{10}e^{-0.025t})$.

(a) When $t = 30$, $N = 100(1-\frac{9}{10}e^{-0.025(30)})$

$= 100(1-\frac{9}{10}e^{-0.75}) \approx 57.48701025$

≈ 57.5 words/min.

(b) $\frac{dN}{dt} = 100\left[0-\frac{9}{10}e^{-0.025t}(-0.025)\right]$

$= 100\left[\frac{9}{10}(0.025)\right]e^{-0.025t} = 2.25e^{-0.025t}$

When $t = 30$; $\frac{dN}{dt} = 2.25e^{-0.025(30)}$

$\approx 1.062824744 \approx 1.06$ (words/min)/hr.

20. $\frac{dN}{dt} = k(100{,}000-N)$, $N(0) = 0$. Then $N(t) = 100{,}000(1-e^{-kt})$. Now $30{,}000 = 100{,}000(1-e^{-10k}) = N(10)$; hence, $k = \frac{-\ln 0.7}{10}$ and $N(t) = 100{,}000(1-e^{\frac{\ln 0.7}{10}t})$.

After 30 days, the number of potential purchasers that have heard about the car

will be $N(30) = 100{,}000(1-(0.7)^3)$

$= 65{,}700.$

21. $N = Ce^{-C_0 e^{-kt}}$

(a) $\frac{dN}{dt} = Ce^{-C_0 e^{-kt}} (C_0 ke^{-kt}) = NC_0 ke^{-kt}$

Now $\frac{C}{N} = e^{C_0 e^{-kt}}$, so $\ln \frac{C}{N} = \ln C - \ln N = C_0 e^{-kt}$.

Therefore, $\frac{dN}{dt} = NC_0 ke^{-kt} = Nk(\ln C - \ln N)$

$= Ng(N).$

(b) When $t=0$, $N=N_0$, so $N_0 = Ce^{-C_0}$

or $\frac{C}{N_0} = e^{C_0}$ or $\ln \frac{C}{N_0} = C_0$.

(c) $\lim_{t\to+\infty} e^{-kt} = 0$, so $\lim_{t\to+\infty} N = Ce^{-C_0(0)}$

$= Ce^0 = C$; hence, $N = C$ is a horizontal asymptote of N.

22. $\frac{dN}{dt} = NC_0 ke^{-kt}$

$\frac{d^2N}{dt^2} = NC_0 k(-ke^{-kt}) + C_0 ke^{-kt}(\frac{dN}{dt}) = 0$

so $-Nk + \frac{dN}{dt} = 0$ or

$-Nk + NC_0 ke^{-kt} = 0$ or $-1 + C_0 e^{-kt} = 0$

$e^{-kt} = \frac{1}{C_0}$ or $t = -\frac{1}{k} \ln \frac{1}{C_0} = \frac{1}{k} \ln C_0$

For $t = \frac{\ln C_0}{k}$, $N = Ce^{-C_0 e^{-\ln C_0}} =$

$Ce^{-C_0 e^{\ln 1/C_0}} = Ce^{-C_0(1/C_0)} = Ce^{-1} = \frac{C}{e}$.

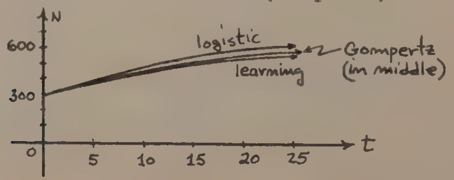
The point of inflection is $(\frac{1}{k} \ln C_0, \frac{C}{e})$.

23. $N_0 = 300$, $C = 700$, $h = 0.07$, $C_0 = \ln \frac{700}{300}$.

$N = 700e^{-\ln \frac{7}{3} \times e^{-0.07t}}$

$= 700e^{-0.8472978604 \quad e^{-0.07t}}$ Thus,

$N \approx 700\, e^{-0.8473(e^{-0.07t})}$ (Gompertz).



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1. $\frac{dy}{dx} = -xy$ or $\frac{dy}{y} = -xdx$; so $\ln|y| = -\frac{x^2}{2} + C$.

$y = e^{(-x^2/2)+C} = C_1 e^{-x^2/2}$, $y = C_1 e^{-x/2} = C_1 d^{-x^2/2}$.

Hence, $y = C_e^{-\frac{x^2}{2}}$.

2. $\frac{ds}{dt} = 3-3s$ or $\frac{ds}{3-3s} = dt$, so $-\frac{1}{3}\ln|1-s| = t+C$

$\ln|1-s| = -3(t+C)$

$|1-s| = e^{-3t-3C} = C_1 e^{-3t}$

$1-s = \pm C_1 e^{-3t}$

$s = 1 - Ce^{-3t}$

3. $\frac{dN}{dt} = 0.1N - 100$ or $\frac{dN}{0.1N-100} = dt$, so

$10 \ln|0.1N - 100| = t+C$

$\ln|0.1N-100| = \frac{1}{10}(t+C)$

$|0.1N-100| = e^{1/10t+1/10C}$

$0.1N - 100 = \pm C_1 e^{t/10}$

$0.1N = 100 + Ce^{t/10}$

$N = 1000 + Ce^{t/10}$

4. $dy = (2y+1)dx$ or $\frac{dy}{2y+1} = dx$, so

$\frac{1}{2} \ln|2y+1| = x+C$

$\ln|2y+1| = 2x+2C$

$|2y+1| = e^{2x+2C} = C_1 e^{2x}$

$2y+1 = \pm C_1 e^{2x}$

$2y = -1 + C_2 e^{2x}$

$y = -\frac{1}{2} + Ce^{2x}$.

5. $\frac{dy}{dt} = -k(y-2)$, $t = 0$, $y = 90$. $\frac{dy}{y-2} = -k$, so

$\ln|y-2| = -kt+C$

$|y-2| = e^{-kt+C}$

$y-2 = C_1 e^{-kt}$

$y = 2 + C_1 e^{-kt}$. Using initial conditio

we have $C_1 = 88$. Thus $y = 2+88e^{-kt}$.

When $t = 10$, $y = 25$, so

$25=2+88e^{-k(10)}$ or $\frac{23}{88} = e^{-10k}$;

$k = -\frac{1}{10} \ln \frac{23}{88}$.

When $y = 10$: $10=2+88e^{-kt}$ or $\frac{8}{88}=\frac{1}{11}=e^{-kt}$

so that $t = -\frac{1}{k} \ln \frac{1}{11} \approx 17.87016$.

Thus, after 7.87 more minutes, the ball will cool to 10°C.

• $\frac{dy}{dt} = -k(y-a)$ $a = ?$

$\frac{dy}{y-a} = -kt$, so $y = a+C_0e^{-kt}$. When $t=0, y=160°$,

so $y = a+(160-a)e^{-kt}$.

When $t = 50$, $y = 100$, so $100 = a + (160-a)e^{-50k}$.

When $t = 100$, $y = 80$, so $80 = a+(160-a)e^{-100k}$.

$e^{-50k} = \frac{100-a}{160-a}$. $e^{-100k} = (e^{-50k})^2$. Thus,

$80 = a+(160-a)(\frac{100-a}{160-a})^2$ or $80-a = \frac{(100-a)^2}{160-a}$,

so $12800-240a+a^2 = 10000-200a+a^2$

$2800 = 40a$

$a = 70°F$.

• $\frac{dy}{dt} = -k(y-70)$,

so $y = 70+C_0e^{-kt}$. When $t = 0, y = 200°$, so

$y = 70+130e^{-kt}$.

When $t = 4$, $y = 175$, so $175 = 70+130e^{-k(4)}$.

$\frac{105}{130} = e^{-4k}$ or $k = -\frac{1}{4} \ln \frac{105}{130}$.

When $t = 7$, $y = ?$, so $y = 70+130e^{-7k}$

$= 70+130(\frac{105}{130})^{7/4} \approx 159.46°C$.

• $\frac{dy}{dt} = -k(y-35)$, so

$y = 35+Ce^{-kt}$.   When $t = 0$, $y = 70$,

so $y = 35+35e^{-kt}$. When $t = 2$, $y = 45$,

so $45 = 35+35e^{-2k}$ or $\frac{10}{35} = \frac{2}{7} = e^{-2k}$.

Thus, $k = -\frac{1}{2} \ln \frac{2}{7}$

When $t = 4$, $y = ?$, so

$y = 35+35e^{-4k} = 35+35(\frac{4}{49}) = 37.86°F$.

9. After t minutes there are $50+(3-2)t=50+t$ gallons of salt water in the tank. Let q be the number of pounds of salt in the tank at time t. Thus, at time t, the concentration of salt in the tank is $\frac{q}{50+t}$ pounds per gallon. Let dt minutes go by. Then $\frac{q}{50+t}(2)dt$ pounds of salt leave the tank; that is, $dq = -\frac{2q}{50+t}dt$, or $\frac{dq}{q} = \frac{-2dt}{50+t}$. Thus, $\int \frac{dq}{q} = -2\int \frac{dt}{50+t}$; that is, $\ln|q| = -2 \ln|50+t|+C$. Exponentiating both sides of the latter equation and using the fact that $t > 0$, we obtain $|q| = |50+t|^{-2}e^C$, or $q = \frac{K}{(50+t)^2}$, where we have put $K = \pm e^C$. When $t = 0$, $q = 10$ pounds, so that $10 = \frac{K}{50^2}$, and $K = 10(50)^2$. It follows that $q = \frac{10(50)^2}{(50+t)^2}$, $(50+t)^2 = \frac{10(50)^2}{q}$, $50+t = 50\sqrt{\frac{10}{q}}$, $t = 50\sqrt{\frac{10}{q}} - 50$ $= 50(\sqrt{\frac{10}{q}} - 1)$. When $q=2$ pounds,

$t = 50(\sqrt{5}-1) \approx 61.80$ minutes.

10. At any time t, there are 50 gallons of water in the tank. Let q be the number of pounds of salt in the tank at time t and let dt minutes go by. Then $2(\frac{1}{30})dt$ pounds of salt go into the tank and $\frac{q}{50}(2)dt$ gallons of salt leave the tank. So $dq = \frac{2}{30}\,dt - \frac{2q}{50}dt$ and $dq=(\frac{5-3q}{75})dt$. Now $\frac{75}{5-3q}dq = dt$ and so $\int \frac{75}{5-3q}dq = \int dt$. Let $u = 5-3q$ and $du = -3dq$. $\int \frac{75}{5-3q}dq =$ $\int \frac{-75\,du}{3u} = -25 \ln|u| + C = -25\ln|5-3q| + C$.

Hence, $-25 \ln|5-3q| = t+k$. When $t=0$, $q=10$, so that $k=-25 \ln 25$. Now $-25 \ln|5-3q| = t-25 \ln 25$. When $q=2$, $t=25(\ln 25-\ln|-1|) = 25 \ln 25$ or $t \approx 80.47$ minutes.

11. Let q be the number of cubic feet of hydrogen sulfide gas in the room at time t. If dt minutes go by, $500dt$ cubic feet of mixture leave the room and the mixture contains $\frac{q}{10,000}$ cubic feet of hydrogen sulfide per cubic foot of mixture; hence, if dt minutes go by, $(500dt)\frac{q}{10,000} = \frac{q}{20}dt$ cubic feet of hydrogen sulfide leave the room. Thus, $dq = -\frac{q}{20}dt$, or $\frac{dq}{q} = -\frac{dt}{20}$. Integrating, we find that $q = q_0 e^{-t/20}$. When $t=0$, there are $10,000(0.01) = 100$ cubic feet of hydrogen sulfide in the room; hence, $q_0 = 100$. When $t=5$ minutes, $q = 100e^{-5/20} = \frac{100}{e^{1/4}}$ cubic feet, so the concentration is $\frac{100}{10,000\, e^{\frac{1}{4}}}(100\%) = \frac{1}{e^{\frac{1}{4}}}\% \approx 0.78\%$.

12. (a) $V\frac{dy}{dt}$ = inflow of pollutant; outflow of pollutant $= RA-Ry$; hence, $\frac{dy}{dt} = \frac{R}{V}(A-y)$.

(b) $\int_0^t \frac{1}{A-y}dy = \frac{R}{V}\int_0^t dt$. After computing the integrals, we get $\frac{A-y(t)}{A-y(0)} = e^{-(R/V)t}$; hence, $y(t) = A-(A-y(0))e^{-(R/V)t}$.

(c) $y(t) = A-(A-y_0)e^{-(R/V)t}$.

13. Let $I(t)$ be the number of people already infected at time t. Then $\frac{dI}{dt} = kI(260,000-I)$; hence, for $I < 260,000$, $\frac{dI}{I(260,000-I)} = \frac{dI}{I} - \frac{dI}{260,000-I} = 260,000\, kdt$,

$\ln I - \ln(260,000-I) = 260,000\, kt + C_1$,

$\ln \frac{I}{260,000-I} = 260,000\, kt + C_1$,

so $\frac{I}{260,000-I} = Ce^{260,000kt}$. Since $I=600$ when $t=0$, we find $C = \frac{3}{1297}$. Using $I = 30,000$ when $t = 10$, we find that $k = \frac{1}{2.6 \times 10^6} \ln \frac{1297}{23}$. Now if $t=30$, we want I: $\frac{I}{260,000-I} = \frac{3}{1297}\left[\exp \frac{1}{10}(\ln \frac{1297}{23})^{30}\right]$, and $I \approx 259,375$ people.

14. (a) Let V be the volume of water in the reservoir at time t. Then V = initial volume + total inflow of water − total outflow of water $= V_0+Rt-rt$; hence, $V = V_0 + (R-r)t$.

(b) $\frac{d}{dt}(V \cdot y) = AR-ry$. By the product rule $V \cdot \frac{dy}{dt} + (R-r)y = AR-ry$, and so $\frac{dy}{dt} = \frac{R}{V}(A-y)$.

(c) $\int_0^t \frac{1}{A-y}dy = R\int_0^t \frac{dt}{V_0+(R-r)t}$.

Hence, $\ln(\frac{y(t)-A}{y(0)-A}) = \frac{R}{r-R} \ln\left[\frac{V_0+(R-r)t}{V_0}\right]$, and then $y(t) = (y(0)-A)\left[\frac{V_0+(R-r)t}{V_0}\right]^{R/(r-R)} + A$.

(d) If $y(0) = y_0$, then $y(t) = A+(y_0-A)(\frac{V}{V_0})^{R/(r-R)} = A+(y_0-A)(\frac{V_0}{V})^{R/(R-r)}$.

15. $\frac{dy}{dx} - 4y = e^{4x}$, $P(x) = -4$.

$\emptyset(x) = e^{\int P(x)dx} = e^{\int(-4)dx} = e^{-4x}$.

Thus, $e^{-4x}(\frac{dy}{dx} - 4y) = e^{4x}(e^{-4x}) = 1$.

So $Dx\left[y\, \emptyset(x)\right] = Dx\left[y\, e^{-4x}\right] = 1$.

Thus, $ye^{-4x} = x+C$.

Thus, $y = xe^{4x}+Ce^{4x} = e^{4x}(x+C)$.

16. $P(x) = -\frac{3}{x}$.

$\emptyset(x) = e^{\int P(x)dx} = e^{\int(-\frac{3}{x})dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3}$.

Thus, $x^{-3}(\frac{dy}{dx} - \frac{3}{x}y) = x^4(x^{-3}) = x$.

So $Dx\left[y\,\phi(x)\right] = Dx\left[y\,x^{-3}\right] = x$.

Hence, $yx^{-3} = \frac{x^2}{2} + C$.

So $y = \frac{x^5}{2} + Cx^3$.

17. $P(t) = 2t$. $\phi(t) = e^{\int P(t)dt} = e^{\int 2t dt} = e^{t^2}$.

Then $e^{t^2}(\frac{dy}{dt} + 2ty) = 2te^{-t^2}(e^{t^2}) = 2t$.

$Dt(y\,\phi(t)) = Dt(y\,e^{t^2}) = 2t$.

So $ye^{t^2} = t^2 + C$ or $y = t^2e^{-t^2} + Ce^{-t^2}$.

18. $\frac{dy}{dx} - xy = \cos x\, e^{x^2/2}$.

$P(x) = -x$. $\phi(x) = e^{\int P(x)dx} = e^{\int(-x)dx}$

$= e^{-x^2/2}$.

$e^{-x^2/2}(\frac{dy}{dx} - xy) = \cos x\, e^{x^2/2}(e^{-x^2/2})$

$= \cos x$.

$Dx\left[y\phi(x)\right] = Dx\left[ye^{-x^2/2}\right] = \cos x$.

o $y\,e^{-x^2/2} = \sin x + C$.

Thus, $y = e^{x^2/2}(\sin x + C)$.

19. $P(t) = \cos t$. $\phi(t) = e^{\int P(t)dt} = e^{\int \cos t\, dt}$

$= e^{\sin t}$.

$e^{\sin t}(\frac{dq}{dt} + \cos t\; q) = e^{-\sin t}e^{\sin t} = 1$.

$D_t\left[q\phi(t)\right] = D_t\left[qe^{\sin t}\right] = 1$.

$qe^{\sin t} = t + C$.

Hence, $q = te^{-\sin t} + Ce^{-\sin t}$.

20. $P(x) = -\frac{2}{x}$. $\phi(x) = e^{\int P(x)dx} = e^{\int(-\frac{2}{x})dx}$

$= e^{-2\ln x} = x^{-2}$.

$x^{-2}(D_xy - \frac{2}{x}y) = x(x^{-2}) = x^{-1}$.

$D_x\left[y\phi(x)\right] = D_x\left[yx^{-2}\right] = x^{-1}$.

So $yx^{-2} = \ln x + C$.

Thus, $y = x^2(\ln x + C)$.

21. (a) $P(t) = \frac{R}{L}$ $\phi(t) = e^{\int P(t)dt} = e^{\int R/L\, dt}$

$= e^{Rt/L}$.

$e^{Rt/L}(\frac{dI}{dt} + \frac{R}{L}I) = \frac{E}{L}e^{Rt/L}$.

$D_t\left[I\phi(t)\right] = D_t\left[Ie^{Rt/L}\right] = \frac{E}{L}e^{Rt/L}$.

$Ie^{Rt/L} = \frac{E}{L}\cdot\frac{L}{R}e^{Rt/L} + C$,

so $I = \frac{E}{R} + e^{-Rt/L}C$.

When $t = 0$, $I = 0$, so $C = -\frac{E}{R}$.

$\lim_{t\to+\infty} I = \frac{E}{R} + 0(C) = \frac{E}{R}$.

(b) $D_t\left[Ie^{Rt/L}\right] = \frac{E}{L}e^{Rt/L}$,

$Ie^{Rt/L} = \int\frac{E}{L}e^{Rt/L}dt$,

$I = \frac{1}{Le^{Rt/L}}\int Ee^{Rt/L}dt$.

22. $\frac{dy}{dx} = e^{-\int P(x)dx}\cdot Q(x)e^{\int P(x)dx}dx$

$-P(x)e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$

$= Q(x) - P(x)\cdot y$.

Hence, $\frac{dy}{dx} + P(x)y = Q(x)$.

23. Let N = the number of defective generators in the warehouse t weeks from now. In t weeks there will be a total of $5000-(200-175)t = 5000-25t$ generators in the warehouse. The fraction of defective generators at the end of t weeks will be $N/(5000-25t)$, so the number of defective generators shipped out in a short period of time dt will be $200[N/(5000-25t)]dt$. During this same period of time $175(0.05)dt = 8.75dt$ defective generators will arrive at the warehouse. Therefore, $dN = 8.75dt - \left[200N/(5000-25t)\right]dt$, or $\frac{dN}{dt} + \frac{200}{5000-25t}N = 8.75$. Solving this differential equation using the integrating factor $\exp(\int\frac{200dt}{5000-25t}) = \exp\left[-8\ln(200-t)\right] = (200-t)^{-8}$, we find that $N = \frac{8.75}{7}(200-t) + (500 - \frac{1750}{7})$.

When $t = 52$, $N = 207.48$, so the percentage of defective generators in the warehouse is $\frac{207.48}{3700} \times 100\% = 5.61\%$.

24. (a) Since the number of workers who have quit at the end of t units of time is $c \cdot t$, the number of skilled workers hired in the same period of time is At and the number of unskilled workers is Bt; the labor force F at time t consists of $F = F_0 + At + Bt - ct = F_0 + kt$, where $k = A+B-c$.

(b) $\frac{dy}{dt}$ = inflow of skilled workers - outflow of skilled workers $= A - \frac{c}{F}y$.

So $\frac{dy}{dt} + \frac{c}{F}y = A$.

(c) Let $\emptyset(t) = \exp\left\{\int \frac{c}{F}dt\right\} = \exp\left\{\int \frac{c}{F_0+kt}dt\right\}$ $= (F_0+kt)^{c/k}$. Then $(F_0+kt)^{c/k}y(t) = \int_0^t A(F_0+kt)^{c/k}dt + F_0^{c/k}y(0)$ and $y(t) = \frac{A}{c+k}(F_0+kt) + \left(y(0) - \frac{AF_0}{c+k}\right)\left(\frac{F_0}{F_0+kt}\right)^{c/k}$.

Review Problem Set, Chapter 7, page 480

1. $f(g(x)) = f(\sqrt[4]{x}) = (\sqrt[4]{x})^4 = x$.
$g(f(x)) = g(x^4) = \sqrt[4]{x^4} = |x| = x$ since $x \geq 0$.

2. $f(g(x)) = f(\sqrt[3]{x-3}) = 3+(\sqrt[3]{x-3})^3 = 3+x-3 = x$.
$g(f(x)) = g(3+x^3) = \sqrt[3]{3+x^3-3} = \sqrt[3]{x^3} = x$.

3. $f(g(x)) = f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{x-1}{x}} = \frac{x}{x-(x-1)}$

$= \frac{x}{x-x+1} = \frac{x}{1} = x$.

$g(f(x)) = g\left(\frac{1}{1-x}\right) = \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}} = \frac{1-1(1-x)}{1}$

$= \frac{1-1+x}{1} = \frac{x}{1} = x$.

4. $f(g(x)) = f\left(\frac{3-\sqrt{1+4x}}{2}\right)$

$= \left(\frac{3-\sqrt{1+4x}}{2}\right)^2 - 3\left(\frac{3-\sqrt{1+4x}}{2}\right) + 2$

$= \frac{9-6\sqrt{1+4x}+1+4x}{4} - \frac{9-3\sqrt{1+4x}}{2} + \frac{8}{4}$

$= \frac{9-6\sqrt{1+4x}+1+4x-18+6\sqrt{1+4x}+8}{4} = \frac{4x}{4} = x$.

$g(f(x)) = g(x^2-3x+2) = \frac{3-\sqrt{1+4(x^2-3x+2)}}{2}$

$= \frac{3-\sqrt{4x^2-12x+9}}{2} = \frac{3-\sqrt{(2x-3)^2}}{2}$

$= \frac{3-|2x-3|}{2} = \frac{3-(3-2x)}{2} = \frac{2x}{2} = x$

since $x \leq \frac{3}{2}$.

5. $f(g(x)) = f\left(\sin^{-1}\left(\frac{1}{x}-1\right)\right) = \frac{1}{1+\sin\left[\sin^{-1}\left(\frac{1}{x}-1\right)\right]}$

$= \frac{1}{1+\frac{1}{x}-1} = \frac{1}{\frac{1}{x}} = x$.

$g(f(x)) = g\left(\frac{1}{1+\sin x}\right) = \sin^{-1}\left[\frac{1}{\frac{1}{1+\sin x}} - 1\right]$

$= \sin^{-1}[1+\sin x-1] = \sin^{-1}[\sin x] = x$.

6. $f(g(x)) = f(\ln(x+\sqrt{x^2+4}) - \ln 2) =$
$e^{[\ln(x+\sqrt{x^2+4})-\ln 2]} - e^{[-\ln(x+\sqrt{x^2+4})+\ln 2]}$

$= (x+\sqrt{x^2+4})e^{-\ln 2} - e^{-\ln(x+\sqrt{x^2+4})}$

$= (x+\sqrt{x^2+4})\left(\frac{1}{2}\right) - \frac{2}{x+\sqrt{x^2+4}}$

$= \frac{x^2+2x\sqrt{x^2+4}+x^2+4-4}{2(x+\sqrt{x^2+4})} = \frac{2x^2+2x\sqrt{x^2+4}}{2(x+\sqrt{x^2+4})}$

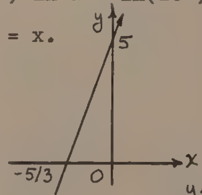
$= \frac{2x(x+\sqrt{x^2+4})}{2(x+\sqrt{x^2+4})} = x$.

$g(f(x)) = \ln(e^x - e^{-x} + \sqrt{(e^x-e^{-x})^2+4}) - \ln 2$

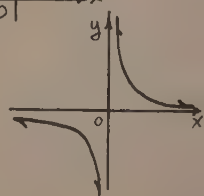
$= \ln(e^x - e^{-x} + e^x + e^{-x}) - \ln 2 = \ln(2e^x) - \ln 2$

$= \ln 2 + x - \ln 2 = x$.

7. $f(x) = 3x+5$.
f is invertible.

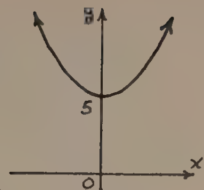


8. $g(x) = \frac{1}{x}$.
g is invertible.



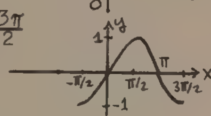
9. $h(x)=3x^2+5$.

h is not invertible.



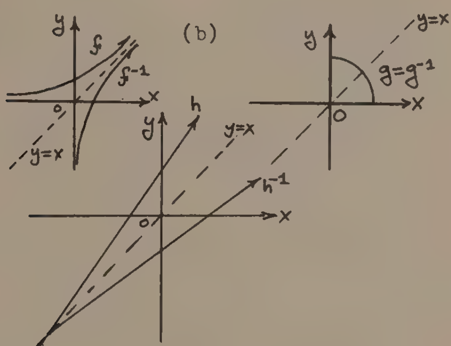
10. $F(x) = \sin x,\ -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

F is not invertible.



11. (a) (b)

(c)



12. If $a = -d$ and $a^2+bc \neq 0$ or if $a = d$ and $b = c = 0$, then $f(x) = \frac{ax+b}{cx+d}$ is its own inverse.

13. $y = 7x-9$.

(a) $x = 7y-9$ or $7y = x+9$, $y = \frac{1}{7}(x+9)$.

Hence, $f^{-1}(x) = \frac{1}{7}(x+9)$.

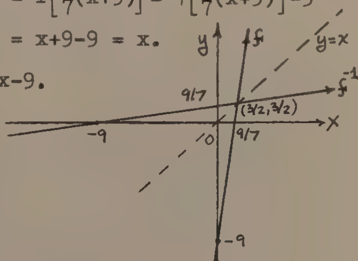
(b) $f^{-1}(f(x)) = f^{-1}(7x-9) = \frac{1}{7}(7x-9+9)$

$= \frac{1}{7} \cdot 7x = x$.

(c) $f(f^{-1}(x)) = f\left[\frac{1}{7}(x+9)\right] = 7\left[\frac{1}{7}(x+9)\right]-9$

$= x+9-9 = x$.

(d) $f(x) = 7x-9$.



14. $y = 1-\ln x$.

(a) $x = 1-\ln y$ or $\ln y = 1-x$ or $y = e^{1-x}$.

Thus, $f^{-1}(x) = e^{1-x}$.

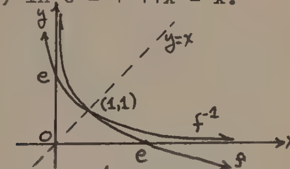
(b) $f^{-1}(f(x)) = f^{-1}(1-\ln x) = e^{1-(1-\ln x)}$

$= e^{\ln x} = x$.

(c) $f(f^{-1}(x)) = f(e^{1-x}) = 1-\ln(e^{1-x})$

$= 1-(1-x)\ln e = 1-1+x = x$.

(d)



15. (a) $y = \frac{4}{x+1}$ or $x = \frac{4}{y+1}$, so $xy+x = 4$

or $y = \frac{4-x}{x}$. Hence, $f^{-1}(x) = \frac{4-x}{x}$.

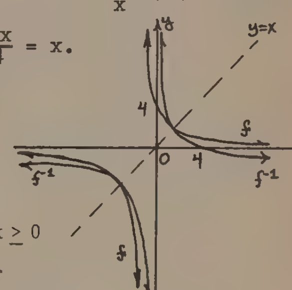
(b) $f^{-1}(f(x)) = f^{-1}\left(\frac{4}{x+1}\right) = \dfrac{4-\frac{4}{x+1}}{\frac{4}{x+1}} = \dfrac{4(x+1)-4}{4}$

$= x+1-1 = x$.

(c) $f(f^{-1}(x)) = f\left(\frac{4-x}{x}\right) = \dfrac{4}{\frac{4-x}{x}+1} = \dfrac{4x}{4-x+x}$

$= \frac{4x}{4} = x$.

(d)



16. $y = e^x+e^{-x}$, $x \geq 0$

(a) $x = e^y+e^{-y}$ or

$xe^y = e^{2y}+1$ or $e^{2y}-xe^y+1 = 0$, so

$e^y = \frac{x\pm\sqrt{x^2-4}}{2}$. Choose $e^y = \frac{x+\sqrt{x^2-4}}{2}$.

$y = \ln\left(\frac{x+\sqrt{x^2-4}}{2}\right) = \ln(x+\sqrt{x^2-4})-\ln 2 = f^{-1}(x)$.

(b) $f^{-1}(f(x)) = f^{-1}(e^x+e^{-x})$

$= \ln(e^x+e^{-x}+\sqrt{(e^x+e^{-x})^2-4}) - \ln 2$

$= \ln(e^x+e^{-x}+\sqrt{(e^x-e^{-x})^2}) - \ln 2$

$= \ln(e^x+e^{-x}+e^x-e^{-x})-\ln 2$

$= \ln 2e^x-\ln 2 = \ln 2+\ln e^x-\ln 2 = x$.

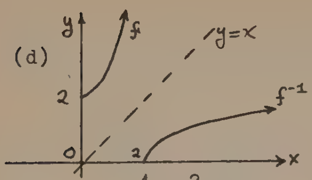
(c) $f(f^{-1}(x)) = f(\ln(x+\sqrt{x^2-4}) -\ln 2)$

$= e^{\ln(x+\sqrt{x^2-4})-\ln 2}+e^{-\ln(x+\sqrt{x^2-4})+\ln 2}$

$= \dfrac{x+\sqrt{x^2-4}}{4} + \dfrac{2}{x+\sqrt{x^2-4}} = \dfrac{x^2+2x\sqrt{x^2-4}+x^2-4+4}{2(x+\sqrt{x^2-4})}$

$= \dfrac{2x^2+2x\sqrt{x^2-4}}{2(x+\sqrt{x^2-4})} = \dfrac{2x(x+\sqrt{x^2-4})}{2(x+\sqrt{x^2-4})} = x$.

(d)



17. $f'(x) = 5x^4+9x^2+7 > 0$ for all x. Hence, by the inverse-function theorem, f is invertible.

18. By the algebraic method, we will solve $y = Ax^2+Bx+C$ for x:

$0 = Ax^2+Bx + (C - y)$,

$$x = \frac{-B \pm\sqrt{B^2 - 4A(C - y)}}{2A}$$

$$= \frac{-B \pm\sqrt{B^2 - 4AC + 4Ay}}{2A}.$$

So $f^{-1}(x) = \frac{-B \pm\sqrt{B^2 - 4AC + 4Ax}}{2A}$, where

$B^2 - 4AC + 4Ax \geq 0$, $x \geq \frac{4AC - B^2}{4A}$.

19. Let $x_1 < x_2$ be two elements in the domain of f^{-1}, and assume that $f^{-1}(x_1) \geq f^{-1}(x_2)$. Since f is increasing and both $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are in the domain of f, we can conclude that $x_2 = f(f^{-1}(x_2)) \leq f(f^{-1}(x_1)) = x_1$. Contradiction. Thus, f^{-1} is also increasing.

20. No. Otherwise, $(f\circ f)(x) = \frac{1}{f^{-1}(f(x))} = \frac{1}{x}$, and we would have the composition of two continuous functions (with domain $\mathbb{R}$) not being a continuous function.

21. $1 = \frac{3}{7}x^5$, $x^5 = \frac{7}{3}$, $x = \sqrt[5]{\frac{7}{3}}$.

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{15}{7}x^4}$. When $y = 1$, $x = \sqrt[5]{\frac{7}{3}}$, so

$\frac{dx}{dy} = \frac{7}{15}\ \frac{1}{\sqrt[5]{(\frac{7}{3})^4}} = \frac{7}{15}\sqrt[5]{(\frac{3}{7})^4}$.

22. $-8 = -\frac{4}{3}x^3$, $x^3 = 6$, $x = \sqrt[3]{6}$.

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-4x^2}$. When $y = -8$, $x = \sqrt[3]{6}$,

so $\frac{dx}{dy} = \frac{1}{-4\sqrt[3]{36}}$.

23. $-1 = \frac{5x}{x+2}$, $-x-2 = 5x$, $6x = -2$, $x = -\frac{1}{3}$.

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{10}{(x+2)^2}} = \frac{(x+2)^2}{10}$.

When $y = -1$, $x = -\frac{1}{3}$, so $\frac{dx}{dy} = \frac{(\frac{5}{3})^2}{10}$

$= \frac{1}{10}\cdot\frac{25}{9} = \frac{5}{18}$.

24. $\frac{\sqrt{3}}{2} = \cos x$, $x = \frac{\pi}{6}$.

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-\sin x}$. When $y = \frac{\sqrt{3}}{2}$, $x = \frac{\pi}{6}$, so

$\frac{dx}{dy} = \frac{1}{-\sin\frac{\pi}{6}} = \frac{1}{-\frac{1}{2}} = -2$.

25. $1 = x + \ln x$, $x = 1$ by inspection

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{1 + \frac{1}{x}} = \frac{x}{x+1}$. When $y = 1$, $x=1$, so

$\frac{dx}{dy} = \frac{1}{1+1} = \frac{1}{2}$.

26. $4 = x^5 + 2x^3 + 1$, $x=1$ by inspection

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{5x^4+6x^2}$. When $y = 4$, $x = 1$, so

$\frac{dx}{dy} = \frac{1}{5+6} = \frac{1}{11}$.

27. $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(-5)} = \frac{1}{7}$.

28. $(f^{-1})'(0.1) = \frac{1}{f'(f^{-1}(0.1))} = \frac{1}{f'(1)} = \frac{1}{3}$.

29. $(f^{-1})'(\pi) = \frac{1}{f'(f^{-1}(\pi))} = \frac{1}{f'(\sqrt{2})} = \frac{1}{\pi}$.

30. (a) $y = -2x^2+8x-5$, $x > 2$.

Consider $2y^2-8y+5+x = 0$, $y > 2$.

$y = \frac{8+\sqrt{64-4(2)(5+x)}}{2(2)} = \frac{8+\sqrt{64-40-8x}}{4}$

$= \frac{8+\sqrt{24-8x}}{4} = \frac{4+\sqrt{6-2x}}{2}$. Thus,

$f^{-1}(x) = \frac{4+\sqrt{6-2x}}{2}$.

(b) $(f^{-1})'(x) = -\frac{1}{2\sqrt{6-2x}}$. $(f^{-1})'(1) = -\frac{1}{4}$

(c) $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(3)}$

$= \frac{1}{-4(3)+8} = -\frac{1}{4}$.

31. (a) 0.3843967745 (b) 1.971536521

(c) 0.3836622700 (d) 1.553343046

(e) 0.9588938924 (f) 1.445468496

(g) -0.8480620790 (h) 1.446441332

(i) 2.711892987 (j) 0.3510036020

(k) 1.107148718 (l) -0.2526802551

32. (a) $\sin^{-1}(-\frac{1}{2}) = u, -\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

$\sin u = -\frac{1}{2}$, $u = -\frac{\pi}{6}$.

(b) $\arccos(-\frac{\sqrt{3}}{2}) = u, 0 \le u \le \pi$.

$\cos u = -\frac{\sqrt{3}}{2}$,

$u = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

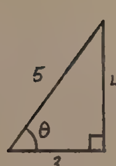
(c) $\arctan\sqrt{3} = u$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

$\tan u = \sqrt{3}$, $u = \frac{\pi}{3}$.

(d) $\text{arcsec}\sqrt{2} = u, 0 \le u \le \pi, u \ne \frac{\pi}{2}$.

$\sec u = \sqrt{2}$, $\cos u = \frac{1}{\sqrt{2}}$,

$u = \frac{\pi}{4}$.

33. 

Let $\theta = \tan^{-1}\frac{4}{3}$,

$\tan\theta = \frac{4}{3}$,

$\cos\theta = \frac{3}{5}$.

34. Let $\theta = \arctan(-\frac{5}{12})$, so $\tan\theta = -\frac{5}{12}$,

$-\frac{\pi}{2} < \theta < 0$. $1 + (-\frac{5}{12})^2 = \sec^2\theta$ or

$\sec^2\theta = (\frac{13}{12})^2$ or $\sec\theta = \frac{13}{12}$, so $\cos\theta = \frac{12}{13}$;

$\frac{\sin\theta}{\cos\theta} = \tan\theta$, so $\sin\theta = \cos\theta\tan\theta =$

$\frac{12}{13}(-\frac{5}{12}) = -\frac{5}{13}$.

35. $\sin(\sin^{-1}(-\frac{12}{13})) = -\frac{12}{13}$.

36. Let $\theta = \arccos(-\frac{3}{5})$, so

$\cos\theta = -\frac{3}{5}$, $\frac{\pi}{2} < \theta < \pi$.

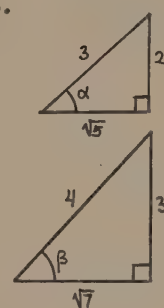
$(-\frac{3}{5})^2 + \sin^2\theta = 1$ or $\sin^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$;

so $\sin\theta = \frac{4}{5}$. Thus, $\tan\theta = \frac{\sin\theta}{\cos\theta} =$

$\frac{4/5}{-\frac{3}{5}} = -\frac{4}{3}$.

37. $\arcsin(\sin\frac{19\pi}{14}) = \arcsin(\sin(\pi+\frac{5\pi}{14}))$

$= -\frac{5\pi}{14}$ since $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$.

38. 

$\sin(\sin^{-1}\frac{2}{3}+\sin^{-1}\frac{3}{4})$.

Let $\alpha = \sin^{-1}\frac{2}{3}$, so

$\sin\alpha = \frac{2}{3}$,

$\cos\alpha = \frac{\sqrt{5}}{3}$.

$\beta = \sin^{-1}\frac{3}{4}$, so

$\sin\beta = \frac{3}{4}$,

$\cos\beta = \frac{\sqrt{7}}{4}$.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

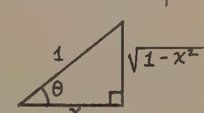
$= \frac{2}{3}(\frac{\sqrt{7}}{4}) + \frac{\sqrt{5}}{3}(\frac{3}{4}) = \frac{\sqrt{7}}{6} + \frac{\sqrt{5}}{4}$.

39. Let $\theta = \arccos x$. Show $\tan\theta = \frac{\sqrt{1-x^2}}{x}$.

For $-1 \le x \le 1$, $\cos\theta = x = \frac{x}{1}$.

$\tan\theta = \frac{\sqrt{1-x^2}}{x}$,

$|x| \le 1$.



40. Let $\theta = \arccos x$, so $\cos\theta = x$, $-1 \le x \le 1$

and $0 \le \theta \le \pi$ or $0 \le \frac{\theta}{2} \le \frac{\pi}{2}$.

$\sin\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-x}{2}}$.

41. $(3\sin t + 2)(2\sin t - 1) = 0$, so

$\sin t = -\frac{2}{3}$ or $\sin t = \frac{1}{2}$.

$t = \sin^{-1}(-\frac{2}{3})$ or $t = \frac{\pi}{6}$;

hat is, $t = -\sin^{-1}\frac{2}{3}$ or $t = \frac{\pi}{6}$.

42. $\cos\beta = \frac{t}{x}$; $x = \frac{t}{\cos\beta}$. $\sin(\alpha - \beta) = \frac{d}{x}$, so

$x = \frac{d}{\sin(\alpha - \beta)}$. Thus,

$$\frac{t}{\cos\beta} = \frac{d}{\sin(\alpha-\beta)}$$

or $d\cos\beta = t\sin(\alpha-\beta)$

$= t(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$;

so $\frac{d}{\cos\alpha} = t\tan\alpha - t\tan\beta$

or $t\tan\beta = t\tan\alpha - d\sec\alpha$.

Then $\tan\beta = \tan\alpha - \frac{d}{t}\sec\alpha$

so $\beta = \arctan(\tan\alpha - \frac{d}{t}\sec\alpha)$.

43. $y = 2\sin^{-1}\frac{x}{3}$. $\frac{dy}{dx} = \frac{2\cdot 1(1/3)}{\sqrt{1-(\frac{x}{3})^2}}$

$= \frac{2}{\sqrt{9-x^2}}$.

44. $g(x) = 4\tan^{-1}x^2$.

$g'(x) = \frac{4(1)}{1+(x^2)^2}(2x) = \frac{8x}{1+x^4}$.

45. $h'(x) = \frac{1}{\sqrt{x-1}\ \sqrt{(x-1)^2-1}}\ \frac{1}{2}(x-1)^{-\frac{1}{2}}$

$= \frac{1}{2}\ \frac{1}{x-1}\ \frac{1}{\sqrt{x-2}}$.

46. $p'(x) = \frac{-1}{\left|\frac{1+x}{1-x}\right|\sqrt{(\frac{1+x}{1-x})^2-1}}\ (\frac{2x}{(1-x)^2})$

$= \left|\frac{1-x}{1+x}\right|\left(\frac{-1}{\sqrt{\frac{4x}{(1-x)^2}}}\right)(\frac{2x}{(1-x)^2})$

$= \left|\frac{1-x}{1+x}\right|\frac{-|1-x|}{2\sqrt{x}}\ \frac{2x}{(1-x)^2} = \frac{-x}{|1+x|\sqrt{x}}$.

47. $F'(t) = \frac{-1}{\sqrt{1-(\sqrt{3t})^2}}\sqrt{3}(\frac{1}{2}t^{-\frac{1}{2}}) = \frac{-\sqrt{3}}{2\sqrt{t}\sqrt{1-3t}}$.

48. $G'(y) = \frac{1}{5}\left[\arccos(y^3+1)\right]^{-4/5}\frac{-1}{\sqrt{1-(y^3+1)^2}}3y^2$

$= \frac{-3y^2}{5\left[\arccos(y^3+1)\right]^{4/5}\sqrt{-y^6-2y^3}}$.

49. $H'(u) = u^2\ \frac{4u^3}{1+u^8} + \tan^{-1}u^4(2u)$

$= \frac{4u^5}{1+u^8} + 2u\tan^{-1}u^4$.

50. $f'(t) = 4(\cot^{-1}t^2)^3\ \frac{-1}{1+t^4}\ (2t) = \frac{-8t(\cot^{-1}t^2)^3}{1+t^4}$.

51. $g'(u) = 2(\csc^{-1}u)\frac{-1}{|u|\sqrt{u^2-1}} = \frac{-2\csc^{-1}u}{|u|\sqrt{u^2-1}}$.

52. $h'(x) = \dfrac{x\left[\frac{2x}{(x^2+1)\sqrt{(x^2+1)^2-1}}\right] - \sec^{-1}(x^2+1)}{x^2}$

$= \frac{2}{(x^2+1)\sqrt{x^4+2x^2}} - \frac{\sec^{-1}(x^2+1)}{x^2}$.

53. $q'(x) = x^3\ \frac{-1}{1+25x^2}(5) + \cot^{-1}5x(3x^2)$

$= \frac{-5x^3}{1+25x^2} + 3x^2\cot^{-1}5x$.

54. $\phi(x) = \dfrac{\sqrt{x^2-1}\ \frac{1}{x^2\sqrt{x^4-1}}(2x) - \sec^{-1}x^2\left[\frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)\right]}{x^2-1}$

$= \dfrac{\frac{2}{x\sqrt{x^2+1}} - \frac{x\sec^{-1}x^2}{\sqrt{x^2-1}}}{x^2-1}$

$= \frac{2\sqrt{x^2-1} - (x^2\sec^{-1}x^2)(\sqrt{x^2+1})}{x\sqrt{x^2+1}(x^2-1)^{3/2}}$.

55. $g'(u) = \left[17+(\sin^{-1}u)^2\right]^{34}\cdot(\frac{1}{\sqrt{1-u^2}})$

$= \frac{\left[17+(\sin^{-1}u)^2\right]^{34}}{\sqrt{1-u^2}}$.

56. $h'(x) = \left[\frac{1-(\tan^{-1}x)^2}{1+(\tan^{-1}x)^2}\right]^{14}\cdot\frac{1}{1+x^2}$.

57. $\cos^{-1}(x+y)+x(-\frac{1}{\sqrt{1-(x+y)^2}})\cdot(1+\frac{dy}{dx}) = 2y\frac{dy}{dx}$,

$\left[-2y - \frac{x}{\sqrt{1-(x+y)^2}}\right]\frac{dy}{dx} = \frac{x}{\sqrt{1-(x+y)^2}} - \cos^{-1}(x+y)$

$\frac{dy}{dx} = \frac{x-\cos^{-1}(x+y)\left[\sqrt{1-(x+y)^2}\right]}{-2y\sqrt{1-(x+y)^2}-x}$

$= \frac{\sqrt{1-(x+y)^2}\cos^{-1}(x+y)-x}{x+2y\sqrt{1-(x+y)^2}}$.

58. $\tan x^2\ \frac{dy}{dx} + y\cdot 2x\sec^2x^2 - y^4 - 4y^3x\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = \frac{y^4-2xy\sec^2x^2}{\tan x^2-4xy^3}$.

59. $3\sin(x-y)+3x\cos(x-y)\left[1-\frac{dy}{dx}\right]=x^2\frac{dy}{dx}+2xy,$

$-3x\cos(x-y)\frac{dy}{dx}-x^2\frac{dy}{dx}=$

$2xy-3\cdot\sin(x-y)-3x\cos(x-y),$

$\frac{dy}{dx}=\frac{3x\cos(x-y)+3\sin(x-y)-2xy}{x^2+3x\cos(x-y)}.$

60. $\frac{y}{1+x^2}+(\tan^{-1}x)\frac{dy}{dx}-\tan^{-1}y-\frac{x}{1+y^2}\frac{dy}{dx}=0,$

$\frac{dy}{dx}=\frac{\tan^{-1}y-\frac{y}{1+x^2}}{\tan^{-1}x-\frac{x}{1+y^2}}=$

$\frac{(1+x^2)(1+y^2)\tan^{-1}y-y(1+y^2)}{(1+x^2)(1+y^2)\tan^{-1}x-x(1+x^2)}=$

$\frac{(1+y^2)[(1+x^2)\tan^{-1}y-y]}{(1+x^2)[(1+y^2)\tan^{-1}x-x]}.$

61. Let $3u = x$, so $3du = dx$.

$\int\frac{dx}{\sqrt{9-x^2}}=\int\frac{3du}{\sqrt{9-9u^2}}=\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C$

$=\sin^{-1}\frac{x}{3}+C.$

62. Let $u=x\sqrt{2}$, $du=dx\sqrt{2}$. $\int(2-u^2)^{-\frac{1}{2}}du=\int\frac{1}{\sqrt{2-u^2}}du$

$=\int\frac{\sqrt{2}\,dx}{\sqrt{2-2x^2}}=\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C=\sin^{-1}\frac{u}{\sqrt{2}}+C.$

63. Let $x = 6u$, $dx = 6du$.

$\int\frac{dx}{x^2+36}=\int\frac{6du}{36u^2+36}=\frac{1}{6}\int\frac{du}{u^2+1}=\frac{1}{6}\tan^{-1}u+C$

$=\frac{1}{6}\tan^{-1}\frac{x}{6}+C.$

64. Let $u = x^2$, $du = 2xdx$.

$\int\frac{2xdx}{1+x^4}=\int\frac{du}{1+u^2}=\tan^{-1}u+C=\tan^{-1}x^2+C.$

65. Let $u=\sqrt{2}\,x^2$, $u^2=2x^4$, $du=2\sqrt{2}x\,dx$.

So $\int\frac{xdx}{\sqrt{9-2x^4}}=\frac{1}{2\sqrt{2}}\int\frac{du}{\sqrt{9-u^2}}=\frac{1}{2\sqrt{2}}\sin^{-1}\frac{u}{3}+C$

$=\frac{1}{2\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}x^2}{3}\right)+C.$

66. Let $V = \sec u$, $dV = \sec u\tan u\,du$.

So $\int\frac{dV}{\sqrt{16-V^2}}=\sin^{-1}\frac{V}{4}+C=\sin^{-1}\left(\frac{\sec u}{4}\right)+C.$

67. Let $7u = x$, $7du = dx$.

$\int\frac{dx}{x\sqrt{x^2-49}}=\int\frac{7du}{7u\sqrt{49u^2-49}}=\frac{1}{7}\int\frac{du}{u\sqrt{u^2-1}}$

$=\frac{1}{7}\sec^{-1}|u|+C=\frac{1}{7}\sec^{-1}\left|\frac{x}{7}\right|+C.$

68. Let $u = t^2$, $t=\sqrt{u}$, $du = 2tdt$.

$\int\frac{dt}{t\sqrt{t^4-1}}=\frac{\frac{du}{2\sqrt{u}}}{\sqrt{u}\sqrt{u^2-1}}=\frac{1}{2}\int\frac{du}{u\sqrt{u^2-1}}$

$=\frac{1}{2}\sec^{-1}|u|+C=\frac{1}{2}\sec^{-1}|t^2|+C.$

69. Let $u = \cos x$, $du = -\sin x\,dx$.

So $\int\frac{\sin x\,dx}{4+\cos^2x}=-\int\frac{du}{4+u^2}=-\frac{1}{2}\tan^{-1}\frac{u}{2}+C$

$=-\frac{1}{2}\tan^{-1}\left(\frac{\cos x}{2}\right)+C.$

70. Let $u=\sin^{-1}x^2$, $du=\frac{2x}{\sqrt{1-x^4}}dx$. So

$\int\frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}dx=\frac{1}{2}\int udu=\frac{1}{2}\cdot\frac{u^2}{2}+C$

$=\frac{(\sin^{-1}x^2)^2}{4}+C.$

71. Let $u=\cot^{-1}v$, $du=-\frac{1}{1+v^2}dv$. So

$\int\frac{\cot^{-1}v\,dv}{1+v^2}=-\int udu=-\frac{u^2}{2}+C$

$=\frac{-(\cot^{-1}v)^2}{2}+C.$

72. Let $u = \cot x$, $du = -\csc^2x\,dx$. So

$\int\frac{-du}{\sqrt{1-u^2}}=-\sin^{-1}u+C=-\sin^{-1}(\cot x)+C.$

73. $\int_0^5\frac{dx}{25+x^2}=\frac{1}{5}\tan^{-1}\frac{x}{5}\Big|_0^5$

$=\frac{1}{5}\left[\tan^{-1}1-\tan^{-1}0\right]=\frac{1}{5}\left[\frac{\pi}{4}-0\right]=\frac{\pi}{20}.$

74. Let $u = x^2$, $du = 2x\,dx$. So $\int_0^{\sqrt{3}}\frac{xdx}{9+x^4}$

$=\frac{1}{2}\int_0^3\frac{du}{9+u^2}=\frac{1}{2}\left(\frac{1}{3}\right)\tan^{-1}\frac{u}{3}\Big|_0^3$

$= \frac{1}{6}(\tan^{-1}1-\tan^{-1}0) = \frac{1}{6}(\frac{\pi}{4} - 0) = \frac{\pi}{24}$.

75. $y' = \frac{a\cos t}{\sqrt{1-a^2\sin^2 t}} - \frac{a\cos(2-t)}{\sqrt{1-a^2\sin^2(2-t)}}$.

Setting $y' = 0$, we have

$$\frac{a\cos t}{\sqrt{1-a^2\sin^2 t}} = \frac{a\cos(2-t)}{\sqrt{1-a^2\sin^2(2-t)}},$$

$a^2\cos^2 t-a^4\cos^2 t\sin^2(2-t) =$

$a^2\cos^2(2-t)-a^4\sin^2 t\cos^2(2-t)$, $\cos^2 t-\cos^2(2-t)$

$= a^2\left[\cos^2 t\sin^2(2-t)-\sin^2 t\cos^2(2-t)\right]$.

Since $\cos^2\alpha-\cos^2\beta = \sin(\alpha+\beta)\sin(\beta-\alpha) =$ $\cos^2\alpha\sin^2\beta-\sin^2\alpha\cos^2\beta$ with $\alpha = t$, $\beta = 2-t$, we can rewrite the above as

$\sin\left[t+(2-t)\right]\sin\left[(2-t)-t\right] =$

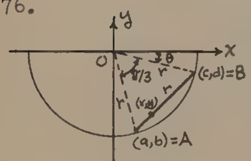
$a^2\sin\left[t+(2-t)\right]\sin\left[(2-t)+t\right]$ or

$(1-a^2)\sin 2\sin(2-2t) = 0$.

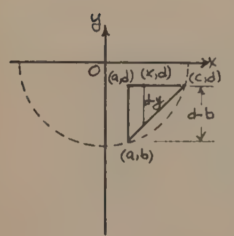
The critical values are those values of t for which $\sin(2-2t) = 0$, that is, $2-2t = n\pi$ or $2t = 2-n\pi$. Thus, the critical values are $t = 1 + \frac{n\pi}{2}$,

$n = 0, \pm 1, \pm 3, \ldots$

76.



(a)



Since $\overline{OA} = \overline{AB} = \overline{OR} = r$ (Figure (a)), then OAB is an equilateral triangle, so that angle BOA $= \frac{\pi}{3}$. In Figure (a), $B=(c,d)=(r\cos\theta,-r\sin\theta)$ and $A=(a,b)=\left[r\cos(\theta+\frac{\pi}{3}),\ -r\sin(\theta+\frac{\pi}{3})\right]$. In Figure (b), the distance from (c,d) to (x,y) is $\frac{3}{4}r$; hence, by similar triangles $\frac{d-y}{d-b} = \frac{(3r/4)}{r} = \frac{3}{4}$. Solving the latter equation for y, we obtain $y=\frac{1}{4}d+\frac{3}{4}b =$

$= \frac{1}{4}\left[-r\sin\theta-3r\sin(\theta+\frac{\pi}{3})\right] =$

$-\frac{r}{4}\left[\sin\theta + 3\sin(\theta + \frac{\pi}{3})\right]$. Therefore, $\frac{dy}{d\theta} = -\frac{r}{4}\left[\cos\theta+3\cos(\theta + \frac{\pi}{3})\right]$, so that for the minimum value of y,

$\cos\theta + 3\cdot\cos(\theta + \frac{\pi}{3}) = 0$, $\cos\theta =$

$-3\cos\theta\cos\frac{\pi}{3} + 3\sin\theta\sin\frac{\pi}{3} =$

$-\frac{3}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta$, or $\sin\theta=\frac{5}{3\sqrt{3}}\cos\theta$.

The inclination angle of $\overline{AB}$ is therefore

$$\tan^{-1}\frac{d-b}{c-a} = \tan^{-1}\left[\frac{-r\sin\theta+r\sin(\theta+\pi/3)}{r\cos\theta-r\cos(\theta+\pi/3)}\right]$$

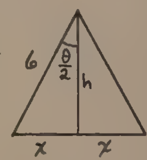
$$= \tan^{-1}\left[\frac{-\sin\theta+\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}}{\cos\theta-\cos\theta\cos\frac{\pi}{3}+\sin\theta\sin\frac{\pi}{3}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta}{\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{3}\cos\theta - \frac{5}{3\sqrt{3}}\cos\theta}{\cos\theta + \frac{5}{3}\cos\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{3} - \frac{5}{3\sqrt{3}}}{1 + \frac{5}{3}}\right] = \tan^{-1}\frac{1}{2\sqrt{3}} \approx 0.28 \text{ radian}$$

77. We want to find $\frac{dA}{dt}$ when $\frac{d\theta}{dt} = 1° = \frac{\pi}{180}$ radian/min.

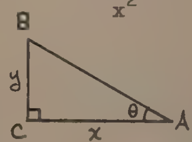


$A = \frac{1}{2}(2x)h = xh$ and $\sin\frac{\theta}{2} = \frac{x}{6}$, $\cos\frac{\theta}{2} = \frac{h}{6}$, and so $A = (6\sin\frac{\theta}{2})(6\cos\frac{\theta}{2})=18\sin\theta$.

Hence, $\frac{dA}{dt} = 18\cos\theta\frac{d\theta}{dt} = 18(\cos\theta)(\frac{\pi}{180})$

$= \frac{\pi\cos\theta}{10}$ square centimeters per min.

78. $\theta = \tan^{-1}\frac{y}{x}$. $|d\theta| = \left|\frac{d(y/x)}{1+\frac{y^2}{x^2}}\right| = \left|\frac{\frac{(xdy-ydx)}{x^2}}{(\frac{x^2+y^2}{x^2})}\right|$



$= \left|\frac{xdy-ydx}{x^2+y^2}\right| = \left|\frac{(\frac{dy}{y} - \frac{dx}{x})}{(\frac{x^2+y^2}{xy})}\right|$

$= \left|\frac{dy}{y} - \frac{dx}{x}\right| \cdot \frac{xy}{x^2+y^2} \leq$

$$\left(\left|\frac{dy}{y}\right| + \left|\frac{dx}{x}\right|\right)\cdot\frac{xy}{x^2+y^2}\ (0.01+0.01)\cdot\frac{xy}{x^2+y^2}.$$

Hence, $|\Delta\theta| \approx |d\theta| \leq \frac{0.02xy}{x^2+y^2}$.

79. $f'(x) = \frac{2x}{x^2+7}$.

80. $f'(t) = \frac{-3 \sin 3t}{\cos 3t} = -3 \tan 3t$.

81. $g'(r) = \frac{1}{r\sqrt{r+2}}\left[\frac{r}{2\sqrt{r+2}} + \sqrt{r+2}\right] = \frac{3r+4}{2r(r+2)}$.

82. $G(u) = 3 \ln(u-1)$, so that $G'(u) = \frac{3}{u-1}$.

83. $g'(x) = x^2\left[\frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}\right]+2x \tan^{-1}(\ln x) =$

$\frac{x}{1+(\ln x)^2} + 2x \tan^{-1}(\ln x)$.

84. $h'(x) = \cos(\ln x)^2 \left[\frac{1}{x^2}(2x)\right] = \frac{2 \cos(\ln x^2)}{x}$.

85. $F'(u) = \dfrac{\frac{u(2)\ln u}{u} - (\ln u)^2}{u^2}$

$= \frac{(\ln u)(2-\ln u)}{u^2}$.

86. $G'(x) = \frac{1}{7}(\ln x)^{-6/7}\left(\frac{1}{x}\right) = \frac{1}{7x \sqrt[7]{(\ln x)^6}}$.

87. $f'(x) = \frac{x}{\sin x}\left[\frac{x \cos x-\sin x}{x^2}\right]= \cot x - \frac{1}{x}$.

88. $H'(x) = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{x}{\sqrt{x^2+1}}\right)$.

89. $g'(x) = -12x^2\cdot e^{-4x^3}$.

90. $H'(x) = xe^{-x}-e^{-x} = e^{-x}(x-1)$.

91. $F'(x) = \frac{1}{\sqrt{1-(e^{-2x})^2}}(-2e^{-2x}) = \frac{-2e^{-2x}}{\sqrt{1-e^{-4x}}}$.

92. $g'(u) = e^u(-\csc^2 e^u)(e^u)+ e^u \cot e^u$

$= e^u\left[-e^u \csc^2 e^u+\cot e^u\right]$.

93. $f'(t) = \left(\frac{e^t-2}{e^t+2}\right)\left[\frac{(e^t-2)(e^t)-(e^t+2)(e^t)}{(e^t-2)^2}\right]$

$= \frac{-4e^t}{(e^t+2)(e^t-2)} = \frac{-4e^t}{e^{2t}-4}$.

94. $H'(x) = \frac{\cos x\ e^{\sin x}}{e^{\sin x}+5}$.

95. $g'(x) = \frac{1}{\sqrt{1-x^4}}(2x)-x(3x^2)e^{x^3}-e^{x^3} =$

$\frac{-2x}{\sqrt{1-x^4}} - e^{x^3}(3x^3+1)$.

96. $g'(x) = e^x \frac{\cos x}{\sin x} + e^x\ln(\sin x)$

$= e^x\left[\cot x+\ln(\sin x)\right]$.

97. $f'(x) =(\sec^2 e^x)(e^x)$.

98. $f'(x) = e^x(2x-2)+e^x(x^2-2x+5) = e^x(x^2+3)$.

99. $f'(x) = 4(3-e^{4x})^3(-4e^{4x})$

$=-16e^{4x}(3-e^{4x})^3$.

100. $g(x) = \tan h\ x$, so that $g'(x) =$

$\sec h^2 x = \frac{4}{(e^x+e^{-x})^2}$.

101. $f'(x) = \left(\frac{e^{-x}+2}{e^x+2}\right)\left[\frac{(e^{-x}+2)(e^x)-(e^x+2)(-e^{-x})}{(e^{-x}+2)^2}\right]$

$= \frac{2e^x+2e^{-x}+2}{(e^x+2)(e^{-x}+2)} = \frac{2(e^x+e^{-x}+1)}{(e^x+2)(e^{-x}+2)}$.

102. $h'(z) = \frac{e^{3z}(-2e^{2z})}{\sqrt{1-(e^{2z})^2}} + 3e^{3z}\cos^{-1} e^{2z}$

$= \frac{-2e^{5z}}{\sqrt{1-e^{4z}}} + 3e^{3z}\cos^{-1} e^{2z}$.

103. $F'(x) = 4ex^{4e-1}$.

104. $g'(x) = -17\pi x^{-17\pi-1}$.

105. $f'(x) = (\ln 5)(-\sin x)5^{\cos x}$

$= -(\ln 5)(\sin x)5^{\cos x}$.

106. $g'(t) = (\ln 3)(2t)e^{t^2+2}$.

107. $f'(x) = (\ln 7)(\cos x^2)(2x)7^{\sin x^2}$.

108. $g'(x) = (x^2+7)(\ln 2)(-5)2^{-5x}+(2x)(2^{-5x})$

$= 2^{-5x}\left[2x-5(x^2+7)\ln 2\right]$.

109. $g'(x) = 5 \ln 3(3^{5x})2^{4x^2}+3^{5x}\ln 2(8x)2^{4x^2}$

$= 2^{4x^2}3^{5x}(5 \ln 3+8x \ln 2)$.

110. $H'(x) = -\sin x\ e^{\cos x}2^{4x}+e^{\cos x}(\ln 2)(4)2^{4x}$

$= 2^{4x}e^{\cos x}(4 \ln 2-\sin x)$.

111. $f'(t) = \dfrac{t\cdot\frac{1}{(\ln 7)t} - \log_7 t}{t^2}$

$= \dfrac{1-(\ln 7)\log_7 t}{t^2 \ln 7} = \dfrac{1-\ln t}{t^2\ln 7}$.

112. $g'(u) = \dfrac{(u+7)}{u(\ln 3)}\left(\dfrac{7}{(u+7)^2}\right) = \dfrac{7}{u(u+7)\ln 3}$.

113. $g'(x) = \frac{1}{4}(\log_{10}x)^{-3/4}\left(\dfrac{1}{(\ln 10)x}\right)$

$= \dfrac{1}{4x(\ln 10)(\log_{10}x)^3}$.

114. $f'(x) = \frac{1}{5}\left[\log_{10}\left(\frac{1+x}{1-x}\right)\right]^{-4/5}\left(\frac{1-x}{1+x}\right)\left(\frac{1}{\ln 10}\right)\left[\frac{+2}{(1-x)^2}\right]$

$= \left[\log_{10}\left(\frac{1+x}{1-x}\right)^{-4/5}\right]\cdot\left[\dfrac{2}{5(1-x^2)\ln 10}\right]$.

115. $g'(x) = (\sin h\ e^{4x})(4e^{4x})$.

116. $H'(s) = \cosh(\sin^{-1}s)\left[\dfrac{1}{\sqrt{1-s^2}}\right]$

$= \dfrac{\cosh(\sin^{-1}s)}{\sqrt{1-s^2}}$.

117. $g'(t) = -\operatorname{csch}(e^{-t})\coth(e^{-t})(-e^{-t})$

$= e^{-t}\operatorname{csch}(e^{-t})\coth(e^{-t})$.

118. $F'(x) = \dfrac{x\ \operatorname{sech}^2(\sin x)\left[\cos x\right]-\tanh(\sin x)}{x^2}$

119. $f'(x) = (-\operatorname{sech} x^2\tanh x^2)(2x)e^{\operatorname{sech} x^2}$.

120. $g'(u) = \sinh u^2(\operatorname{sech}^2 3u)(3)+\cosh u^2$

$\cdot\ (2u)(\tanh 3u)$.

121. $g'(t) = \dfrac{1}{\tanh t+\operatorname{sech} t}\left[\operatorname{sech}^2 t-\operatorname{sech} t\ \tanh t\right]$

$= \dfrac{\operatorname{sech} t(\operatorname{sech} t-\tanh t)}{\tanh t+\operatorname{sech} t}$.

122. $f'(x) = \dfrac{1}{1+(\sinh x^3)^2}\cdot(\cosh x^3)(3x^2)$

$= \dfrac{3x^2\cosh x^3}{\cosh^2 x^3} = 3x^2\ \operatorname{sech} x^3$.

123. $g'(x) = \dfrac{1}{\sqrt{1+(3x+1)^2}}(3) = \dfrac{3}{\sqrt{9x^2+6x+2}}$.

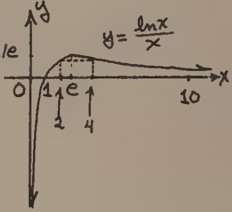
124. $g'(u) = \dfrac{1}{1-(e^{5u})^2}(5e^{5u}) = \dfrac{5e^{5u}}{1-e^{10u}}$.

125. $f'(x) = \dfrac{1}{1-(e^x)^2}(e^x) = \dfrac{e^x}{1-e^{2x}}$.

126. $F'(x) = \dfrac{1}{1-(e^{-x^2})^2}\cdot(-2x)e^{-x^2}$

$= \dfrac{-2xe^{-x^2}}{1-e^{-2x^2}}$.

127. No. To aid in sketching the function $\frac{\ln x}{x}$, we look at the first derivative: $D_x\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} = 0$



for $x=e$. When $x<e$, $\ln x < 1$, and $D_x\left(\frac{\ln x}{x}\right) > 0$; for $x > e$, $\ln x > 1$, so that $D_x\left(\frac{\ln x}{x}\right) < 0$. Hence, $(e,\frac{1}{e}) \approx (2.72, 0.37)$ is the point where a maximum occurs. We can see there is an x other than 2 for which $\frac{\ln x}{x} = \frac{\ln 2}{2}$; in fact, $\frac{\ln 4}{4} = \frac{\ln 2}{2}$.

128. Yes. Since the function is monotone increasing on $(0,1]$, each negative value of y corresponds to exactly one value of x on this interval.

129. $\ln y = x\ln 3x$, $\frac{1}{y}\frac{dy}{dx} = \ln 3x+x\left(\frac{3}{3x}\right)$

$= \ln 3x+1$, $\frac{dy}{dx} = (3x)^x(\ln 3x+1)$.

130. $\ln y = x^3\ln x$, $\frac{1}{y}\frac{dy}{dx} = x^3\left(\frac{1}{x}\right)+3x^2\ln x$

$= x^2+3x^2\ln x$, $\frac{dy}{dx} = x^{x^3}(x^2)(1+3\ln x)$

$= x^{2+x^3}(1+3\ln x)$.

131. $\ln y = x^2\ln(\sin x)$, $\frac{1}{y}\frac{dy}{dx} = x^2\frac{\cos x}{\sin x} +$

$2x\ln(\sin x)$, $\frac{dy}{dx} = (\sin x)^{x^2}\left[x^2\cot x +\right.$

$\left.2x\ln(\sin x)\right]$.

132. $\ln y = 2x\ln(\cosh x)$, $\frac{1}{y}\frac{dy}{dx} = 2x\frac{\sinh x}{\cosh x} +$

$2\ln(\cosh x)$, $\frac{dy}{dx} = (\cosh x)^{2x}(2\left[x\tanh x+\right.$

$\left.\ln(\cosh x)\right])$.

133. $\ln y = x^3\ln(\tanh^{-1}x)$, $\frac{1}{y}\frac{dy}{dx} =$

$$\frac{x^3(\frac{1}{1-x^2})}{(\tanh^{-1}x)} + 3x^2\ln(\tanh^{-1}x),$$

$$\frac{dy}{dx} = (\tanh^{-1}x)^{x^3}\left[\frac{x^3}{(\tanh^{-1}x)(1-x^2)} + 3x^2\ln(\tanh^{-1}x)\right].$$

134. $\ln y = \cos^{-1}x \ln x$, $\frac{1}{y}\frac{dy}{dx} = \frac{\cos^{-1}x}{x} + \frac{-\ln x}{\sqrt{1-x^2}}$, $\frac{dy}{dx} = x^{\cos^{-1}x}\left[\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right]$.

135. $\ln y = \ln(\cos x)+\frac{1}{3}\ln(1+\sin^2 x)-5\ln(\sin x)$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{\cos x} + \frac{\sin x \cos x}{3(1+\sin^2 x)} - \frac{5\cos x}{\sin x},$$

$$\frac{dy}{dx} = \frac{\cos x \sqrt[3]{1+\sin^2 x}}{\sin^5 x}\left[-\tan x + \frac{\sin 2x}{3(1+\sin^2 x)} - 5\cot x\right].$$

136. $\ln y = 2\ln x+\ln(\sin x) - \frac{1}{2}\ln(1-3\tan x \sec 2x)$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{-3\sec^2 x \sec 2x - 3\tan x(2\sec 2x \tan 2x)}{2(1-3\cdot\tan x \sec 2x)},$$

$$\frac{dy}{dx} = \frac{x^2 \sin x}{\sqrt{1-3\tan x \cdot \sec 2x}}\left[\frac{2}{x} + \cot x + \frac{3\sec 2x(\sec^2 x+\tan x\cdot\tan 2x)}{2(1-3\cdot\tan x \sec 2x)}\right].$$

137. $\ln y=2\ln x+3\ln(x+5)+\ln(\sin 2x)-\ln(\sec 3x)$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{3}{x+5} + \frac{2\cos 2x}{\sin 2x} - \frac{3\sec 3x \tan 3x}{\sec 3x},\quad \frac{dy}{dx} = \frac{x^2(x+5)^3\sin 2x}{\sec 3x}\left[\frac{2}{x} + \frac{3}{x+5} + 2\cot 2x - 3\tan 3x\right].$$

138. $\ln y = \ln x+\ln(\cot x)-\ln(x+1)-2\ln(x+3) - 4\ln(x+7)$, $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{-\csc^2 x}{\cot x} - \frac{1}{x+1} - \frac{2}{x+3} - \frac{4}{x+7}$, $\frac{dy}{dx} = \frac{x\cot x}{(x+1)(x+3)^2(x+7)^4}\left[\frac{1}{x} - \frac{1}{\sin x \cos x} - \frac{1}{x+1} - \frac{2}{x+3} - \frac{4}{x+7}\right]$.

139. $f(x) = a^x$, so that $f'(x)$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}.$$

Hence, $f'(0) = \lim_{h\to 0}\frac{a^h-a^0}{h} = \lim_{h\to 0}\frac{a^h-1}{h}$.

But we know that $f'(0) = a^0\ln a = \ln a$.

Hence, $\lim_{h\to 0}\frac{a^h-1}{h} = \ln a$.

140. For $t \neq 1$, $p(t) = \int_a^b x^t dx = \frac{x^{t+1}}{t+1}\Big|_a^b = \frac{b^{t+1}-a^{t+1}}{t+1}$; and $p(-1) = \int_a^b \frac{1}{x}dx = \ln|x|\Big|_a^b = \ln b - \ln a$. Now $\lim_{t\to -1}\frac{b^{t+1}-a^{t+1}}{t+1} =$

$$\lim_{t\to -1}\frac{\frac{(\frac{b}{a})^{t+1}-1}{t+1}}{a^{t+1}} = \lim_{h\to 0}\left[\frac{(\frac{b}{a})^h-1}{h}\right](a^h)$$

$$= \lim_{h\to 0}\frac{(\frac{b}{a})^h-1}{h}\cdot\lim_{h\to 0}a^h = \ln(\tfrac{b}{a})\cdot 1 \text{ (by}$$

Problem 139) $= \ln b - \ln a = p(-1)$. Hence p is continuous at the number -1.

141. $e^{4y}+4xe^{4y}\frac{dy}{dx}+\cos y - x\sin y\frac{dy}{dx} = 0$,

$$\frac{dy}{dx} = \frac{e^{4y}+\cos y}{x(\sin y=4e^{4y})} = \frac{2}{x^2(\sin y-4e^{4y})}.$$

(See original statement of problem.)

142. $2^x\frac{dy}{dx}+y(\ln 2)(2^x)+e^y+xe^y\frac{dy}{dx} = 0$,

$$\frac{dy}{dx} = \frac{-\left[y\ln 2(2^x)+e^y\right]}{2^x+xe^y}.$$

143. $\sinh(x-y)\left[1 - \frac{dy}{dx}\right]+\cosh(x+y)\left[1+\frac{dy}{dx}\right] = 0$,

$$\frac{dy}{dx}\left[\cosh(x+y)-\sinh(x-y)\right] = -\left[\sinh(x-y) + \cosh(x+y)\right],$$

$$\frac{dy}{dx} = \frac{\sinh(x-y)+\cosh(x+y)}{\sinh(x-y)-\cosh(x+y)}.$$

144. $\log_{10}(x+y)+\log_{10}(x-y) = 2$,

$\log_{10}(x+y)(x-y) = 2$, $\log_{10}(x^2-y^2) = 2$.

So $\frac{2x-2y\frac{dy}{dx}}{(\ln 10)(x^2-y^2)} = 0$, and $2x-2y\frac{dy}{dx} = 0$;

hence, $\frac{dy}{dx} = \frac{x}{y}$.

145. $D_x y = \frac{1}{5+(\ln x)^3} \cdot D_x \ln x = \frac{1}{x[5+(\ln x)]^3}$.

146. $D_x y = \frac{1}{3+(\cosh x)^2} D_x \cosh x = \frac{\sinh x}{3+\cosh^2 x}$.

147. Let $u = 8+3x$, so that $du = 3dx$. So
$\int \frac{dx}{8+3x} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C =$
$\frac{1}{3}\ln|8+3x| + C$.

148. Put $u = \ln x$, so that $du = \frac{1}{x}dx$. So
$\int \frac{\sin(\ln x)}{x}dx = \int \sin u\, du = -\cos u + C$
$= -\cos(\ln x) + C$.

149. Put $u = \ln x$, so that $du = \frac{1}{x}dx$. So
$\int \frac{\ln x}{x}dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$.

150. Put $u = x^2-4$, so that $du = 2x\, dx$.
So $\int xe^{x^2-4}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$
$= \frac{1}{2}e^{x^2-4} + C$.

151. Put $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}dx$. So
$\int e^{\sqrt{x}} \frac{dx}{\sqrt{x}} = 2\int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$.

152. Put $u = e^x$, so that $du = e^x dx$. So
$\int \frac{e^x dx}{\sqrt{1-e^{2x}}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(e^x) + C$.

153. Put $u = e^x$, so that $du = e^x dx$. So
$\int \frac{e^x dx}{\cos^2 e^x} = \int \frac{du}{\cos^2 u} = \int \sec^2 u\, du =$
$\tan u + C = \tan(e^x) + C$.

154. Put $u = \frac{1}{x}$, so that $du = -\frac{1}{x^2}dx$. So
$\int \frac{\pi^{1/x}}{x^2}dx = -\int \pi^u du = \frac{-\pi^u}{\ln \pi} + C = \frac{-\pi^{1/x}}{\ln \pi} + C$.

155. $\int 2^x \cdot 5^x\, dx = \int 10^x dx = \frac{10^x}{\ln 10} + C$.

156. Let $u = 3^{2x}$, so that $du = 2(\ln 3)(3)^{2x}dx$,
so $3^{2x}dx = \frac{du}{2\ln 3}$. So $\int 3^{2x}\cos(3^{2x})dx =$
$\frac{1}{2\ln 3}\int \cos u\, du = \frac{1}{2\ln 3}\sin u + C =$
$\frac{1}{2\ln 3}\sin(3^{2x}) + C$.

157. Put $u = 1 + \frac{1}{x}$, so that $du = -\frac{1}{x^2}dx$.
$\int_1^4 \frac{\frac{1}{x^2}}{1+\frac{1}{x}}dx = -\int_2^{5/4} \frac{du}{u} = \ln|u| \Big|_{5/4}^{2}$
$= \ln 2 - \ln\frac{5}{4} = \ln\frac{8}{5}$.

158. Put $u = \cosh 5x$, so that $du = 5\sinh 5x dx$.
So $\int e^{\cosh 5x}\sinh 5x\, dx = \frac{1}{5}\int e^u du =$
$\frac{1}{5}e^u + C = \frac{1}{5}e^{\cosh 5x} + C$. Hence,
$\int_0^{1/5} e^{\cosh 5x}\sinh 5x\, dx = \frac{1}{5}e^{\cosh 5x}\Big|_0^{1/5}$
$= \frac{1}{5}(e^{\cosh 1} - e^{\cosh 0}) = \frac{1}{5}(e^{\cosh 1} - e)$.

159. Put $u = \coth x$, so that $du = -\operatorname{csch}^2 x dx$.
So $\int \operatorname{csch}^2 x \coth x\, dx = -\int u du =$
$\frac{u^2}{2} + C = \frac{-\coth^2 x}{2} + C$.

160. Put $u = \sinh^{-1}x$, so that $du = \frac{1}{\sqrt{1+x^2}}dx$.
So $\int \frac{(\sinh^{-1}x)^5}{\sqrt{1+x^2}}dx = \int u^5 du = \frac{u^6}{6} + C$
$= \frac{(\sinh^{-1}x)^6}{6} + C$.

161. Put $u = \cosh^{-1}x$, so that $du = \frac{1}{\sqrt{x^2-1}}dx$.
So $\int \frac{e^{\cosh^{-1}x}}{\sqrt{x^2-1}}dx = \int e^u du = e^u + C$
$= e^{\cosh^{-1}x} + C$ or $x+\sqrt{x^2-1} + C$.

162. $\int \frac{dx}{\sqrt{16+x^2}} = \sinh^{-1}\frac{x}{4} + C$.

163. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + C$.

164. Put $u = 3x$, so that $du = 3dx$. So
$\int \frac{dx}{16-9x^2} = \frac{1}{3}\int \frac{du}{16-u^2} =$

$$\frac{1}{12}\begin{cases} \tanh^{-1}\frac{u}{4} + C & |u| < a \\ \coth^{-1}\frac{u}{4} + C, & |u| > a > 0 \end{cases} =$$

$$\frac{1}{12}\begin{cases} \tanh^{-1}\frac{3x}{4} + C, & |3x| < 4 \\ \coth^{-1}\frac{3x}{4} + C, & |3x| > 4 \end{cases}$$

165. Put $u = 2x$, so that $du = 2dx$. So

$$\int \frac{dx}{x\sqrt{16-4x^2}} = \frac{1}{2}\int \frac{du}{\frac{u}{2}\sqrt{16-u^2}} = -\frac{1}{4}\operatorname{sech}^{-1}\frac{|u|}{4}+C$$

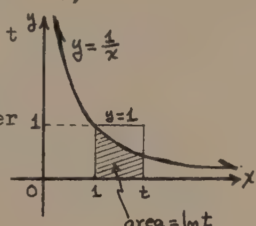
$$= -\frac{1}{4}\operatorname{sech}^{-1}\frac{|2x|}{4} + C = -\frac{1}{4}\operatorname{sech}^{-1}\frac{|x|}{2} + C.$$

166. Put $u = 2x$, so that $du = 2dx$. So

$$\int \frac{dx}{x\sqrt{16-4x^2}} = \frac{1}{2}\int \frac{du}{\frac{u}{2}\sqrt{16+u^2}} = -\frac{1}{4}\operatorname{csch}^{-1}\frac{|u|}{4}+C$$

$$= -\frac{1}{4}\operatorname{csch}^{-1}\frac{|2x|}{4}+C = -\frac{1}{4}\operatorname{csch}^{-1}\frac{|x|}{2} + C.$$

167. Since $\ln t < 0$ when $t < 1$, then clearly $t > \ln t$ for $0 < t < 1$. For $t > 1$, the area under the curve $y=1$ from $x=1$ to $x=t$ is greater than (or equal for $t=1$) the area under the curve $y = \frac{1}{x}$ on the same interval. Hence,



$\int_1^t 1dx \geq \int_1^t \frac{1}{x}\,dx$; integrating, we get $t-1 \geq \ln t$. But $t > t-1$. Hence, $t > \ln t$ for all positive values of t.

168. By Problem 167, $t > \ln t$, so that $e^t > e^{\ln t}$, $t > 0$, and so $e^t > t$. Hence, $(e^t)^n > t^n$ for all positive integers n and $t > 0$. Now, let $t = \frac{x}{n}$, $x > 0$. Therefore, $e^{x/n\ n} > (\frac{x}{n})^n$ and $e^x > \frac{x^n}{n^n}$, so that $\frac{e^x}{x^n} > \frac{1}{n^n} = (\frac{1}{n})^n$.

169. (a) In Problem 168, put $n=2$, so that $\frac{e^x}{x^2} > (\frac{1}{2})^2$. So $\frac{e^x}{x} > \frac{x}{4}$.

(b) Since $\frac{e^x}{x} > \frac{x}{4}$ and $\lim_{x\to+\infty} \frac{x}{4} = +\infty$, then $\lim_{x\to+\infty} \frac{e^x}{x} = +\infty$.

(c) In Problem 168, put $n=3$, so that $\frac{e^x}{x^3} > (\frac{1}{3})^3$, and so $\frac{e^x}{x^2} > \frac{x}{27}$.

(d) Since $\frac{e^x}{x^2} > \frac{x}{27}$ and $\lim_{x\to+\infty} \frac{x}{27} = +\infty$, then $\lim_{x\to+\infty} \frac{e^x}{x^2} = +\infty$.

170. (a) According to Problem 168, $\frac{e^x}{x^{n+1}} > (\frac{1}{n+1})^{n+1}$ holds for all positive integers n, $x > 0$. So $\frac{e^x}{x^n} > \frac{x}{(n+1)^{n+1}}$.

Now $\lim_{x\to+\infty} \frac{x}{(n+1)^{n+1}} = +\infty$, so that $\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$.

(b) $\lim_{x\to+\infty} x^n e^{-x} = \lim_{x\to+\infty} \frac{1}{\frac{e^x}{x^n}}$

$$= \frac{1}{\lim_{x\to+\infty} \frac{e^x}{x^n}} = 0.$$

171. Let $f(t) = \ln t$ on $[1,1+x]$. By the mean value theorem, there exists t_1 with $1 < t_1 < 1+x$ such that $\ln(1+x)-\ln 1 = [(1+x)-1]f'(t_1)$; that is, $\ln(1+x) = \frac{x}{t_1}$.

Now $1 > \frac{1}{t_1} > \frac{1}{1+x}$, and since $x > 0$, $x > \frac{x}{t_1} > \frac{x}{1+x}$. Hence, $\frac{x}{1+x} < \ln(1+x) < x$.

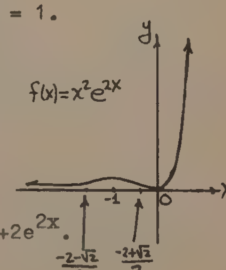
172. $\lim_{x\to0} \frac{\sinh x}{x} = \lim_{x\to0} \frac{e^x-e^{-x}}{2x} =$

$$\lim_{x\to0} \frac{e^x-1-e^{-x}+1}{2x} = \frac{1}{2}\lim_{x\to0}\left[\left(\frac{e^x-1}{x}\right)+\left(\frac{e^{-x}-1}{-x}\right)\right]$$

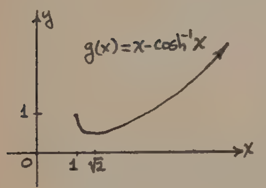
$$= \frac{1}{2}\left[\lim_{x\to0} \frac{e^x-1}{x} + \lim_{-x\to0} \frac{e^{-x}-1}{-x}\right]$$

$$= \frac{1}{2}(\ln e+\ln e) = \frac{1}{2}(2) = 1.$$

173. $f'(x) = 2xe^{2x}+2x^2e^{2x} = 2xe^{2x}(1+x) = 0$ when $x=0$ or $x=-1$. $f''(x) = 2e^{2x}+4xe^{2x}+4xe^{2x}+4x^2e^{2x} = 4x^2e^{2x}+8xe^{2x}+2e^{2x}$. $f''(0) = 2 > 0$. The

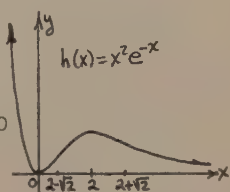


relative minimum is 0 at $x = 0$. $f''(-1) = e^{-2}(-8+4+2) < 0$. The relative maximum is $\frac{1}{e^2}$ at $x = -1$. $(-1,\frac{1}{e^2}) \approx (-1,0.14)$. $f''(x) = 0$ when $2e^{2x}(2x^2+4x+1) = 0$ for $x = \frac{-2\pm\sqrt{2}}{2}$; that is, $x \approx -1.71$ or $x \approx -0.29$. So $(-1.71, 0.10)$ is an inflection point; $(-0.29, 0.05)$ is an inflection point, since f is concave upward at x=0 and concave downward at $x = -1$; and $f''(-2) > 0$, so that f is concave upward at $x = -2$.

174. 

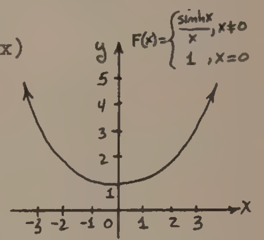
$g(x)=x-\cosh^{-1}x$ is defined for $x \geq 1$. $g'(x) = 1-\frac{1}{\sqrt{x^2-1}} = 0$ for $x=\sqrt{2}$. $g''(x) = -\frac{x}{(x^2-1)^{3/2}}$; $g''(\sqrt{2}) < 0$, and there is a relative minimum at $x = \sqrt{2}$. The relative minimum is $g(\sqrt{2}) \approx 0.53$. $g''(x) = 0$ when $x = 0$, but 0 is not in the domain of g; hence there are no inflection points.

175. $h'(x)=-x^2e^{-x}+2xe^{-x}= e^{-x}(2x-x^2)=e^{-x}x(2-x)=0$ when x=0 or when x=2. $h''(x)=-2xe^{-x}+x^2e^{-x}+2e^{-x}-2xe^{-x}=e^{-x}(x^2-4x+2)$. Now $h''(0) > 0$, so h has a relative minimum at 0, namely, 0; $h''(2) < 0$, so h has a relative maximum at 2, namely, $h(2) = \frac{4}{e^2} \approx 0.54$. $h''(x) = 0$ for $x^2-4x+2=0$, that is, when $x=2+\sqrt{2}$ or $x=2-\sqrt{2}$. Corresponding to these values are inflection points,



since h is concave upward at x=0, concave downward at x=2, and concave upward at x=4. Since $\lim_{x\to+\infty} x^2e^{-x}=0$ then y=0 is an asymptote.

176. F is continuous at 0, since $\lim_{x\to 0} \frac{\sinh x}{x} = 1=F(0)$ by Problem 172. $F'(x) = \frac{x\cosh x-\sinh x}{x^2}$, $x\neq 0$ and $F'(x)=0$ for x=0. Now since $x > \tanh x$ for $x > 0$ then $x\cosh x > \sinh x$ and so $x\cosh x-\sinh x > 0$. Hence, $F'(x) > 0$ for $x > 0$, so that F is increasing for positive x. If $x < 0$, then $x < \tanh x$ or $x<\frac{\sinh x}{\cosh x}$, so $x\cosh x < \sinh x$; that is, $x\cosh x - \sinh x < 0$. And so F is decreasing for negative x. Hence, (0,1) is a relative minimum. We note, also, that F is an even function, with $f(1) = f(-1) \approx 1.18$, $f(2) = f(-2) \approx 1.81$, $f(3)=f(-3) \approx 3.34$.



177. By the method of circular disks, $V = \int_1^{e^3} \pi(\frac{1}{\sqrt{x}})^2dx = \pi\int_1^{e^3}\frac{1}{x}dx = \pi\ln|x|\Big|_1^{e^3} \pi(\ln e^3-\ln 1) = \pi(3)$
$= 3\pi$ cubic units.

178. $A = \int_0^{2\sqrt{2}} \frac{4}{\sqrt{4x^2+9}}dx$. Now, we let u=2x, so that du = 2dx. So $\int\frac{4dx}{\sqrt{4x^2+9}} = \frac{1}{2}\int\frac{4du}{\sqrt{u^2+9}}$
$= 2\sinh^{-1}\frac{u}{3} + C = 2\sinh^{-1}\frac{2x}{3}+ C$.
Hence, $A = \int_0^{2\sqrt{2}} \frac{4}{\sqrt{4x^2+9}}dx = 2\sinh^{-1}\frac{2x}{3}\Big|_0^{2\sqrt{2}}$

$= 2\sinh^{-1}\frac{4\sqrt{2}}{3} - 0 = 2\ln\left[\frac{4\sqrt{2}}{3}+\sqrt{\left(\frac{4\sqrt{2}}{3}\right)^2+1}\right]$

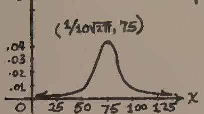
$= 2\ln\left(\frac{4\sqrt{2}}{3}+\frac{\sqrt{41}}{3}\right) \approx 2(1.39)$

$= 2.78$ square units.

179. $f'(x) = \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}}\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right].$

Critical number: $x = \mu$.

We have maximum at $x = \mu$; the maximum is $f(\mu) = \frac{1}{\sigma\sqrt{2+1}}e^0 = \frac{1}{\sigma\sqrt{2\pi}}$.



$f(x) = \frac{1}{10\sqrt{2\pi}}\exp\left[\frac{-(x-75)^2}{200}\right]$

180. (a) $T = \frac{N}{k} + N(1-0.95^k)$;

$T' = -\frac{N}{k^2} - N(0.95)^k\ln(0.95) = 0$

provided $k^2(0.95)^k = -\frac{1}{\ln(0.95)}$; so $k=5$.

(b) With $p = 0.8$, we solve

$-\frac{N}{k^2} - N(0.8)^k\ln(0.8) = 0$; that is,

$k^2(0.8)^k = -\frac{1}{\ln(0.8)}$; $k = 3$.

(c) With $p = 0.7$, we want

$k^2(0.7)^k = -\frac{1}{\ln(0.7)}$; $k = 3$.

181. (a) At the end of 12 years, \$30,000 invested at 10% and compounded continuously will be worth $\$(30{,}000)e^{\frac{10(12)}{100}}$, that is, $\$30{,}000e^{6/5} \approx \$99{,}603.51$.

(b) At the end of 12 years, \$0.01 doubled every six months will be worth $\$0.01(2)^{24} \approx \$167{,}772.16$. You decide which plan you would prefer.

182. (a) Put $g(x) = \ln x$, so that $g'(x) = \frac{1}{x}$.

By the mean value theorem, $g(1+\frac{x}{a}) - g(\frac{x}{a}) = \left[(1+\frac{x}{a})-\frac{x}{a}\right]g'(t)$ holds for some t with $\frac{x}{a} \le t \le 1+\frac{x}{a}$. Therefore, $\ln(1+\frac{x}{a})-\ln(\frac{x}{a}) = \frac{1}{t}$.

(b) By part (a), $\frac{1}{t} = \ln(1+\frac{x}{a}) - \ln(\frac{x}{a}) = \ln\left[\frac{1+\frac{x}{a}}{\frac{x}{a}}\right] = \ln(1+\frac{a}{x})$. Since $t \le 1+\frac{x}{a}$, then

$\frac{1}{1+\frac{x}{a}} \le \frac{1}{t} = \ln(1+\frac{a}{x})$; that is, $\ln(1+\frac{a}{x}) \ge \frac{1}{1+\frac{x}{a}} = \frac{a}{a+x}$.

(c) $D_x(1+\frac{a}{x})^x = D_x e^{x\ln(1+\frac{a}{x})} =$

$e^{x\ln(1+\frac{a}{x})}D_x\left[x\ln(1+\frac{a}{x})\right] =$

$(1+\frac{a}{x})^x\left[\ln(1+\frac{a}{x}) + x\frac{1}{1+\frac{a}{x}}(-\frac{a}{x^2})\right] =$

$(1+\frac{a}{x})^x\left[\ln(1+\frac{a}{x}) - \frac{a}{a+x}\right] \ge 0$ by part (b).

Since $f'(x) = D_x(1+\frac{a}{x})^x \ge 0$, then f is increasing.

183. $\$(1+\frac{a}{x})^x$ is the value at the end of one year of one dollar invested at 100a% interest per year compounded x times during the year. The fact that $(1+\frac{a}{x})^x$ increases as x increases simply means that the value of the investment at the end of the year will increase if the compounding of interest occurs more often.

184. $S = p(1+\frac{r}{n})^{nt-1} + p(1+\frac{r}{n})^{nt-2} + \cdots + p(1+\frac{r}{n}) =$

$p\left[\frac{(1+\frac{r}{n})^{nt}-1}{n/r}\right]$. Hence, $p = \frac{Sr}{n\left[(1+\frac{r}{n})^{nt}-1\right]}$.

185. If the half-life of a substance is T, then $k = \frac{\ln 2}{T}$. When $T=2$, $k = -\frac{\ln 2}{2}$.

Now, since $q = q_0e^{kt}$, it follows that $e^{kt} = \frac{q}{q_0}$, so that $kt = \ln(\frac{q}{q_0})$ and $t = \frac{1}{k}\ln(\frac{q}{q_0})$.

Putting $q = \frac{1}{10}q_0$, so that $\frac{q}{q_0} = \frac{1}{10}$, we obtain $t = \frac{-2}{\ln 2}\ln(\frac{1}{10}) = \frac{2\ln 10}{\ln 2} \approx 6.64$ hours.

186. We want to find q_2 when $t_2=1$, and when

$q_0=150{,}000$ at $t_0=0$, and $q_1=900{,}000$ at $t_1=2$. So $q=(150{,}000)(\frac{900{,}000}{150{,}000})^{\frac{1}{2}} = (150{,}000)(6)^{\frac{1}{2}}$. Hence, $q_2 \approx 367{,}423.46$ bacteria.

187. At $t_0=0$, $x_0=200$; at $t_1=1$, $x_1=160$; $a=80$. We want to find t when $x=110$. We know $x-a = (x_0-a)e^{kt}$. We find k by the formula $k = \frac{1}{t_1-t_0}\ln(\frac{x_1-a}{x_0-a})=\frac{1}{1-0}\ln(\frac{160-80}{200-80})$.

$k = \ln\frac{2}{3}$. Hence, $\frac{x-a}{x_0-a}=e^{(\ln\frac{2}{3})t} = (\frac{2}{3})t$. So

$t = \dfrac{\ln(\frac{x-a}{x_0-a})}{\ln 2/3}$. When $x=110$ and $x_0=200$,

$t = \dfrac{\ln(\frac{30}{120})}{\ln 2/3} = \dfrac{\ln \frac{1}{4}}{\ln \frac{2}{3}} \approx 3.42$. So

approximately 2.42 minutes later, the coffee will have cooled to 110°F.

188. We have $q=q_0e^{-at}$; hence, $dq=-aq_0e^{-at}dt= -aqdt$. Therefore, in dt units of time, $aqdt$ atoms of substance A are transformed into atoms of substance B. Similarly, in dt units of time, $by \cdot dt$ atoms of substance B decompose; hence, the net change in the number y atoms of substance B in dt units of time is given by $dy=aqdt-bydt$. Therefore, $\frac{dy}{dt} + by = aq = aq_0e^{-at}$, as was to be shown.

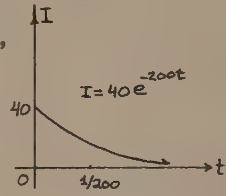
189. By Problem 188, $y=Ce^{-bt}+\frac{aq_0}{b-a}e^{-at}$ is a solution of the differential equation $\frac{dy}{dt}+by=aq_0e^{-at}$. Now, $y=0$ when $t=0$, so that $0=C+\frac{aq_0}{b-a}$, and $C=\frac{-aq_0}{b-a}$. Hence,

(a) $y=\frac{-aq_0}{b-a}e^{-bt}+\frac{aq_0}{b-a}e^{-at}=\frac{aq_0}{a-b}(e^{-bt}-e^{-at})$.

(b) $q=q_0e^{-at}$, so that $\frac{y}{q}=\frac{a}{a-b}\left[e^{(a-b)t}-1\right]$; therefore, $(\frac{a-b}{a})(\frac{y}{q})+1=e^{(a-b)t}$, so that $(a-b)t=\ln\left[1+(\frac{a-b}{a})(\frac{y}{q})\right]$, and

$t = \frac{1}{a-b}\ln\left[1+(\frac{a-b}{a})(\frac{y}{q})\right]$.

190. (a) $\frac{dI}{dt} = \frac{-I}{RC}$, so that $\frac{dI}{I} = -\frac{dt}{RC}$.



Hence, $\ln I=\frac{-t}{RC}+K$, and $I=e^{-t/RC}e^{K}$. When $t=0$, $I=I_0$. Hence, $I_0=e^{K}$. Therefore, $I=I_0e^{-t/RC}$.

(b) $I=40e^{\frac{-t}{1/200}}= 40e^{-200t}$.

191. (a) $1700 = \dfrac{500N}{500+(N-500)e^{-0.62(2)}}$ or

$5N = 17(500)+17(N-500)e^{-1.24}$ or

$N(5-17e^{-1.24}) = 17(500)(1-e^{-1.24})$ or

$N = \dfrac{17(500)(1-e^{-1.24})}{5-17e^{-1.24}} \approx 75{,}064$

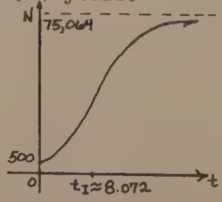
(b) When $t = 10$,

$N = \dfrac{500(75{,}064)}{500+(75{,}064-500)e^{-6.2}} = 57{,}624$.

(c) $t=\frac{1}{k}\ln\frac{C-N_0}{N_0} = \frac{1}{0.62}\ln\frac{75{,}064-500}{500}$

≈ 8.07 years.

(d)



$N=\dfrac{C}{1+C_0e^{-kt}}$,

$C=75{,}064$
$C_0 \approx 149$
$k \approx 0.62$

192. Let dt units of time elapse, so that $10dt$ pounds of pollutant go into the lake. Now suppose there are G cubic feet of water in the lake. (Notice that G remains constant.) Let q be the number of pounds of pollutant in the lake at time t. In dt minutes,

100dt cubic feet of water run out of the lake carrying $\frac{100q}{G}dt$ pounds of pollutant. So the net change in the amount of pollutant after dt minutes is $dq=10dt-\frac{100q}{G}dt$, or $\frac{dq}{dt}=10-\frac{100q}{G}$. Letting $u=10-\frac{100q}{G}$, we have $du=\frac{-100}{G}dq$ and $\frac{du}{dt}=\frac{-100}{G}u$. Now from page 463, the solution to this differential equation is $u=u_0e^{-100/G\ t}$, or $10-\frac{100q}{G}=u_0e^{-100/G\ t}$. When $t=0$, $q=0$ so that $u_0=10$. Hence, $1-\frac{10q}{G}=e^{-100/G\ t}$ and $q=\frac{G-Ge^{-100/G\ t}}{10}$. At time $t=t_1$, $q=\frac{1}{20}G$. So $\frac{1}{20}G=\frac{G-Ge^{-100/G\ t_1}}{10}$ and $\frac{1}{2}=1-e^{-100/G\ t_1}$ or $e^{-100/G\ t_1}=\frac{1}{2}$, $\frac{100}{G}t_1=\ln 2$ and so $G=\frac{100t_1}{\ln 2}$. Now $t_1=5.2596 \times 10^6$ minutes. Hence,

$$G=\frac{5.2596 \times 10^8}{\ln 2}=7.588 \times 10^8 \text{ cu. feet.}$$

8 TECHNIQUES OF INTEGRATION

Problem Set 8.1, page 489

1. $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1 - \sin^2 x)\cos x\,dx$. Put $u = \sin x$, so that $du = \cos x\,dx$. So $\int (1 - \sin^2 x)\cos x\,dx = \int (1 - u^2)du = u - \frac{u^3}{3} + C =$ $\sin x - \frac{\sin^3 x}{3} + C$. Hence, $\int \cos^3 x\,dx =$ $\sin x - \frac{\sin^3 x}{3} + C$.

2. $\int \sin^3 4x\,dx = \int \sin^2 4x \sin 4x\,dx =$ $\int (1 - \cos^2 4x)\sin 4x\,dx$. Put $u = \cos 4x$, so that $du = -4 \sin 4x\,dx$. So $\int \sin^3 4x\,dx = -\frac{1}{4}\int (1 - u^2)du =$ $-\frac{1}{4}(u - \frac{u^3}{3}) + C = \frac{1}{4}(\frac{\cos^3 4x}{3} - \cos 4x) + C$.

3. $\int \sin^5 2t\,dt = \int \sin^4 2t \sin 2t\,dt =$ $\int (1 - \cos^2 2t)^2 \sin 2t\,dt$. Put $u = \cos 2t$, $du =$ $-2 \sin 2t$. So $\int \sin^5 2t\,dt = -\frac{1}{2}\int (1 - u^2)^2 du =$ $-\frac{1}{2}\int (1 - 2u^2 + u^4)du = -\frac{1}{2}(u - \frac{2}{3}u^3 + \frac{1}{5}u^5) + C =$ $-\frac{1}{2}(\cos 2t - \frac{2}{3}\cos^3 2t + \frac{1}{5}\cos^5 2t) + C$.

4. $\int \cos^5 3v\,dv = \int \cos^4 3v \cos 3v\,dv =$ $\int (1 - \sin^2 3v)^2 \cos 3v\,dv$. Putting $u = \sin 3v$, we have $du = 3 \cos 3v\,dv$, so that $\int \cos^5 3v\,dv =$ $\frac{1}{3}\int (1 - u^2)^2 du = \frac{1}{3}\int (1 - 2u^2 + u^4)du = \frac{1}{3}(u - \frac{2u^3}{3} +$ $\frac{u^5}{5}) + C = \frac{\sin 3v}{3} - \frac{2}{9}\sin^3 3v + \frac{\sin^5 3v}{15} + C$.

5. $\int \sin^7 2x \cos^3 2x\,dx = \int \sin^7 2x \cos^2 2x \cos 2x\,dx =$ $\int \sin^7 2x(1 - \sin^2 2x)\cos 2x\,dx$. Putting $u = \sin 2x$, so that $du = 2 \cos 2x\,dx$, we have $\sin^7 2x \cos^3 2x\,dx$ $\frac{1}{2} u^7(1 - u^2)du = \frac{1}{2}(u^7 - u^9)du = \frac{1}{2}(\frac{u^8}{8} - \frac{u^{10}}{10}) + C =$ $\frac{\sin^8 2x}{16} - \frac{\sin^{10} 2x}{20} + C$.

6. $\int \cos^3 x \sin^3 x\,dx = \int \cos^3 x \sin^2 x \sin x\,dx =$ $\int \cos^3 x(1 - \cos^2 x)\sin x\,dx$. Putting $u = \cos x$, we have $du = -\sin x\,dx$, so that $\int \cos^3 x \sin^3 x\,dx =$ $-\int u^3(1 - u^2)du = \int (u^5 - u^3)du = \frac{u^6}{6} - \frac{u^4}{4} + C =$ $\frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$.

7. $\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cos x\,dx =$ $\int \sin^2 x(1 - \sin^2 x)\cos x\,dx$. Let $u = \sin x$, so that $du = \cos x\,dx$. Then $\int \sin^2 x \cos^3 x\,dx =$ $\int u^2(1 - u^2)du = \int (u^2 - u^4)du = \frac{u^3}{3} - \frac{u^5}{5} + C =$ $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$.

8. $\int \sin^3 4x \cos^2 4x\,dx = \int \sin^2 4x \cos^2 4x \sin 4x\,dx =$ $\int (1 - \cos^2 4x)(\cos^2 4x)\sin 4x\,dx$. Put $u = \cos 4x$, so that $du = -4 \sin 4x\,dx$. Hence, $\int \sin^3 4x \cos^2 4x\,dx =$ $-\frac{1}{4}\int (1 - u^2)u^2 du = -\frac{1}{4}\int (u^2 - u^4)du = -\frac{1}{4}(\frac{u^3}{3} - \frac{u^5}{5}) + C$ $-\frac{1}{4}(\frac{\cos^3 4x}{3} - \frac{\cos^5 4x}{5}) + C$.

9. $\int \sin^2 3x\,dx = \frac{1}{2}\int (1 - \cos 6x)dx = \frac{1}{2}(x - \frac{1}{6}\sin 6x) + C$ $\frac{6x - \sin 6x}{12} + C$.

10. $\int \cos^2 \frac{x}{2}\,dx = \frac{1}{2}\int (1 + \cos x)dx = \frac{1}{2}(x + \sin x) + C$.

11. $\int \sin^2 \frac{t}{2}\,dt = \frac{1}{2}(1 - \cos t)dt = \frac{t - \sin t}{2} + C$.

2. $\int \cos^4 2x\,dx = \int (\cos^2 2x)^2 dx = \int\left[\frac{1}{2}(1 + \cos 4x\right]^2 dx =$
$\frac{1}{4}\int(1 + 2\cos 4x + \cos^2 4x)dx = \frac{1}{4}(x + \frac{2}{4}\sin 4x) +$
$\frac{1}{4}\int\frac{1}{2}(1 + \cos 8x)dx = \frac{1}{4}(x + \frac{1}{2}\sin 4x) +$
$\frac{1}{8}(x + \frac{1}{8}\sin 8x) + C = \frac{3}{8}x + \frac{8\sin 4x + \sin 8x}{64} + C.$

3. $\int \sin^6 u\,du = \int(\sin^2 u)^3 du = \int\frac{1}{8}(1 - \cos 2u)^3 du =$
$\frac{1}{8}\int(1 - 3\cos 2u + 3\cos^2 2u - \cos^3 2u)du =$
$\frac{1}{8}(u - \frac{3}{2}\sin 2u) + \frac{3}{8}\int\frac{(1 + \cos 4u)}{2}du -$
$\frac{1}{8}\int(1 - \sin^2 2u)\cos 2u\,du = \frac{1}{8}(u - \frac{3}{2}\sin 2u) + \frac{3}{16}u +$
$\frac{3}{64}\sin 4u - \frac{1}{16}\int(1 - t^2)dt$, where $t = \sin 2u$ =
$\frac{5u}{16} - \frac{3}{16}\sin 2u + \frac{3}{64}\sin 4u - \frac{\sin 2u}{16} + \frac{\sin^3 2u}{48} + C =$
$\frac{5}{16}u - \frac{1}{4}\sin 2u + \frac{3}{64}\sin 4u + \frac{\sin^3 2u}{48} + C.$

4. $\int \sin^2\pi t\cos^2\pi t\,dt = \int\frac{1}{2}(1 - \cos 2\pi t)\frac{1}{2}(1 + \cos 2\pi t)dt =$
$\frac{1}{4}\int(1 - \cos^2 2\pi t)dt = \frac{1}{4}\int\sin^2 2\pi t\,dt =$
$\frac{1}{4}\int\frac{1}{2}(1 - \cos 4\pi t)dt = \frac{1}{8}(t - \frac{\sin 4\pi t}{4\pi}) + C.$

5. $\int \sin 5x\cos 2x\,dx = \frac{1}{2}\int(\sin 7x + \sin 3x)dx =$
$-\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + C.$

6. $\int \sin 4x\cos 2x\,dx = \frac{1}{2}\int(\sin 6x + \sin 2x)dx =$
$-\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + C.$

7. $\int \cos 4x\cos 3x\,dx = \frac{1}{2}\int(\cos x + \cos 7x)dx =$
$\frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C.$

8. $\int \sin 3t\cos 5t\,dt = \frac{1}{2}\int[\sin 8t + \sin(-2t)]dt =$
$-\frac{1}{16}\cos 8t + \frac{1}{4}\cos 2t + C.$

9. $\int \sin 7u\sin 3u\,du = \frac{1}{2}\int(\cos 4u - \cos 10u)du =$
$\frac{1}{8}\sin 4u - \frac{1}{20}\sin 10u + C.$

0. $\int \cos 8v\cos 4v\,dv = \frac{1}{2}\int(\cos 4v + \cos 12v)dv =$
$\frac{1}{8}\sin 4v + \frac{1}{24}\sin 12v + C.$

1. $\int \sin^5 x\cos^2 x\,dx = \int\sin^4 x\cos^2 x\sin x\,dx =$
$\int(1 - \cos^2 x)^2\cos^2 x\sin x\,dx$. Put $u = \cos x$, so that $du = -\sin x\,dx$. Hence, $\int\sin^5 x\cos^2 x\,dx =$
$-\int(1 - u^2)^2u^2du = -\int(u^2 - 2u^4 + u^6)du =$
$-(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}) + C = -(\frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7}) + C =$
$\frac{2\cos^5 x}{5} - \frac{\cos^3 x}{5} - \frac{\cos^7 x}{7} + C.$

22. $\int \sin^4 2x\cos^5 2x\,dx = \int\sin^4 2x\cos^4 2x\cos 2x\,dx =$
$\int\sin^4 2x(1 - \sin^2 2x)^2\cos 2x\,dx$. Let $u = \sin 2x$, so that $du = 2\cos 2x\,dx$. Hence, $\int\sin^4 2x\cos^5 2x\,dx =$
$\frac{1}{2}\int u^4(1 - u^2)^2du = \frac{1}{2}\int(u^4 - 2u^6 + u^8)du =$
$\frac{1}{2}(\frac{u^5}{5} - \frac{2}{7}u^7 + \frac{u^9}{9}) + C = \frac{1}{2}(\frac{\sin^5 2x}{5} - \frac{2}{7}\sin^7 2x +$
$\frac{\sin^9 2x}{9}) + C.$

23. $\int\frac{\sin^3 x}{\cos^4 x}dx = \int\frac{(1 - \cos^2 x)\sin x}{\cos^4 x}dx$. Put $u = \cos x$, so that $du = -\sin x\,dx$. Hence, $\int\frac{\sin^3 x}{\cos^4 x}dx =$
$-\int\frac{1 - u^2}{u^4}du = \int(-u^{-4} + u^{-2})du = \frac{1}{3}u^{-3} - u^{-1} + C =$
$\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C.$

24. $\int \sin^2 3x\cos^4 3x\,dx = \int(\frac{1 - \cos 6x}{2})(\frac{1 + \cos 6x}{2})^2dx =$
$\frac{1}{8}\int[1 + \cos 6x - \cos^2 6x - (\cos 6x)(1 - \sin^2 6x)]dx =$
$\frac{1}{8}\int(\frac{1}{2} + \cos 6x - \frac{\cos 12x}{2})dx - \frac{1}{48}\int(1 - u^2)du,$
where $u = \sin 6x$ = $\frac{x}{16} + \frac{\sin 6x}{48} - \frac{\sin 12x}{192} - \frac{\sin 6x}{48} +$
$\frac{\sin^{-3}6x}{144} + C = \frac{x}{16} - \frac{\sin 12x}{192} + \frac{\sin^3 6x}{144} + C.$

25. $\int \cos^6 4x\,dx = \int(\frac{1 + \cos 8x}{2})^3dx = \frac{1}{8}\int(1 + 3\cos 8x +$
$3\cos^2 8x + \cos^3 8x)dx = \frac{x}{8} + \frac{3\sin 8x}{64} + \frac{3x}{16} +$
$\frac{3\sin 16x}{256} + \frac{1}{64}\int(1 - u^2)du$, where $u = \sin 8x$ =
$\frac{5x}{16} + \frac{\sin 8x}{16} + \frac{3\sin 16x}{256} - \frac{\sin^3 8x}{192} + C.$

26. $\int\sqrt[3]{\sin^2 3x}\cos^5 3x\,dx = \int\sqrt[3]{\sin^2 3x}\cos^4 3x\cos 3x\,dx =$
$\int\sqrt[3]{\sin^2 3x}(1 - \sin^2 3x)^2\cos 3x\,dx$. Let $u = \sin 3x$, so that $du = 3\cos 3x\,dx$. Hence,
$\int\sqrt[3]{\sin^2 3x}\cos^5 3x\,dx = \frac{1}{3}\int u^{2/3}(1 - u^2)^2du =$
$\frac{1}{3}\int u^{2/3}(1 - 2u^2 + u^4)du = \frac{1}{3}\int(u^{2/3} - 2u^{8/3} + u^{14/3})du =$
$\frac{1}{3}(\frac{3}{5}u^{5/3} - \frac{6}{11}u^{11/3} + \frac{3}{17}u^{17/3}) + C =$
$\frac{1}{3}(\frac{3}{5}\sin^{5/3}3x - \frac{6}{11}\sin^{11/3}3x + \frac{3}{17}\sin^{17/3}3x) + C.$

27. $\int_{\pi/4}^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \int_{\pi/4}^{\pi/2} \frac{(1 - \sin^2 x)\cos x\,dx}{\sqrt{\sin x}}$. Let $u = \sin x$, so that $du = \cos x\,dx$. Hence, $\int \frac{(1 - \sin^2 x)}{\sqrt{\sin x}} \cos x\,dx = \int (\sin^{-1/2} x - \sin^{3/2} x)\cos x\,dx = \int (u^{-1/2} - u^{3/2})\,du = 2u^{1/2} - \frac{2}{5}u^{5/2} + C = 2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{5/2} + C$. So $\int_{\pi/4}^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \left(2\sqrt{\sin x} - \frac{2}{5}\sin^{5/2} x\right)\Big|_{\pi/4}^{\pi/2} = \left(2\sqrt{\sin \frac{\pi}{2}} - \frac{2}{5}\sin^{5/2}\frac{\pi}{2}\right) - \left(2\sqrt{\sin\frac{\pi}{4}} - \frac{2}{5}\sin^{5/2}\frac{\pi}{4}\right) = \frac{8}{5} - 2\,\frac{\sqrt{2}}{2} + \frac{2}{5}\left(\frac{\sqrt{2}}{2}\right)^{5/2}$.

28. $\int_0^{\pi/3} \sin^2 3x \cos^5 3x\,dx = \int_0^{\pi/3} \sin^2 3x\,(1 - \sin^2 3x)^2 \cos 3x\,dx$. Let $u = \sin 3x$, so that $du = 3\cos 3x\,dx$. Hence, $\int \sin^2 3x\,(1 - \sin^2 3x)^2 \cos 3x\,dx = \frac{1}{3}\int u^2(1 - u^2)^2\,du = \frac{1}{3}\int (u^2 - 2u^4 + u^6)\,du = \frac{1}{3}\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C = \frac{1}{3}\left(\frac{\sin^3 3x}{3} - \frac{2}{5}\sin^5 3x + \frac{\sin^7 3x}{7}\right) + C$. Hence, $\int_0^{\pi/3} \sin^2 3x \cos^5 3x\,dx = \frac{1}{3}\left(\frac{\sin^3 3x}{3} - \frac{2}{5}\sin^5 3x + \frac{\sin^7 3x}{7}\right)\Big|_0^{\pi/3} = 0$.

29. $\int_0^{1/2} \sqrt[4]{\sin \pi t}\,\cos^3 \pi t\,dt = \int_0^{1/2} \sqrt[4]{\sin \pi t}\,(1 - \sin^2 \pi t)\cos \pi t\,dt$. Put $u = \sin \pi t$, so that $du = \pi \cos \pi t\,dt$. Now $\int \sqrt[4]{\sin \pi t}\,(1 - \sin^2 \pi t)\cos \pi t\,dt = \frac{1}{\pi}\int u^{1/4}(1 - u^2)\,du = \frac{1}{\pi}\left(\frac{4}{5}u^{5/4} - \frac{4}{13}u^{13/4}\right) + C = \frac{1}{\pi}\left(\frac{4}{5}\sin^{5/4}\pi t - \frac{4}{13}\sin^{13/4}\pi t\right) + C$. Hence, $\int_0^{1/2} \sqrt[4]{\sin \pi t}\,\cos^3 \pi t\,dt = \frac{1}{\pi}\left(\frac{4}{5}\sin^{5/4}\pi t - \frac{4}{13}\sin^{13/4}\pi t\right)\Big|_0^{1/2} = \frac{1}{\pi}\left(\frac{4}{5}\cdot 1 - \frac{4}{13}\cdot 1\right) = \frac{32}{65\pi}$.

30. $\int_{1/4}^{1/2} \frac{\cos^5 \pi u}{\sin^2 \pi u}\,du = \int_{1/4}^{1/2} \frac{(1 - \sin^2 \pi u)^2}{\sin^2 \pi u}\cos \pi u\,du$. Let $v = \sin \pi u$, so that $dv = \pi \cos \pi u\,du$. So $\int \frac{(1 - \sin^2 \pi u)^2}{\sin^2 \pi u}\cos \pi u\,du = \frac{1}{\pi}\int \frac{1 - 2v^2 + v^4}{v^2}\,du = \frac{1}{\pi}\int (v^{-2} - 2 + v^2)\,du = \frac{1}{\pi}\left(-v^{-1} - 2v + \frac{v^3}{3}\right) + C = \frac{1}{\pi}\left(-\frac{1}{\sin \pi u} - 2\sin \pi u + \frac{\sin^3 \pi u}{3}\right) + C$. Hence, $\int_{1/4}^{1/2} \frac{\cos^5 \pi u}{\sin^2 \pi u}\,du = \frac{1}{\pi}\left(-\frac{1}{\sin \pi u} - 2\sin \pi u + \frac{\sin^3 \pi u}{3}\right)\Big|_{1/4}^{1/2} = \frac{1}{\pi}\left[\left(-\frac{1}{1} - 2(1) + \frac{1}{3}\right) - \left(-\sqrt{2} - \sqrt{2} + \frac{\sqrt{2}}{12}\right)\right] = \frac{25\sqrt{2} - 32}{12\pi}$.

31. $\int_0^{\pi/8} \sin^4 2x \cos^2 2x\,dx = \frac{1}{2}\int_0^{\pi/4} \sin^4 u \cos^2 u\,du$, where $u = 2x$. Thus, $\int_0^{\pi/8} \sin^4 2x \cos^2 2x\,dx = \frac{1}{2}\int_0^{\pi/4} \left(\frac{1 - \cos 2u}{2}\right)^2\left(\frac{1 + \cos 2u}{2}\right)du = \frac{1}{16}\int_0^{\pi/4} (1 - \cos 2u)(1 - \cos^2 2u)\,du = \frac{1}{16}\int_0^{\pi/4} (1 - \cos 2u)\sin^2 2u\,du = \frac{1}{16}\int_0^{\pi/4} \sin^2 2u\,du - \frac{1}{16}\int_0^{\pi/4} \sin^2 2u \cos 2u\,du = \frac{1}{16}\int_0^{\pi/4} \frac{1 - \cos 4u}{2} - \frac{1}{16}\int_0^1 \frac{1}{2}v^2\,dv$, where $v = \sin 2u$. Thus, $\int_0^{\pi/8} \sin^4 2x \cos^2 2x\,dx = \frac{1}{16}\left(\frac{u}{2} - \frac{\sin 4u}{8}\right)\Big|_0^{\pi/4} - \left(\frac{1}{32}\,\frac{v^3}{3}\right)\Big|_0^1 = \frac{1}{16}\left(\frac{\pi}{8}\right) - \frac{1}{32}\left(\frac{1}{3}\right) = \frac{3\pi - 4}{384}$.

32. $\int_0^{\pi} \sin^8 x\,dx = \int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right)^4 dx = \frac{1}{16}\int_0^{\pi} (1 - 4\cos 2x + 6\cos^2 2x - 4\cos^3 2x + \cos^4 2x)$ Now $\int \cos 2x\,dx = \frac{\sin 2x}{2} + C$, $\int \cos^2 2x\,dx = \int \frac{1 + \cos 4x}{2}\,dx = \frac{x}{2} + \frac{\sin 4x}{8} + C$, $\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)\cos 2x\,dx = \int \cos 2x\,dx - \int \sin^2 2x \cos 2x\,dx$, so that $\int \cos^3 2x\,dx = \frac{\sin 2x}{2} - \frac{1}{2}\cdot\frac{\sin^3 2x}{3} + C$ and, by Problem 20, $\int \cos^4 2x\,dx = \frac{3x}{8} + \frac{8\sin 4x + \sin 8x}{64} + C$. Hence, $\int_0^{\pi} \sin^8 x\,dx = \frac{1}{16}\left[x - 4\,\frac{\sin 2x}{2} + 6\left(\frac{x}{2} + \frac{\sin 4x}{8}\right) - 4\left(\frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}\right) + \frac{3x}{8} + \frac{8\sin 4x + \sin 8x}{64}\right]\Big|_0^{\pi} = \frac{1}{16}\left(\pi + 3\pi + \frac{3\pi}{8}\right) = \frac{35\pi}{128}$.

33. $\int_0^1 \sin 2\pi x \cos 3\pi x\,dx = \frac{1}{2}\int_0^1 [\sin 5\pi x + \sin(-\pi x)]$ $\left[-\frac{1}{10\pi}\cos 5\pi x + \frac{1}{2\pi}\cos \pi x\right]\Big|_0^1 = \left[\frac{1}{10\pi} - \frac{1}{2\pi}\right] - \left[-\frac{1}{10\pi} + \frac{1}{2\pi}\right]$

34. $\int_0^5 \cos\frac{2\pi x}{5}\cos\frac{7\pi x}{5}\,dx = \frac{1}{2}\int_0^5(\cos \pi x + \cos\frac{9\pi x}{5})dx =$
$(\frac{1}{2\pi}\sin \pi x + \frac{5}{18\pi}\sin\frac{9\pi x}{5})\Big|_0^5 = 0.$

35. $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m - n)x +$
$\cos(m + n)x]dx.$

Case 1. Suppose that $m = n$. Then
$\int_{-\pi}^{\pi}\cos mx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1 + \cos 2nx)dx =$
$\frac{1}{2}(x + \frac{\sin 2nx}{2n})\Big|_{-\pi}^{\pi} = \frac{1}{2}[(\pi + 0) - (-\pi + 0)] = \pi.$

Case 2. Suppose that $m \neq n$. Then
$\int_{-\pi}^{\pi}\cos mx \cos nx\,dx =$
$\frac{1}{2}\left[\frac{\sin(m - n)x}{m - n} + \frac{\sin(m + n)x}{m + n}\right]\Big|_{-\pi}^{\pi}.$
$\frac{1}{2}\left[(\frac{\sin(m - n)\pi}{m - n} + \frac{\sin(m + n)\pi}{m + n}) - (-\frac{\sin(m - n)\pi}{m - n} -\right.$
$\left.\frac{\sin(m + n)\pi}{m + n})\right] = 0$ since $m - n$ and $m + n$ are integers.

36. $\int_{-\pi}^{\pi}\sin mx \sin nx\,dx =$
$\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m - n)x - \cos(m + n)x]dx.$

Case 1. $m = n$. $\frac{1}{2}\int_{-\pi}^{\pi}(\cos 0 - \cos 2nx)dx =$
$\frac{1}{2}\int_{-\pi}^{\pi}(1 - \cos 2nx)dx = \frac{1}{2}(x - \frac{\sin 2nx}{2n})\Big|_{-\pi}^{\pi} =$
$\frac{1}{2}(\pi - 0 - (-\pi) + 0) = \pi.$

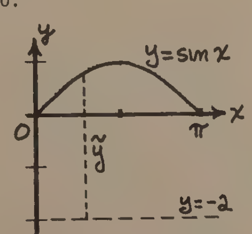
Case 2. $m \neq n$. $\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m - n)x - \cos(m + n)x]dx =$
$\frac{1}{2}(\frac{\sin(m - n)x}{m - n} - \frac{\sin(m + n)x}{m + n})\Big|_{-\pi}^{\pi} = \frac{1}{2}(\frac{\sin(m - n)\pi}{m - n} -$
$\frac{\sin(m + n)\pi}{m + n} + \frac{\sin(m - n)\pi}{m - n} - \frac{\sin(m + n)\pi}{m - n}) =$
$\frac{\sin(m - n)\pi}{m - n} - \frac{\sin(m + n)\pi}{m + n} = 0 - 0 = 0$, since $m-n$ and $m + n$ are integers.

37. $\int_{-\pi}^{\pi}\sin nx \cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(n + m)x +$
$\sin(n - m)x]dx$. Therefore, if $n \neq m$,
$\int_{-\pi}^{\pi}\sin nx \cos mx\,dx = \frac{1}{2}\left[-\frac{\cos(n + m)x}{n + m} -\right.$
$\left.\frac{\cos(n - m)x}{n - m}\right]\Big|_{-\pi}^{\pi} = \frac{1}{2}\left[-\frac{\cos(n + m)\pi}{n + m} - \frac{\cos(n - m)\pi}{n - m} +\right.$
$\left.\frac{\cos(n + m)(-\pi)}{n + m} + \frac{\cos(n - m)(-\pi)}{n - m}\right] = 0$, since cosine is an even function and $\cos t(-\pi) = \cos t\pi$. If $m = n$, we also have $\int_{-\pi}^{\pi}\sin nx \cos mx\,dx =$
$\int_{-\pi}^{\pi}\sin nx \cos nx\,dx = \frac{1}{n}\cdot\frac{\sin^2 nx}{2}\Big|_{-\pi}^{\pi} = 0.$

38. (1) $\sin(s + t) = \sin s \cos t + \sin t \cos s$ and $\sin(s - t) = \sin s \cos t - \sin t \cos s$; hence, adding the two equations, we obtain $\sin(s + t) + \sin(s - t) = 2 \sin s \cos t$. Therefore, $\sin s \cos t =$
$\frac{1}{2}\sin(s + t) + \frac{1}{2}\sin(s - t).$
(2) $\cos(s - t) = \cos s \cos t + \sin s \sin t$ and $\cos(s + t) = \cos s \cos t - \sin s \sin t$; hence, adding the two equations, we obtain $\cos(s - t) + \cos(s + t) = 2 \cos s \cos t$. Therefore, $\cos s \cos t =$
$\frac{1}{2}\cos(s - t) + \frac{1}{2}\cos(s + t).$
(3) Subtracting the second equation in (2) from the first, we obtain $\cos(s - t) - \cos(s + t) =$
$2 \sin s \sin t$. Therefore, $\sin s \sin t =$
$\frac{1}{2}\cos(s - t) - \frac{1}{2}\cos(s + t).$

39. $V = \pi\int_0^{\pi}(\sin x)^2dx = \pi\int_0^{\pi}\frac{1 - \cos 2x}{2}dx =$
$\frac{\pi}{2}(x - \frac{\sin 2x}{2x})\Big|_0^{\pi} = \frac{\pi^2}{2}$ cubic units.

40.



$V = \int_0^{\pi}\pi\tilde{y}^2\,dx =$
$\pi\int_0^{\pi}(\sin x + 2)^2dx =$
$\pi\int_0^{\pi}[\frac{1}{2} - \frac{1}{2}\cos 2x +$
$4\sin x + 4]dx =$
$\pi[\frac{9}{2}x - \frac{1}{4}\sin 2x -$
$4\cos x]\Big|_0^{\pi} = \frac{9}{2}\pi^2 + 8\pi$
cubic units.

41. $A = \int_0^{2\pi}\cos^2 x\,dx = \frac{1}{2}\int_0^{2\pi}(1 + \cos 2x)dx =$
$\frac{1}{2}(x + \frac{\sin 2x}{2})\Big|_0^{2\pi} = \pi$ square units.

42. If $y = x - \frac{\pi}{2}$, then $\int_0^{\pi} \cos^n x\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(y - \frac{\pi}{2})]^n dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin y)^n dy$. Since n is an odd integer, we have $[\sin(-y)]^n = [-\sin y]^n = -[\sin y]^n$; that is, $[\sin y]^n$ is an odd function of y. It follows that $\int_{-\pi/2}^{\pi/2} [\sin y]^n dy = 0$; hence, $\int_0^{\pi} \cos^n x\, dx = 0$.

43. $\frac{ds}{dt} = \sin^2 \pi t$, so that $s = \int \sin^2 \pi t\, dt = \frac{1}{2}\int (1 - \cos 2\pi t)dt = \frac{1}{2}(t - \frac{\sin 2\pi t}{2\pi}) + C$. $s = 0$ when $t = 0$. Hence, $0 = \frac{1}{2}(0 - 0) + C$; so $C = 0$. $s = \frac{2\pi t - \sin 2\pi t}{4\pi}$. When $t = 8$, $s = \frac{16\pi - \sin 16\pi}{4\pi} = 4$.

44. Choose a fixed value for m. Now,
$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\sum_{n=1}^{N} a_n \sin nx)\sin mx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\sum_{n=1}^{N} a_n \sin nx \sin mx)dx = \frac{1}{\pi}\sum_{n=1}^{N} \int_{-\pi}^{\pi} a_n \sin nx \sin mx\, dx = \frac{1}{\pi}\sum_{n=1}^{N} a_n \int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \frac{1}{\pi} a_m \cdot \pi = a_m$ since all the terms in the summation are 0 except when $n = m$.

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1. Put $u = 4x$, so that $du = 4\, dx$. So $\int \cot 4x\, dx = \frac{1}{4}\int \cot u\, du = \frac{1}{4} \ln|\sin u| + C = \frac{1}{4} \ln|\sin 4x| + C$.

2. Put $u = \frac{x}{2}$, so that $du = \frac{1}{2}\, dx$. So $\int \tan \frac{x}{2}\, dx = 2\int \tan u\, du = -2 \ln|\cos u| + C = 2 \ln|\sec u| + C = 2 \ln\left|\sec \frac{x}{2}\right| + C$.

3. Put $u = 3x$, so that $du = 3\, dx$. So $\int \frac{dx}{\cos 3x} = \frac{1}{3}\int \frac{du}{\cos u} = \frac{1}{3}\int \sec u\, du = \frac{1}{3} \ln|\sec u + \tan u| + C = \frac{1}{3} \ln|\sec 3x + \tan 3x| + C$.

4. Put $u = \frac{x}{5}$, so that $du = \frac{1}{5}\, dx$. So $\int \csc \frac{x}{5}\, dx = 5\int \csc u\, du = 5 \ln|\csc u - \cot u| + C = 5 \ln|\csc \frac{x}{5} - \cot \frac{x}{5}| + C$.

5. $\int \tan^2 \frac{2x}{3}\, dx = \int (\sec^2 \frac{2x}{3} - 1)dx = \int \sec^2 \frac{2x}{3}\, dx - \int dx$. Put $u = \frac{2x}{3}$, so that $du = \frac{2}{3}\, dx$. Then $\int \tan^2 \frac{2x}{3}\, dx = \int \frac{3}{2} \sec^2 u\, du - \int dx = \frac{3}{2} \tan u - x + C = \frac{3}{2} \tan \frac{2x}{3} - x + C$.

6. $\int \cot^3 5x\, dx = \int \cot^2 5x \cot 5x\, dx = \int (\csc^2 5x - 1)\cot 5x\, dx = \int \csc^2 5x \cot 5x\, dx - \int \cot 5x\, dx$. In order to evaluate the first integral, put $u = \cot 5x$, so that $du = -5 \csc^2 5x\, dx$. Hence, $\int \cot^3 5x\, dx = \int -\frac{1}{5} u\, du - \frac{1}{5} \ln|\sin 5x| + C = -\frac{u^2}{10} - \frac{\ln|\sin 5x|}{5} + C = -\frac{1}{10}(\cot^2 5x + 2 \ln|\sin 5x|) +$

7. $\int \cot^4 4x\, dx = \int \cot^2 4x \cot^2 4x\, dx = \int (\csc^2 4x - 1)\cot^2 4x\, dx = \int \csc^2 4x \cot^2 4x\, dx - \int (\csc^2 4x - 1)dx = \int \csc^2 4x \cot^2 4x\, dx + \frac{1}{4} \cot 4x + x + C$. Now put $u = \cot 4x$, so that $du = -4 \csc^2 4x\, dx$. Hence, $\int \cot^4 4x\, dx = -\frac{1}{4}\int u^2 du + \frac{1}{4} \cot 4x + x + C = -\frac{\cot^3 4x}{12} + \frac{1}{4} \cot 4x + x + C$.

8. $\int \tan^3 \frac{\pi t}{2}\, dt = \int \tan^2 \frac{\pi t}{2} \tan \frac{\pi t}{2}\, dt = \int (\sec^2 \frac{\pi t}{2} - 1) \tan \frac{\pi t}{2}\, dt = \int \sec^2 \frac{\pi t}{2} \tan \frac{\pi t}{2}\, dt - \int \tan \frac{\pi t}{2}\, dt$. Put $u = \tan \frac{\pi t}{2}$, so that $du = \frac{\pi}{2} \sec^2 \frac{\pi t}{2}\, dt$. Hence, $\int \tan^3 \frac{\pi t}{2}\, dt = \frac{2}{\pi}\int u\, du - \frac{2}{\pi} \ln|\sec \frac{\pi t}{2}| + C = \frac{\tan^2 \frac{\pi t}{2}}{\pi} - \frac{2}{\pi} \ln|\sec \frac{\pi t}{2}| + C$.

9. $\int \csc^4 3t\, dt = \int \csc^2 3t \csc^2 3t\, dt = \int (1 + \cot^2 3t)\csc^2 3t\, dt$. Put $u = \cot 3t$, so that $du = -3 \csc^2 3t\, dt$. Hence, $\int \csc^4 3t\, dt = -\int \frac{1}{3}(1 + u^2)du = -\frac{1}{3}(u + \frac{u^3}{3}) + C = -\frac{1}{3}(\cot 3t + \frac{\cot^3 3t}{3}) + C$.

10. $\int \sec^6 2x\, dx = \int \sec^4 2x \sec^2 2x\, dx = \int (1 + \tan^2 2x)^2 \sec^2 2x\, dx$. Put $u = \tan 2x$, so that

$du = 2\sec^2 2x\,dx$. So $\int \sec^6 2x\,dx = \frac{1}{2}\int (1 + u^2)^2 du = \frac{1}{2}\int (1 + 2u^2 + u^4)du = \frac{1}{2}(u + \frac{2}{3}u^3 + \frac{1}{5}u^5) + C = \frac{1}{2}(\tan 2x + \frac{2}{3}\tan^3 2x + \frac{1}{5}\tan^5 2x) + C.$

11. Put $u = \tan 2t$, so that $du = 2\sec^2 2t\,dt$. Hence, $\int \tan^4 2t \sec^2 2t\,dt = \int \frac{1}{2}u^4 du = \frac{1}{10}u^5 + C = \frac{1}{10}(\tan^5 2t) + C.$

12. $\int \cot^4 3x \csc^4 3x\,dx = \int \cot^4 3x \csc^2 3x \csc^2 3x\,dx = \int (\cot^6 3x + \cot^4 3x)\csc^2 3x\,dx$. Put $u = \cot 3x$, so that $du = -3\csc^2 3x\,dx$. Hence, $\int \cot^4 3x \csc^4 3x\,dx = \int (u^6 + u^4)(-\frac{1}{3})du = -\frac{u^7}{21} - \frac{u^5}{15} + C = -\frac{\cot^7 3x}{21} - \frac{\cot^5 3x}{15} + C.$

13. $\int \tan^3 5x \sec^5 5x\,dx = \int \tan^2 5x \sec^4 5x \tan 5x \sec 5x\,dx = \int (\sec^2 5x - 1)\sec^4 5x \tan 5x \sec 5x\,dx$. Put $u = \sec 5x$, so that $du = 5\sec 5x \tan 5x\,dx$. Hence, $\int \tan^3 5x \sec^5 5x\,dx = \frac{1}{5}\int (u^6 - u^4)du = \frac{1}{5}(\frac{u^7}{7} - \frac{u^5}{5}) + C = \frac{\sec^7 5x}{35} - \frac{\sec^5 5x}{25} + C.$

14. $\int \cot^3 \frac{\pi x}{2} \csc^3 \frac{\pi x}{2}\,dx = \int \cot^2 \frac{\pi x}{2} \csc^2 \frac{\pi x}{2} \cot \frac{\pi x}{2}\,dx = \int (1 + \csc^2 \frac{\pi x}{2}) \csc^2 \frac{\pi x}{2} \cot \frac{\pi x}{2} \csc \frac{\pi x}{2}\,dx$. Put $u = \csc \frac{\pi x}{2}$, so that $du = -\frac{\pi}{2}\csc \frac{\pi x}{2} \cot \frac{\pi x}{2}\,dx$. Hence, $\int \cot^3 \frac{\pi x}{2} \csc^3 \frac{\pi x}{2}\,dx = -\frac{2}{\pi}\int (u^2 + u^4)du = -\frac{2}{\pi}(\frac{u^3}{3} + \frac{u^5}{5}) + C = -\frac{2}{\pi}(\csc^3 \frac{\pi x}{2} + \csc^5 \frac{\pi x}{2}) + C.$

15. $\int (\tan 2x + \cot 2x)^2 dx = \int (\tan^2 2x + 2\tan 2x \cot 2x + \cot^2 2x)dx = \int (\tan^2 2x + 1 + 1 + \cot^2 2x)dx = \int (\sec^2 2x + \csc^2 2x)dx = \frac{1}{2}\tan 2x - \frac{1}{2}\cot 2x + C.$

16. $\int (\sec 3x + \tan 3x)^2 dx = \int (\sec^2 3x + 2\sec 3x \tan 3x + \tan^2 3x)dx = \frac{1}{3}\tan 3x + \frac{2}{3}\sec 3x + \int (\sec^2 3x - 1)dx = \frac{2}{3}\tan 3x + \frac{2}{3}\sec 3x - x + C.$

17. $\int \frac{\sec^4 t}{\sqrt{\tan t}}\,dt = \int \frac{\sec^2 t\,(\tan^2 t + 1)}{\sqrt{\tan t}}\,dt$. Put $u = \tan t$, so that $du = \sec^2 t\,dt$. Hence, $\int \frac{\sec^4 t}{\sqrt{\tan t}}\,dt = \int \frac{u^2 + 1}{\sqrt{u}}\,du = \int (u^{3/2} + u^{-1/2})du = \frac{2}{5}u^{5/2} + 2u^{1/2} + C = \frac{2}{5}\tan^{5/2} t + 2\tan^{1/2} t + C.$

18. $\int \frac{\tan^3 3x}{\sqrt{\sec 3x}}\,dx = \int \frac{\tan^2 3x \tan 3x}{\sqrt{\sec 3x}}\,dx = \int \frac{(\sec^2 3x - 1)\tan 3x \sec 3x}{\sqrt{\sec 3x}\,(\sec 3x)}\,dx$. Put $u = \sec 3x$, so that $du = 3\sec 3x \tan 3x\,dx$. Hence, $\int \frac{\tan^3 3x}{\sqrt{\sec 3x}}\,dx = \frac{1}{3}\int \frac{u^2 - 1}{u^{3/2}}\,du = \frac{1}{3}(u^{1/2} - u^{-3/2})du = \frac{1}{3}(\frac{2}{3}u^{3/2} + 2u^{-1/2}) + C = \frac{2}{9}\sec^{3/2} 3x + \frac{2}{3}\sec^{-1/2} 3x + C.$

19. $\int \tan^3 7x \sec^4 7x\,dx = \int \tan^3 7x \sec^2 7x \sec^2 7x\,dx = \int \tan^3 7x\,(1 + \tan^2 7x)\sec^2 7x\,dx$. Put $u = \tan 7x$, so that $du = 7\sec^2 7x\,dx$. Hence, $\int \tan^3 7x \sec^4 7x\,dx = \frac{1}{7}\int (u^3 + u^5)du = \frac{1}{7}(\frac{u^4}{4} + \frac{u^6}{6}) + C = \frac{\tan^4 7x}{28} + \frac{\tan^6 7x}{42} + C.$

20. $\int (\frac{\tan x}{\cos x})^4 dx = \int \tan^4 x \sec^4 x\,dx = \int \tan^4 x\,(1 + \tan^2 x)\sec^2 x\,dx$. Put $u = \tan x$, so that $du = \sec^2 x\,dx$. Hence, $\int (\frac{\tan x}{\cos x})^4 dx = \int (u^4 + u^6)du = \frac{u^5}{5} + \frac{u^7}{7} + C = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.$

21. $\int \cot 3x \csc^3 3x\,dx = \int \cot 3x \csc 3x \csc^2 3x\,dx$. Put $u = \csc 3x$, so that $du = -3\csc 3x \cot 3x\,dx$. So $\int \cot 3x \csc^3 3x\,dx = -\frac{1}{3}\int u^2 du = -\frac{u^3}{9} + C = -\frac{\csc^3 3x}{9} + C.$

22. $\int \cot^{7/2} 2x \csc^4 2x\,dx = \int \cot^{7/2} 2x \csc^2 2x \csc^2 2x\,dx = \int \cot^{7/2} 2x\,(1 + \cot^2 2x)\csc^2 2x\,dx$. Put $u = \cot 2x$, so that $du = -2\csc^2 2x\,dx$. Then $\int \cot^{7/2} 2x \csc^4 2x\,dx = -\frac{1}{2}\int (u^{7/2} + u^{1/2})du = -\frac{1}{2}(\frac{2}{9}u^{9/2} + \frac{2}{3}u^{3/2}) + C = \frac{1}{9}\cot^{9/2} 2x + \frac{1}{3}\cot^{3/2} 2x + C.$

23. $\int \tan^3 5x \sec 5x\,dx = \int \tan^2 5x \tan 5x \sec 5x\,dx = \int (\sec^2 5x - 1)\tan 5x \sec 5x\,dx$. Put $u = \sec 5x$, so that $du = 5\sec 5x \tan 5x\,dx$. Hence $\int \tan^3 5x \sec 5x\,dx = \frac{1}{5}\int (u^2 - 1)du = \frac{1}{5}(\frac{u^3}{3} - u) + C = \frac{\sec^3 5x}{15} - \frac{\sec 5x}{5} + C.$

24. $\int \cot^3 \frac{x}{2} \csc^3 \frac{x}{2}\,dx = \int \cot^2 \frac{x}{2} \csc^2 \frac{x}{2} \cot \frac{x}{2} \csc \frac{x}{2}\,dx = \int (\csc^2 \frac{x}{2} - 1)\csc^2 \frac{x}{2} \cot \frac{x}{2} \csc \frac{x}{2}\,dx$. Put $u =$

$\csc \frac{x}{2}$, so that $du = -\frac{1}{2} \csc \frac{x}{2} \cot \frac{x}{2}\, dx$. Hence,

$\int \cot^3 \frac{x}{2} \csc^3 \frac{x}{2}\, dx = -2\int (u^4 - u^2)\,du = -2(\frac{u^5}{5} - \frac{u^3}{3}) + C =$

$\frac{2}{3} \csc^3 \frac{x}{2} - \frac{2}{5} \csc^5 \frac{x}{2} + C.$

25. $\int \tan^3 2x \sqrt{\sec 2x}\, dx =$

$\int \tan^2 2x \sqrt{\sec 2x} \tan 2x \frac{\sec 2x}{\sec 2x}\, dx =$

$\int (\sec^2 2x - 1) \sec^{-1/2} x \tan 2x \sec 2x\, dx$. Put $u =$ $\sec 2x$, so that $du = 2 \sec 2x \tan 2x\, dx$. So

$\int \tan^3 2x \sqrt{\sec 2x}\, dx = \frac{1}{2}\int (u^{3/2} - u^{-1/2})\,du =$

$\frac{1}{2}(\frac{2}{5}u^{5/2} - 2u^{1/2}) + C = \frac{1}{5} \sec^{5/2} 2x - \sec^{1/2} 2x + C.$

26. $\int \sqrt{\tan 7x} \sec^4 7x\, dx = \int \sqrt{\tan 7x} \sec^2 7x \sec^2 7x\, dx =$

$\int \sqrt{\tan 7x}\, (1 + \tan^2 7x) \sec^2 7x\, dx$. Put $u = \tan 7x$, so that $du = 7 \sec^2 7x\, dx$. Hence,

$\int \sqrt{\tan 7x} \sec^4 7x\, dx = \frac{1}{7}\int (u^{1/2} + u^{5/2})\,du =$

$\frac{1}{7}(\frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2}) + C = \frac{1}{7}(\frac{2}{3} \tan^{3/2} 7x + \frac{2}{7} \tan^{7/2} 7x) + C.$

27. $\int \tan^5 x \sec^7 x\, dx = \int \tan^4 x \sec^6 x \tan x \sec x\, dx =$

$\int (\sec^2 x - 1)^2 \sec^6 x \tan x \sec x\, dx$. Put $u = \sec x$, so that $du = \sec x \tan x\, dx$. Hence,

$\int \tan^5 x \sec^7 x\, dx = \int (u^2 - 1)^2 u^6\, du =$

$\int (u^{10} - 2u^8 + u^6)\,du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C = \frac{\sec^{11} x}{11} -$

$\frac{2}{9} \sec^9 x + \frac{\sec^7 x}{7} + C.$

28. $\int \frac{\csc^4 2\pi x}{\cot^2 2\pi x}\, dx = \int \frac{\csc^2 2\pi \csc^2 2\pi x}{\cot^2 2\pi x}\, dx =$

$\int \frac{(1 + \cot^2 2\pi x) \csc^2 2\pi x}{\cot^2 2\pi x}\, dx$. Put $u = \cot 2\pi x$, so that $du = -2\pi \csc^2 2\pi x\, dx$. Hence, $\int \frac{\csc^4 2\pi x}{\cot^2 2\pi x}\, dx =$

$-\frac{1}{2\pi} \int \frac{1 + u^2}{u^2}\, du = -\frac{1}{2\pi} \int (u^{-2} + 1)\,du =$

$-\frac{1}{2\pi}(-u^{-1} + u) + C = \frac{1}{2\pi \cot 2\pi x} - \frac{\cot 2\pi x}{2\pi} + C.$

29. Put $u = \cot 8x$, so that $du = -8 \csc^2 8x\, dx$. Hence,

$\int \frac{\csc^2 8x}{\cot^4 8x}\, dx = -\frac{1}{8}\int \frac{du}{u^4} = -\frac{1}{8} \frac{u^{-3}}{(-3)} + C = \frac{1}{24u^3} + C =$

$\frac{1}{24 \cot^3 8x} + C.$

30. $\int \sec^3 2x \tan^5 2x\, dx = \int \sec^2 2x \tan^4 2x \sec 2x \tan 2x\, dx =$

$\int \sec^2 2x\, (\sec^2 2x - 1)^2 \sec 2x \tan 2x\, dx$. Put $u =$ $\sec 2x$, so that $du = 2 \sec 2x \tan 2x\, dx$. Hence,

$\int \sec^3 2x \tan^5 2x\, dx = \frac{1}{2}\int u^2 (u^2 - 1)^2\, du =$

$\frac{1}{2}\int (u^6 - 2u^4 + u^2)\,du = \frac{1}{2}(\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}) + C =$

$\frac{1}{2}(\frac{\sec^7 2x}{7} - \frac{2}{5} \sec^5 2x + \frac{1}{3} \sec^3 2x) + C.$

31. Let $u = x^2$, so that $du = 2x\, dx$. $\int x \cot^3 x^2 \csc^3 x^2\, dx =$

$\frac{1}{2}\int \cot^3 u \csc^3 u\, du = \frac{1}{2}\int \cot^2 u \csc^2 u (\cot u \csc u)\,du.$

Now let $v = \csc u$, so that $dv = -\csc u \cot u\, du$.

Thus $\int \csc^3 dv = -\frac{1}{2}\int (v^2 - 1)v^2\, dv = \frac{1}{2}\int (v^2 - v^4)\,dv =$

$\frac{1}{2}(\frac{v^3}{3} - \frac{v^5}{5}) + C = \frac{1}{6} \csc^3 x^2 - \frac{1}{10} \csc^5 x^2 + C.$

32. Let $u = x^4$, so that $du = 4x^3 dx$. $\int x^3 \tan^5 x^4 \sec^7 x^4\, dx =$

$\frac{1}{4}\int \tan^5 u \sec^7 u\, du = \frac{1}{4}\int \tan^4 u \sec^6 u (\sec u \tan u)\,du.$

Let $v = \sec u$, so $dv = \sec u \tan u\, du$. Thus,

$\int \tan^5 x^4 \sec^7 x^4\, du = \frac{1}{4}\int (v^2 - 1)^2 v^6\, dv =$

$\frac{1}{4}\int (v^{10} - 2v^8 + v^6)\,dv = \frac{\sec^{11} x^4}{44} - \frac{\sec^9 x^4}{18} + \frac{\sec^7 x^4}{28} + C$

33. $\int_{\pi/6}^{\pi/9} \cot 3x\, dx = \frac{1}{3} \ln|\sin 3x| \Big|_{\pi/6}^{\pi/9} =$

$\frac{1}{3}\left[\ln(\sin \frac{\pi}{3}) - \ln(\sin \frac{\pi}{2})\right] = \frac{1}{3}(\ln \frac{\sqrt{3}}{2} - \ln 1) =$

$\frac{1}{3} \ln \frac{\sqrt{3}}{2}.$

34. $\int_{\pi/8}^{\pi/6} 5 \sec 2x\, dx = \frac{5}{2} \ln|\sec 2x + \tan 2x| \Big|_{\pi/8}^{\pi/6} =$

$\frac{5}{2}\left[\ln|\sec \frac{\pi}{3} + \tan \frac{\pi}{3}| - \ln|\sec \frac{\pi}{4} + \tan \frac{\pi}{4}|\right] =$

$\frac{5}{2}(\ln|2 + \sqrt{3}| - \ln|\sqrt{2} + 1|) = \frac{5}{2} \ln \frac{2 + \sqrt{3}}{1 + \sqrt{2}}.$

35. $\int_{\pi/4}^{\pi/2} \cot^4 x \csc^4 x\, dx = \int_{\pi/4}^{\pi/2} \cot^4 x (1 + \cot^2 x) \csc^2 x\, dx$

Put $u = \cot x$, so that $du = -\csc^2 x\, dx$. Note that $u = 1$ when $x = \frac{\pi}{4}$ and $u = 0$ when $x = \frac{\pi}{2}$; hence,

$\int_{\pi/4}^{\pi/2} \cot^4 x \csc^4 x\, dx = \int_1^0 u^4 (1 + u^2)(-1)\,du =$

$\int_0^1 (u^4 + u^6)\,du = (\frac{u^5}{5} + \frac{u^7}{7})\Big|_0^1 = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}.$

36. $\int_0^{\pi/4} \tan^5 x\, dx = \int_0^{\pi/4} \tan^4 x \tan x\, dx =$

$\int_0^{\pi/4} \frac{\tan^4 x}{\sec x} \sec x \tan x\, dx =$

$\int_0^{\pi/4} \frac{(\sec^2 x - 1)^2}{\sec x} \sec x \tan x\, dx$. Put $u = \sec x$, so that $du = \sec x \tan x\, dx$, $u = 1$ when $x = 0$, and $u = \sqrt{2}$ when $x = \frac{\pi}{4}$. Thus, $\int_0^{\pi/4} \tan^5 x\, dx =$

$\int_1^{\sqrt{2}} \frac{(u^2 - 1)^2}{u} du = \int_1^{\sqrt{2}} \frac{u^4 - 2u^2 + 1}{u} du =$

$\int_1^{\sqrt{2}} (u^3 - 2u + \frac{1}{u}) du = (\frac{u^4}{4} - u^2 + \ln|u|)\Big|_1^{\sqrt{2}} =$

$(1 - 2 + \ln\sqrt{2}) - (\frac{1}{4} - 1 + \ln 1) = \ln\sqrt{2} - \frac{1}{4}$.

7. $A = 2\int_0^{\pi/4} 5\tan^2 x\, dx = 10\int_0^{\pi/4} (\sec^2 x - 1) dx =$

$10(\tan x - x)\Big|_0^{\pi/4} = 10(\tan\frac{\pi}{4} - \frac{\pi}{4}) = 10(1 - \frac{\pi}{4}) =$

$5(\frac{4 - \pi}{2})$ square units.

8. $A = 2\int_0^{\pi/3} \sec x\, dx = 2\ln|\sec x + \tan x|\Big|_0^{\pi/3} =$

$2(\ln|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}| - \ln|\sec 0 + \tan 0|) =$

$2(\ln(2 + \sqrt{3}) - \ln 1) = 2\ln(2 + \sqrt{3})$ square units.

9. $V = \pi\int_0^{\pi/3} (\sec^2 x)^2 dx = \pi\int_0^{\pi/3} \sec^2 x(1 + \tan^2 x) dx =$

$\pi\left[\tan x + \frac{\tan^3 x}{3}\right]\Big|_0^{\pi/3} = \pi(\tan\frac{\pi}{3} + \frac{\tan^3\frac{\pi}{3}}{3}) =$

$\pi(\sqrt{3} + \frac{(\sqrt{3})^3}{3}) = 2\pi\sqrt{3}$ cubic units. Notice that $\int \sec^2 x \tan^2 x\, dx$ was obtained by letting $u = \tan x$, $du = \sec^2 x\, dx$, so that $\int u^2 du = \frac{u^3}{3} + C = \frac{\tan^3 x}{3} + C$.

0. (a) $\int \cot u\, du = \int \frac{\cos u}{\sin u} du$. Let $v = \sin u$, so that $dv = \cos u\, du$. Then $\int \frac{dv}{v} = \ln|v| + C = \ln|\sin u| + C$.

(b) $\int \csc u\, du = \int \csc u \frac{(\csc u - \cot u)}{(\csc u - \cot u)} du$. Put $v = \csc u - \cot u$, so that $dv = (-\csc u \cot u + \csc^2 u) dv$. Hence, $\int \csc u\, du = \int \frac{dv}{v} = \ln|v| + C =$ $\ln|\csc u - \cot u| + C$.

1. The arc length is given by

$\int_{\pi/4}^{\pi/2} \sqrt{1 + \left[\frac{1}{\sin x}(\cos x)\right]^2} dx = \int_{\pi/4}^{\pi/2} \sqrt{1 + \cot^2 x}\, dx =$

$\int_{\pi/4}^{\pi/2} \csc x\, dx = \ln|\csc x - \cot x|\Big|_{\pi/4}^{\pi/2} =$

$\ln|\csc\frac{\pi}{2} - \cot\frac{\pi}{2}| - \ln|\csc\frac{\pi}{4} - \cot\frac{\pi}{4}| = \ln(1 - 0) -$ $\ln(\sqrt{2} - 1) = -\ln(\sqrt{2} - 1)$ unit.

2. Let $m = 2t$ for some positive integer. Then

$\int \tan^m x \sec^n x\, dx = \int [\tan^2 x]^t \sec^n x\, dx =$

$\int [\sec^2 x - 1]^t \sec^n x\, dx = \int \sum_{j=0}^{t} (-1)^{t-j}\binom{t}{j} \sec^{2j+n} x\, dx =$

$\sum_{j=0}^{t} (-1)^{t-j}\binom{t}{j} \int \sec^{2j+n} x\, dx$. Since n is odd, $2j + n$ is odd, and $\int \tan^m x \sec^n x\, dx =$

$\sum_j \int (-1)^{t-j}\binom{t}{j} \sec^{2j+n} x\, dx$.

43. $D_x \ln|\tan\frac{u}{2}| = \frac{1}{\tan\frac{u}{2}} (\sec^2\frac{u}{2})(\frac{1}{2}) = \frac{\frac{1}{2\cos^2\frac{u}{2}}}{\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}} =$

$\frac{1}{2\cos\frac{u}{2}\sin\frac{u}{2}} = \frac{1}{\sin 2(\frac{u}{2})} = \frac{1}{\sin u} = \csc u$. Hence,

$\int \csc u\, du = \ln|\tan\frac{u}{2}| + C$. Now $\tan\frac{u}{2} = \frac{\sin\frac{u}{2}}{\cos\frac{u}{2}} =$

$\frac{\sqrt{1 - \cos u}}{\sqrt{1 + \cos u}} \cdot \frac{\sqrt{1 + \cos u}}{\sqrt{1 + \cos u}} = \pm\frac{\sin u}{1 + \cos u} \cdot \frac{(1 - \cos u)}{(1 - \cos u)} =$

$\pm\frac{(\sin u - \sin u \cos u)}{\sin^2 u} = \pm(\frac{1}{\sin u} - \frac{\cos u}{\sin u}) =$

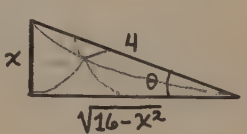
$\pm(\csc u - \cot u)$. So $|\tan\frac{u}{2}| = |\csc u - \cot u|$.

Problem Set 8.3, page 498

1. Put $x = 4\sin\theta$, so that $dx = 4\cos\theta\, d\theta$. Thus, $\int \frac{dx}{x^2\sqrt{16 - x^2}} = \int \frac{4\cos\theta\, d\theta}{16\sin^2\theta(4\cos\theta)} = \frac{1}{16}\int \csc^2\theta\, d\theta =$

$\frac{1}{16}(-\cot\theta) + C =$

$\frac{-\sqrt{16 - x^2}}{16x} + C$.

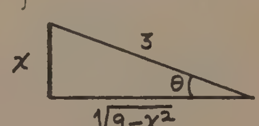


2. Put $x = 3\sin\theta$, so that $dx = 3\cos\theta\, d\theta$. Thus,

$\int \frac{\sqrt{9 - x^2}}{x^2} dx = \int \frac{3\cos\theta(3\cos\theta) d\theta}{9\sin^2\theta} =$

$\int \frac{1 - \sin^2\theta}{\sin^2\theta} d\theta = \int \csc^2\theta\, d\theta - \int d\theta = -\cot\theta - \theta + C =$

$\frac{-\sqrt{9 - x^2}}{x} - \sin^{-1}\frac{x}{3} + C$.

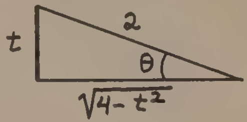


3. Put $t = 2\sin\theta$, so that $dt = 2\cos\theta\, d\theta$. Thus, $\int \frac{dt}{t^4\sqrt{4 - t^2}} = \int \frac{2\cos\theta\, d\theta}{16\sin^4\theta(2\cos\theta)} = \frac{1}{16}\int \csc^4\theta\, d\theta =$

$\frac{1}{16}\int \csc^2\theta(\cot^2\theta + 1)d\theta$. Now put $u = \cot\theta$, so that $du = -\csc^2\theta\, d\theta$. Thus, $\int\frac{dt}{t^4\sqrt{4-t^2}} = -\frac{1}{16}\int(u^2+1)du =$

$-\frac{1}{16}\left(\frac{u^3}{3} + u\right) + C =$

$-\frac{1}{16}\left(\frac{\cot^3\theta}{3} + \cot\theta\right) + C =$

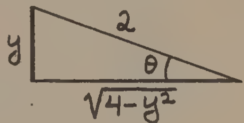


$-\frac{1}{16}\left[\frac{1}{3}\left(\frac{\sqrt{4-t^2}}{t}\right)^3 + \frac{\sqrt{4-t^2}}{t}\right] + C.$

4. Put $y = 2\sin\theta$, so that $dy = 2\cos\theta\, d\theta$. Thus,

$\int\frac{y^3}{\sqrt{4-y^3}}dy = \int\frac{8\sin^3\theta}{2\cos\theta}(2\cos\theta\, d\theta) =$

$\int 8\sin\theta(1-\cos^2\theta)d\theta.$

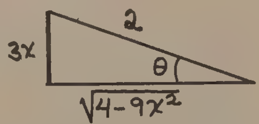


Now put $u = \cos\theta$, so that $du = -\sin\theta\, d\theta$. Now

$\int\frac{y^3}{\sqrt{4-y^2}}dy = \int -8(1-u^2)du = \frac{8}{3}u^3 - 8u + C =$

$\frac{8}{3}\cos^3\theta - 8\cos\theta + C = \frac{(\sqrt{4-y^2})^3}{3} - 4\sqrt{4-y^2} + C.$

5. Put $3x = 2\sin\theta$, so that $3\, dx = 2\cos\theta\, d\theta$.



Thus, $\int\frac{x^2\, dx}{\sqrt{4-9x^2}} =$

$\int\frac{\frac{4}{9}\sin^2\theta(\frac{2}{3}\cos\theta\, d\theta)}{2\cos\theta} = \frac{4}{27}\int\sin^2\theta\, d\theta =$

$\frac{4}{27}\int\frac{1-\cos 2\theta\, d\theta}{2} = \frac{2}{27}(\theta - \frac{\sin 2\theta}{2}) + C =$

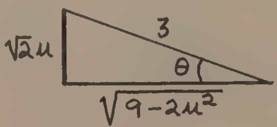
$\frac{2}{27}[\sin^{-1}\frac{3x}{2} - (\sin\theta)(\cos\theta)] + C =$

$\frac{2}{27}\left[\sin^{-1}\frac{3x}{2} - \frac{3x(\sqrt{4-9x^2})}{4}\right] + C.$

6. Put $\sqrt{2}u = 3\sin\theta$, so that $\sqrt{2}\, du = 3\cos\theta\, d\theta$.

Thus, $\int\sqrt{9-2u^2}\, du = \int\sqrt{9-9\sin^2\theta}\left(\frac{3}{\sqrt{2}}\cos\theta\right) d\theta =$

$\frac{3}{\sqrt{2}}\int\cos^2\theta\, d\theta =$



$\frac{3}{\sqrt{2}}\int\frac{1+\cos 2\theta}{2}d\theta =$

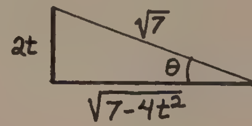
$\frac{3}{2\sqrt{2}}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C =$

$\frac{3}{2\sqrt{2}}\left(\sin^{-1}\frac{\sqrt{2}}{3}u + \frac{\sin\theta\cos\theta}{2}\right) + C =$

$\frac{3}{2\sqrt{2}}\left(\sin^{-1}\frac{\sqrt{2}}{3}u + \frac{\sqrt{2}u\sqrt{9-2u^2}}{18}\right) + C.$

7. Put $2t = \sqrt{7}\sin\theta$, so that $2\, dt = \sqrt{7}\cos\theta\, d\theta$.

Thus, $\int\frac{\sqrt{7-4t^2}}{t^4} =$



$\frac{7}{2}\int\frac{\cos\theta\cdot\cos\theta\, d\theta}{\frac{49}{16}\sin^4\theta} =$

$\frac{8}{7}\int\frac{1-\sin^2\theta}{\sin^4\theta}d\theta =$

$\frac{8}{7}\left[\int\csc^4\theta\, d\theta - \int\csc^2\theta\, d\theta\right] =$

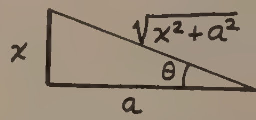
$\frac{8}{7}\left[\int\csc^2\theta(\cot^2\theta+1)d\theta + \cot\theta\right] + C =$

$\frac{8}{7}\left(\frac{-\cot^3\theta}{3} - \cot\theta + \cot\theta\right) + C =$

$\frac{-(\sqrt{7-4t^2})^3}{21t^3} + C$, where the integral was obtained by the technique used in Problem 2.

8. Put $x = a\tan\theta$, so that $dx = a\sec^2\theta = d\theta$. Thus,

$\int\frac{dx}{x^2(a^2+x^2)^{\frac{3}{2}}}$, $a > 0 =$



$\int\frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta\, a^3\sec^3\theta} =$

$\frac{1}{a^4}\int\frac{\cos\theta}{\frac{\sin^2\theta}{\cos^2\theta}}d\theta = \frac{1}{a^4}\int\frac{\cos^3\theta}{\sin^2\theta}d\theta = \frac{1}{a^4}\int\frac{\cos^2\theta\cdot\cos\theta\, d\theta}{\sin^2\theta}$

$\frac{1}{a^4}\int\frac{(1-\sin^2\theta)}{\sin^2\theta}\cos\theta\, d\theta$. Let $u = \sin\theta$, so that $du = \cos\theta\, d\theta$. Hence, $\int\frac{dx}{x^2(a^2+x^2)^{\frac{3}{2}}} =$

$\frac{1}{a^4}\int(u^{-2}-1)d\theta = \frac{1}{a^4}(-\frac{1}{u} - u) + C = -\frac{1}{a^4}\left(\frac{\sin^2\theta+1}{\sin\theta}\right) + C$

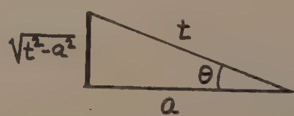
$-\frac{1}{a^4}\left(\frac{\frac{x^2}{x^2+a^2}+1}{\frac{x}{\sqrt{x^2+a^2}}}\right) + C = \frac{1}{a^4}\left(\frac{2x^2+a^2}{x\sqrt{x^2+a^2}}\right) + C.$

9. Put $t = a\sec\theta$, so that $dt = a\sec\theta\tan\theta\, d\theta$.

Thus, $\int\frac{t\, dt}{\sqrt{t^2-a^2}} = \int\frac{a\sec\theta\cdot a\sec\theta\tan\theta\, d\theta}{a\tan\theta} =$

$a\int\sec^2\theta\, d\theta =$

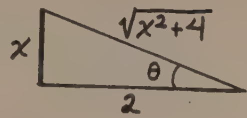
$a\tan\theta + C =$



$a\frac{\sqrt{t^2-a^2}}{a} + C =$

$\sqrt{t^2-a^2} + C.$

10. Put $x = 2 \tan \theta$, so that $dx = 2 \sec^2\theta \, d\theta$. Thus,



$\int \frac{x^3 \, dx}{\sqrt{x^2 + 4}} =$

$\int \frac{8 \tan^3\theta (2 \sec^2\theta) d\theta}{2 \sec \theta} =$

$8\int \tan^3\theta \sec \theta \, d\theta =$

$8\int \tan^2\theta \sec \theta \tan \theta \, d\theta = 8\int (\sec^2\theta - 1)\sec \theta \tan \theta \, d\theta.$

Now, let $u = \sec \theta$, so that $du = \sec \theta \tan \theta \, d\theta$ and

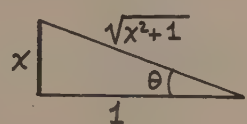
$\int \frac{x^3 dx}{\sqrt{x^2 + 4}} = 8\int (u^2 - 1)du = 8\left(\frac{u^3}{3} - u\right) + C =$

$8\left(\frac{\sec^3\theta}{3} - \sec \theta\right) + C = 8\left[\frac{(\sqrt{x^2 + 4})^3}{24} - \frac{\sqrt{x^2 + 4}}{2}\right] + C =$

$\frac{(\sqrt{x^2 + 4})^3}{3} - 4\sqrt{x^2 + 4} + C = \frac{1}{3}\sqrt{x^2 + 4}\,(x^2 - 8) + C.$

11. Put $x = \tan \theta$, so that

$dx = \sec^2\theta \, d\theta$. Thus,



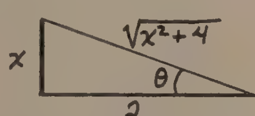
$\int \frac{dx}{x^2\sqrt{1 + x^2}} =$

$\int \frac{\sec^2\theta \, d\theta}{\tan^2\theta (\sec \theta)} = \int \frac{\sec \theta \, d\theta}{\tan^2\theta} = \int \frac{\cos \theta}{\sin^2\theta} d\theta =$

$\int \csc \theta \cot \theta \, d\theta = -\csc \theta + C = \frac{-\sqrt{x^2 + 1}}{x} + C.$

12. Put $x = 2 \tan \theta$, so that

$dx = 2 \sec^2\theta \, d\theta$. Thus,



$\int \frac{dx}{x^4 \sqrt{4 + x^2}} =$

$\int \frac{2 \sec^2\theta \, d\theta}{16 \tan^4\theta \, 2 \sec \theta} = \frac{1}{8}\int \frac{\sec \theta \, d\theta}{\tan^4\theta} = \frac{1}{8}\int \frac{\cos^3\theta}{\sin^4\theta} d\theta =$

$\frac{1}{8}\int \frac{1 - \sin^2\theta}{\sin^4\theta} \cos \theta \, d\theta$. Now let $u = \sin \theta$, so that

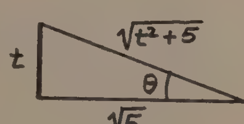
$du = \cos \theta \, d\theta$. Hence, $\int \frac{dx}{x^4 \sqrt{4 + x^2}} =$

$\frac{1}{8}\int (u^{-4} - u^{-2})du = \frac{1}{8}\left(\frac{-1}{3u^3} + \frac{1}{u}\right) + C =$

$\frac{1}{8}(\csc \theta - \csc^3\theta) + C = \frac{1}{8}\left(\frac{\sqrt{x^2 + 4}}{x} - \frac{(x^2 + 4)^{\frac{3}{2}}}{x^3}\right) + C.$

13. Put $t = \sqrt{5} \tan \theta$, so that

$dt = \sqrt{5} \sec^2\theta \, d\theta$. Now



$\int \frac{dt}{t \sqrt{t^2 + 5}} =$

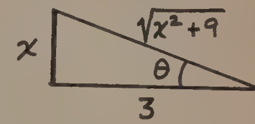
$\int \frac{\sqrt{5} \sec^2\theta \, d\theta}{\sqrt{5} \tan \theta \sqrt{5} \sec \theta} =$

$\frac{1}{\sqrt{5}} \int \frac{\sec \theta \, d\theta}{\tan \theta} = \frac{1}{\sqrt{5}} \int \csc \theta \, d\theta =$

$\frac{1}{\sqrt{5}} \ln|\csc \theta - \cot \theta| + C = \frac{1}{\sqrt{5}}\left|\ln \frac{\sqrt{t^2 + 5} - \sqrt{5}}{t}\right| + C.$

14. Put $x = 3 \tan \theta$, so

that $dx = 3 \sec^2\theta \, d\theta$.



Thus, $\int \frac{dx}{(x^2 + 9)^2} =$

$\int \frac{3 \sec^2\theta \, d\theta}{81 \sec^4\theta} = \frac{1}{27} \int \frac{d\theta}{\sec^2\theta} = \frac{1}{27} \int \cos^2\theta \, d\theta =$

$\frac{1}{27} \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{27}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C =$

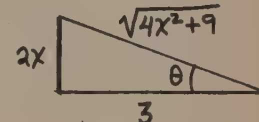
$\frac{1}{27}\left(\frac{\theta}{2} + \frac{2 \sin \theta \cos \theta}{4}\right) + C = \frac{1}{54} (\theta + \sin \theta \cos \theta) + C =$

$\frac{1}{54}\left(\tan^{-1} \frac{x}{3} + \frac{x}{\sqrt{x^2 + 9}} \cdot \frac{3}{\sqrt{x^2 + 9}}\right) + C =$

$\frac{1}{54}\left(\tan^{-1} \frac{x}{3} + \frac{3x}{x^2 + 9}\right) + C.$

15. Put $2x = 3 \tan \theta$, so that

$2 \, dx = 3 \sec^2\theta \, d\theta$. Thus,



$\int \frac{7x^3 dx}{(4x^2 + 9)^{\frac{3}{2}}} =$

$\int \frac{7\left(\frac{27}{8}\right) \tan^3\theta \left(\frac{3}{2} \sec^2\theta\right)}{27 \sec^3\theta} d\theta = \frac{21}{16} \int \frac{\tan^3\theta}{\sec \theta} d\theta =$

$\frac{21}{16} \int \frac{\sin^3\theta}{\cos^2\theta} d\theta = \frac{21}{16} \int \frac{(1 - \cos^2\theta) \sin \theta \, d\theta}{\cos^2\theta}$. Now put

$u = \cos \theta$, so that $du = -\sin \theta \, d\theta$. Hence,

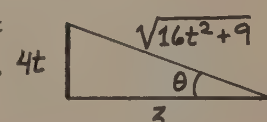
$\int \frac{7x^3 dx}{(4x^2 + 9)^{\frac{3}{2}}} = \frac{21}{16} \int (-u^{-2} + 1)du =$

$\frac{21}{16}\left(\frac{1}{\cos \theta} + \cos \theta\right) + C = \frac{21}{16}\left(\frac{\sqrt{4x^2 + 9}}{3} + \frac{3}{\sqrt{4x^2 + 9}}\right) + C =$

$\frac{7}{8}\left[\frac{2x^2 + 9}{\sqrt{4x^2 + 9}}\right] + C.$

16. Put $4t = 3 \tan \theta$, so that

$4 \, dt = 3 \sec^2\theta \, d\theta$. Thus,



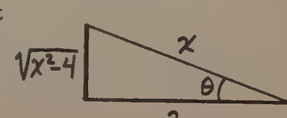
$\int \frac{dt}{\sqrt{16t^2 + 9}} =$

$\int \frac{\frac{3}{4} \sec^2\theta \, d\theta}{3 \sec \theta} =$

$\frac{1}{4} \ln|\sec \theta + \tan \theta| + C = \frac{1}{4} \ln\left|\frac{\sqrt{16t^2 + 9} + 4t}{3}\right| + C.$

17. Put $x = 2 \sec \theta$, so that

$dx = 2 \sec \theta \tan \theta \, d\theta$.

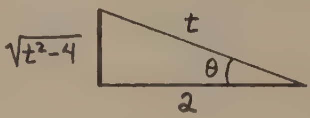


Thus, $\int \frac{dx}{x^2\sqrt{x^2 - 4}} =$

$\int \frac{2 \sec \theta \tan \theta \, d\theta}{4 \sec^2\theta \, (2 \tan \theta)} = \frac{1}{4} \int \cos \theta \, d\theta = \frac{1}{4} \sin \theta + C =$
$\frac{\sqrt{x^2 - 4}}{4x} + C.$

18. Put $t = 2 \sec \theta$, so that

$dt = 2 \sec \theta \tan \theta \, d\theta.$



Thus, $\int \frac{dt}{t^4 \sqrt{t^2 - 4}} =$

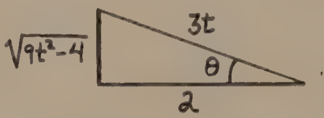
$\int \frac{2 \sec \theta \tan \theta \, d\theta}{16 \sec^4\theta (2 \tan \theta)} =$

$\frac{1}{16} \int \cos^3\theta \, d\theta = \frac{1}{16} \int (1 - \sin^2\theta)\cos \theta \, d\theta.$ Now let $u = \sin \theta$, so that $du = \cos \theta \, d\theta$. Hence,

$\int \frac{dt}{t^4 \sqrt{t^2 - 4}} = \frac{1}{16} \int (1 - u^2)du = \frac{1}{16}\left(u - \frac{u^3}{3}\right) + C =$

$\frac{1}{16} \left(\sin \theta - \frac{\sin^3\theta}{3}\right) + C = \frac{1}{16}\left(\frac{\sqrt{t^2 - 4}}{t} - \frac{(t^2 - 4)^{\frac{3}{2}}}{3t^3}\right) + C.$

19. Put $3t = 2 \sec \theta$, so that $3 \, dt = 2 \sec \theta \tan \theta \, d\theta$.



Thus, $\int \frac{dt}{\sqrt{9t^2 - 4}} =$

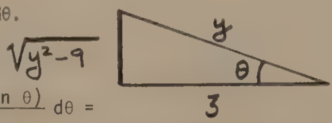
$\int \frac{\frac{2}{3} \sec \theta + \tan \theta \, d\theta}{2 \tan \theta} =$

$\frac{1}{3} \int \sec \theta \, d\theta =$

$\frac{1}{3} \ln|\sec \theta + \tan \theta| + C = \frac{1}{3} \ln \left|\frac{3t + \sqrt{9t^2 - 4}}{2}\right| + C.$

20. Put $y = 3 \sec \theta$, so that

$dy = 3 \sec \theta \tan \theta \, d\theta.$

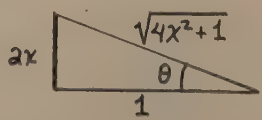


Thus, $\int \frac{\sqrt{y^2 - 9}}{y} dy =$

$\int \frac{3 \tan \theta \, (3 \sec \theta \tan \theta)}{3 \sec \theta} d\theta =$

$\int 3 \tan^2\theta \, d\theta = 3\int (\sec^2\theta - 1) \, d\theta = 3 \tan \theta - 3\theta + C =$
$\sqrt{y^2 - 9} - 3 \sec^{-1} \frac{y}{3} + C.$

21. Put $2x = \tan \theta$, so that $dx = \frac{1}{2} \sec^2\theta \, d\theta$. Now,



$\int \frac{dx}{\sqrt{4x^2 + 1}} =$

$\frac{1}{2} \int \frac{\sec^2\theta \, d\theta}{\sec \theta} =$

$\frac{1}{2} \ln|\sec \theta + \tan \theta| + C =$

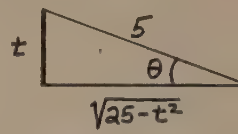
$\frac{1}{2} \ln|\sqrt{4x + 1} + 2x| + C.$ Thus, $\int_{3/8}^{2/3} \frac{dx}{\sqrt{4x^2 + 1}} =$

$\frac{1}{2} \ln\left|\sqrt{4x^2 + 1} + 2x\right| \Big|_{3/8}^{2/3} = \frac{1}{2} \ln \left(\frac{5}{3} + \frac{4}{3}\right) -$

$\frac{1}{2} \ln\left(\frac{5}{4} + \frac{3}{4}\right) = \frac{1}{2}(\ln 3 - \ln 2) = \frac{1}{2} \ln \frac{3}{2}.$

22. Put $t = 5 \sin \theta$, so that

$dt = 5 \cos \theta \, d\theta.$ Thus,



$\int \sqrt{25 - t^2} dt =$

$\int 5 \cos \theta \, (5 \cos \theta) \, d\theta =$

$\frac{25}{2} \int (1 + \cos 2\theta)d\theta = \frac{25}{2} \left(\theta + \frac{\sin 2\theta}{2}\right) + C =$

$\frac{25}{2} (\theta + \sin \theta \cos \theta) + C = \frac{25}{2} \sin^{-1} \frac{t}{5} + \frac{t}{5} \frac{\sqrt{25 - t^2}}{5} +$

Hence, $\int_3^4 \sqrt{25 - t^2} \, dt = \frac{25}{2}\left(\sin^{-1} \frac{t}{5} + \frac{t\sqrt{25 - t^2}}{25}\right)\Big|_3^4 =$

$\frac{25}{2}\left(\sin^{-1} \frac{4}{5} + \frac{12}{25} - \sin^{-1} \frac{3}{5} - \frac{12}{25}\right) =$

$\frac{25}{2}\left(\sin^{-1} \frac{4}{5} - \sin^{-1} \frac{3}{5}\right).$

23. In (1), for $-4 < x < 4$, $x \neq 0$, $D_x \left[\frac{-\sqrt{16 - x^2}}{16 x} + C\right]$

$\frac{\frac{16x(-\frac{1}{2})(-2x)}{\sqrt{16 - x^2}} + \sqrt{16 - x^2}(16)}{(16x)^2} = \frac{16x^2 + 16(16 - x^2)}{16^2x^2 \sqrt{16 - x^2}} =$

$\frac{16^2}{16^2x^2 \sqrt{16}} \quad \frac{1}{x^2 \sqrt{16 - x^2}}.$ In (13), for $t \neq 0$,

$D_t\left[\frac{1}{\sqrt{5}} \ln \left|\frac{\sqrt{t^2 + 5} - \sqrt{5}}{t}\right| + C\right] = \frac{t}{\sqrt{5}(\sqrt{t^2 + 5} - \sqrt{5})} \cdot$

$\left[\frac{\frac{t(2t)}{2 \sqrt{t^2 + 5}} - (\sqrt{t^2 + 5} - \sqrt{5}) \cdot 1}{t^2}\right] =$

$\frac{t}{\sqrt{5} \, (\sqrt{t^2 + 5} - \sqrt{5})} \cdot \left[\frac{t^2 - (\sqrt{t^2 + 5} - \sqrt{5})\sqrt{t^2 + 5}}{t^2 \sqrt{t^2 + 5}}\right] =$

$\frac{t^2 - t^2 - 5 + \sqrt{5} \sqrt{t^2 + 5}}{\sqrt{5}t \sqrt{t^2 + 5}(\sqrt{t^2 + 5} - \sqrt{5})} =$

$\frac{\sqrt{t^2 + 5} - \sqrt{5}}{t\sqrt{t^2 + 5}(\sqrt{t^2 + 5} - \sqrt{5})} = \frac{1}{t\sqrt{t^2 + 5}}$

In (19), for $|t| > \frac{2}{3}$, $D_t\left[\frac{1}{3} \ln\left|\frac{3t + \sqrt{9t^2 - 4}}{2}\right| + C\right]$

$\frac{1}{3} \left(\frac{2}{3t + \sqrt{9t^2 - 4}}\right)\left(\frac{1}{2}\right)\left(3 + \frac{9t}{\sqrt{9t^2 - 4}}\right) =$

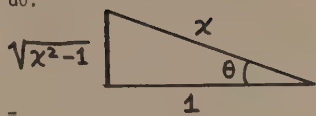
$\frac{\sqrt{9t^2 - 4} + 3t}{\left[3t + \sqrt{9t^2 - 4}\right] \, (\sqrt{9t^2 - 4})} = \frac{1}{\sqrt{9t^2 - 4}}.$

24. (a) Put $u = x^2 - 1$, so that du $2x$ dx. Then

$\int \frac{x \, dx}{\sqrt{x^2 - 1}} = \frac{1}{2}\int \frac{du}{u^{\frac{1}{2}}} = \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{x^2 - 1} + C.$

(b) Put $x = \sec\theta$, so that $dx = \sec\theta\tan\theta\, d\theta$.

So $\int \frac{x\,dx}{\sqrt{x^2 - 1}} =$



$\int \frac{\sec\theta \sec\theta\tan\theta}{\tan\theta}\, d\theta =$

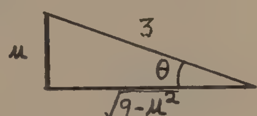
$\int \sec^2\theta\, d\theta = \tan\theta + C = \sqrt{x^2 - 1} + C.$

5. Completing the square on $5 - 4t - t^2$, we have $5 - (4t + t^2) = 5 + 4 - (2 + t)^2 = 9 - (2 + t)^2$.

Now we put $u = 2 + t$, so that $du = dt$. Thus,

$\int \frac{dt}{(5 - 4t - t^2)^{\frac{3}{2}}} = \int \frac{du}{(9 - u^2)^{\frac{3}{2}}}$. Now let $u = 3\sin\theta$,

so that $du = 3\cos\theta\, d\theta$.



So $\int \frac{du}{(9 - u^2)^{\frac{3}{2}}} =$

$\int \frac{3\cos\theta\, d\theta}{27\cos^3\theta} = \frac{1}{9}\int \sec^2\theta\, d\theta = \frac{1}{9}\tan\theta + C =$

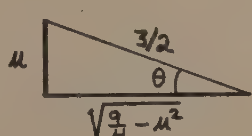
$\frac{1}{9}\frac{u}{\sqrt{9 - u^2}} + C$. Thus, $\int \frac{dt}{(5 - 4t - t^2)^{\frac{3}{2}}} =$

$\frac{2 + t}{9\sqrt{5 - 4t - t^2}} + C.$

6. Completing the square on $2 - x - x^2$, we have $2 - (x + x^2) = 2 + \frac{1}{4} - \left(\frac{1}{2} + x\right)^2$. Now put $u = \frac{1}{2} + x$,

so that $du = dx$. Thus, $\int \frac{x\,dx}{\sqrt{2 - x - x^2}} = \int \frac{u - \frac{1}{2}}{\sqrt{\frac{9}{4} - u^2}}\, du.$

Now let $u = \frac{3}{2}\sin\theta$, so that $du = \frac{3}{2}\cos\theta\, d\theta$.



So $\int \frac{u - \frac{1}{2}}{\sqrt{\frac{9}{4} - u^2}}\, du =$

$\int \frac{\left(\frac{3}{2}\sin\theta - \frac{1}{2}\right)}{\frac{3}{2}\cos\theta}\left(\frac{3}{2}\cos\theta\right) d\theta \quad \int\left(\frac{3}{2}\sin\theta - \frac{1}{2}\right) d\theta =$

$-\frac{3}{2}\cos\theta - \frac{1}{2}\theta + C = -\frac{1}{2}\left[\left(\frac{3\sqrt{\frac{9}{4} - u^2}}{\frac{3}{2}}\right) + \sin^{-1}\frac{2u}{3}\right] + C.$

Hence, $\int \frac{x\,dx}{\sqrt{2 - x - x^2}} = -\frac{1}{2}\left(2\sqrt{2 - x - x^2} + \sin^{-1}\frac{(1 + 2x)}{3}\right) + C.$

7. Completing the square on $t^2 + 3t + 4$, we have

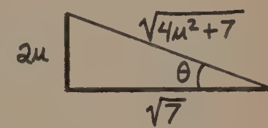
$t^2 + 3t + 4 = t^2 + 3t + \frac{9}{4} - \frac{9}{4} + 4 = (t + \frac{3}{2})^2 + \frac{7}{4}$.

Now put $u = t + \frac{3}{2}$, so that $du = dt$. Thus,

$\int \frac{2t}{(t^2 + 3t + 4)^2}\, dt = \int \frac{2u - 3}{(u^2 + \frac{7}{4})^2}\, du$. Now put $u =$

$\frac{\sqrt{7}}{2}\tan\theta$, so that

$du = \frac{\sqrt{7}}{2}\sec^2\theta\, d\theta$.



Thus, $\int \frac{2u \cdot 3}{(u^2 + \frac{7}{4})^2}\, du =$

$\int \frac{(\sqrt{7}\tan\theta - 3)\frac{\sqrt{7}}{2}}{\frac{49}{16}\sec^4\theta}\sec^2\theta\, d\theta =$

$\int (\frac{8}{7}\sin\theta\cos\theta - \frac{24\sqrt{7}}{49}\cos^2\theta)d\theta = \frac{8}{7}\frac{\sin^2\theta}{2} -$

$\frac{24\sqrt{7}}{49}\int \frac{1 + \cos 2\theta}{2}\, d\theta = \frac{4}{7}\sin^2\theta - \frac{12\sqrt{7}}{49}\left(\theta + \frac{\sin 2\theta}{2}\right) + C =$

$\frac{4}{7}\left(\frac{4u^2}{4u^2 + 7}\right) - \frac{12\sqrt{7}}{49}\left(\tan^{-1}\frac{2u}{\sqrt{7}} + \frac{2u}{\sqrt{4u^2 + 7}} \cdot \frac{\sqrt{7}}{\sqrt{4u^2 + 7}}\right) + C.$

Hence, $\int \frac{2t}{(t^2 + 3t + 4)^2} = \frac{4(t + \frac{3}{2})^2}{7(t^2 + 3t + 4)} -$

$\frac{12\sqrt{7}}{49}\tan^{-1}\left(\frac{2t + 3}{\sqrt{7}}\right) - \frac{6t + 9}{7(t^2 + 3t + 4)} + C =$

$\frac{4t^2 + 12t + 9 - 6t - 9}{7(t^2 + 3t + 4)} - \frac{12\sqrt{7}}{49}\tan^{-1}\left(\frac{2t + 3}{\sqrt{7}}\right) + C =$

$\frac{4t^2 + 6t}{7(t^2 + 3t + 4)} - \frac{12\sqrt{7}}{49}\tan^{-1}\left(\frac{2t + 3}{\sqrt{7}}\right) + C.$

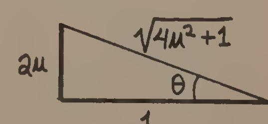
28. Completing the square on $2t^2 - 6t + 5$, we have $2(t^2 - 3t) + 5 = 2(t^2 - 3t + \frac{9}{4}) + 5 - \frac{9}{4} =$

$2(t - \frac{3}{2})^2 + \frac{1}{2}$. Now put $u = t - \frac{3}{2}$, so that $du = dt$.

Thus, $\int \frac{dt}{\sqrt{2t^2 - 6t + 5}} = \int \frac{du}{\sqrt{2u^2 + \frac{1}{2}}}$. Put $\sqrt{2}u = \tan\theta$,

$\sqrt{2}\, du = \frac{1}{\sqrt{2}}\sec^2\theta\, d\theta$.

So $\int \frac{du}{\sqrt{2u^2 + \frac{1}{2}}} =$



$\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\frac{1}{\sqrt{2}}\sec\theta} =$

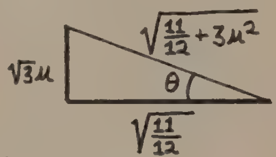
$\frac{\sqrt{2}}{2}\int \sec\theta\, d\theta = \frac{\sqrt{2}}{2}\ln|\sec\theta + \tan\theta| + C =$

$\frac{\sqrt{2}}{2}\ln|\sqrt{4u^2 + 1} + 2u| + C$. Hence, $\int \frac{dt}{\sqrt{2t^2 - 6t + 5}} =$

$\frac{\sqrt{2}}{2}\ln|\sqrt{4t^2 - 12t + 10} + 2t - 3| + C.$

29. Completing the square on $1 - x + 3x^2$, we have $3x^2 - x + 1 = 3(x^2 - \frac{x}{3}) + 1 = 3(x^2 - \frac{x}{3} + \frac{1}{36}) - \frac{3}{36} + 1 = 3(x - \frac{1}{6})^2 + \frac{11}{12}$. Now, put $u = x - \frac{1}{6}$, so that $x = u + \frac{1}{6}$, $dx = du$, and

$\int \frac{x\,dx}{\sqrt{1 - x + 3x^2}} = \int \frac{(u + \frac{1}{6})du}{\sqrt{3u^2 + \frac{11}{12}}}$. Now, put $\sqrt{3}u = \sqrt{\frac{11}{12}}\tan\theta$, so that $u = \frac{\sqrt{11}}{6}\tan\theta$, $du = \frac{\sqrt{11}}{6}\sec^2\theta\,d\theta$, and $3u^2 + \frac{11}{12} = \frac{11}{12}(\tan^2\theta + 1) = \frac{11}{12}\sec^2\theta$. Thus,



$\int \frac{x\,dx}{\sqrt{1 - x + 3x^2}} =$

$\int \frac{\frac{\sqrt{11}}{6}\tan\theta + \frac{1}{6}}{\frac{\sqrt{11}}{\sqrt{12}}\sec\theta}\frac{\sqrt{11}}{6}\sec^2\theta\,d\theta =$

$\frac{\sqrt{3}}{18}\int(\sqrt{11}\sec\theta\tan\theta + \sec\theta)d\theta =$

$\frac{\sqrt{3}}{18}(\sqrt{11}\sec\theta + \ln|\sec\theta + \tan\theta|) + C =$

$\frac{\sqrt{3}}{18}\left(\sqrt{11}\frac{\sqrt{\frac{11}{12} + 3u^2}}{\sqrt{\frac{11}{12}}} + \ln\left|\frac{\sqrt{\frac{11}{12} + 3u^2}}{\sqrt{\frac{11}{12}}} + \frac{\sqrt{3}u}{\sqrt{\frac{11}{12}}}\right|\right) + C =$

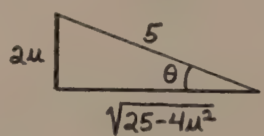
$\frac{\sqrt{3}}{18}\left[\sqrt{12}\sqrt{3x^2 - x + 1} + \ln\left|\sqrt{\frac{12}{11}}\left(\sqrt{3x^2 - x + 1} + \sqrt{3}(x - \frac{1}{6})\right)\right|\right] + C = \frac{\sqrt{3}}{18}\left[2\sqrt{3}\sqrt{3x^2 - x + 1} + \ln\sqrt{\frac{12}{11}} + \ln\left|\sqrt{3x^2 - x + 1} + \sqrt{3}x - \frac{\sqrt{3}}{6}\right|\right] + C =$

$\frac{1}{3}\sqrt{3x^2 - x + 1} + \frac{\sqrt{3}}{18}\ln\left|\sqrt{3x^2 - x + 1} + \sqrt{3}x - \frac{\sqrt{3}}{6}\right| + C$,

where the constant $\frac{\sqrt{3}}{18}\ln\sqrt{\frac{12}{11}}$ has been absorbed into the constant of integration.

30. $6 - x - x^2 = 6 - (x + x^2) = 6 + \frac{1}{4} - (x^2 + x + \frac{1}{4}) = \frac{25}{4} - (x + \frac{1}{2})^2$. Put $u = x + \frac{1}{2}$, so that $du = dx$.

Thus, $\int \frac{x\,dx}{\sqrt{6 - x - x^2}} =$



$\int \frac{(u - \frac{1}{2})}{\sqrt{\frac{25}{4} - u^2}}\,du.$

Now put $u = \frac{5}{2}\sin\theta$, so that $du = \frac{5}{2}\cos\theta\,d\theta$. So,

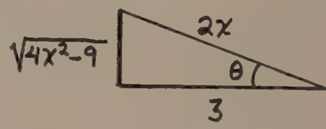
$\int \frac{(u - \frac{1}{2})}{\sqrt{\frac{25}{4} - u^2}}\,du = \int \frac{(\frac{5}{2}\sin\theta - \frac{1}{2})\frac{5}{2}\cos\theta\,d\theta}{\frac{5}{2}\cos\theta} =$

$-\frac{5}{2}\cos\theta - \frac{1}{2}\theta + C = -\frac{1}{2}(\sqrt{25 - 4u^2} + \sin^{-1}\frac{2u}{5}) + C =$

$-\frac{1}{2}(\sqrt{6 - x - x^2} + \sin^{-1}\frac{2x + 1}{5}) + C.$

31. Put $2x = 3\sec\theta$, so that $2\,dx = 3\sec\theta\tan\theta\,d\theta$.

Thus, $\int \frac{dx}{(4x^2 - 9)^{\frac{3}{2}}} =$



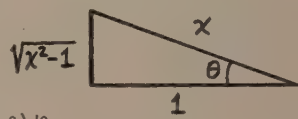
$\int \frac{\frac{3}{2}\sec\theta\tan\theta\,d\theta}{(3\tan\theta)^3} =$

$\frac{1}{18}\int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \frac{1}{18}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$. Put $u = \sin\theta$, so that $du = \cos\theta\,d\theta$. Now, $\int \frac{dx}{(4x^2 - 9)^{\frac{3}{2}}} = \frac{1}{18}\int \frac{du}{u^2} =$

$\frac{1}{18}(-\frac{1}{u}) + C = -\frac{1}{18\sin\theta} + C = -\frac{x}{9\sqrt{4x^2 - 9}} + C.$

32. Put $x = \sec\theta$, so that

$dx = \sec\theta\tan\theta\,d\theta$.



Thus, $\int x^3\sqrt{x^2 - 1}\,dx =$

$\int \sec^3\theta\tan\theta(\sec\theta\tan\theta)d\theta =$

$\int \sec^4\theta\tan^2\theta\,d\theta = \int(1 + \tan^2\theta)\tan^2\theta\sec^2\theta\,d\theta$. Now let $u = \tan\theta$, so that $du = \sec^2\theta\,d\theta$. Hence,

$\int x^3\sqrt{x^2 - 1}\,dx = \int(u^2 + u^4)du = \frac{u^3}{3} + \frac{u^5}{5} + C =$

$\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C = \frac{(x^2 - 1)^{\frac{3}{2}}}{3} + \frac{(x^2 - 1)^{\frac{5}{2}}}{5} + C.$

33. Let $u = t - 2$, then $\int \frac{t\,dt}{t^2 - 4t + 8} = \int \frac{(u + 2)du}{u^2 + 4} =$

$\int \frac{u\,du}{u^2 + 4} + \int \frac{2\,du}{u^2 + 4} = \frac{1}{2}\ln(u^2 + 4) + \tan^{-1}\frac{u}{2} + C =$

$\frac{1}{2}\ln(t^2 - 4t + 8) + \tan^{-1}\frac{t - 2}{2} + C.$

34. Let $u = 9 + x^2$, so $du = 2x\,dx$. Then $\int \frac{2x\,dx}{\sqrt{9 + x^2}} =$

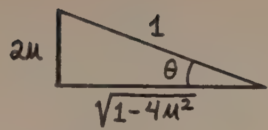
$\int \frac{du}{u^{\frac{1}{2}}} = 2u^{\frac{1}{2}} + C = 2\sqrt{9 + x^2} + C.$

35. Completing the square on $-3 + 8x - 4x^2$, we have $-3 - 4(-2x + x^2) = -3 + 4 - 4(1 - 2x + x)^2 = 1 - 4(1 - x)^2$. Now put $u = 1 - x$, so that $du = -dx$.

Thus, $\int \frac{dx}{\sqrt{-3 + 8x - 4x^2}} = \int \frac{-du}{\sqrt{1 - 4u^2}}$. Now let

$2u = \sin\theta$, so that

$2\,du = \cos\theta\,d\theta$.



So $\displaystyle\int \frac{-du}{\sqrt{1-4u^2}} =$

$\displaystyle\int \frac{-\frac{1}{2}\cos\theta\,d\theta}{\cos\theta} = -\frac{1}{2}\theta + C = -\frac{1}{2}\sin^{-1}2u + C =$

$\displaystyle\frac{\sin^{-1}(-2u)}{2} + C$. Thus, $\displaystyle\int \frac{dx}{\sqrt{-3+8x-4x^2}} =$

$\displaystyle\frac{\sin^{-1}(2x-2)}{2} + C$.

36. $\displaystyle\int \frac{t-1}{(t^2-2t+1)^2}\,dt = \int \frac{t-1}{(t-1)^4}\,dt = \int \frac{dt}{(t-1)^3}$.
Hence, $\displaystyle\int \frac{t-1}{(t^2-2t+1)^2}\,dt = -\frac{1}{2}(t-1)^{-2} + C =$

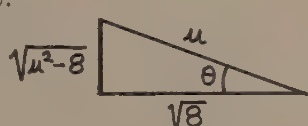
$\displaystyle -\frac{1}{2(t-1)^2} + C$.

37. $\displaystyle\int \frac{x\,dx}{\sqrt{4-x^2}} = -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{4-x^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{4-x^2} + C$.

38. Completing the square on $x^2 + 6x + 1$, we have
$x^2 + 6x + 1 = x^2 + 6x + 9 - 9 + 1 = (x+3)^2 - 8$.
Now put $u = x + 3$, so that $du = dx$. Thus,
$\displaystyle\int \frac{3}{(x^2+6x+1)^2}\,dx = \int \frac{3\,du}{(u^2-8)^2}$. Now let $u =$

$\sqrt{8}\sec\theta$, so that

$du = \sqrt{8}\sec\theta\tan\theta\,d\theta$.



Thus, $\displaystyle\int \frac{3\,du}{(u^2-8)^2} =$

$\displaystyle\int \frac{3\sqrt{8}\sec\theta\tan\theta\,d\theta}{64\tan^2\theta} =$

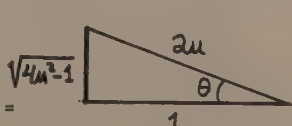
$\displaystyle\frac{3\sqrt{8}}{64}\int \csc\theta\,d\theta = \frac{3\sqrt{8}}{64}\ln|\csc\theta - \cot\theta| + C =$

$\displaystyle\frac{3\sqrt{8}}{64}\ln\left|\frac{u}{\sqrt{u^2-8}} - \frac{\sqrt{8}}{\sqrt{u^2-8}}\right| + C$. Hence,

$\displaystyle\int \frac{3}{(x^2+6x+1)^2}\,dx = \frac{3\sqrt{8}}{64}\ln\left|\frac{x+3-\sqrt{8}}{\sqrt{x^2+6x+1}}\right| + C$.

39. Completing the square on $2 - 3x + x^2$, we have
$x^2 = 3x + \frac{9}{4} - \frac{9}{4} + 2 = (x - \frac{3}{2})^2 - \frac{1}{4}$. Now put $u =$
$x - \frac{3}{2}$, so that $du = dx$. Thus, $\displaystyle\int \sqrt{2-3x+x^2}\,dx =$
$\displaystyle\int \sqrt{u^2 - \frac{1}{4}}\,du$. Now let $u = \frac{1}{2}\sec\theta$, so that $du =$

$\frac{1}{2}\sec\theta\tan\theta\,d\theta$.



So $\displaystyle\int \sqrt{u^2 - \frac{1}{4}}\,du =$

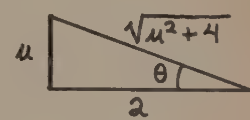
$\displaystyle\int \frac{1}{2}\tan\theta\sec\theta\tan\theta\,d\theta =$

$\displaystyle\frac{1}{2}\int (\sec^3\theta - \sec\theta)\,d\theta =$

$\displaystyle\frac{1}{2}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right] + C = \frac{1}{4}\left(2u\sqrt{4u^2-1} - \ln|2u + \sqrt{4u^2-1}|\right) + C$. Hence, $\displaystyle\int \sqrt{2-3x+x^2}\,dx =$
$\frac{1}{4}\,2(x - \frac{3}{2})2\sqrt{2-3x+x^2} - \frac{1}{4}\ln|2x - 3 + 2\sqrt{2-3x+x^2}| + C =$
$(x - \frac{3}{2})\sqrt{2-3x+x^2} - \frac{1}{4}\ln|2x - 3 + 2\sqrt{2-3x+x^2}| + C$.

40. Put $u = 3e^{-x}$, so that $du = -3e^{-x}dx$. Thus,
$\displaystyle\int \frac{e^{-x}dx}{\sqrt{4+9e^{-2x}}} = \int \frac{-\frac{1}{3}\,du}{\sqrt{4+u^2}}$. Now let $u = 2\tan\theta$, so

that $du = 2\sec^2\theta\,d\theta$.



So $\displaystyle\int \frac{e^{-x}dx}{\sqrt{4+9e^{-2x}}} =$

$\displaystyle -\frac{1}{3}\int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} =$

$\displaystyle -\frac{1}{3}\int \sec\theta\,d\theta =$

$\displaystyle -\frac{1}{3}\ln|\sec\theta+\tan\theta| + C = -\frac{1}{3}\ln\left|\frac{\sqrt{u^2+4}+u}{2}\right| + C =$

$\displaystyle -\frac{1}{3}\ln\left(\frac{\sqrt{4+9e^{-2x}} + 3e^{-x}}{2}\right) + C$, or

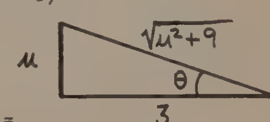
$\displaystyle \frac{1}{3}\ln\left(\frac{2}{\sqrt{4+9e^{-2x}} + 3e^{-x}}\right) + C$.

41. $\displaystyle\int \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx = \int \frac{du}{\sqrt{u}}$, $u = x^2 + 2x + 2$. Hence
$\displaystyle\int \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx = 2\sqrt{x^2+2x+2} + C$.

42. Put $u = \tan t$, so that $du = \sec^2 t\,dt$. Thus,
$\displaystyle\int \frac{\sec^2 t}{(\tan^2 t + 9)^{3/2}}\,dt = \int \frac{du}{(u^2+9)^{3/2}}$. Now put

$u = 3\tan\theta$, so that

$du = 3\sec^2\theta\,d\theta$.



Thus, $\displaystyle\int \frac{\sec^2 t}{(\tan^2 t + 9)^{3/2}}\,dt =$

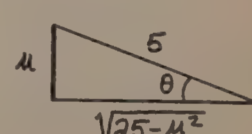
$\displaystyle\int \frac{du}{(u^2+9)^{3/2}} = \int \frac{3\sec^2\theta\,d\theta}{3^3\sec^3\theta} = \frac{1}{9}\int \cos\theta\,d\theta =$

$\displaystyle\frac{1}{9}\sin\theta + C = \frac{u}{9\sqrt{u^2+9}} + C = \frac{\tan t}{9\sqrt{\tan^2 t + 9}} + C$.

43. Put $u = \cos v$, so that $du = -\sin v\,dv$. Thus,
$\displaystyle\int \frac{\sin v}{(25-\cos^2 v)^{\frac{3}{2}}}\,dv = -\int \frac{du}{(25-u^2)^{\frac{3}{2}}}$. Now put

$u = 5\sin\theta$, so that

$du = 5\cos\theta\,d\theta$. Hence,



$\displaystyle\int \frac{\sin v\,dv}{(25-\cos^2 v)^{\frac{3}{2}}} =$

$-\int \frac{du}{(25-u^2)^{\frac32}} = -\int \frac{5\cos\theta\, d\theta}{125\cos^3\theta} = -\frac{1}{25}\int \sec^2\theta\, d\theta =$

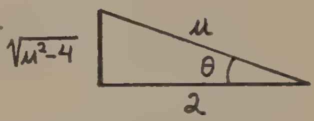
$-\frac{1}{25}\tan\theta + C = \frac{-u}{25\sqrt{25-u^2}} + C = \frac{-\cos v}{25\sqrt{25-\cos^2 v}} + C.$

44. Put $u = \ln t$, so that $du = \frac{dt}{t}$. Thus

$\int \frac{dt}{t\left[(\ln t)^2-4\right]^{\frac32}} = \int\frac{du}{(u^2-4)^{\frac32}}$. Now put

$u = 2\sec\theta$, so that

$du = 2\sec\theta\tan\theta\, d\theta$.



$\int \frac{dt}{t\left[(\ln t)^2-4\right]^{\frac32}} =$

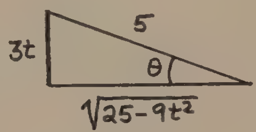
$\int\frac{du}{(u^2-4)^{\frac32}} = \int\frac{2\sec\theta\tan\theta\,d\theta}{8\tan^3\theta} = \frac14\int\frac{\cos\theta}{\sin^2\theta}d\theta$. Now put

$v = \sin\theta$. Thus, $\frac14\int\frac{\cos\theta}{\sin^2\theta}d\theta = \frac14\int\frac{dv}{v^2} = -\frac14\csc\theta + C =$

$-\frac14\csc(\sec^{-1}(\frac{\ln t}{2})) + C.$

45. Put $3t = 5\sin\theta$, so that

$3\,dt = 5\cos\theta\,d\theta$. Thus,



$\int\frac{t^2dt}{(25-9t^2)^{\frac32}} =$

$\int\frac{\frac{25}{9}\sin^2\theta\,\frac53\cos\theta\,d\theta}{5^3\cos^3\theta} = \frac{1}{27}\int\frac{1-\cos^2\theta}{\cos^2\theta}d\theta =$

$\frac{1}{27}(\tan\theta-\theta)d\theta = \frac{1}{27}\left(\frac{3t}{\sqrt{25-9t^2}} - \sin^{-1}\frac{3t}{5}\right)+C.$

Hence, $\int_0^1 \frac{t^2\,dt}{(25-9t^2)^{\frac32}} = \frac{1}{27}\left(\frac{3t}{\sqrt{25-9t^2}} - \sin^{-1}\frac{3t}{5}\right)\Big|_0^1 =$

$\frac{1}{27}\left(\frac34 - \sin^{-1}\frac35\right).$

46. $\int\frac{dz}{e^z-e^{-z}} = \int\frac{e^zdz}{e^{2z}-1}$. Put $u = e^z$, so that $du =$ e^zdz. Thus, $\int\frac{e^zdz}{e^{2z}-1} = \int\frac{du}{u^2-1}$. Now let $u = \sec\theta$, so that $du = \sec\theta\tan\theta\,d\theta$. So $\int\frac{du}{u^2-1} =$

$\int\frac{\sec\theta\tan\theta\,d\theta}{\tan^2\theta} = \int\frac{1}{\sin\theta}d\theta = \int\csc\theta\,d\theta =$

$\ln|\csc\theta-\cot\theta| + C = \ln\left|\frac{u}{\sqrt{u^2-1}} - \frac{1}{\sqrt{u^2-1}}\right| + C =$

$\ln\left|\frac{e^z-1}{\sqrt{e^{2z}-1}}\right| + C$. Thus, $\int_{\ln 2}^{\ln 3}\frac{dz}{e^z-e^{-z}} =$

$\ln\left|\frac{e^z-1}{\sqrt{e^{2z}-1}}\right|\Bigg|_{\ln 2}^{\ln 3} = \ln\left(\frac{3-1}{\sqrt{9-1}}\right) - \ln\left(\frac{2-1}{\sqrt{4-1}}\right) =$

$\ln\frac{2}{\sqrt8} - \ln\frac{1}{\sqrt3}.$

47. Let $t = 3\sec\theta$, so that $dt = 3\sec\theta\tan\theta\,d\theta$.

Thus, $\int\frac{\sqrt{t^2-9}}{t}dt = \int\frac{3\tan\theta}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta =$

$3\int\tan^2\theta\,d\theta = 3\int(\sec^2\theta-1)d = 3\tan\theta - 3\theta + C =$

$\sqrt{t^2-9} - 3\sec^{-1}\frac t3 + C$. Hence, $\int_3^6\frac{\sqrt{t^2-9}}{t}dt =$

$(\sqrt{t^2-9} - 3\sec^{-1}\frac t3)\Big|_3^6 = (3\sqrt3 - 3\sec^{-1}2) -$

$(0 - 3\sec^{-1}1) = 3\sqrt3 - 3(\frac\pi3) = 3\sqrt3-\pi.$

48. Let $u = 9y^2 + 12y + 3$, then

$\int_0^1(3y+2)\sqrt{9y^2+12y+3}\,dy = \frac16\int_3^{24}\sqrt u\,du =$

$\frac16\,\frac{u^{3/2}}{3/2}\Big|_3^{24} = \frac19\cdot u^{3/2}\Big|_3^{24} = \frac19(48\sqrt6 - 3\sqrt3) =$

$\frac13(16\sqrt6-\sqrt3).$

49. $A = \int_4^5\frac{45}{\sqrt{16x^2-175}}dx$. Put $4x = \sqrt{175}\sec\theta$, so that $4\,dx = \sqrt{175}\sec\theta\tan\theta\,d\theta$. Therefore,

$\int\frac{45}{\sqrt{16x^2-175}}dx = \int\frac{\frac{45}{4}\sqrt{175}}{\sqrt{175}\tan\theta}\sec\theta\tan\theta\,d\theta =$

$\frac{45}{4}\int\sec\theta\,d\theta = \frac{45}{4}\ln|\sec\theta+\tan\theta| + C =$

$\frac{45}{4}\ln(\frac{4x}{\sqrt{175}} + \frac{\sqrt{16x^2-175}}{\sqrt{175}}) + C$. Thus,

$A = \frac{45}{4}\ln(\frac{4x+\sqrt{16x^2-175}}{\sqrt{175}})\Big|_4^5 = \frac{45}{4}[\ln(\frac{20+15}{\sqrt{175}}) -$

$\ln(\frac{16+9}{\sqrt{175}})] = \frac{45}{4}\ln(\frac75)$ square units.

50. Let $y = 2\sinh t$, $dy = 2\cosh t\,dt$, and

$\int\frac{y^2\,dy}{(4+y^2)^{5/2}} = \int\frac{4\sinh^2 t}{(4+4\sinh^2t)^{5/2}}\cdot 2\cosh t\,dt =$

$\frac14\int\frac{\sinh^2t\cosh t}{(1+\sinh^2t)^{5/2}}dt = \frac14\int\frac{\sinh^2t}{\cosh^4t}dt =$

$\frac14\int\tanh^2 t\,\text{sech}^2t\,dt = \frac14\int\tanh^2t\,\frac{d}{dt}(\tanh t)dt =$

$\frac{1}{12}\tanh^3t + C$. Hence, $\int\frac{y^2\,dy}{(4+y^2)^{5/2}} =$

$\frac{1}{12}\tanh^3(\sinh^{-1}(\frac y2)) + C.$

51. $V = \pi\int_0^4 y^2dx = \pi\int_0^4 \frac{x^2}{(x^2+16)^3}\,dx$. (Let $x = 4\tan\theta$, so that $dx = 4\sec^2\theta\, d\theta$) $= \pi\int_0^{\pi/4} \frac{16\tan^2\theta\cdot 4\sec^2\theta}{(16\tan^2\theta + 16)^3} =$

$\frac{\pi}{64}\int_0^{\pi/4} \frac{\tan^2\theta\sec^2\theta}{\sec^6\theta}\,d\theta = \frac{\pi}{64}\int_0^{\pi/4}\sin^2\theta\cos^2\theta\, d\theta =$

$\frac{\pi}{256}\int_0^{\pi/4}(1 - \cos^2 2\theta)d\theta = \frac{\pi}{256}\int_0^{\pi/4}[1 - \frac{1}{2}(1 \cos 4\theta)]d\theta =$

$\frac{\pi}{256}\int_0^{\pi/4}(\frac{1}{2} - \frac{1}{2}\cos 4\theta)d\theta = \frac{\pi}{256}[\frac{\theta}{2} - \frac{\sin 4\theta}{8}]\Big|_0^{\pi/4} =$

$\frac{\pi}{256}[\frac{\pi}{8}] = \frac{\pi^2}{2048}$.

52. By the definition of $\sin^{-1}$, $\sin\theta = \frac{u}{a}$ and $u = a\sin\theta$.

(a) $\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{\cos^2\theta}$. Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\cos\theta > 0$. Hence, $\sqrt{a^2 - u^2} = a\cos\theta$.

(b) Since $u = a\sin\theta$, then $du = a\cos\theta\, d\theta$.

(c) $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{u/a} = \frac{a}{u}$, provided $u \neq 0$.

(d) $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{a^2-u^2}}{a}} = \frac{a}{\sqrt{a^2 - u^2}}$.

(e) $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{u/a}{\frac{\sqrt{a^2-u^2}}{a}} = \frac{u}{\sqrt{a^2 - u^2}}$.

(f) $\cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{a^2 - u^2}}{u}$, provided $u \neq 0$.

53. When $u = a\sin\theta$, we get the same relationships that we did from a right triangle, so that it is not really necessary to assume that u is positive.

54. Suppose $a > 0$ and suppose that we wish to integrate an expression involving $\sqrt{a^2 + u^2}$, where u is real. Then we let $\theta = \tan^{-1}\frac{u}{a}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then by definition of inverse tangent, $\tan\theta = \frac{u}{a}$.

(i) $\sqrt{a^2 + u^2} = \sqrt{a^2 + a^2\tan^2\theta} = a\sqrt{\sec^2\theta}$. Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\sec\theta > 0$. Thus, $\sqrt{a^2 + u^2} = a\sec\theta$.

(ii) Since $u = a\tan\theta$, then $du = a\sec^2\theta\, d\theta$.

(iii) Since $\sec\theta = \frac{\sqrt{a^2 + u^2}}{a}$, then $\cos\theta = \frac{a}{\sqrt{a^2 + u^2}}$.

(iv) Now $\frac{\sin\theta}{\cos\theta} = \tan\theta$, then $\sin\theta = \frac{u}{a}(\frac{a}{\sqrt{a^2 + u^2}}) = \frac{u}{\sqrt{a^2 + u^2}}$.

(v) Since $\tan\theta = \frac{u}{a}$, then $\cot\theta = \frac{a}{u}$, $u \neq 0$.

These relationships are those displayed on a right triangle where $\tan\theta = \frac{u}{a}$, whether or not u is positive.

55. $S = \int_{\ln 1/2}^{\ln 3/5}\sqrt{1 + (\frac{e^x}{\sqrt{1 - e^{2x}}})^2}\,dx =$

$\int_{\ln 1/2}^{\ln 3/5}\frac{1}{\sqrt{1 - e^{2x}}}\,dx$. Put $u = e^x$, so that $du = e^x dx$.

Thus $S = \int_{1/2}^{3/5}\frac{du}{u\sqrt{1 - u^2}}$. Now let $u = \sin\theta$, so that $du = \cos\theta\, d\theta$. Now $S = \int_{\pi/6}^{\sin^{-1}3/5}\frac{\cos\theta\, d\theta}{\sin\theta\cos\theta} =$

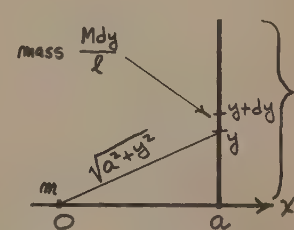
$\int_{\pi/6}^{\sin^{-1}3/5}\csc\theta\, d\theta\ \ln|\csc\theta - \cot\theta|\Big|_{\pi/6}^{\sin^{-1}3/5} =$

$\ln|\csc(\sin^{-1}\frac{3}{5}) - \cot(\sin^{-1}\frac{3}{5})| - \ln\csc|\frac{\pi}{6} - \cot\frac{\pi}{6}| =$

$\ln|\frac{5}{3} - \frac{4}{3}| - \ln|2 - \sqrt{3}| = \ln(\frac{1}{3}) - \ln(2 - \sqrt{3}) =$

$\ln\left(\frac{1/3}{2 - \sqrt{3}}\right) = \ln\left(\frac{2 + \sqrt{3}}{3(2 - \sqrt{3})(2 + \sqrt{3})}\right) = \ln\left(\frac{2 + \sqrt{3}}{3}\right)$ units.

56. The infinitesimal force on m, because of the portion of the wire between y and $y + dy$, is directed along the hypotenuse of the indicated right triangle and has magnitude



$\frac{Gm(\frac{M\,dy}{\ell})}{a^2 + y^2}$. The horizontal component of this force is given by $dF_v\ \frac{Gm(\frac{M\,dy}{\ell})}{a^2 + y^2}\cdot\frac{a}{\sqrt{a^2 + y^2}}$. Hence,

$F_v = \int_0^\ell \frac{Gm\,M\,a\,dy}{\ell(a^2 + y^2)^{3/2}}\ \ \frac{Gm\,M\,a}{\ell}\int_0^\ell\frac{dy}{(a^2 + y^2)^{3/2}}$.

Now put $y = a\tan\theta$, so that $dy = a\sec^2\theta\, d\theta$. So

$\int_1^\ell\frac{dy}{(a^2 + y^2)^{3/2}} = \int_0^{\tan^{-1}\frac{\ell}{a}}\frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} =$

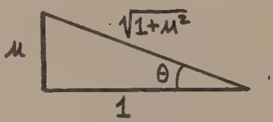
$\int_0^{\tan^{-1}\frac{\ell}{a}}\frac{1}{a^2}\cos\theta\, d\theta = \frac{1}{a^2}\sin\theta\Big|_0^{\tan^{-1}\frac{\ell}{a}} =$

$\frac{1}{a^2}\left(\frac{\ell}{\sqrt{a^2+\ell^2}}\right)$. Thus $F_v = \frac{Gm\,M\,a}{\ell}\left[\frac{1}{a^2}\left(\frac{\ell}{\sqrt{a^2+\ell^2}}\right)\right]$

$\frac{Gm\,M}{a\sqrt{a^2+\ell^2}}$.

57. $x^2dy - \sqrt{x^2-9}\,dx = 0$, so $dy = \frac{\sqrt{x^2-9}}{x^2}$ and $y = \int\frac{\sqrt{x^2-9}}{x^2}\,dx + C_1$. Put $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$. Thus, $\int\frac{\sqrt{x^2-9}}{x^2}\,dx = \int\frac{3\tan\theta\,(3\sec\theta\tan\theta)}{9\sec^2\theta}\,d\theta = \int\frac{\tan^2\theta}{\sec\theta}\,d\theta = \int(\sec\theta-\cos\theta)d\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + C = \ln|\frac{x}{3} + \tan(\sec^{-1}(\frac{x}{3}))| - \sin(\sec^{-1}(\frac{x}{3})) + C$. Hence, $y = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| - \frac{1}{|x|}\sqrt{x^2-9} + C$.

58. $S = 2\pi\int x\sqrt{1+(x')^2}\,dy$, and so $S = 2\pi\int_0^2 e^y\sqrt{1+e^{2y}}\,dy$. Put $u = e^y$, so that $du = e^y dy$. Thus, $\int e^y\sqrt{1+e^{2y}}\,dy = \int\sqrt{1+u^2}\,du$. Now let $u = \tan\theta$, so that $du = \sec^2\theta\,d\theta$. Thus, $\int\sqrt{1+u^2}\,du = \int\sec\theta\cdot\sec^2\theta\,d\theta = \int\sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}\sqrt{1+u^2}\cdot u + \frac{1}{2}\ln|\sqrt{1+u^2}+u| + C$. So $S = \left[\frac{\sqrt{1+e^{2y}}\cdot e^y}{2} + \frac{1}{2}\ln(\sqrt{1+e^{2y}}+e^y)\right]\Big|_0^2 = \frac{1}{2}[\sqrt{1+e^4}(e^2) + \ln(\sqrt{1+e^4}+e^2) - \sqrt{2} - \ln(\sqrt{2}+1)] = \frac{1}{2}\left[e^2\sqrt{1+e^4} - \sqrt{2} + \ln\left(\frac{\sqrt{1+e^4}+e^2}{\sqrt{2}+1}\right)\right]$.



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1. Put $u = x$ and $dv = \cos 2x\,dx$, so that $du = dx$ and $v = \frac{1}{2}\sin 2x$. Thus, $\int x\cos 2x\,dx = uv - \int v\,du = \frac{1}{2}x\sin 2x - \int\frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.

2. Put $u = x$ and $dv = \sin kx\,dx$, so that $du = dx$ and $v = -\frac{1}{k}\cos kx$. Thus, $\int x\sin kx\,dx = uv - \int v\,du = -\frac{1}{k}x\cos kx - \int -\frac{1}{k}\cos kx\,dx = -\frac{1}{k}x\cos kx + \frac{1}{k^2}\sin kx + C$.

3. Put $u = x$ and $dv = e^{3x}dx$, so that $du = dx$ and $v = \frac{1}{3}e^{3x}$. Thus, $\int xe^{3x}dx = uv - \int v\,du = \frac{1}{3}xe^{3x} - \int\frac{1}{3}e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$.

4. Put $u = x$ and $dv = e^{-4x}dx$, so that $du = dx$ and $v = -\frac{1}{4}e^{-4x}$. Thus, $\int xe^{-4x}dx = uv - \int v\,du = -\frac{1}{4}xe^{-4x} - \int -\frac{1}{4}e^{-4x}dx = -\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C$.

5. Put $u = \ln 5x$ and $dv = dx$, so that $du = \frac{1}{x}dx$ and $v = x$. Thus, $\int\ln 5x\,dx = uv - \int v\,du = x\ln 5x - \int x(\frac{1}{x})dx = x\ln 5x - x + C$.

6. Put $u = \ln 2x$ and $dv = x\,dx$, so that $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$. Then $\int x\ln 2x\,dx = \int u\,dv = uv - \int v\,du = \frac{x^2}{2}\ln 2x - \int\frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2}{2}\ln 2x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln 2x - \frac{x^2}{4} + C$.

7. Put $u = \cos^{-1}x$ and $dv = dx$, so that $du = -\frac{1}{\sqrt{1-x^2}}dx$ and $v = x$. Thus, $\int\cos^{-1}x\,dx = uv - \int v\,du = x\cos^{-1}x - \int -\frac{x}{\sqrt{1-x^2}}\,dx$. Now let $y = 1-x$ so that $dy = -2x\,dx$. So $\int\frac{-x}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int\frac{dy}{y^{1/2}} = y^{1/2} + C = \sqrt{1-x^2} + C$. Hence, $\int\cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2} + C$.

8. Put $u = \ln(x^2)$ and $dv = x^3dx$, so that $du = \frac{2}{x}dx$ and $v = \frac{x^4}{4}$. Thus, $\int x^3\ln(x^2)\,dx = uv - \int v\,du = \frac{x^4}{4}\ln(x^2) - \int\frac{x^4}{4}(\frac{2}{x})\,dx = \frac{x^4}{2}\ln x - \frac{x^4}{8} + C$.

9. Put $u = \sec^{-1}x$, $dv = dx$, so that $du = \frac{dx}{x\sqrt{x^2-1}}$ and $v = x$. Thus, $\int\sec^{-1}x\,dx = x\sec^{-1}x - \int\frac{dx}{\sqrt{x^2-1}}$. Let $x = \sec\theta$, $dx = \sec\theta\tan\theta$, $\sqrt{x^2-1} = \tan\theta$. So $\int\frac{dx}{\sqrt{x^2-1}} = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| = \ln|x+\sqrt{x^2-1}|$. Hence, $\int\sec^{-1}x\,dx = x\sec^{-1}x -$

$\ln|x + \sqrt{x^2 - 1}| + C.$

10. Put $u = \sin^{-1}3x$ and $dv = dx$, so that $du = \frac{3}{\sqrt{1 - 9x^2}}\,dx$ and $v = x$. Thus, $\int \sin^{-1}3x\,dx = uv - \int v\,du =$ $x\sin^{-1}3x - \int \frac{3x}{\sqrt{1 - 9x^2}}\,dx$. Now let $y = 1 - 9x^2$, so that $dy = -18x\,dx$. Then $\int \frac{3x}{\sqrt{1 - 9x^2}}\,dx = -\frac{1}{6}\int \frac{dy}{y^{1/2}} =$ $-\frac{1}{3}y^{1/2} + C = -\frac{1}{3}\sqrt{1 - 9x^2} + C$. Hence, $\int \sin^{-1}3x\,dx =$ $x\sin^{-1}3x + \frac{1}{3}\sqrt{1 - 9x^2} + C.$

11. Put $u = t$ and $dv = \sec t\tan t\,dt$, so that $du = dt$ and $v = \sec t$. Thus, $\int t\sec t\tan t\,dt = uv -$ $\int v\,du = t\sec t - \int \sec t\,dt = t\sec t -$ $\ln|\sec t + \tan t| + C.$

12. Put $u = \tan^{-1}x$ and $dv = dx$, so that $du = \frac{1}{1 + x^2}\,dx$ and $v = x$. Thus, $\int \tan^{-1}x\,dx = uv - \int v\,du =$ $x\tan^{-1}x - \int \frac{x}{1 + x^2}\,dx$. Now let $y = 1 + x^2$, so that $dy = 2x\,dx$. So $\int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\int \frac{dy}{y} = \frac{1}{2}\ln|y| + C.$ Hence, $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C.$

13.

$$\begin{array}{lll}
\underline{u} & \underline{v'} & \\
x^2 & \sin 3x & \\
2x & -\frac{1}{3}\cos 3x & \xrightarrow{+} +x^2\left(-\frac{1}{3}\cos 3x\right) \\
2 & -\frac{1}{9}\sin 3x & \xrightarrow{-} -2x\left(-\frac{1}{9}\sin 3x\right) \\
0 & \frac{1}{27}\cos 3x & \xrightarrow{+} +2\left(\frac{1}{27}\cos 3x\right).
\end{array}$$

Therefore, $\int x^2\sin 3x\,dx = -\frac{x^2}{3}\cos 3x + \frac{2}{9}x\sin 3x +$ $\frac{2}{27}\cos 3x + C.$

14.

$$\begin{array}{lll}
\underline{u} & \underline{v'} & \\
x^2 & \sin^2 x = \frac{1 - \cos 2x}{2} & \\
2x & \frac{x}{2} - \frac{\sin 2x}{4} & \xrightarrow{+} +x^2\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) \\
2 & \frac{x^2}{4} + \frac{\cos 2x}{8} & \xrightarrow{-} -2x\left(\frac{x^2}{4} + \frac{\cos 2x}{8}\right) \\
0 & \frac{x^3}{12} + \frac{\sin 2x}{16} & \xrightarrow{+} +2\left(\frac{x^3}{12} + \frac{\sin 2x}{16}\right)
\end{array}$$

Thus, $\int x^2\sin^2 x\,dx = \frac{x^3}{2} - \frac{x^2\sin 2x}{4} - \frac{x^3}{2} - \frac{x\cos 2x}{4} +$ $\frac{x^3}{6} + \frac{\sin 2x}{8} + C = \frac{x^3}{6} - \frac{x^2\sin 2x}{4} - \frac{x\cos 2x}{4} +$ $\frac{\sin 2x}{8} + C.$

15.

$$\begin{array}{lll}
\underline{u} & \underline{v'} & \\
3x^2 - 2x + 1 & \cos x & \\
6x - 2 & \sin x & \xrightarrow{+} +(3x^2 - 2x + 1)\sin x \\
6 & -\cos x & \xrightarrow{-} -(6x - 2)(-\cos x) \\
0 & -\sin x & \xrightarrow{+} +(6)(-\sin x)
\end{array}$$

Therefore, $\int (3x^2 - 2x + 1)\cos x\,dx =$ $(3x^2 - 2x - 5)\sin x + (6x - 2)\cos x + C.$

16.

$$\begin{array}{lll}
\underline{u} & \underline{v'} & \\
x^2 - 3x + 2 & e^{-x} & \\
2x - 3 & -e^{-x} & \xrightarrow{+} +(x^2 - 3x + 2)(-e^{-x}) \\
2 & e^{-x} & \xrightarrow{-} -(2x - 3)e^{-x} \\
0 & -e^{-x} & \xrightarrow{+} +(2)(-e^{-x})
\end{array}$$

Therefore, $\int (x^2 - 3x + 2)e^{-x}dx = -(x^2 - 3x + 2)e^{-x} -$ $(2x - 3)e^{-x} - 2e^{-x} + C = -(x^2 - x + 1)e^{-x} + C.$

17.

$$\begin{array}{lll}
\underline{u} & \underline{v'} & \\
\frac{x^2}{2} + x & e^{2x} & \\
x + 1 & \frac{1}{2}e^{2x} & \xrightarrow{+} \frac{1}{2}\left(\frac{x^2}{2} + x\right)e^{2x} \\
1 & \frac{1}{4}e^{2x} & \xrightarrow{-} -\frac{(x + 1)}{4}e^{2x} \\
0 & (1/8)e^{2x} & \xrightarrow{+} \frac{1}{8}e^{2x}
\end{array}$$

Therefore, $\int \left(\frac{x^2}{2} + x\right)e^{2x}dx = e^{2x}\left[\frac{1}{2}\left(\frac{x^2}{2} + x\right) -\right.$ $\left.\frac{1}{4}(x + 1) + \frac{1}{8}\right] + C = \left[\frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8}\right]e^{2x} + C.$

18.

$$\begin{array}{lll}
\underline{u} & \underline{v'} & \\
x^2 & \sec^2 x\tan x & \\
2x & \frac{1}{2}\sec^2 x & \xrightarrow{+} +x^2\left(\frac{1}{2}\sec^2 x\right) \\
2 & \frac{1}{2}\tan x & \xrightarrow{-} -2x\left(\frac{1}{2}\tan x\right) \\
0 & \frac{1}{2}\ln|\sec x| & \xrightarrow{+} +2\left(\frac{1}{2}\ln|\sec x|\right)
\end{array}$$

Therefore, $\int x^2\sec^2 x\tan x\,dx = \frac{1}{2}x^2\sec^2 x - x\tan x +$ $\ln|\sec x| + C.$

19. Put $u = e^{-x}$ and $dv = \cos 2x\,dx$, so that $du =$ $-e^{-x}dx$ and $v = \frac{1}{2}\sin 2x$. Thus, $\int e^{-x}\cos 2x\,dx =$ $uv - \int v\,du = \frac{e^{-x}}{2}\sin 2x + \frac{1}{2}\int \sin 2x\,e^{-x}\,dx$. Now

put $u_1 = e^{-x}$ and dv_1 sin 2x dx, so that $du_1 = -e^{-x}dx$ and $v_1 = -\frac{1}{2}\cos 2x$. Hence, $\int \sin 2x\, e^{-x}\, dx = u_1 v_1 - \int v_1 du_1 = \frac{-e^{-x}}{2}\cos 2x - \int \frac{1}{2}\cos 2x\, e^{-x}\, dx$. Hence, $\int e^{-x}\cos 2x\, dx = \frac{e^{-x}}{2}\sin 2x - \frac{1}{4}e^{-x}\cos 2x - \frac{1}{4}\int \cos 2x\, e^{-x}dx + C$. Therefore, $\frac{5}{4}\int e^{-x}\cos 2x\, dx = \frac{e^{-x}}{2}\sin 2x - \frac{1}{4}e^{-x}\cos 2x + C$, and so $\int e^{-x}\cos 2x\, dx = \frac{2}{5}e^{-x}\sin 2x - \frac{1}{5}e^{-x}\cos 2x + C = \frac{e^{-x}}{5}(2\sin 2x - \cos 2x) + C$.

20. Put $u = e^{2x}$ and $dv = \sin x\, dx$, so that $du = 2e^{2x}dx$ and $v = -\cos x$. Then $\int e^{2x}\sin x\, dx = uv - \int v\, du = -e^{2x}\cos x + 2\int \cos x\, e^{2x}dx$. Now put $u_1 = e^{2x}$ and $dv_1 = \cos x\, dx$ so that $du_1 = 2e^{2x}dx$ and $v_1 = \sin x$. Thus, $\int \cos x\, e^{2x}dx = u_1 v_1 - \int v_1 du_1 + C = e^{2x}\sin x - \int 2\sin x\, e^{2x}dx + C$. Therefore, $\int e^{2x}\sin x\, dx = -e^{2x}\cos x + 2[e^{2x}\sin x - 2\int \sin x\, e^{2x}\, dx] + C$, and so $5\int e^{2x}\sin x\, dx = -e^{2x}\cos x + 2e^{2x}\sin x + C$ and $\int e^{2x}\sin x\, dx = -\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x + C$.

21. $\int \csc^3 x\, dx = \int \csc x \csc^2 x\, dx$. Put $u = \csc x$ and $dv = \csc^2 x\, dx$, so that $du = -\csc x \cot x\, dx$ and $v = -\cot x$. Thus, $\int \csc^3 x\, dx = uv - \int v\, du = -\csc x \cot x - \int \cot^2 x \csc x\, dx = -\csc x \cot x - \int \csc^3 x\, dx + \csc x\, dx$. So $2\int \csc^3 x\, dx = -\csc x \cot x + \ln|\csc x - \cot x| + C$. Therefore, $\int \csc^3 x\, dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$.

22. Put $u = e^{ax}$ and $dv = \sin(b\, x)dx$, so that $du = ae^{ax}\, dx$ and $v = -\frac{1}{b}\cos(b\, x)$. Thus, $\int e^{ax}\sin(b\, x)dx = uv - \int v\, du = e^{ax}(-\frac{1}{b})\cos b\, x + \frac{a}{b}\int e^{ax}\cos bx\, dx$. Now put $u_1 = e^{ax}$ and $dv_1 = \cos(b\, x)dx$, so that $du_1 = ae^{ax}\, dx$ and $v_1 = \frac{1}{b}\sin(b\, x)$. So $\int e^{ax}\cos(b\, x)dx = u_1 v_1 - \int v_1 du_1 = \frac{1}{b}e^{ax}\sin(b\, x) - \int \frac{a}{b}\sin(b\, x)e^{ax}\, dx$. Hence, $\int e^{ax}\sin(b\, x)dx = -\frac{e^{ax}}{b}\cos(b\, x) + \frac{a}{b^2}e^{ax}\sin(b\, x) - \frac{a^2}{b^2}\int \sin(b\, x)e^{ax}\, dx$ and $\frac{b^2 + a^2}{b^2}\int e^{ax}\sin(b\, x)dx = \frac{e^{ax}}{b^2}(a\sin(b\, x) - b\cos(b\, x)) + C$.

23. Put $x^2 = t$, so that $2x\, dx = dt$. Thus, $\int x^3 e^{x^2}dx = \frac{1}{2}\int te^t\, dt$. Now put $u = t$ and $dv = e^t\, dt$, so that $du = dt$ and $v = e^t$. Thus, $\int te^t\, dt = uv - \int v\, du = te^t - \int e^t dt = te^t - e^t + C$. Therefore, $\int x^3 e^{x^2}\, dx = \frac{1}{2}[te^t - e^t] + C = \frac{e^{x^2}}{2}(x^2 - 1) + C$.

24. Put $2x^2 = t$, so that $4x\, dx = dt$. Thus, $\int x^3 \sin 2x^2\, dx = \int \frac{1}{8}t\sin t\, dt$. Now put $u = t$ and $dv = \sin t\, dt$, so that $du = dt$ and $v = -\cos t$. Hence, $\int t\sin t\, dt = uv - \int v\, du = -t\cos t + \int \cos t\, dt = -t\cos t + \sin t + C$. Therefore, $\int x^3 \sin 2x^2\, dx = \frac{1}{8}(-t\cos t + \sin t) + C = \frac{1}{8}(-2x^2\cos 2x^2 + \sin 2x^2) + C$.

25. If $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$ and $\int \sqrt{1 + x^2}\, dx = \int \sec\theta \sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta$. Integrating by parts as in Example 8 of the present section, we have $\int \sqrt{1 + x^2}\, dx = \int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{x\sqrt{1 + x^2}}{2} + \frac{1}{2}\ln|\sqrt{1 + x^2} + x| + C$.

26. Let $u = \sin x$; then $\int \cos x \tan^{-1}(\sin x)dx = \int \tan^{-1}u\, du = u\tan^{-1}u - \int \frac{u}{1 + u^2}\, du = u\tan^{-1}u - \frac{1}{2}\ln(1 + u^2) + C$. Hence, $\int \cos x\tan^{-1}(\sin x)dx = \sin x\tan^{-1}(\sin x) - \frac{1}{2}\ln(1 + \sin^2 x) + C$.

27.
$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ 2x - 1 & e^{-x} & \\ 2 & -e^{-x} & \xrightarrow{+} \quad +(2x - 1)(-e^{-x}) \\ 0 & e^{-x} & \xrightarrow{-} \quad -2e^{-x} \end{array}$$

Therefore, $\int (2x - 1)e^{-x}dx = -e^{-x}[2x + 1] + C$.

28.
$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ x & \sinh x & \\ 1 & \cosh x & \xrightarrow{+} \quad x\cosh x \\ 0 & \sinh x & \xrightarrow{-} \quad -\sinh x \end{array}$$

Hence, $\int x\sinh x\, dx = x\cosh x - \sinh x + C$.

9. $\int xe^x \sin x\,dx = -xe^x\cos x + \int xe^x\cos x\,dx +$
$\int e^x\cos x\,dx = -xe^x\cos x + xe^x\sin x - \int xe^x\sin x\,dx -$
$\int e^x\sin x + \int e^x\cos x\,dx$. Hence, $\int xe^x\sin x\,dx =$
$\frac{1}{2}xe^x(\sin x - \cos x) + \frac{1}{2}\int e^x(\cos x - \sin x)\,dx =$
$\frac{1}{2}xe^x(\sin x - \cos x) + \frac{1}{2}e^x\sin x - \ e^x\sin x\,dx =$
$\frac{1}{2}xe^x(\sin x - \cos x) + \frac{1}{2}e^x\sin x -$
$\frac{1}{2}e^x(\sin x - \cos x) + C = \frac{1}{2}xe^x(\sin x - \cos x) +$
$\frac{1}{2}e^x\cos x + C$.

0. $\int \ln(1 + t^2)dt = t\ln(1 + t^2) - \int\frac{2t^2}{1 + t^2}\,dt =$
$t\ln(1 + t^2) - 2\int(\sec^2\theta - 1)d\theta$, where $t = \tan\theta$. So
$t\ln(1 + t^2) - 2(t - \theta) + C = t\ln(1 + t^2) - 2t +$
$2\tan^{-1}t + C$.

1. $\int(\ln x)^2dx = x(\ln x)^2 - 2\int\ln x\,dx = x(\ln x)^2 -$
$2x\ln x + 2x + C$.

2. $\int\sin\sqrt{x}\,dx = 2\int u\sin u\,du$, $u^2 = x$. Hence,
$\int\sin\sqrt{x}\,dx = 2[-u\cos u + \sin u] + C =$
$-2\sqrt{x}\cos\sqrt{x} - 2\sin\sqrt{x} + C$.

3. $\int x\csc^2x\,dx = \int x\frac{d}{dx}(-\cot x)dx = -x\cot x +$
$\int\cot x\,dx + C = -x\cot x + \ln|\sin x| + C$.

4. Let $u = \cosh^{-1}x$, $du = dx$. Then $\int\cosh^{-1}x\,dx =$
$x\cosh^{-1}x - \int\frac{x\,dx}{\sqrt{x^2 - 1}} = x\cosh^{-1}x - \frac{1}{2}\int\frac{2x\,dx}{\sqrt{x^2 - 1}} =$
$x\cosh^{-1}x - \frac{1}{2}\int\frac{du}{\sqrt{u}} = x\cosh^{-1}x + \sqrt{x^2 - 1} + C$.

5. Let $u = x^2$, $du = 2x\,dx$. Then $\int\frac{x^3}{\sqrt{1 - x^2}}\,dx =$
$\frac{1}{2}\int\frac{u\,du}{\sqrt{1 - u}} = -x^2\sqrt{1 - x^2} + \int 2x\sqrt{1 - x^2}\,dx$. Hence,
$\int\frac{x^3}{\sqrt{1 - x^2}}\,dx = -x^2\sqrt{1 - x^2} - \frac{2}{3}(1 - x^2)^{3/2} + C =$
$-\sqrt{1 - x^2}\,(x^2 + \frac{2}{3} - \frac{2}{3}x^2) + C = -\frac{1}{3}\sqrt{1 - x^2}\,(x^2 + 2) + C$.

5. If $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$.
$\int\frac{x^2}{\sqrt{x^2 - 1}}\,dx = \int\frac{\sec^2\theta}{\tan\theta}\sec\theta\tan\theta\,d\theta = \int\sec^3\theta\,d\theta$.
Integrating by parts as in Example 8 of the present section, we have $\int\frac{x^2}{\sqrt{x^2 - 1}}\,dx = \int\sec^3\theta\,d\theta =$
$\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta\tan\theta| + C = \frac{1}{2}x\sqrt{x^2 - 1} +$
$\frac{1}{2}\ln|x + \sqrt{x^2 - 1}| + C$.

37. Put $x^4 = t$, so that $4x^3\,dx = dt$. Now $\int x^{11}\cos x^4dx =$
$\frac{1}{4}\int t^2\cos t\,dt$. Now use the tabular method:

u	v'		
t^2	$\cos t$		
$2t$	$\sin t$	$\xrightarrow{+}$	$+ t^2\sin t$
2	$-\cos t$	$\xrightarrow{-}$	$- (2t)(-\cos t)$
0	$-\sin t$	$\xrightarrow{+}$	$+ 2(-\sin t)$

Thus, $\int t^2\cos t\,dt = t^2\sin t + 2t\cos t - 2\sin t + C$.
Therefore, $\int x^{11}\cos x^4\,dx = \frac{1}{4}(t^2\sin t + 2t\cos t -$
$2\sin t) + C = \frac{1}{4}(x^8\sin x^4 + 2x^4\cos x^4 - 2\sin x^4) +$
C.

38. Put $t = \sqrt{x}$, so that $dt = \frac{1}{2\sqrt{x}}\,dx$. Thus,
$\int x^{3/2}\cos\sqrt{x}\,dx = \int x^2\cos\sqrt{x}\,\frac{1}{\sqrt{x}}\,dx = 2\int t^4\cos t\,dt$.
Now we use the tabular method:

u	v'		
t^4	$\cos t$		
$4t^3$	$\sin t$	$\xrightarrow{+}$	$+ t^4\sin t$
$12t^2$	$-\cos t$	$\xrightarrow{-}$	$- 4t^3(-\cos t)$
$24t$	$-\sin t$	$\xrightarrow{+}$	$+ 12t^2(-\sin t)$
24	$\cos t$	$\xrightarrow{-}$	$- 24t\cos t$
0	$\sin t$	$\xrightarrow{+}$	$+ 24\sin t$

Hence, $\int t^4\cos t\,dt = t^4\sin t + 4t^3\cos t -$
$12t^2\sin t - 24t\cos t + 24\sin t + C$. Therefore,
$\int x^{3/2}\cos\sqrt{x}\,dx = 2[x^2\sin\sqrt{x} + 4x\sqrt{x}\cos\sqrt{x} -$
$12x\sin\sqrt{x} - 24\sqrt{x}\cos\sqrt{x} + 24\sin\sqrt{x}] + C$.

39. We use the tabular method to find $\int 4x^2\sin 3x\,dx$.

u	v'		
$4x^2$	$\sin 3x$		
$8x$	$-\frac{1}{3}\cos 3x$	$\xrightarrow{+}$	$+ 4x^2(-\frac{1}{3}\cos 3x)$
8	$-\frac{1}{9}\sin 3x$	$\xrightarrow{-}$	$- 8x(-\frac{1}{9}\sin 3x)$
0	$+\frac{1}{27}\cos 3x$	$\xrightarrow{+}$	$8(\frac{1}{27}\cos 3x)$

Thus, $\int_0^{\pi/9} 4x^2\sin 3x\,dx = [-\frac{4}{3}x^2\cos 3x + \frac{8}{9}x\sin 3x +$

$\frac{8}{27}\cos 3x]\Big|_0^{\frac{\pi}{9}} = -\frac{4}{3}\frac{\pi^2}{81}\cos\frac{\pi}{3} + \frac{8}{9}\frac{\pi}{9}\sin\frac{\pi}{3} + \frac{8}{27}\cos\frac{\pi}{3} - \frac{8}{27} = -\frac{2}{243}\pi^2 + \frac{4\pi}{81}\sqrt{3} - \frac{4}{27} = \frac{12\pi\sqrt{3} - 2\pi^2 - 36}{243}$.

40. Put $t = 1 + x$, so that $dt = dx$. Thus, $\int_0^1 \frac{x\,e^x}{(1+x)^2}\,dx = \int_1^2 \frac{(t-1)e^{t-1}}{t^2}\,dt = \int_1^2 \frac{e^{t-1}}{t}\,dt - \int_1^2 \frac{e^{t-1}}{t^2}\,dt$. Now to evaluate the first integral, put $u = \frac{1}{t}$ and $dv = e^{t-1}dt$, so that $du = -\frac{1}{t^2}\,dt$ and $v = e^{t-1}$. So $\int_1^2 \frac{e^{t-1}}{1}\,dt = (uv)\Big|_{t=1}^{t=2} - \int_1^2 v\,du = \frac{e^{t-1}}{t}\Big|_{t=1}^{t=2} - \int_1^2 -\frac{e^{t-1}}{t^2}\,dt$. Therefore, $\int_0^1 \frac{xe^x}{(1+x)^2} = \left[\frac{e^{t-1}}{t}\Big|_1^2 + \int_1^2 \frac{e^t}{t^2}\,dt\right] - \int_1^2 \frac{e^{t-1}}{t^2}\,dt = \frac{e^{t-1}}{t}\Big|_1^2 = \frac{e}{2} - 1$.

41. By problem 9, $\int \sec^{-1}x\,dx = x\sec^{-1}x - \ln|x + \sqrt{x^2 - 1}| + C$. Thus, $\int_2^3 \sec^{-1}x\,dx = (x\sec^{-1}x - \ln|x + \sqrt{x^2-1}|)\Big|_2^3 = 3\sec^{-1}3 - \ln(3 + \sqrt{8}) - 2\sec^{-1}2 + \ln(2 + \sqrt{3}) = 3\sec^{-1}3 - \frac{2\pi}{3} + \ln\frac{2+\sqrt{3}}{3+\sqrt{8}}$.

42. By problem 7, $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2} + C$. Thus, $\int_{-1}^1 \cos^{-1}x\,dx = (x\cos^{-1}x - \sqrt{1-x^2})\Big|_{-1}^1 = \cos^{-1}1 - \sqrt{1-1} + \cos^{-1}(-1) + \sqrt{1-1} = \pi$.

43. We evaluate $\int(5x^2 - 3x + 1)\sin x\,dx$ by the tabular method.

u	v'	
$5x^2 - 3x + 1$	$\sin x$	
$10x - 3$	$-\cos x$	$\xrightarrow{+}$ $+\ (5x^2 - 3x + 1)(-\cos x)$
10	$-\sin x$	$\xrightarrow{-}$ $-\ (10x - 3)(-\sin x)$
0	$\cos x$	$\xrightarrow{+}$ $+\ (10)\cos x$

Thus, $\int_0^{\frac{\pi}{4}} (5x^2 - 3x + 1)\sin x\,dx = [-(5x^2 - 3x + 1)\cos x + \sin x(10x - 3) + 10\cos x]\Big|_0^{\frac{\pi}{4}} = -(\frac{5\pi^2}{16} - \frac{3\pi}{4} + 1)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(\frac{10\pi}{4} - 3) + \frac{10\sqrt{2}}{2} + 1 + 0 + 10 = \sqrt{2}\,(3 + \frac{13\pi}{8} - \frac{5\pi^2}{32}) - 9$.

44. Let $u = \sin(\ln x)$ and $dv = dx$, so that $du = \frac{1}{x}\cos(\ln x)\,dx$ and $v = x$. Thus, $\int_1^e \sin(\ln x)\,dx$ $uv\Big|_1^e - \int_1^e v\,du = x\sin(\ln x)\Big|_1^e - \int_1^e \cos(\ln x)dx.$ Now let $U = \cos(\ln x)$ and $dV = dx$, so that $dU = \frac{-1}{x}\sin(\ln x)\,dx$ and $V = x$. Thus, $\int_1^e \cos(\ln x)dx$ $UV\Big|_1^e - \int_1^e V\,dU = x\cos(\ln x)\Big|_1^e + \int_1^e \sin(\ln x)\,d$ Therefore, $\int_1^e \sin(\ln x)\,dx = x\sin(\ln x)\Big|_1^e - [x\cos(\ln x)\Big|_1^e + \int_1^e \sin(\ln x)\,dx]$, or $2\int_1^e \sin(\ln x)\,dx = x\sin(\ln x)\Big|_1^e - x\cos(\ln x)\Big|$ Thus, $\int_1^e \sin(\ln x)\,dx = \frac{1}{2}[e\sin(\ln e) - 1\sin(\ln$ $e\cos(\ln e) + 1\cos(\ln 1)] = \frac{1}{2}[e\sin 1 - \sin 0 -$ $e\cos 1 + 1\cos 0] = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2} = \frac{e}{2}(\frac{\pi}{2} - 0) + \frac{1}{2} = \frac{1}{2}(\frac{e\pi}{2} + 1)$.

45. $\int_0^{\pi/2} x\sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(x - x\cos 2x)dx = \frac{1}{4}x^2\Big|_0^{\pi/}$ $\frac{1}{4}x\sin 2x\Big|_0^{\pi/2} - \frac{1}{8}\cos 2x\Big|_0^{\pi/2} = \frac{\pi^2}{16} + \frac{1}{4} = \frac{\pi^2 + 4}{16}$.

46. Let $u^2 = x$, $dx = 2u\,du$. Then $\int x\tan^{-1}\sqrt{x}\,dx = \int 2u^3\tan^{-1}u\,du = \frac{1}{2}u^4\tan^{-1}u - \frac{1}{2}\int\frac{u^4}{1+u^2}\,du$. If we let $u = \tan\theta$, then $\int\frac{u^4}{1+u^2}\,du = \int\tan^4\theta\,d\theta = \int\tan^2\theta(\sec^2\theta - 1)d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + C$. Hence, $\int_0^3 x\tan^{-1}\sqrt{x}\,dx = [\frac{1}{2}x^2\tan^{-1}\sqrt{x} - \frac{1}{6}x\sqrt{x} + \frac{1}{2}\sqrt{x} - \frac{1}{2}\tan^{-1}\sqrt{x}]\Big|_0^3 = \frac{4\pi}{3}$.

47.

u	v'	
x^4	$\cos 2x$	
$4x^3$	$\frac{1}{2}\sin 2x$	$\xrightarrow{+}$ $\frac{1}{2}x^4\sin 2x$
$12x^2$	$-\frac{1}{4}\cos 2x$	$\xrightarrow{-}$ $-x^3\cos 2x$
$24x$	$-\frac{1}{8}\sin 2x$	$\xrightarrow{+}$ $-\frac{3}{2}x^2\sin 2x$
24	$\frac{1}{16}\cos 2x$	$\xrightarrow{-}$ $\frac{3}{2}x\cos 2x$
0	$\frac{1}{32}\sin 2x$	$\xrightarrow{+}$ $\frac{3}{4}\sin 2x$

Therefore, $\int x^4 \cos 2x\,dx = \frac{1}{2}x^4 \sin 2x + x^3 \cos 2x - \frac{3}{2}x^2 \sin 2x - \frac{3}{2}x \cos 2x + \frac{3}{4}\sin 2x + C.$

8.

$$\begin{array}{lll} \underline{u} & \underline{v} & \\ x^3 - 2x^2 + x & e^x & \\ 3x^2 - 4x + 1 & e^x & \xrightarrow{+} (x^3 - 2x^2 + x)e^x \\ 6x - 4 & e^x & \xrightarrow{-} (3x^2 - 4x + 1)e^x \\ 6 & e^x & \xrightarrow{+} (6x - 4)e^x \\ 0 & e^x & \xrightarrow{-} 6e^x \end{array}$$

Therefore, $\int (x^3 - 2x^2 + x)e^x dx = (x^3 - 5x^2 + 11x - 11)e^x + C.$

9.

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ t^4 & e^{-t} & \\ 4t^3 & -e^{-t} & \xrightarrow{+} + t^4(-e^{-t}) \\ 12t^2 & e^{-t} & \xrightarrow{-} - 4t^3(e^{-t}) \\ 24t & -e^{-t} & \xrightarrow{+} 12t^2(-e^{-t}) \\ 24 & e^{-t} & \xrightarrow{-} -24t(e^{-t}) \\ 0 & -e^{-t} & \xrightarrow{+} 24(-e^{-t}) \end{array}$$

Thus, $\int t^4 e^{-t} dt = -e^{-t}(t^4 + 4t^3 + 12t^2 + 24t + 24) + C.$

0.

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ x^5 - x^3 + x & e^{-x} & \\ 5x^4 - 3x^2 + 1 & -e^{-x} & \xrightarrow{+} + (x^5 - x^3 + x)(-e^{-x}) \\ 20x^3 - 6x & e^{-x} & \xrightarrow{-} - (5x^4 - 3x^2 + 1)e^{-x} \\ 60x^2 - 6 & -e^{-x} & \xrightarrow{+} + (20x^3 - 6x)(-e^{-x}) \\ 120x & e^{-x} & \xrightarrow{-} - (60x^2 - 6)e^{-x} \\ 120 & -e^{-x} & \xrightarrow{+} + 120x\,(-e^{-x}) \\ 0 & e^{-x} & \xrightarrow{-} - 120\,(e^{-x}) \end{array}$$

Therefore, $\int (x^5 - x^3 + x)e^{-x}dx = -e^{-x}(x^5 - x^3 + x + 5x^4 - 3x^2 + 1 + 20x^3 - 6x + 60x^2 - 6 + 120x + 120) + C = -e^{-x}(x^5 + 5x^4 + 19x^3 + 57x^2 + 115x + 115) + C.$

1. We use the tabular method of repeated integration by parts:

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ x^2 & f'''(x) & \\ 2x & f''(x) & \xrightarrow{+} + x^2 f''(x) \\ 2 & f'(x) & \xrightarrow{-} - 2x\, f'(x) \\ 0 & f(x) & \xrightarrow{+} + 2\, f(x) \end{array}$$

Thus, $\int_0^a x^2 f'''(x)dx = [x^2 f''(x) - 2x\, f'(x) + 2f(x)]\Big|_0^a = a^2 f''(a) - 2a\, f'(a) + 2f(a) - 2f(0).$

52. $\int F_1(x)F_2(x)dx = uv - \int v\,du = F_1(x)[G(x) + C_0] - \int (G(x) + C_0)F_1'(x)\,dx = F_1(x)G(x) + F_1(x)C_0 - \int F_1'(x)G(x)\,dx - \int C_0F_1'(x)\,dx = F_1(x)G(x) + F_1(x)C_0 - \int F_1'(x)G(x)\,dx - C_0F_1(x) + C_1 = F_1(x)G(x) - \int F_1'(x)C(x)\,dx + C_1.$

53. Put $u = f(x)$ and $dv = dx$, so that $du = f'(x)\,dx$ and $v = x$. Thus, $\int f(x)\,dx = uv - \int v\,du = x\,f(x) - \int x\,f'(x)\,dx.$

54. About the y axis: $V_y = \pi\int_0^\pi 2x \sin x\,dx = 2\pi[\sin x - x\cos x]\Big|_0^\pi = 2\pi^2$. About the x axis: $V_x = \pi\int_0^\pi \sin^2 x\,dx = \pi\int_0^\pi \frac{1 - \cos 2x}{2}\,dx = \frac{\pi}{2}[x - \frac{1}{2}\sin 2x]\Big|_0^\pi = \frac{\pi^2}{2}$. Hence, $V_y = 4V_x$.

55. Let $u = \sec^3 x$ and $dv = \sec^2 x\,dx$. Then $du = (3\sec^2 x)(\sec x \tan x)dx$, $v = \tan x$. Thus, $\int \sec^5 x\,dx = \sec^3 x \tan x - 3\int \tan^2 x \sec^3 x\,dx = \sec^3 x \tan x - 3\int \sec^5 x + 3\int \sec^3 x\,dx$. Hence, $\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\,dx$. Now, by Example 8 in the text, $\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C_1$. Hence, $\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C.$

56. Let $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$. Then $\int \sqrt{a^2 + x^2}\,dx = a^2\int \sqrt{1 + \tan^2\theta}\,\sec^2\theta\,d\theta = a^2\int \sec^3\theta\,d\theta = \frac{a^2}{2}\sec\theta\tan\theta + \frac{a^2}{2}\ln|\sec\theta + \tan\theta| + C$. Therefore, $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln\left|\frac{\sqrt{a^2 + x^2} + x}{a}\right| + C.$

57.

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ x^2 & g''(x) & \\ 2x & g'(x) & \xrightarrow{+} + x^2 g(x) \\ 2 & g(x) & \xrightarrow{-} - 2x\, g(x) \\ 0 & \int g(x)dx & \xrightarrow{+} + 2\int g(x)dx \end{array}$$

Therefore, $\int x^2 f(x)\,dx = x^2 g'(x) - 2x\,g(x) + 2\int g(x)\,dx$.

58. $\int\sqrt{2 - 3x + x^2}\,dx = \int\sqrt{(x - 3/2)^2 - \frac{1}{4}}\,dx = \int\sqrt{u^2 - \frac{1}{4}}\,du$, where $u = x - 3/2$. But, by Problem 56, $\int\sqrt{u^2 - \frac{1}{4}}\,du = \frac{u}{2}\sqrt{u^2 - \frac{1}{4}} - \frac{1}{8}\ln\left|u + \sqrt{u^2 - \frac{1}{4}}\right| + C$. Hence, $\int\sqrt{2 - 3x + x^2}\,dx = \frac{2x - 3}{4}\sqrt{2 - 3x + x^2} - \frac{1}{8}\ln\left|\frac{2x - 3}{2} + \sqrt{2 - 3x + x^2}\right| + C$.

59. $A = \int_0^a xe^{-x}dx$. Put $u = x$ and $dv = e^{-x}\,dx$ so that $du = dx$ and $v = -e^{-x}$. Thus, $A = \int_0^a xe^{-x}\,dx = [-xe^{-x}]\Big|_0^a - \int_0^a -e^{-x}\,dx = -ae^{-a} + (-e^{-x})\Big|_0^a = -ae^{-a} - e^{-a} + 1 = 1 - (a + 1)e^{-a}$. Now, $f'(x) = e^{-x} - xe^{-x}$, so that $f'(x) = 0$ when $x = 1$. For $x < 1$, $f'(x) > 0$ and for $x > 1$, $f'(x) < 0$; hence, f takes on its maximum value when $x = 1$. Thus, $a = 1$, so that $A = 1 - \frac{2}{e}$ square units.

60. $V = \pi\int_0^a (xe^{-x})^2\,dx = \pi\int_0^1 x^2e^{-2x}\,dx$. By the tabular method,

$$\begin{array}{lcl}
u & v' & \\
x^2 & e^{-2x} & \\
2x & -\frac{1}{2}e^{-2x} & \xrightarrow{+} -\frac{x^2}{2}e^{-2x} \\
2 & \frac{1}{4}e^{-2} & \xrightarrow{-} -\frac{x}{2}e^{-2x} \\
0 & -\frac{1}{8}e^{-2x} & \xrightarrow{+} -\frac{1}{4}e^{-2x}
\end{array}$$

Therefore, $V = \pi[-\frac{x^2}{2}e^{-2x} + \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}]\Big|_0^1 = \frac{\pi}{4}[1 - e^{-2}]$.

61. If $a \neq -1$, then $\int x^a \ln x\,dx = \frac{1}{a + 1}x^{a+1}\ln x - \int\frac{1}{a + 1}x^{a+1}\cdot\frac{1}{x}\,dx = \frac{1}{a + 1}x^{a+1}\ln x - \frac{1}{(a + 1)^2}x^{a+1} + C$. If $a = -1$, then $\int x^a\ln x\,dx = \frac{1}{2}(\ln x)^2 + C$. Hence,

$$\int x^a \ln x\,dx = \begin{cases} (\frac{1}{a + 1})x^{a+1}[\ln x - \frac{1}{a + 1}] + C, & \text{if } a \neq -1 \\ \frac{(\ln x)^2}{2} + C, & \text{if } a = -1. \end{cases}$$

62. Want to show $\int u\,dv = uv - u_1v_1 + u_2v_2 - \cdots \mp u_nv_n \pm \int u_{n+1}dv_{n+1}$ for all n, where + is used if n is odd, and − (minus) is used if n is even; $du = u_1$ and $du_{i-1} = u_i$, $i \geq 2$; also $\int v = v_1 + C$ and $\int v_{i-1} = v_i + C$, $i \geq 2$. Now, for $n = 1$, the tabular method for integration is just $\int u\,dv = uv - \int u_1dv_1 = uv - \int v\,du$, which is true by integration by parts. Now assume the method holds when $n = k$. Thus, $\int u\,dv = uv - u_1v_1 + u_2v_2 - \cdots \mp \int u_kdv_k$. If k is odd, we use the +; if k is even, we use the −. Now $\int u_kdv_k = u_kv_k - \int v_kdu_k$ by integration by parts; and from above, we have $\int v_kdu_k = \int u_{k+1}dv_{k+1}$. Thus, after substituting, we obtain $\int u\,dv = uv - u_1v_1 + u_2v_2 - \cdots \mp u_kv_k \pm \int u_{k+1}dv_{k+1}$. So by mathematical induction, the tabular method of integration is true for all n.

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1. $\frac{x + 1}{x(x - 2)} = \frac{A}{x} + \frac{B}{x - 2}$. By the short method of substitution, $\frac{0 + 1}{0 - 2} = A$ or $A = -\frac{1}{2}$; $\frac{2 + 1}{2} = B$ or $B = \frac{3}{2}$. Thus, $\int\frac{x + 1}{x(x - 2)}\,dx = \int\frac{-1/2}{x}\,dx + \int\frac{3/2}{x - 2}\,dx = -\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x - 2| + C = \ln|x - 2|^{3/2} + \ln|x|^{-1/2} + C = \ln\frac{|x - 2|^{3/2}}{|x|^{1/2}} + C$.

2. $\frac{x + 3}{x^2 - x - 2} = \frac{x + 3}{(x - 2)(x + 1)} = \frac{A}{x - 2} + \frac{B}{x + 1}$. By the short method of substitution, $A = 5/3$, $B = -\frac{2}{3}$. Thus, $\int\frac{x + 3}{x^2 - x - 2}\,dx = \frac{5}{3}\ln|x - 2| - \frac{2}{3}\ln|x + 1| + C$.

3. $\frac{31x - 9}{6y^2 - y - 2} = \frac{31x - 9}{(3y - 2)(2y + 1)} = \frac{A}{3y - 2} + \frac{B}{2y + 1}$. By the short method of substitution, $A = 5$, $B = 7$. Thus, $\int\frac{31x - 9}{6y^2 - y - 2}\,dy = \frac{5}{3}\ln|3y - 2| + \frac{7}{2}\ln|2y + 1| + C$.

4. $\frac{11t + 17}{2t^2 + 7t - 4} = \frac{11t + 17}{(2t - 1)(t + 4)} = \frac{A}{(2t - 1)} + \frac{B}{(t + 4)}$. Then $A = 5$, $B = -3$. Thus, $\int\frac{11t + 17}{2t^2 + 7t - 4}\,dt =$

$\frac{5}{2}\ln|2t - 1| - 3\ln|t + 4| + C.$

5. $\frac{4t^2 - 3t - 4}{t^3 - t^2 - 2t} = \frac{4t^2 - 3t - 4}{t(t - 2)(t + 1)} = \frac{A}{t} + \frac{B}{t - 2} + \frac{C}{t + 1}.$

Then A = 2, B = 1, C = 1. Thus, $\int \frac{4t^2 - 3t - 4}{t^3 - t^2 - 2t}\,dt =$

$2\ln|t| + \ln|t - 2| + \ln|t + 1| + C.$

6. $\frac{8x + 7}{2x^2 + 3x + 1} = \frac{8x + 7}{(2x + 1)(x + 1)} = \frac{A}{2x + 1} + \frac{B}{x + 1}$. By the short method of substitution, $\frac{8(-\frac{1}{2}) + 7}{-\frac{1}{2} + 1} = A$ or

$A = 6$; $\frac{-8 + 7}{-2 + 1} = B$ or $B = 1$. Thus,

$\int \frac{8x + 7}{(2x^2 + 3x + 1)}\,dx = \int \frac{6}{2x + 1}\,dx + \int \frac{1}{x + 1}\,dx =$

$\frac{6}{2}\ln|2x + 1| + \ln|x + 1| + C = \ln (x+1)(2x+1)^3 + C.$

7. $\frac{2x + 1}{x^3 + x^2 - 2x} = \frac{2x + 1}{x(x + 2)(x - 1)} = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x - 1}.$

Now, $\frac{2(0) + 1}{(0 + 2)(0 - 1)} = A$ or $A = -\frac{1}{2}$: $\frac{-4 + 1}{-2(-2 - 1)} = B$ or $B = -\frac{1}{2}$; $\frac{2 + 1}{1(1 + 2)} = C$ or $C = 1$. Therefore,

$\int \frac{2x + 1}{x^3 + x^2 - 2x}\,dx = \int \frac{-\frac{1}{2}}{x}\,dx + \int \frac{-\frac{1}{2}}{x + 2}\,dx + \int \frac{1}{x - 1}\,dx =$

$-\frac{1}{2}\ln|x| - \frac{1}{2}\ln|x + 2| + \ln|x - 1| + C =$

$\ln \frac{|x - 1|}{|x|^{1/2}|x + 2|^{1/2}} + C = \ln \frac{|x - 1|}{\sqrt{|x(x + 2)|}} + C.$

8. $\frac{3z + 1}{z(z^2 - 4)} = \frac{3z + 1}{z(z + 2)(z - 2)} = \frac{A}{z} + \frac{B}{z + 2} + \frac{C}{z - 2}.$

Here, $\frac{0 + 1}{(0 + 2)(0 - 2)} = A$ or $A = -\frac{1}{4}$; $\frac{-6 + 1}{-2(-2 - 2)} = B$ or $B = -\frac{5}{8}$; $\frac{7}{2(2 + 2)} = C$ or $C = \frac{7}{8}$. Thus,

$\int \frac{3z + 1}{z(z^2 - 4)}\,dz = \int \frac{-1/4}{z}\,dz + \int \frac{-5/8}{z + 2}\,dz + \int \frac{7/8}{z - 2}\,dz =$

$-\frac{1}{4}\ln|z| - \frac{5}{8}\ln|z + 2| + \frac{7}{8}\ln|z - 2| + C =$

$\ln \frac{|z - 2|^{7/8}}{|z + 2|^{5/8}|z|^{1/4}} + C.$

9. $\frac{1}{x^3 - x} = \frac{1}{x(x + 1)(x - 1)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 1}.$

Thus $\frac{1}{(0 + 1)(0 - 1)} = A$ or $A = -1$; $\frac{1}{-1(-1 - 1)} = B$ or $B = \frac{1}{2}$; $\frac{1}{1(1 + 1)} = C$ or $C = \frac{1}{2}$. Thus, $\int \frac{1}{x^3 - x}\,dx =$

$\int \frac{-1}{x}\,dx + \int \frac{\frac{1}{2}}{x + 1}\,dx + \int \frac{\frac{1}{2}}{x - 1}\,dx = -\ln|x| +$

$\frac{1}{2}\ln|x + 1| + \frac{1}{2}\ln|x - 1| + C =$

$\ln \frac{|x + 1|^{1/2}|x - 1|^{1/2}}{|x|} + C = \ln \frac{\sqrt{|x^2 - 1|}}{|x|} + C.$

10. $\frac{t + 7}{(t + 1)(t - 1)(t - 3)} = \frac{A}{t + 1} + \frac{B}{t - 1} + \frac{C}{t - 3}.$

Now, $\frac{-1 + 7}{(-1 - 1)(-1 - 3)} = A$ or $A = \frac{3}{4}$;

$\frac{1 + 7}{(1 + 1)(1 - 3)} = B$ or $B = -2$; $\frac{3 + 7}{(3 + 1)(3 - 1)} = C$ or $C = \frac{5}{4}$. Thus, $\int \frac{t + 7}{(t + 1)(t - 1)(t - 3)}\,dt = \int \frac{3/4}{t + 1}\,dt + \int \frac{-2}{t - 1}\,dt + \int \frac{5/4}{t - 3}\,dt = \frac{3}{4}\ln|t + 1| - 2\ln|t - 1| + \frac{5}{4}\ln|t - 3| + C = \ln \frac{|t + 1|^{3/4}|t - 3|^{5/4}}{(t - 1)^2} + C.$

11. $\frac{x^2}{x^2 - x - 6} = 1 + \frac{x + 6}{(x - 3)(x + 2)}$. Now,

$\frac{x + 6}{(x - 3)(x + 2)} = \frac{A}{x - 3} + \frac{B}{x + 2}$, where $\frac{3 + 6}{3 + 2} = A$ or $A = \frac{9}{5}$ and $\frac{-2 + 6}{(-2 - 3)} = B$ or $B = -\frac{4}{5}$. Thus,

$\int \frac{x^2}{x^2 - x - 6}\,dx = \int 1\,dx + \int \frac{9/5}{x - 3}\,dx + \int \frac{-4/5}{x + 2}\,dx =$

$x + \frac{9}{5}\ln|x - 3| - \frac{4}{5}\ln|x + 2| + C = x + \ln\left|\frac{(x - 3)^{9/5}}{(x + 2)^{4/5}}\right| + C.$

12. Dividing numerator by denominator, we have

$\frac{x^3 + 2x^2 - 3x + 1}{x^3 + 3x^3 + 2x} = 1 + \frac{-x^2 - 5x + 1}{x(x + 2)(x + 1)}$

$\frac{-x^2 - 5x + 1}{x(x + 2)(x + 1)} = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x + 1}$. Thus,

$\frac{0 - 0 + 1}{(0 + 2)(0 + 1)} = A$ or $A = \frac{1}{2}$; $\frac{-4 + 10 + 1}{(-2)(-2 + 1)} = B$ or $B = \frac{7}{2}$; $\frac{-1 + 5 + 1}{(-1)(-1 + 2)} = C$ or $C = -5$. Hence,

$\int \frac{x^3 + 2x^2 - 3x + 1}{x^3 + 3x^2 + 2x}\,dx = \int 1\,dx + \int \frac{\frac{1}{2}}{x}\,dx + \int \frac{\frac{7}{2}}{x + 2}\,dx + \int \frac{-5}{x + 1}\,dx = x + \frac{1}{2}\ln|x| + \frac{7}{2}\ln|x + 2| - 5\ln|x + 1| + C = x + \ln \frac{|x|^{1/2}|x + 2|^{7/2}}{|x + 1|^5} + C.$

13. Dividing numerator by denominator, we have

$\frac{x^3 + x^2 - 9x - 3}{x^2 + x - 12} = x + \frac{3x - 3}{x^2 + x - 12} = x + \frac{3x - 3}{(x + 4)(x - 3)}$. $\frac{3x - 3}{(x + 4)(x - 3)} = \frac{A}{x + 4} + \frac{B}{x - 3}.$

By the short method of substitution, $A = -\frac{15}{7}$, $B = \frac{6}{7}$. Thus, $\int \frac{x^3 + x^2 - 9x - 3}{x^2 + x - 12}\,dx =$

$\frac{1}{2}x^2 - \frac{15}{7}\ln|x+4| + \frac{6}{7}\ln|x-3| + C = \frac{1}{2}x^2 + \ln\left|\frac{(x-3)^{6/7}}{(x+4)^{15/7}}\right| + C.$

14. $\frac{x}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$. By the short method of substitution, $\frac{1}{(1+1)(1+2)} = A$ or $A = \frac{1}{6}$; $\frac{-1}{(-1-1)(-1+2)} = B$ or $1 = \frac{1}{2}$. Thus, $\int \frac{x}{(x-1)(x+1)(x+2)}\,dx = \int \frac{1/6}{x-1}\,dx + \int \frac{1/2}{x+1}\,dx + \int \frac{-2/3}{x+2}\,dx = \frac{1}{6}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{2}{3}\ln|x+2| + C = \ln\frac{|x-1|^{1/6}|x+1|^{1/2}}{(x+2)^{2/3}} + C.$

15. Dividing numerator by denominator, we have

$$\begin{array}{r} x+2 \\ x^2+3x-10 \overline{)\,x^3+5x^2-4x-20} \\ \underline{x^3+3x^2-10x} \\ 2x^2+6x-20 \\ \underline{2x^2+6x-20} \end{array}$$

Thus, $\int \frac{x^3+5x^2-4x-20}{x^2+3x-10}\,dx = \int (x+2)\,dx = \frac{x^2}{2} + 2x + C.$

16. $\frac{x^4+2x^3+1}{x^3-x^2-2x} = x + 3 + \frac{5x^2+6x+1}{x^3-x^2-2x}$. Now $\frac{5x^2+6x+1}{x^3-x^2-2x} = \frac{(5x+1)(x+1)}{x(x-2)(x+1)} = \frac{5x+1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$. We have $\frac{0+1}{0-2} = A$ or $A = -\frac{1}{2}$; $\frac{10+1}{2} = B$ or $B = \frac{11}{2}$. Thus, $\int \frac{x^4+2x^3+1}{x^3-x^2-2x}\,dx = \int \frac{-\frac{1}{2}}{x}dx + \int \frac{\frac{11}{2}}{x-2}\,dx = -\frac{1}{2}\ln|x| + \frac{11}{2}\ln|x-2| + C = \ln\frac{|x-2|^{11/2}}{|x|^{1/2}} + C.$

17. $\frac{5x^2-7x+8}{x^3+3x^2-4x} = \frac{5x^2-7x+8}{x(x+4)(x-1)} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-1}$. Here, $\frac{0-0+8}{(0+4)(0-1)} = A$ or $A = -2$; $\frac{5(16)+28+8}{-4(4-1)} = B$ or $B = \frac{29}{5}$; $\frac{5-7+8}{1(1+4)} = C$ or $C = \frac{6}{5}$. Thus, $\int \frac{5x^2-7x+8}{x^3+3x^2-4x}\,dx = \int \frac{-2}{x}\,dx + \int \frac{\frac{29}{5}}{x+4}\,dx + \int \frac{\frac{6}{5}}{x-1}\,dx = -2\ln|x| + \frac{29}{5}\ln|x+4| + \frac{6}{5}\ln|x-1| + C = \ln\left|\frac{(x+4)^{29/5}(x-1)^{6/5}}{x^2}\right| + C$

18. Dividing numerator by denominator, we have $\frac{x^3+5x^2-x-22}{x^2+3x-10} = x + 2 + \frac{3x-2}{x^2+3x-10}$. Now, $\frac{3x-2}{x^2+3x-10} = \frac{3x-2}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$. Thus, $\frac{-15-2}{-5-2} = A$ or $A = \frac{17}{7}$; $\frac{6-2}{2+5} = B$ or $\frac{4}{7} = B$. So $\int \frac{x^3+5x^2-x-22}{x^2+3x-10}\,dx = \int (x+2)\,dx + \int \frac{17/7}{x+5}\,dx + \int \frac{4/7}{x-2}\,dx = \frac{x^2}{2} + 2x + \frac{17}{7}\ln|x+5| + \frac{4}{7}\ln|x-2| + \ldots$ $\frac{x^2}{2} + 2x + \ln|(x+5)^{17/7}(x-2)^{4/7}| + C.$

19. $\frac{x^2}{x^2+x-6} = 1 + \frac{6-x}{x^2+x-6}$. Now $\frac{6-x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$; here $\frac{6+3}{-3-2} = A$ or $A = -\frac{9}{5}$ and $\frac{6-2}{2+3} = B$ or $\frac{4}{5} = B$. Thus, $\int \frac{x^2}{x^2+x-6}\,dx = \int 1\,dx \ldots \int \frac{-9/5}{x+3}\,dx + \int \frac{4/5}{x-2}\,dx = x - \frac{9}{5}\ln|x+3| + \frac{4}{5}\ln|x-2| + C = x + \ln\left|\frac{(x-2)^{4/5}}{(x+3)^{9/5}}\right| + C.$

20. By long division $\frac{5x^3-6x^2-68x-16}{x^3-2x^2-8x} = 5 + \frac{4x^2-28x-16}{x^3-2x^2-8x}$. $\frac{4x^2-28x-16}{x(x-4)(x+2)} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+2}$; hence, $A = 2$, $B = -\frac{8}{3}$, $C = \frac{14}{3}$. Therefore, $\int \frac{5x^3-6x^2-68x-16}{x^3-2x^2-8x}\,dx = 5x + 2\ln|x| - \frac{8}{3}\ln|x-4| + \frac{14}{3}\ln|x+2| + C = 5x + \ln\frac{x^2|x+2|^{14/3}}{|x-4|^{8/3}} + C.$

21. $\frac{y^3-4y-1}{y(y-1)^3} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{(y-1)^2} + \frac{D}{(y-1)^3}$; $A = 1$ and $D = -4$. Now $y^3 - 4y - 1 = (y-1)^3 + B(y-1)^2 + C(y-1)y - 4y = (1+B)y^3 + (-3-2B+C)y^2 + (3+B-C-4)y - 1$. Hence, $B = 0$, $C = 3$. Thus, $\int \frac{y^3-4y-1}{y(y-1)^3}\,dy = \int \left[\frac{1}{y} + \frac{3}{(y-1)^2} - \frac{4}{(y-1)^3}\right] dy =$

$\ln|y| - \frac{3}{y-1} + \frac{2}{(y-1)^2} + C.$

22. $\frac{2x}{(x+2)(x^2-1)} = \frac{2x}{(x+2)(x+1)(x-1)} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{x-1}$. Here, $\frac{-4}{(-2+1)(-2-1)} = A$ or $A = -\frac{4}{3}$; $\frac{-2}{(-1+2)(-1-1)} = B$ or $B = 1$; $\frac{2}{(1+2)(1+1)} = C$ or $C = \frac{1}{3}$. Therefore, $\int \frac{2x}{(x+2)(x^2-1)}\,dx = \int \frac{-\frac{4}{3}}{x+2}\,dx + \int \frac{1}{x+1}\,dx + \int \frac{\frac{1}{3}}{x-1}\,dx = -\frac{4}{3}\ln|x+2| + \ln|x+1| + \frac{1}{3}\ln|x-1| + C = \ln\left|\frac{(x+1)(x-1)^{1/3}}{(x+2)^{4/3}}\right| + C.$

23. $\frac{2z+3}{z^2(4z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{4z+1}$. We can find B and C by the short method of substitution. Thus, $\frac{0+3}{0+1} = B$ or $B = 3$; $\frac{-\frac{1}{2}+3}{(-\frac{1}{4})^2} = C$ or $C = 40$. Now, $2z + 3 = Az(4z+1) + 3(4z+1) + 40z^2$. So $0 = 4A + 40$, so that $A = -10$. Now, $\int \frac{2z+3}{z^2(4z+1)}\,dz = \int \frac{-10}{z}\,dz + \int \frac{3}{z^2}dz + \int \frac{40}{4z+1}\,dz = -10\ln|z| - \frac{3}{z} + 10\ln|4z+1| + C = -\frac{3}{z} + \ln\left(\frac{4z+1}{z}\right)^{10} + C.$

24. $\frac{x^2+1}{(x+3)(x^2+4x+4)} = \frac{x^2+1}{(x+3)(x+2)^2} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$. Here, $\frac{9+1}{(-3+2)^2} = A$ or $A = 10$; $\frac{4+1}{-2+3} = C$ or $C = 5$. Thus, $x^2 + 1 = 10(x+2)^2 + B(x+3)(x+2) + 5(x+3)$, and so $1 = 10 + B$ and $B = -9$. Therefore, $\int \frac{x^2+1}{(x+3)(x^2+4x+4)}\,dx = \int \frac{10}{x+3}\,dx + \int \frac{-9}{x+2}\,dx + \int \frac{5}{(x+2)^2}\,dx = 10\ln|x+3| - 9\ln|x+2| - \frac{5}{x+2} + C = \ln\left|\frac{(x+3)^{10}}{(x+2)^9}\right| - \frac{5}{x+2} + C$, where the last integral was obtained by putting $u = x + 2$.

25. $\frac{x+3}{(x+1)^2(x+7)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+7}$. Here, $\frac{-1+3}{-1+7} = B$ or $B = \frac{1}{3}$; $\frac{-7+3}{(-7+1)^2} = C$ or $C = -\frac{1}{9}$. Hence, $x + 3 = A(x+1)(x+7) + \frac{1}{3}(x+7) - \frac{1}{9}(x+1)^2$ and so $0 = A - \frac{1}{9}$ and $A = 1/9$. Thus, $\int \frac{x+3}{(x+1)^2(x+7)}\,dx = \int \frac{\frac{1}{9}}{x+1}\,dx + \int \frac{\frac{1}{3}}{(x+1)^2}\,dx + \int \frac{-\frac{1}{9}}{x+7}\,dx = \frac{1}{9}\ln|x+1| - \frac{1}{3(x+1)} - \frac{1}{9}\ln|x+7| + C = \ln\left|\frac{x+1}{x+7}\right|^{1/9} - \frac{1}{3(x+1)} + C.$

26. $\frac{x+4}{(x^2+2x+1)(x-1)^2} = \frac{x+4}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$. Now, $\frac{-1+4}{(-1-1)^2} = B$ or $B = \frac{3}{4}$; $\frac{1+4}{(1+1)^2} = D$ or $D = \frac{5}{4}$. Thus, $x + 4 = A(x+1)(x-1)^2 + \frac{3}{4}(x-1)^2 + C(x+1)^2(x-1) + \frac{5}{4}(x+1)^2$. $x + 4 = A(x^3 - x^2 - x + 1) + \frac{3}{4}(x^2 - 2x + 1) + C(x^3 + x^2 - x - 1) + \frac{5}{4}(x^2+2x+1)$. So $0 = A + C$; $0 = -A + \frac{3}{4} + C + \frac{5}{4} = -A + C + 2$. Adding, $2C + 2 = 0$, so that $C = -1$; so $A = 1$. Hence, $\int \frac{x+4}{(x^2+2x+1)(x-1)^2}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{\frac{3}{4}}{(x+1)^2}\,dx + \int \frac{-1}{x-1}\,dx + \int \frac{\frac{5}{4}}{(x-1)^2}\,dx = \ln|x+1| - \frac{-3}{4(x+1)} - \ln|x-1| - \frac{5}{4(x-1)} + C = \ln\left|\frac{x+1}{x-1}\right| - \frac{3}{4(x+1)} - \frac{5}{4(x-1)} + C.$

27. $\frac{4x^2 - 7x + 10}{(x+2)(3x-2)^2} = \frac{A}{x+2} + \frac{B}{3x-2} + \frac{C}{(3x-2)^2}$. Here, $\frac{16+14+10}{(-6-2)^2} = A$ or $A = \frac{5}{8}$; $\frac{4(\frac{4}{9}) - \frac{14}{3} + 10}{\frac{2}{3}+2} = C$ or $C = \frac{8}{3}$. Now, $4x^2 - 7x + 10 = \frac{5}{8}(3x-2)^2 + B(x+2)(3x-2) + \frac{8}{3}(x+2)$, so that $4 = \frac{45}{8} + 3B$ and $B = -\frac{13}{24}$. Hence, $\int \frac{4x^2-7x+10}{(x+2)(3x-2)^2}\,dx = \int \frac{\frac{5}{8}}{x+2}\,dx + \int \frac{-\frac{13}{24}}{3x-2}\,dx + \int \frac{\frac{8}{3}}{(3x-2)^2}\,dx = \frac{5}{8}\ln|x+2| - \frac{13}{24}\left(\frac{1}{3}\right)\ln|3x-2| + \frac{8}{3}\left(\frac{1}{3}\right) - \frac{1}{3x-2} + C = \ln\frac{|x+2|^{5/8}}{|3x-2|^{13/72}} - \frac{8}{9(3x-2)} + C.$

28. $\frac{x^3 - 3x^2 + 5x - 12}{(x - 1)^2(x^2 - 3x - 4)} = \frac{x^3 - 3x^2 + 5x - 12}{(x - 1)^2(x - 4)(x + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x - 4} + \frac{D}{x + 1}$. Now

$\frac{1 - 3 + 5 - 12}{(1 - 4)(1 + 1)} = B$ or $B = \frac{3}{2}$; $\frac{64 - 48 + 20 - 12}{9(5)} = C$ or $C = \frac{8}{15}$; $\frac{-1 - 3 - 5 - 12}{(-2)^2(-5)} = D$ or $D = \frac{21}{20}$. Thus,

$x^3 - 3x^2 + 5x - 12 = A(x - 1)(x - 4)(x + 1) + B(x - 4)(x + 1) + C(x - 1)^2(x + 1) + D(x - 1)^2(x - 4)$. $1 = A + C + D$. So $A = 1 - \frac{8}{15} - \frac{21}{20} = -\frac{7}{12}$. Therefore, $\int \frac{x^3 - 3x^2 + 5x - 12}{(x - 1)^2(x^2 - 3x - 4)} dx =$

$\int \frac{-\frac{7}{12}}{x - 1} dx + \int \frac{(\frac{3}{2})}{(x - 1)^2} dx + \int \frac{(\frac{8}{15})}{x - 4} dx + \int \frac{(\frac{21}{20})}{x + 1} dx =$

$-\frac{7}{12} \ln |x - 1| - \frac{3}{2} (\frac{1}{x - 1}) + \frac{8}{15} \ln |x - 4| + \frac{21}{20} \ln |x + 1| + C = \ln \frac{|x - 4|^{8/15}|x + 1|^{21/20}}{|x - 1|^{7/12}} - \frac{3}{2(x - 1)} + C.$

29. $\frac{4z^2}{(z - 1)^2(z^2 - 4z + 3)} = \frac{4z^2}{(z - 1)^2(z - 3)(z - 1)} = \frac{A}{z - 1} + \frac{B}{(z - 1)^2} + \frac{C}{(z - 1)^3} + \frac{D}{z - 3}$. Now $\frac{4}{1 - 3} =$

C or $C = -2$; $\frac{36}{(3 - 1)^3} = D$ or $D = \frac{9}{2}$. $4z^2 =$

$A(z - 1)^2(z - 3) + B(z - 1)(z - 3) + C(z - 3) + D(z - 1)^3$. $4z^2 = A(z^3 - 5z^2 + 7z - 3) + B(z^2 - 4z + 3) + C(z - 3) + D(z^3 - 3z^2 + 3z - 1)$.

$0 = A + D$; $4 = -5A + B - 3D$. Since $D = \frac{9}{2}$, then $A = -\frac{9}{2}$. Thus, $4 = \frac{45}{2} + B - \frac{27}{2}$ or $B = -5$. So

$\int \frac{4z^2}{(z - 1)^2(z^2 - 4z + 3)} dz = \int \frac{(-\frac{9}{2})}{z - 1} dz + \int \frac{(-5)}{(z - 1)^2} dz + \int \frac{(-2)}{(z - 1)^3} dz + \int \frac{(\frac{9}{2})}{z - 3} dz = -\frac{9}{2} \ln |z - 1| - 5\left(-\frac{1}{z - 1}\right) + \frac{1}{(z - 1)^2} + \frac{9}{2} \ln |z - 3| + C =$

$\ln \left|\frac{z - 3}{z - 1}\right|^{\frac{9}{2}} + \frac{1}{(z - 1)^2} + \frac{5}{z - 1} + C.$

30. $\frac{t + 2}{(t^2 - 1)(t + 3)^2} = \frac{t + 2}{(t + 1)(t - 1)(t + 3)^2} = \frac{A}{t + 1} + \frac{B}{t - 1} + \frac{C}{t + 3} + \frac{D}{(t + 3)^2}$. Now $\frac{-1 + 2}{(-1 - 1)(-1 + 3)^2} =$ or $A = -\frac{1}{8}$; $\frac{1 + 2}{(1 + 1)(1 + 3)^2} = B$ or $B = \frac{3}{32}$;

$\frac{-3 + 2}{(-3 + 1)(-3 - 1)} = D$ or $D = -\frac{1}{8}$. Thus, $t + 2 = -\frac{1}{8}(t - 1)(t + 3)^2 + \frac{3}{32}(t + 1)(t + 3)^2 + C(t + 3)(t - 1)(t + 1) - \frac{1}{8}(t + 1)(t - 1)$.

$0 = \frac{5}{8} + \frac{21}{32} + 3C - \frac{1}{8}$ since there is no t^2 term.

$0 = -24 + 21 + 96C$. $96C = 3$. $C = \frac{1}{32}$. So,

$\int \frac{t + 2}{(t^2 - 1)(t + 3)^2} dt = \int \frac{(-\frac{1}{8})}{t + 1} dt + \int \frac{(\frac{3}{32})}{t - 1} dt + \int \frac{(\frac{1}{32})}{t + 3} dt + \int \frac{(-\frac{1}{8})}{(t + 3)^2} dt = -\frac{1}{8} \ln |t + 1| + \frac{3}{32} \ln |t - 1| + \frac{1}{32} \ln|t + 3| + \frac{1}{8(t + 3)} + C =$

$\ln \frac{|t - 1|^{\frac{3}{32}}|t + 3|^{\frac{1}{32}}}{|t + 1|^{\frac{1}{8}}} + \frac{1}{8(t + 3)} + C.$

31. $\frac{x}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}$. Thus, $\frac{-1}{1} = A$ and $\frac{-2}{-1} = B$. Hence, $\int_2^4 \frac{x\, dx}{(x + 1)(x + 2)} = \int_2^4 \frac{-1}{x + 1} dx + \int_2^4 \frac{2}{x + 2} dx = -\ln |x + 1|\Big|_2^4 + 2 \ln |x + 2|\Big|_2^4 =$

$-\ln (5) + \ln (3) + \ln 6^2 - \ln 4^2 = \ln \frac{27}{20}$.

32. $\frac{5t^2 - 3t + 18}{t(3 + t)(3 - t)} = \frac{A}{t} + \frac{B}{3 + t} + \frac{C}{3 - t}$. Thus, $\frac{18}{9} = 2 = A$; $\frac{72}{-3(6)} = -4 = B$; $\frac{54}{18} = 3 = C$. Hence,

$\int_1^2 \frac{5t^2 - 3t + 18}{t(9 - t^2)} dt = \int_1^2 \frac{2}{t} dt + \int_1^2 \frac{-4}{3 + t} dt + \int_1^2 \frac{3}{3 - t} dt = \ln t^2\Big|_1^2 -4 \ln (3 + t)\Big|_1^2 + \ln (3 - t)^3\Big|_1^2 = \ln 4 - 4 \ln 5 + 4 \ln 4 - \ln 8 =$

$5 \ln 4 - 4 \ln 5 - \ln 8 = \ln \frac{128}{625}$.

33. $\frac{4t^5 - 3t^4 - 6t^3 + 4t^2 + 6t - 1}{(t - 1)(t^2 - 1)} = 4t^2 + t - 1 + \frac{4t}{(t - 1)(t^2 - 1)}$. Now $\frac{4t}{(t - 1)(t^2 - 1)} =$

$\frac{4t}{(t - 1)(t - 1)(t + 1)} = \frac{A}{t - 1} + \frac{B}{(t - 1)^2} + \frac{C}{t + 1}$.

$B = \frac{4}{1+1} = 2$, $C = \frac{-4}{(-2)^2} = -1$. $4t = A(t^2 - 1) + B(t + 1) + C(t^2 - 2t + 1) = (A + C)t^2 + (B - 2C)t + (-A + B + C) = (A - 1)t^2 + 4t + (1 - A)$; hence, $A = 1$. Thus, $\int_2^3 \frac{4t^5 - 3t^4 - 6t^3 + 4t^2 + 6t - 1}{(t - 1)(t^2 - 1)}\,dt = \int_2^3 (4t^2 + t - 1)dt + \int_2^3 \frac{dt}{t - 1} + \int_2^3 \frac{2\,dt}{(t - 1)^2} - \int_2^3 \frac{dt}{t + 1} = \left(\frac{4}{3} + 3 + \frac{t2}{2} - t\right)\Big|_2^3 + \ln|t - 1|\Big|_2^3 - \frac{2}{t - 1}\Big|_2^3 - \ln|t + 1|\Big|_2^3 = \left(36 + \frac{9}{2} - 3\right) - \left(\frac{32}{3} + 2 - 2\right) + \ln 2 - 1 + 2 - \ln 4 + \ln 3 = \frac{167}{6} + \ln\frac{3}{2}$.

34. $\frac{x^5 + 3x^4 - 4x^3 - x^2 + 11x + 12}{x^2(x^2 + 5x + 4)} = x - 2 + \frac{2x^3 + 7x^2 + 11x + 12}{x^2(x + 4)(x + 1)}$. $\frac{2x^3 + 7x^2 + 11x + 12}{x^2(x + 4)(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 4} + \frac{D}{x + 1}$. $B = \frac{12}{(4)(1)} = 3$, $C = \frac{2(-4)^3 + (-4)^2 + 11(-4) + 12}{(-4)^2(-4 + 1)} = 1$, $D = \frac{-2 + 7 - 11 + 12}{3} = 2$. $2x^3 + 7x^2 + 11 + 12 = A(x^3 + 5x^2 + 4x) + B(x^2 + 5x + 4) + C(x^3 + x^2) + D(x^3 + 4x^2) = A(x^3 + 5x^2 + 4x) + 3(x^2 + 5x + 4) + (x^3 + x^2) + 2(x^3 + 4x^2) = (A + 3)x^3 + (5A + 12)x^2 + (4A + 15)x + 12$. Therefore, $A = -1$, and we have $\int_1^2 \frac{x^5 + 3x^4 - 4x^3 - x^2 + 11x + 12}{x^2(x^2 + 5x + 4)}\,dx = \int_1^2 (x - 2)dx + \int_1^2 \frac{(-1)}{x}\,dx + \int_1^2 \frac{3}{x^2}\,dx + \int_1^2 \frac{1}{x + 4}\,dx + \int_1^2 \frac{2}{x + 1}\,dx = \left(\frac{x^2}{2} - 2x\right)\Big|_1^2 - \ln|x|\Big|_1^2 - \frac{3}{x}\Big|_1^2 + \ln|x + 4|\Big|_1^2 + 2\ln|x + 1|\Big|_1^2 = -\frac{1}{2} - \ln 2 + \ln 1 - \frac{3}{2} + 3 + \ln 6 - \ln 5 + 2\ln 3 - 2\ln 2 = 1 + \ln\frac{27}{20}$.

35. $\int_3^5 \frac{x^2 - 2}{(x - 2)^2}\,dx = \int_3^5 1\,dx + \int_3^5 \frac{4x - 6}{(x - 2)^2}\,dx$. Now, $\frac{4x - 6}{(x - 2)^2} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2}$. $\frac{8 - 6}{1} = 2 = B$. Thus, $4x - 6 = A(x - 2) + 2$. $4 = A$. Therefore, $\int_3^5 \frac{x^2 - 2}{(x - 2)^2}\,dx = x\Big|_3^5 + \int_3^5 \frac{4\,dx}{x - 2} + \int_3^5 \frac{2}{(x - 2)^2}\,dx = 5 - 3 + 4\ln|x - 2|\Big|_3^5 - \frac{2}{x - 2}\Big|_3^5 = 2 + 4\ln 3 - \frac{2}{3} + 2 = \frac{10}{3} + 4\ln 3$.

36. $\int_1^2 \frac{2z^3 + 1}{z(z + 1)^2}\,dz = \int_1^2 2\,dz + \int_1^2 \frac{-4z^2 - 2z + 1}{z(z + 1)^2}\,dz$. Now, $\frac{-4z^2 - 2z + 1}{z(z + 1)^2} = \frac{A}{z} + \frac{B}{z + 1} + \frac{C}{(z + 1)^2}$. Thus, $\frac{1}{1} = A$; $\frac{-4 + 2 + 1}{-1} = 1 = C$. Also, $-4z^2 - 2z + 1 = A(z + 1)^2 + B(z)(z + 1) + Cz$. Thus, $-4 = A + B$, and so $B = -5$. Hence, $\int_1^2 \frac{2z^3 + 1}{z(z + 1)^2}\,dz = 2z\Big|_1^2 + \int_1^2 \frac{1}{z}\,dz + \int_1^2 \frac{-5}{z + 1}\,dz + \int_1^2 \frac{1}{(z + 1)^2}\,dz = 2 + \ln z\Big|_1^2 - 5\ln(z + 1)\Big|_1^2 - \frac{1}{z + 1}\Big|_1^2 = 2 + \ln 2 - 5\ln\frac{3}{2} - \frac{1}{3} + \frac{1}{2} = \frac{13}{6} + \ln 2 - 5\ln\frac{3}{2}$.

37. $\int\frac{ax + b}{cx + d}\,dx = \int\frac{a}{c}\,dx + \int\frac{bc - ad}{c(cx + d)}\,dx = \frac{a}{c}x + \frac{bc - ad}{c}\int\frac{1}{cx + d}\,dx$. To evaluate the integral, put $u = cx + d$, so that $du = c\,dx$. Hence, $\int\frac{1}{cx + d}\,dx = \frac{1}{c}\int\frac{1}{u}\,du = \frac{1}{c}\ln|u| + K = \frac{\ln|cx + d|}{c} + K$. Thus, $\int\frac{ax + b}{cx + d}\,dx = \frac{a}{c}x + \frac{bc - ad}{c^2}\ln|cx + d| + K$.

38. (a) For $a \neq b$, $\frac{1}{(x - a)(x - b)} = \frac{A}{x - a} + \frac{B}{x - b}$. Hence, $A = \frac{1}{a - b}$ and $B = \frac{1}{b - a}$. So $\int\frac{dx}{(x - a)(x - b)} = \int\frac{1}{(a - b)(x - a)}\,dx + \int\frac{1}{(b - a)(x - b)}\,dx = \frac{1}{a - b}\ln|x - a| + \frac{1}{b - a}\ln|x - b| + C = \frac{1}{a - b}(\ln|x - a| - \ln|x - b|) + C = \frac{1}{a - b}\ln\left|\frac{x - a}{x - b}\right| + C$.

(b) If $a = b$, then $\int\frac{dx}{(x - a)(x - b)} = \int\frac{dx}{(x - a)^2}$. Put $u = x - a$, so that $du = dx$ and $\int\frac{dx}{(x - a)^2} = \int\frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{x - a} + C$. Hence, $\int\frac{dx}{(x - a)(x - b)} = -\frac{1}{x - a} + C$, $a = b$.

(c) Yes. To see that it is so, put $u = \frac{1}{x - b}$ and

put $n = \frac{1}{a-b}$. Now, $\int \frac{dx}{(x-a)(x-b)} =$

$\frac{1}{a-b} \ln \left|\frac{x-a}{x-b}\right| + C = \frac{1}{a-b} \ln \left|1 - \frac{a-b}{x-b}\right| + C =$

$n \ln \left|1 - \frac{u}{n}\right| + C = \ln \left|1 - \frac{u}{n}\right|^n + C$. Notice that

$n \to +\infty$ as $a \to b$. Thus, $\lim_{a \to b} \int \frac{dx}{(x-a)(x-b)} =$

$\lim_{n\to+\infty} \ln \left|1 - \frac{u}{n}\right|^n + C = \ln \lim_{n\to+\infty} \left|1 - \frac{u}{n}\right|^n + C =$

$\ln e^{-u} + C = -u + C = \frac{-1}{x-b} + C = \int \frac{dx}{(x-b)^2}$.

39. (a) $x^2 - x - 6 = x^2 - x + \frac{1}{4} - \frac{1}{4} - 6 = (x - \frac{1}{2})^2 - \frac{25}{4}$.

Put $u = x - \frac{1}{2}$, so that $du = dx$ and $\int \frac{x+1}{x^2 - x - 6} dx =$

$\int \frac{u + \frac{3}{2}}{u^2 - \frac{25}{4}} du = \int \frac{u\,du}{u^2 - \frac{25}{4}} + \frac{3}{2} \int \frac{1}{(u - \frac{5}{2})(u + \frac{5}{2})} du =$

$\frac{1}{2} \ln \left|u^2 - \frac{25}{4}\right| + \frac{3}{2} \int \frac{1}{(u - \frac{5}{2})(u + \frac{5}{2})} du$. Now

$\frac{1}{(u - \frac{5}{2})(u + \frac{5}{2})} = \frac{A}{u - \frac{5}{2}} + \frac{B}{u + \frac{5}{2}}$; $A = \frac{1}{(\frac{5}{2} + \frac{5}{2})} = \frac{1}{5}$

and $B = \frac{1}{-\frac{5}{2} - \frac{5}{2}} = -\frac{1}{5}$. Hence, $\int \frac{x+1}{x^2 - x - 6} dx =$

$\frac{1}{2} \ln \left|u^2 - \frac{25}{4}\right| + \frac{3}{2} \int \frac{(\frac{1}{5})\,du}{u - \frac{5}{2}} + \frac{3}{2} \int \frac{(-\frac{1}{5})\,du}{u + \frac{5}{2}} =$

$\frac{1}{2} \ln \left|u^2 - \frac{25}{4}\right| + \frac{3}{10} \ln \left|u - \frac{5}{2}\right| - \frac{3}{10} \ln \left|u + \frac{5}{2}\right| + C =$

$\frac{1}{2} \ln |x^2 - x - 6| + \frac{3}{10} \ln |x - 3| - \frac{3}{10} \ln |x + 2| + C =$

$\ln \frac{|x^2 - x - 6|^{\frac{1}{2}} |x - 3|^{\frac{3}{10}}}{|x + 2|^{\frac{3}{10}}} + C =$

$\ln \frac{(|x - 3||x + 2|)^{\frac{1}{2}} |x - 3|^{\frac{3}{10}}}{|x + 2|^{\frac{3}{10}}} + C =$

$\ln(|x - 3|^{\frac{4}{5}} |x + 2|^{\frac{1}{5}}) + C = \ln|(x - 3)^{\frac{4}{5}} (x + 2)^{\frac{1}{5}}| + C$.

(b) $\frac{x+1}{(x-3)(x+2)} = \frac{D}{x-3} + \frac{E}{x+2}$; $D = \frac{3+1}{3+2} = \frac{4}{5}$,

$E = \frac{-2+1}{-2-3} = \frac{1}{5}$, so that $\int \frac{x+1}{(x-3)(x+2)} dx =$

$\frac{4}{5} \int \frac{dx}{x-3} + \frac{1}{5} \int \frac{dx}{x+2} = \frac{4}{5} \ln |x - 3| + \frac{1}{5} \ln|x+2| + C =$

$\ln (|x - 3|^{\frac{4}{5}} |x + 2|^{\frac{1}{5}}) + C = \ln|(x - 3)^{\frac{4}{5}} (x + 2)^{\frac{1}{5}}| + C$.

40. $\frac{x+c}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$. $\frac{a+c}{1} = B$. Now,

$x + c = A(x - a) + B = A(x - a) + (a + c)$. So

$1 = A$. Thus, $\int \frac{x+c}{(x-a)^2} dx = \int \frac{1}{x-a} dx +$

$\int \frac{a+c}{(x-a)^2} dx = \ln |x - a| + (a + c)(-\frac{1}{x-a}) + K =$

$\ln |x - a| - \frac{(a+c)}{x-a} + K$.

41. $A = \int_0^4 \frac{4-x}{(x+2)^2} dx$. Now $\frac{4-x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$

and so $\frac{4+2}{1} = B$. Thus, $4 - x = A(x + 2) + 6$, and

$A = -1$. Therefore, $A = \int_0^4 \frac{-1}{x+2} dx + \int_0^4 \frac{6}{(x+2)^2} dx =$

$-\ln |x + 2| \Big|_0^4 + -\frac{6}{x+2} \Big|_0^4 = -\ln 6 + \ln 2 - 1 + 3 =$

$2 + \ln \frac{1}{3} = 2 - \ln 3$ square unit.

42. $V = \pi \int_0^4 \left[\frac{4-x}{(x+2)^2}\right]^2 dx = \pi \int_0^4 \frac{x^2 - 8x + 16}{(x+2)^4} dx$.

$\frac{x^2 - 8x + 16}{(x+2)^4} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} +$

$\frac{D}{(x+2)^4}$. Now $\frac{4 + 16 + 16}{1} = D$ or $D = 36$. Thus,

$x^2 - 8x + 16 = A(x + 2)^3 + B(x + 2)^2 + C(x+2) + 36$.

$0 = A$; $1 = 6A + B$, so that $B = 1$; $-8 = 12A + 4B + C$,

so that $C = -12$. Therefore, $V = \pi\left[\int_0^4 \frac{1}{(x+2)^2} dx + \int_0^4 \frac{-12}{(x+2)^3} dx + \int_0^4 \frac{36}{(x+2)^4} dx\right] =$

$\pi\left[-\frac{1}{x+2}\Big|_0^4 + \frac{6}{(x+2)^2}\Big|_0^4 + \frac{-12}{(x+2)^3}\Big|_0^4\right] =$

$\pi(-\frac{1}{6} + \frac{1}{2} + \frac{1}{6} - \frac{3}{2} - \frac{1}{18} + \frac{3}{2}) = \frac{4\pi}{9}$ cubic units.

43. $\int \frac{dx}{q - ax^2} = \frac{1}{a} \int \frac{dx}{\frac{q}{a} - x^2}$. Now put $w = \sqrt{\frac{q}{a}}$. $\frac{1}{(\frac{q}{a} - x^2)} =$

$\frac{1}{(w + x)(w - x)} = \frac{A}{w + x} + \frac{B}{w - x}$. Thus, $\frac{1}{2w} = A$ and

$\frac{1}{2w} = B$. Thus, $\frac{1}{a} \int \frac{1}{(\frac{q}{a} - x^2)} dx =$

$\frac{1}{a}\left[\int \frac{\frac{1}{2w}}{w + x} dx + \int \frac{\frac{1}{2w}}{w - x} dx\right] = \frac{1}{a}\left[\frac{1}{2w} \ln |w + x| - \frac{1}{2w} \ln |w - x|\right] + C = \frac{1}{a}\left[\frac{1}{2w} \ln \left|\frac{w + x}{w - x}\right|\right] + C =$

$\frac{1}{2a\sqrt{\frac{q}{a}}} \ln\left|\frac{\sqrt{q} + \sqrt{a}\,x}{\sqrt{q} - \sqrt{a}\,x}\right| + C = \frac{1}{2\sqrt{aq}} \ln \left|\frac{(\sqrt{q} + \sqrt{a}\,x)^2}{q - ax^2}\right| + C.$

44. First call $1 + b = h$ and $1 - b = e$. Thus, $\frac{dy}{(1 - hy)(1 - ey)} = ak\,dt$. And so $\int \frac{dy}{(1 - hy)(1 - ey)} = a\,kt + C_1$. Now $\frac{1}{(1 - hy)(1 - ey)} = \frac{A}{1 - hy} + \frac{B}{1 - ey}$, so that $A = \frac{1}{1 - \frac{e}{h}}$ and $B = 1 - \frac{h}{e}$. Therefore,

$\int \frac{dy}{(1 - hy)(1 - ey)} = \int \frac{1 - \frac{e}{h}}{1 - hy}\,dy + \int \frac{1 - \frac{h}{e}}{1 - ey}\,dy =$

$(1 - \frac{e}{h})(-\frac{1}{h}) \ln|1 - hy| + (1 - \frac{h}{e})(-\frac{1}{e}) \ln|1 - ey| + C_2 =$

$(1 - \frac{1 - b}{1 + b})(-\frac{1}{1 + b}) \ln|1 - (1 + b)y| +$

$(1 - \frac{1 + b}{1 - b})(-\frac{1}{1 - b}) \ln|1 - (1 - b)y| + C_2$. Hence,

$(\frac{-2b}{1 + b}) \ln|1 - (1 + b)y| + \frac{2b}{1 - b} \ln|1 - (1 - b)y| =$

$akt + C$ or $\ln \frac{|1 - (1 - b)y|^{\frac{2b}{1-b}}}{|1 - (1 + b)y|^{\frac{2b}{1+b}}} = akt + C.$

45. $\frac{dx}{(a - x)^4} = kdt$, and so $\int \frac{dx}{(a - x)^4} = kt + C_1$. Put $u = a - x$, so that $du = -dx$. Hence, $\int \frac{dx}{(a - x)^4} =$

$\int -\frac{du}{u^4} = \frac{1}{3u^3} + C_2 = \frac{1}{3(a - x)^3} + C_2$. Therefore,

$\frac{1}{3(a - x)^3} = kt + C$. $(a - x)^3 = \frac{1}{3(kt + C)}$.

$a - x = \frac{1}{\sqrt[3]{3(kt + C)}}$ and $x = a - \frac{1}{\sqrt[3]{3(kt + C)}}$.

46. $S = \int_0^{\frac{1}{3}} \sqrt{1 + (\frac{-2x}{1 - x^2})^2}\,dx = \int_0^{\frac{1}{3}} \sqrt{\frac{x^4 + 2x^2 + 1}{(1 - x^2)^2}}\,dx =$

$\int_0^{\frac{1}{3}} \frac{x^2 + 1}{1 - x^2}\,dx = \int_0^{\frac{1}{3}} - 1\,dx + \int_0^{\frac{1}{3}} \frac{2}{1 - x^2}\,dx$. Now

$\frac{2}{1 - x^2} = \frac{A}{1 + x} + \frac{B}{1 - x}$. $\frac{2}{1 + 1} = 1 = A$ and

$\frac{2}{1 + 1} = 1 = B$. Thus, $S = -x \Big|_0^{\frac{1}{3}} + \int_0^{\frac{1}{3}} \frac{1}{1 + x}\,dx +$

$\int_0^{\frac{1}{3}} \frac{1}{1 - x}\,dx = -\frac{1}{3} + \ln|1 + x| \Big|_0^{\frac{1}{3}} - \ln|1 - x| \Big|_0^{\frac{1}{3}} =$

$-\frac{1}{3} + \ln 4 - \ln 2 = \ln 2 - \frac{1}{3}$.

47. $C(x) = \int \frac{400x^2 + 1300x - 900}{x(x - 1)(x + 3)}\,dx =$

$100 \int \frac{4x^2 + 13x - 9}{x(x - 1)(x + 3)}\,dx$, where $C(2) = 47$. Now,

$\frac{4x^2 + 13x - 9}{x(x - 1)(x + 3)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{D}{x + 3}$. Thus,

$\frac{-9}{-3} = 3 = A$; $\frac{4 + 13 - 9}{1(4)} = 2 = B$; $\frac{36 - 39 - 9}{(-3)(-4)} = -1 = D$.

So $C(x) = 100\left[\int \frac{3}{x}\,dx + \int \frac{2}{x - 1}\,dx - \int \frac{1}{x + 3}\,dx\right] =$

$100\,(3 \ln|x| + 2 \ln|x - 1| - \ln|x + 3|) + K$,

$C(x) = 100 \ln \frac{x^3(x - 1)^2}{|x + 3|} + K$. When $x = 2$, $C(x) = 47$.

Hence, $47 = 100 \ln \frac{8}{5} + K$ and $K = 47 - 100 \ln \frac{8}{5}$.

Therefore, $C(x) = 100 \ln \frac{x^3(x - 1)^2}{x + 3} + 47 - 100 \ln \frac{8}{5} =$

$100 \ln \frac{5x^3(x - 1)^2}{8(x + 3)} + 47$.

48. The partial fractions decomposition of a rational function whose denominator factors completely into linear factors consists of a sum of expressions of the form $\frac{k}{(ax + b)^n}$, where $n = 1, 2, 3, \ldots$.

When $n = 1$, $\int \frac{k}{ax + b}\,dx = \frac{k}{a} \ln|ax + b| + c$. When $n > 1$, then we put $u = ax + b$, so that $du = a\,dx$ and $\int \frac{k}{(ax + b)^n}\,dx = \int \frac{\frac{k}{a}\,du}{u^n} = \frac{k}{a}(\frac{1}{1 - n})u^{-n+1} + C =$

$\frac{k}{a(1 - n)u^{n-1}} + C$; this latter function is a rational function.

49. Since $\frac{1}{(A - y)(B + y)} = \frac{a}{A - y} + \frac{b}{B + y}$ with $a = \frac{1}{A + B}$ and $b = \frac{1}{A + B}$, we have that

$\int \frac{1}{(A - y)(B + y)} \frac{dy}{dt}\,dt = b \ln|B + y| - a \ln|A - y| + C_1 =$

$\ln\left|\frac{(B + y)^b}{(A - y)^a}\right| + C_1$. Therefore, $\frac{(B + y)^b}{(A - y)^a} = K_0 e^{kt}$,

and $\left(\frac{B + y}{A - y}\right)^{\frac{1}{A+B}} = K_0 e^{kt}$. So $y = \frac{A K_0 e^{k(A+B)t} - B}{(1 + K_0 e^{k(A+B)t})}$.

50. $dt = \frac{dq}{(Bq - A)q}$, $t = \int \frac{dq}{(Bq - A)q}$. Now $\frac{1}{(Bq - A)q} =$

$\frac{D}{Bq - A} + \frac{E}{q}$, $D = \frac{1}{(\frac{A}{B})} = \frac{B}{A}$, and $E = \frac{1}{(-A)} = -\frac{1}{A}$;

hence, $t = \frac{B}{A} \int \frac{dq}{Bq - A} - \frac{1}{A} \int \frac{dq}{q}$. Thus,

$t = \frac{B}{A}(\frac{1}{B} \ln|Bq - A|) - \frac{1}{A} \ln|q| + k_1$, so that

$At + k = \ln\left|\frac{Bq - A}{q}\right|$, where $k = -Ak_1$. Exponentiating both sides of the latter equation, we obtain $e^{At+k} = \left|\frac{Bq - A}{q}\right|$, so that $\frac{Bq - A}{q} =$

$\pm e^k e^{At} = Ce^{At}$, where $C = \pm e^k$. Thus, $Bq - A = qCe^{At}$, $(B - Ce^{At})q = A$; so $q = \frac{A}{B - Ce^{At}}$.

Suppose that $q = q_0$ when $t = 0$, so that $q_0 = \frac{A}{B - C}$, and $C = B - \frac{A}{q_0}$.

51. Since $\frac{1}{p(1 - p)} = \frac{1}{p} + \frac{1}{1 - p}$, we have that $\int k\,dt = \int \frac{1}{p(1 - p)} \frac{dp}{dt}\,dt = \ln\left(\frac{p}{1 - p}\right) + C_1$. Hence, $kt + C_2 = \ln\left(\frac{p}{1 - p}\right) + C_1$. $e^{kt}e^{C_2} = \frac{p}{1 - p} e^{C_1}$. $e^{kt} \cdot C_3 = \frac{p}{1 - p}$, where $C_3 = \frac{e^{C_2}}{e^{C_1}}$. $e^{kt}C_3 - e^{kt}C_3 p = p$, $p(1 + e^{kt}C_3) = e^{kt}C_3$, $p = \frac{C_3}{\frac{1}{e^{kt}} + C_3} = \frac{C_3}{e^{-kt} + C_3} = \frac{1}{1 + Ce^{-kt}}$, where $C = \frac{1}{C_3}$.

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1. $\frac{1}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}$. Here, A can be found by the short method of substitution as $\frac{1}{1^2 + 4} = A$, or $A = \frac{1}{5}$. Now $\frac{1}{(x - 1)(x^2 + 4)} = \frac{(\frac{1}{5})}{(x - 1)} + \frac{Bx + C}{x^2 + 4}$, and multiplying by $(x - 1)(x^2 + 4)$ on both sides, we get $1 = \frac{1}{5}(x^2 + 4) + (Bx + C)(x - 1) = (\frac{1}{5} + B)x^2 + (C - B)x + (\frac{4}{5} - C)$. Thus, $0 = \frac{1}{5} + B$, $0 = C - B$, $1 = \frac{4}{5} - C$. Hence, $C = -\frac{1}{5}$ and $B = -\frac{1}{5}$. So $\int \frac{5}{(x - 1)(x^2 + 4)}\,dx = 5\int \frac{(\frac{1}{5})}{x - 1}\,dx + 5\int \frac{-\frac{1}{5}x - \frac{1}{5}}{x^2 + 4}\,dx = 5\int \frac{(\frac{1}{5})}{x - 1}\,dx - 5\int \frac{(\frac{1}{5}x)}{x^2 + 4}\,dx - 5\int \frac{(\frac{1}{5})}{x^2 + 4}\,dx = \ln|x - 1| - (\frac{1}{2})\ln(x^2 + 4) - (\frac{1}{2})\tan^{-1}\frac{x}{2} + C$, where the second integral is obtained by putting $u = x^2 + 4$.

2. $\int \frac{x^5 + 9x^3 + 1}{x^3 + 9x}\,dx = \int x^2\,dx + \int \frac{1}{x^3 + 9x}\,dx$. Now, $\frac{1}{x(x^2 + 9)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 9}$. $\frac{1}{0 + 9} = \frac{1}{9} = A$. Now, $1 = \frac{1}{9}x^2 + 1 + Bx^2 + Cx$, so that $0 = \frac{1}{9} + B$ and $0 = C$. Thus $B = -\frac{1}{9}$. So $\int \frac{x^5 + 9x^3 + 1}{x^3 + 9x}\,dx = \frac{x^3}{3} + \int \frac{\frac{1}{9}}{x}\,dx + \int \frac{-\frac{1}{9}x}{x^2 + 9}\,dx = \frac{x^3}{3} + \frac{1}{9}\ln|x| - \frac{1}{9}(\frac{1}{3})\tan^{-1}\frac{x}{3} + C = \frac{x^3}{3} + \ln|x|^{1/9} - \frac{1}{27}\tan^{-1}\frac{x}{3} + C$.

3. $\frac{x + 3}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$. Now $\frac{0 + 3}{0 + 1} = 3 = A$. Thus, $x + 3 = 3x^2 + 3 + Bx^2 + Cx$. So $0 = 3 + B$ and $B = -3$; $1 = C$. Hence, $\int \frac{x + 3}{x(x^2 + 1)}\,dx = \int \frac{3}{x}\,dx + \int \frac{-3x}{x^2 + 1}\,dx + \int \frac{1}{x^2 + 1}\,dx = 3\ln|x| - \frac{3}{2}\ln|x^2 + 1| + \tan^{-1}x + C = \ln\frac{|x^3|}{(x^2 + 1)^{3/2}} + \tan^{-1}x + C$.

4. $\frac{1}{y^4 - 16} = \frac{1}{(y + 2)(y - 2)(y^2 + 4)} = \frac{A}{y + 2} + \frac{B}{y - 2} + \frac{Cy + D}{y^2 + 4}$. Now $\frac{1}{1} = A$; $\frac{1}{1} = B$; $1 = (y - 2)(y^2 + 4) + (y + 2)(y^2 + 4) + (Cy + D)(y^2 - 4)$; $0 = 1 + 1 + C$ and so $C = -2$. $0 = -2 + 2 + D$ and so $D = 0$. Hence, $\int \frac{dy}{y^4 - 16} = \int \frac{1}{y + 2}\,dy + \int \frac{1}{y - 2}\,dy + \int \frac{-2y}{y^2 + 4}\,dy = \ln|y + 2| + \ln|y - 2| - \ln|y^2 + 4| + C = \ln\left|\frac{(y + 2)(y - 2)}{y^2 + 4}\right| + C = \ln\left|\frac{y^2 - 4}{y^2 + 4}\right| + C$.

5. $\frac{3x^2 + x - 2}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{2x^2 + 1}$. By the short method of substitution, $A = 1$; by equating coefficients, $B = 2$ and $C = 3$. $\frac{3x^2 + x - 2}{(x - 1)(x^2 + 1)} = \frac{1}{x - 1} + \frac{2x + 3}{x^2 + 1}$, $\int \frac{3x^2 + x - 2}{(x - 1)(x^2 + 1)}\,dx = \int \frac{dx}{x - 1} + \int \frac{2x}{x^2 + 1}\,dx + 3\int \frac{dx}{x^2 + 1} = \ln|x - 1| + \ln|x^2 + 1| + 3\tan^{-1}x + C = \ln[|x - 1|(x^2 + 1)] + 3\tan^{-1}x + C$.

6. $\frac{7x^2 + 6x + 5}{x(x^2 + x + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}$; $A = 5$ by the short

method of substitution; by equating coefficients $B = 2$ and $C = 1$. $\int \frac{7x^2 + 6x + 5}{x(x^2 + x + 1)}\,dx = \int \frac{5}{4}\,dx + \int \frac{2x + 1}{x^2 + x + 1}\,dx = 5 \ln |x| + \ln |x^2 + x + 1| + C = \ln[|x|^5(x^2 + x + 1)] + C$. [Note that the substitution $u = x^2 + x + 1$ was used to evaluate the second integral.]

. $\frac{x}{x^4 - 1} = \frac{x}{(x + 1)(x - 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 1}$. Now, $\frac{-1}{-2(2)} = \frac{1}{4} = A$; $\frac{1}{2(2)} = \frac{1}{4} = B$; therefore, $x = \frac{1}{4}(x - 1)(x^2 + 1) + \frac{1}{4}(x + 1)(x^2 + 1) + (Cx + D)(x^2 - 1)$. The coefficient of x^3 is $0 = \frac{1}{4} + \frac{1}{4} + C$ and so $C = -\frac{1}{2}$; also, $0 = -\frac{1}{4} + \frac{1}{4} + D$ and $D = 0$. Thus, $\int \frac{4x}{x^4 - 1}\,dx = 4 \int \frac{(\frac{1}{4})}{x + 1}\,dx + 4 \int \frac{(\frac{1}{4})}{x - 1}\,dx + 4 \int \frac{(-\frac{1}{2}x)}{x^2 + 1}\,dx = \ln |x + 1| + \ln|x - 1| - \ln (x^2 + 1) + C = \ln(\frac{|x^2 - 1|}{x^2 + 1}) + C$.

Put $u = 2x^2 - 12x + 19$, so that $du = (4x - 12)dx = 4(x - 3)dx$. Thus, $\int \frac{x - 3}{2x^2 - 12x + 19}\,dx = \int \frac{1}{4}\frac{du}{u} = \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |2x^2 - 12x + 19| + C$.

$\frac{2t^2 - t + 1}{t(t^2 + 25)} = \frac{A}{t} + \frac{Bt + C}{t^2 + 25}$. Thus, $\frac{0 - 0 + 1}{0 + 25} = \frac{1}{25} = A$. Now $2t^2 - t + 1 = \frac{1}{25}(t^2 + 25) + Bt^2 + Ct$. So $2 = \frac{1}{25} + B$ and $B = \frac{49}{25}$; $-1 = C$. Therefore, $\int \frac{2t^2 - t + 1}{t(t^2 + 25)}\,dt = \int \frac{(\frac{1}{25})}{t}\,dt + \int \frac{(\frac{49}{25}t)}{t^2 + 25}\,dt + \int \frac{(-1)}{t^2 + 25}\,dt = \frac{1}{25} \ln |t| + \frac{49}{50} \ln |t^2 + 25| - \frac{1}{5} \tan^{-1} \frac{t}{5} + C = \ln|t^{1/25}(t^2 + 25)^{49/50}| - \frac{1}{5} \tan^{-1} \frac{t}{5} + C$.

. Since $2u^3 - u^2 + 8u - 4 = u^2(2u - 1) + 4(2u - 1) = (2u - 1)(u^2 + 4)$, then $\frac{u^2 - u - 21}{2u^3 - u^2 + 8u - 4} = \frac{A}{2u - 1} + \frac{Bu + C}{u^2 + 4}$. Hence, by the short method of substitution, $A = -5$; by equating coefficients, $B = 3$ and $C = 1$. $\int \frac{u^2 - 2 - 21}{2u^3 - u^2 + 8u - 4}\,du = -\frac{5}{2} \ln |2u - 1| + \frac{3}{2} \ln |u^2 + 4| + \frac{1}{2} \tan^{-1} (\frac{u}{2}) + C$.

11. $\frac{16}{x(x^2 + 4)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2 + 4)^2}$. $\frac{16}{16} = 1 = A$. Now $16 = (x^2 + 4)^2 + (Bx + C)(x)(x^2 + 4) + (Dx + E)x$. The coefficient of x^4 is $0 = 1 + B$ and $B = -1$; the coefficient of x^2 is $0 = 8 + 4B + D$ so $D = -4$. Also, $0 = C$ and $0 = 4C + E$, and so $E = 0$. Therefore, $\int \frac{16}{x(x^2 + 4)^2}\,dx = \int \frac{1}{x}\,dx + \int \frac{-x}{x^2 + 4}\,dx + \int \frac{-4x}{(x^2 + 4)^2}\,dx = \ln |x| - \frac{1}{2} \ln (x^2 + 4) + \frac{2}{x^2 + 4} + C = \ln\left|\frac{x}{\sqrt{x^2 + 4}}\right| + \frac{2}{x^2 + 4} + C$, where the third integration is obtained by putting $u = x^2 + 4$ and so forth.

12. $\frac{2x^3 + 9}{x^4 + x^3 + 12x^2} = \frac{2x^3 + 9}{x^2(x^2 + x + 12)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + x + 12}$. $\frac{9}{12} = \frac{3}{4} = B$. Now $2x^3 + 9 = Ax(x^2 + x + 12) + \frac{3}{4}(x^2 + x + 12) + (Cx + D)x^2$. Thus, $12A + \frac{3}{4} = 0$ so that $A = -\frac{1}{16}$; $A + C = 0$, so $C = \frac{1}{16}$; $A + \frac{3}{4} + D = 0$ so that $D = -\frac{11}{16}$. Therefore,

$$\int \frac{2x^3 + 9}{x^4 + x^3 + 12x^2}\,dx = \int \frac{(-\frac{1}{16})}{x}\,dx + \int \frac{(\frac{3}{4})}{x^2}\,dx + \int \frac{(\frac{x}{16} - \frac{11}{16})}{x^2 + x + 12}\,dx = -\frac{1}{16} \ln |x| - \frac{3}{4x} + \frac{1}{16}\int \frac{x - 11}{(x + \frac{1}{2})^2 + \frac{47}{4}}.$$

Now put $u = x + \frac{1}{2}$, so that $du = dx$. So

$$\int \frac{x - 11}{(x + \frac{1}{2})^2 + \frac{47}{4}}\,dx = \int \frac{u - \frac{23}{2}}{u^2 + \frac{47}{4}}\,du = \int \frac{u}{u^2 + \frac{47}{4}}\,du - \frac{23}{2}\int \frac{du}{u^2 + \frac{47}{4}} = \frac{1}{2} \ln (u^2 + \frac{47}{4}) - \frac{23}{2}(\frac{2}{\sqrt{47}})\tan^{-1} \frac{2u}{\sqrt{47}} + C.$$

Thus, $\int \frac{2x^3 + 9}{x^4 + x^3 + 12x^2}\,dx = -\frac{1}{16} \ln |x| - \frac{3}{4x} + \frac{1}{32} \ln (x^2 + x + 12) - \frac{23}{\sqrt{47}} \tan^{-1} \frac{2x + 1}{\sqrt{47}} + C$.

13. $\frac{15y^2 - 4y + 12}{3y^3 - y^2 + 12y - 4} = \frac{3}{3y - 1} + \frac{4y}{y^2 + 4}$. [Note that $3y^3 - y^2 + 12y - 4 = y^2(3y - 1) + 4(3y - 1) = (3y - 1)(y^2 + 4)$.] $\int \frac{15y^2 - 4y + 12}{3y^3 - y^2 + 12y - 4}\,dy =$ $\ln|3y - 1| + 2\ln|y^2 + 4| + C =$ $\ln|3y - 1|(y^2 + 4)^2 + C$.

14. $\int \frac{x^3 + 2x^2 + 7x + 2}{x^2 + 2x + 5}\,dx = \int x\,dx + \int \frac{2x + 2}{x^2 + 2x + 5}\,dx =$ $\frac{x^2}{2} + \ln(x^2 + 2x + 5) + C$, where we used the substitution $u = x^2 + 2x + 5$ to evaluate the second integral.

15. $\frac{6x^2 - 8x - 1}{(x - 2)(2x^2 - 3x + 5)} = \frac{A}{x - 2} + \frac{Bx + C}{2x^2 - 3x + 5}$,

$A = \frac{24 - 16 - 1}{8 - 6 + 5} = \frac{7}{7} = 1$. Hence, $6x^2 - 8x - 1 = (2x^2 - 3x + 5) + (Bx + C)(x - 2) = (2 + B)x^2 + (-3 - 2B + C)x + (5 - 2C)$. Thus, $6 = 2 + B$, so that $B = 4$, and $-1 = 5 - 2C$, so that $C = 3$. We therefore have $\int \frac{6x^2 - 8x - 1}{(x - 2)(2x^2 - 3x + 5)}\,dx = \int \frac{dx}{x - 2} + 4\int \frac{x\,dx}{2x^2 - 3x + 5} + 3\int \frac{dx}{2x^2 - 3x + 5} = \ln|x - 2| + 2\ln|2x^2 - 3x + 5| + 3\int \frac{dx}{2x^2 - 3x + 5}$. Now

$\int \frac{dx}{2x^2 - 3x + 5} = \int \frac{dx}{2(x^2 - \frac{3}{2}x + \frac{9}{16}) + \frac{31}{8}} =$ $\int \frac{dx}{2(x - \frac{3}{4})^2 + \frac{31}{8}}$. Put $u = \sqrt{2}(x - \frac{3}{4})$, so that $du = \sqrt{2}\,dx$ and $\int \frac{dx}{2x^2 - 3x + 5} = \frac{1}{\sqrt{2}} \int \frac{du}{u^2 + \frac{31}{8}} =$

$\frac{1}{\sqrt{2}}\left(\sqrt{\frac{8}{31}} \tan^{-1} \sqrt{\frac{8}{31}}\, u\right) + C =$

$\frac{2}{\sqrt{31}} \tan^{-1}\left[\sqrt{\frac{8}{31}}\, \sqrt{2}(x - \frac{3}{4})\right] + C =$

$\frac{2}{\sqrt{31}} \tan^{-1}\left[\frac{4}{\sqrt{31}}(x - \frac{3}{4})\right] + C$. Hence,

$\int \frac{6x^2 - 8x - 1}{(x - 2)(2x^2 - 3x + 5)}\,dx =$

$\ln[|x - 2|(2x^2 - 3x + 5)^2] + \frac{6}{\sqrt{31}} \tan^{-1}\left[\frac{4}{\sqrt{31}}(x - \frac{3}{4})\right] + C$.

16. $\frac{17}{(y - 2)(y^2 + 4y + 5)} = \frac{A}{y - 2} + \frac{By + C}{y^2 + 4y + 5}$. So $\frac{17}{4 + 8 + 5} = 1 = A$. Now $17 = (y^2 + 4y + 5) + (By + C)(y - 2)$. $0 = 1 + B$, so that $B = -1$; $0 = 4$ $2B + C$, so that $C = -6$. $\int \frac{17}{(y - 2)(y^2 + 4y + 5)}\,dy$ $\int \frac{1}{y - 2}\,dy - \int \frac{y + 6}{y^2 + 4y + 5}\,dy = \ln|y - 2| -$ $\int \frac{y + 6}{(y + 2)^2 + 1}\,dy$. Let $u = y + 2$, $du = dy$, $y = u - 2$. So $\int \frac{17}{(y - 2)(y^2 + 4y + 5)}\,dy = \ln|y - 2$ $\int \frac{u + 4}{u^2 + 1}\,du = \ln|y - 2| - \frac{1}{2}\ln(u^2 + 1) -$ $4\tan^{-1}u + C = \ln \frac{|y - 2|}{\sqrt{y^2 + 4y + 5}} - 4\tan^{-1}(y + 2) + C$

17. By long division first, we obtain

$\frac{2x^4 - 7x^3 + 31x^2 - 45x + 46}{2x^3 - 7x^2 + 11x - 10} = x + \frac{20x^2 - 35x + 46}{2x^3 - 7x^2 + 11x - 10} = x + \frac{20x^2 - 35x + 46}{(x - 2)(2x^2 - 3x + 5)}$.

Now, $\frac{20x^2 - 35x + 46}{(x - 2)(2x^2 - 3x + 5)} = \frac{A}{x - 2} + \frac{Bx + C}{2x^2 - 3x + 5}$,

where by the short method of substitution, $\frac{80 - 70 + 46}{8 - 6 + 5} = A = 8$. Now $20x^2 - 35x + 46 = 8(2x^2 - 3x + 5) + (Bx + C)(x - 2)$; collecting terms and equating coefficients, we get $B = 4$ and $C = -3$. Thus, $\int x\,dx + \int \frac{8}{x - 2}\,dx + \int \frac{4x - 3}{2x^2 - 3x + 5}\,dx$ $\frac{x^2}{2} + 8\ln|x - 2| + \ln|2x^2 - 3x + 5| + C$. Therefore, $\int \frac{2x^4 - 7x^3 + 31x^2 - 45x + 46}{2x^3 - 7x^2 + 11x - 10}\,dx = \frac{1}{2}x^2 +$ $\ln[|x - 2|^8(2x^2 - 3x + 5)] + C$.

18. Put $u = x^3 + 4x^2 + 6x + 4$, so that $du = 3x^2 + 8x +$ and $\int \frac{3x^2 + 8x + 6}{x^3 + 4x^2 + 6x + 4}\,dx = \int \frac{du}{u} = \ln|u| + C =$ $\ln|x^3 + 4x^2 + 6x + 4| + C$.

19. $\frac{5t^3 - 3t^2 + 2t - 1}{t^2(t^2 + 9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct + D}{t^2 + 9} =$

$\frac{0 - 0 + 0 - 1}{9} = -\frac{1}{9} = B$. Thus, $5t^3 - 3t^2 + 2t - 1$

$At(t^2 + 9) - \frac{1}{9}(t^2 + 9) + t^2(Ct + D)$. Now $5 = A + C$ and $9A = 2$ so $A = \frac{2}{9}$ and $C = \frac{43}{9}$. $-3 = -\frac{1}{9} + D$, so $D = -\frac{26}{9}$. Therefore, $\int \frac{5t^3 - 3t^2 + 2t - 1}{t^2(t^2 + 9)}\, dt =$

$$\int \frac{(\frac{2}{9})}{t}\, dt + \int \frac{(-\frac{1}{9})}{t^2}\, dt + \int \frac{\frac{43}{9}t - \frac{26}{9}}{t^2 + 9}\, dt = \frac{2}{9}\ln|t| +$$

$$\frac{1}{9t} + \frac{1}{9}\int \frac{43t - 26}{t^2 + 9}\, dt = \frac{2}{9}\ln|t| + \frac{1}{9t} + \frac{43}{18}\ln|t^2 + 9| -$$

$$\frac{26}{9}\left(\frac{1}{3}\right)\tan^{-1}\frac{t}{3} + C = \ln|t^{2/9}(t^2 + 9)^{43/18}| +$$

$$\frac{1}{9t} - \frac{26}{27}\tan^{-1}\frac{t}{3} + C.$$

The integrand is already a partial fraction. Put $x = \tan\theta$, so that $x^2 = \tan^2\theta$ and $dx = \sec^2\theta\, d\theta$.

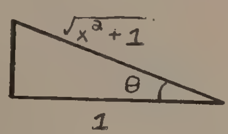
Thus, $\int \frac{dx}{(x^2 + 1)^3} = \int \frac{\sec^2\theta\, d\theta}{\sec^6\theta} = \int \cos^4\theta\, d\theta =$

$$\int \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta =$$

$$\frac{1}{4}(\theta + \sin 2\theta) + \frac{1}{4}\int \left(\frac{1 + \cos 4\theta}{2}\right) d\theta =$$

$$\frac{1}{4}\theta + \frac{1}{4}\sin 2\theta +$$

$$\frac{1}{8} + \frac{\sin 4\theta}{32} + C =$$



$$\frac{3}{8} + \frac{1}{2}\sin\theta\cos\theta +$$

$$\frac{2\sin 2\theta\cos 2\theta}{32} + C = \frac{3\theta}{8} + \frac{1}{2}\sin\theta\cos\theta +$$

$$\frac{\sin\theta\cos\theta\cos 2\theta}{8} + C = \frac{3}{8}\tan^{-1}x + \frac{1}{2}\frac{x}{1 + x^2} +$$

$$\frac{x}{8(1 + x^2)}(\cos^2\theta - \sin^2\theta) + C = \frac{3}{8}\tan^{-1}x +$$

$$\frac{x}{2(1 + x^2)} + \frac{x(1 - x^2)}{8(1 + x^2)^2} + C.$$

$$\frac{x^3 + 4}{x^2(x^2 + 1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} + \frac{Ex + G}{(x^2 + 1)^2}.$$

$\frac{0 + 4}{0 + 1} = 4 = B$; thus, $x^3 + 4 = Ax(x^2 + 1)^2 + 4(x^2 + 1)^2 + (Cx + D)(x^2)(x^2 + 1) + (Ex + G)x^2$. $x^3 + 4 = A(x^5 + 2x^3 + x) + 4(x^4 + 2x^2 + 1) + C(x^5 + x^3) + D(x^4 + x^2) + Ex^3 + Gx^2$. $0 = A + C$. $0 = 4 + D$, so that $D = -4$. $1 = 2A + C + E$. $0 = 8 + D + G$. $0 = A$. Thus, $C = 0$ and $E = 1$. $G = -4$. Therefore, $\int \frac{x^3 + 4}{x^2(x^2 + 1)^2}\, dx = \int \frac{4}{x^2}\, dx +$

$$\int \frac{-4}{x^2 + 1}\, dx + \int \frac{x - 4}{(x^2 + 1)^2}\, dx = -\frac{4}{x} - 4\tan^{-1}x -$$

$$\frac{1}{2(x^2 + 1)} - \int \frac{4}{(x^2 + 1)^2}\, dx = -\frac{4}{x} - 4\tan^{-1}x -$$

$$\frac{1}{2(x^2 + 1)} - 4\left[\frac{\tan^{-1}x}{2} + \frac{x}{2(x^2 + 1)}\right] + C = -\frac{4}{x} -$$

$6\tan^{-1}x - \frac{1}{2(x^2 + 1)} - \frac{2x}{(x^2 + 1)} + C$, where we used the trigonometric substitution $x = \tan\theta$ to evaluate the last integral.

22. $\frac{2y^2}{y^4 + y^3 + 12y^2} = \frac{2y^2}{y^2(y^2 + y + 12)} = \frac{2}{y^2 + y + 12}$.

Now $\int \frac{2y^2}{y^4 + y^3 + 12y^2}\, dy = \int \frac{2}{y^2 + y + 12}\, dy =$

$\int \frac{2}{(y + \frac{1}{2})^2 + \frac{47}{4}}\, dy$. Now put $u = y + \frac{1}{2}$, $du = dy$.

Thus, $\int \frac{2}{u^2 + \frac{47}{4}}\, dy = 2\left(\frac{2}{\sqrt{47}}\right)\tan^{-1}\frac{2u}{\sqrt{47}} + C$. Hence,

$$\int \frac{2y^2}{y^4 + y^3 + 12y^2}\, dy = \frac{4}{\sqrt{47}}\tan^{-1}\frac{(2y + 1)}{\sqrt{47}} + C.$$

23. $\int \frac{x^5 + 4x^3 + 3x^2 - x + 2}{x^5 + 4x^3 + 4x}\, dx = \int 1\, dx +$

$\int \frac{3x^2 - 5x + 2}{x^5 + 4x^3 + 4x}\, dx$. $\frac{3x^2 - 5x + 2}{x^5 + 4x^3 + 4x} = \frac{3x^2 - 5x + 2}{x(x^2 + 2)^2} =$

$\frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}$. Now $\frac{0 - 0 + 2}{4} = \frac{1}{2} = A$;

$3x^2 - 5x + 2 = \frac{1}{2}(x^2 + 2)^2 + (Bx + C)x(x^2 + 2) + (Dx + E)x$. $0 = \frac{1}{2} + B$ and $B = -\frac{1}{2}$; $0 = C$; $3 = 2 + 2B + D$, and $D = 2$; $-5 = 2C + E$, so that $E = -5$.

Thus, $\int \frac{x^5 + 4x^3 + 3x^2 - x + 2}{x^5 + 4x^3 + 4x}\, dx = x + \int \frac{(\frac{1}{2})}{x}\, dx +$

$$\int \frac{(-\frac{1}{2})x}{x^2 + 2}\, dx + \int \frac{2x}{(x^2 + 2)^2}\, dx + \int \frac{(-5)}{(x^2 + 2)^2}\, dx =$$

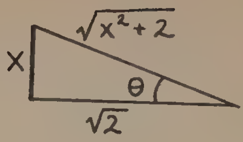
$$x + \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2 + 2) - \frac{1}{x^2 + 2} +$$

$\int \frac{-5}{(x^2 + 2)^2}\, dx$. Put $x = \sqrt{2}\tan\theta$, so that $dx = \sqrt{2}\sec^2\theta\, d\theta$. Thus, $\int \frac{1}{(x^2 + 2)^2}\, dx = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{4\sec^4\theta} =$

$$\frac{\sqrt{2}}{4}\cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\int \frac{1 + \cos 2\theta}{2}\, d\theta = \frac{\sqrt{2}}{8}\left(\theta + \frac{\sin 2\theta}{2}\right) + C =$$

$\frac{\sqrt{2}}{8}\left(\tan^{-1}\frac{x}{\sqrt{2}} + \frac{\sqrt{2}\,x}{x^2 + 2}\right) + C$, where $\sin 2\theta =$

$2 \sin\theta \cos\theta$ and



Hence, $\int \frac{x^5 + 4x^3 + 3x^2 - x + 2}{x^5 + 4x^3 + 4x}\,dx = x +$

$\ln \frac{|x|^{\frac{1}{2}}}{(x^2+2)^{\frac{1}{4}}} - \frac{1}{x^2+2} - \frac{5\sqrt{2}}{8}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{5x}{4(x^2+2)} + C.$

24. $\frac{4x^2}{(x-1)^2(x^2-x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} +$

$\frac{Cx+D}{x^2-x+1}$. First, $\frac{4}{1} = B$. Now $4x^2 =$

$A(x-1)(x^2-x+1) + 4(x^2-x+1) +$

$(Cx+D)(x-1)^2$. Equating coefficients and solving simultaneous equations, we get $A = 4$, $C = -4$, and $D = 0$. $\int \frac{4x^2}{(x-1)^2(x^2-x+1)}\,dx =$

$\int \frac{4}{x-1}\,dx + \int \frac{4}{(x-1)^2}\,dx + \int \frac{-4x}{x^2-x+1}\,dx =$

$4\ln|x-1| - \frac{4}{x-1} + \int \frac{-4x}{(x-\frac{1}{2})^2 + \frac{3}{4}}\,dx$. Now put

$u = x - \frac{1}{2}$, so that $du = dx$. Thus,

$\int \frac{-4x}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx = \int \frac{-4u-2}{u^2+\frac{3}{4}}\,du = -2\ln|u^2+\frac{3}{4}| -$

$\frac{2}{\sqrt{3}}\tan^{-1}\frac{2u}{\sqrt{3}} + C$. Therefore,

$\int \frac{4x^2}{(x-1)^2(x^2-x+1)}\,dx = \ln\frac{(x-1)^4}{(x^2-x+1)^2} -$

$\frac{4}{x-1} - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C.$

25. $\frac{4(t+1)}{t(t^2+2t+2)^2} = \frac{1}{t} - \frac{t+2}{t^2+2t+2} - \frac{2t}{(t^2+2t+2)^2}$

by partial fractions. Hence, $\int \frac{4(t+1)}{t(t^2+2t+2)^2}\,dt =$

$\ln|t| - \int \frac{t+2}{t^2+2t+2}\,dt - \int \frac{2t}{(t^2+2t+2)^2}\,dt.$

Now put $u = t + 1$, so that $du = dt$ and $t^2 + 2t + 2 =$ $(t+1)^2 + 1 = u^2 + 1$. Then $\int \frac{t+2}{t^2+2t+2}\,dt =$

$\int \frac{u+1}{u^2+1}\,du = \frac{1}{2}\ln|u^2+1| + \tan^{-1}u + C_1 =$

$\frac{1}{2}\ln(t^2+2t+2) + \tan^{-1}(t+1) + C_1$. Also,

$\int \frac{2t}{(t^2+2t+2)^2}\,dt = \int \frac{2u-2}{(u^2+1)^2}\,du =$

$\int \frac{2u}{(u^2+1)^2}\,du - \int \frac{2}{(u^2+1)^2}\,du = \frac{-1}{u^2+1} -$

$2\left[\frac{\tan^{-1}u}{2} + \frac{u}{2(u^2+1)}\right] + C_2$, where the first integral is obtained by letting $V = u^2 + 1$, and the second from Example 9, Section 8.6 (page 522).

Thus, $\int \frac{2t}{(t^2+2t+1)^2}\,dt = \frac{-1}{t^2+2t+2} - \tan^{-1}(t+$

$\frac{t+1}{t^2+2t+2} + C_2$. We substitute back and get

$\int \frac{4(t+1)}{t(t^2+2t+1)^2} = \ln|t| - \frac{1}{2}\ln(t^2+2t+2) -$

$\tan^{-1}(t+1) + \frac{1}{t^2+2t+2} + \tan^{-1}(t+1) +$

$\frac{t+1}{t^2+2t+2} + C = \ln\frac{|t|}{(t^2+2t+2)^{1/2}} +$

$\frac{t+2}{t^2+2t+2} + C.$

26. $\frac{1}{x^3+3x^2+7x+5} = \frac{1}{(x+1)(x^2+2x+5)} = \frac{A}{x+1} +$

$\frac{Bx+C}{x^2+2x+5}$. $A = \frac{1}{1-2+5} = \frac{1}{4}$. Thus,

$1 = A(x^2+2x+5) + (Bx+C)(x+1) = (A+B)x^2 +$

$(2A+B+C)x + (5A+C)$, so that $0 = A + B$ and

$B = -A = -\frac{1}{4}$. Also, $1 = 5A + C$, so that $C = 1 -$

$5A = 1 - \frac{5}{4} = -\frac{1}{4}$. Therefore, $\frac{dx}{x^3+3x^2+7x+5} =$

$\frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{x+1}{x^2+2x+5}\,dx = \frac{1}{4}\ln|x+1| +$

$\frac{1}{8}\int \frac{(2x+2)\,dx}{x^2+2x+5} = \frac{1}{4}\ln|x+1| + \frac{1}{8}\ln(x^2+2x+5) + C$

$\frac{2}{8}\ln|x+1| + \frac{1}{8}\ln(x^2+2x+5) + C =$

$\frac{1}{8}\ln\frac{(x+1)^2}{x^2+2x+5} + C = \frac{1}{8}\ln\frac{x^2+2x+1}{x^2+2x+5} + C.$

27. $\frac{t+10}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$. $\frac{-1+10}{1+1} = \frac{9}{2} = A$

$t + 10 = A(t^2+1) + (Bt+C)(t+1)$, and so

$0 = A + B$ and $B = -\frac{9}{2}$; $1 = B + C$, so that $C = \frac{11}{2}$.

$\int_0^3 \frac{t+10}{(t+1)(t^2+1)}\,dt = \int_0^3 \frac{(\frac{9}{2})}{t+1}\,dt +$

$$\int_0^3 \frac{-\frac{9}{2}t + \frac{11}{2}}{t^2 + 1}\,dt = \frac{9}{2}\ln|t+1|\Big|_0^3 - \frac{1}{2}\Big[\frac{9}{2}\ln(t^2+1)\Big|_0^3 - 11\tan^{-1} t\Big|_0^3\Big] = \frac{9}{2}\ln 4 - \frac{9}{4}\ln 10 + \frac{11}{2}\tan^{-1}3.$$

$\frac{1}{8x^3 + 27} = \frac{1}{(2x+3)(4x^2-6x+9)} = \frac{A}{2x+3} + \frac{Bx+C}{4x^2-6x+9}$. Now $\frac{1}{4(\frac{9}{4}) + 9 + 9} = \frac{1}{27} = A$;

$1 = A(4x^2 - 6x + 9) + (Bx + C)(2x + 3)$.

$0 = 4A + 2B$, and so $B = -\frac{2}{27}$. Also, $0 = -6A + 3B + 2C$, and so $C = \frac{6}{27}$. Thus, $\int_0^1 \frac{dx}{8x^3+27} = \int_0^1 \frac{(1/27)}{2x+3}\,dx + \int_0^1 \frac{-\frac{2}{27}x + \frac{6}{27}}{4x^2 - 6x + 9}\,dx = \frac{1}{27}\ln|2x+3|\Big|_0^1 - \frac{2}{27}\int_0^1 \frac{x-3}{4(x-\frac{3}{2})^2 + \frac{27}{4}}\,dx$. Now put $u = x - \frac{3}{2}$, so that $du = dx$. Therefore, $\int_0^1 \frac{x-3}{4(x-\frac{3}{2})^2 + \frac{27}{4}}\,dx =$

$$\int_{-3/2}^{-1/2} \frac{u - \frac{3}{2}}{4u^2 + \frac{27}{4}}\,du = \frac{1}{8}\ln\Big(4u^2 + \frac{27}{4}\Big)\Big|_{-3/2}^{-1/2} - \frac{3}{8}\Big(\frac{2}{\sqrt{27}}\Big)\tan^{-1}\frac{2u}{\sqrt{27}}\Big|_{-3/2}^{-1/2} = \frac{1}{8}\Big[\ln\frac{31}{4} - \ln\frac{63}{4}\Big] - \frac{3}{4\sqrt{27}}\Big[\tan^{-1}\Big(-\frac{1}{\sqrt{27}}\Big) - \tan^{-1}\Big(-\frac{1}{\sqrt{3}}\Big)\Big].$$

$\frac{8x}{(2x+1)(4x^2+1)} = \frac{A}{2x+1} + \frac{Bx+C}{4x^2+1}$. Now $\frac{-4}{1+1} = -2 = A$. $8x = A(4x^2+1) + (Bx+C)(2x+1)$, so that $0 = 4A + 2B$ and $B = 4$; $8 = B + 2C$, and so $C = 2$. Thus, $\int_0^{1/2} \frac{8x\,dx}{(2x+1)(4x^2+1)} = \int_0^{1/2} \frac{-2}{2x+1}\,dx + \int_0^{1/2} \frac{4x+2}{4x^2+1}\,dx = -\ln|2x+1|\Big|_0^{1/2} + \frac{1}{2}\ln(4x^2+1)\Big|_0^{1/2} + \tan^{-1}2x\Big|_0^{1/2} = -\ln 2 + \frac{1}{2}\ln 2 + \tan^{-1}1 = \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi - 2\ln 2}{4}$.

$\frac{4}{x^3 + 4x} = \frac{4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$. Thus, $\frac{4}{0+4} = 1 = A$ and $4 = A(x^2+4) + (Bx+C)x$, and so $0 = A + B$ and $B = -1$; $0 = C$. Thus, $\int_1^2 \frac{4}{x^3+4x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_1^2 \frac{-x}{x^2+4}\,dx = \ln|x|\Big|_1^2 - \frac{1}{2}\ln(x^2+4)\Big|_1^2 = \ln 2 - \frac{1}{2}\ln 8 + \frac{1}{2}\ln 5 = \ln 2 - \frac{3}{2}\ln 2 + \frac{1}{2}\ln 5 = -\frac{1}{2}\ln 2 + \frac{1}{2}\ln 5 = \ln\sqrt{\frac{5}{2}}$.

31. $\frac{1 - x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$. $\frac{1-0}{0+1} = 1 = A$; so $1 - x^2 = A(x^2+1) + (Bx+C)x$. $-1 = A + B$ and so $B = -2$; $C = 0$. Thus, $\int_1^2 \frac{1-x^2}{x(x^2+1)}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_1^2 \frac{-2x}{x^2+1}\,dx = \ln|x|\Big|_1^2 - \ln|x^2+1|\Big|_1^2 = \ln 2 - \ln 5 + \ln 2 = \ln\frac{4}{5}$.

32. $\frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$. Thus, $\frac{1 - 1 + 2 - 1 + 2}{(1+2)^2} = \frac{1}{3} = A$ and $x^4 - x^3 + 2x^2 - x + 2 = \frac{1}{3}(x^2+2)^2 + (Bx+C)(x-1)(x^2+2) + (Dx+E)(x-1)$. Equating coefficients, we get $B = \frac{2}{3}$, $C = -\frac{1}{3}$, $D = -1$, and $E = 0$. Thus, $\int_2^5 \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2}\,dx =$

$$\int_2^5 \frac{\frac{1}{3}}{x-1}\,dx + \int_2^5 \frac{\frac{2}{3}x - \frac{1}{3}}{x^2+2}\,dx + \int_2^5 \frac{-x}{(x^2+2)^2}\,dx =$$

$$\frac{1}{3}\ln(x-1)\Big|_2^5 + \frac{1}{3}[\ln(x^2+2)\Big|_2^5 - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\Big|_2^5 + \frac{1}{2(x^2+2)}\Big|_2^5 = \frac{1}{3}(\ln 4) + \frac{1}{3}(\ln 27 - \ln 6) -$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\frac{5}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{2}{\sqrt{2}} + \frac{1}{54} - \frac{1}{12} = \frac{1}{3}\ln 2 + \frac{1}{3}\ln 9 - \frac{1}{\sqrt{2}}\tan^{-1}\frac{5}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{2}{\sqrt{2}} - \frac{7}{108} =$$

$$\ln 18^{1/3} - \frac{7}{108} + \frac{1}{\sqrt{2}}\Big(\tan^{-1}\frac{2}{\sqrt{2}} - \tan^{-1}\frac{5}{\sqrt{2}}\Big).$$

33. Put $u = \sin x$, so that $du = \cos x\,dx$. Then $\int \frac{\cos x\,dx}{\sin^3 x + \sin^2 x + 9\sin x + 9} = \int \frac{du}{u^3 + u^2 + 9u + 9} = \int \frac{dt}{(u+1)(u^2+9)}$.

34. Put $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\,dx$. Then $\int \frac{dx}{x\sqrt{x} + x + 1} =$ $\int \frac{dx}{\sqrt{x}(x + \sqrt{x} + \frac{1}{\sqrt{x}})} = \int \frac{2\,du}{u^2 + u + \frac{1}{u}} = \int \frac{2u}{u^3 + u^2 + 1}\,du.$

35. Put $u = e^x$, so that $du = e^x dx$. Then $\int \frac{3e^{2x} + 2e^x - 2}{e^{3x} - 1}\,dx = \int \frac{3e^{2x} + 2e^x - 2}{e^x(e^{3x} - 1)}(e^x)\,dx =$ $\int \frac{3u^2 + 2u - 2}{u(u^3 - 1)}\,du.$

36. Put $u = e^x$, so that $du = e^x dx$. Then $\int \frac{3e^{3x} + e^x + 3}{(e^{2x} + 1)^2}\,dx = \int \frac{(3e^{3x} + e^x + 3)}{e^x(e^{2x} + 1)^2}\,e^x dx =$ $\int \frac{3u^3 + u + 3}{u(u^2 + 1)^2}\,du.$

37. $\frac{ax^3 + bx^2 + cx + d}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$. Thus, $ax^3 + bx^2 + cx + d = (Ax + B)(x^2 + 1) + Cx + D.$ Now $a = A$; $b = B$; $c = A + C$, so that $C = c - a$; $d = B + D$, so that $D = d - b$. Therefore, $\frac{ax^3 + bx^2 + cx + d}{(x^2 + 1)^2} = \frac{ax + b}{x^2 + 1} + \frac{(c - a)x + (d - b)}{(x^2 + 1)^2}.$

38. $\int \frac{ax^3 + bx^2 + cx + d}{(x^2 + 1)^2}\,dx = \int \frac{ax + b}{x^2 + 1}\,dx + \int \frac{(c - a)x}{(x^2 + 1)^2}dx +$ $\int \frac{d - b}{(x^2 + 1)^2}\,dx = \frac{a}{2}\ln(x^2 + 1) + b\tan^{-1}x +$ $\frac{c - a}{2}\left(-\frac{1}{x^2 + 1}\right) + (d - b)\left[\frac{\tan^{-1}x}{2} + \frac{x}{2(x^2 + 1)}\right] + C =$ $\frac{1}{2}\left[a\ln(x^2 + 1) + (b + d)\tan^{-1}x + \frac{(a - c) + (d - b)x}{(x^2 + 1)}\right] + C$, where the last integral was obtained using the substitution $x = \tan\theta$.

39. (a) Put $u = x^5 + 2x^3 + x$, so that $du = (5x^4 + 6x^2 + 1)dx$. Thus, $\int \frac{5x^4 + 6x^2 + 1}{x^5 + 2x^3 + x}\,dx =$ $\int \frac{du}{u} = \ln|u| + C = \ln|x^5 + 2x^3 + x| + C.$

(b) $\frac{5x^4 + 6x^2 + 1}{x^5 + 2x^3 + x} = \frac{(5x^2 + 1)(x^2 + 1)}{x(x^2 + 1)^2} = \frac{5x^2 + 1}{x(x^2 + 1)}.$

Thus, $\frac{5x^2 + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$. $1 = A$; $5x^2 + 1 = x^2 + 1 + Bx^2 + Cx$. Hence, $5 = 1 + B$, so that $B = 4$; $0 = C$. Therefore, $\int \frac{5x^4 + 6x^2 + 1}{x^5 + 2x^3 + x}\,dx = \int \frac{1}{x}\,dx +$ $\int \frac{4x}{x^2 + 1}\,dx = \ln|x| + 2\ln(x^2 + 1) + C =$ $\ln|x(x^2 + 1)^2| + C = \ln|x^5 + 2x^3 + x| + C.$

40. Put $t = au^2 + q$, so that $dt = 2au\,du$. For $k \neq 1$, we have $\int \frac{u\,du}{(au^2 + q)^k} = \frac{1}{2a}\int t^{-k}dt = \frac{1}{2a(1 - k)}t^{-k+1} +$ $C = \frac{1}{2a(1 - k)(au^2 + q)^{k-1}} + C$. For $k = 1$, $\int \frac{u\,du}{(au^2 + q)} =$ $\frac{1}{2a}|\ln au^2 + q| + C$. Hence, $\int \frac{u\,du}{(au^2 + q)^k} =$

$$\begin{cases} \frac{1}{2a}\ln|au^2 + q| + C \text{ for } k = 1 \\ \frac{1}{2a(1 - k)(au^2 + q)^{k-1}} + C \text{ for } k \neq 1. \end{cases}$$

41. $ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + C =$ $a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$. Put $u = x + \frac{b}{2a}$ and $q = \frac{4ac - b^2}{4a}$. Now $\frac{q}{a} = \frac{4ac - b^2}{4a^2}$. Since $b^2 - 4ac < 0$, then $b^2 - 4ac > 0$. Also $4a^2 > 0$. Therefore, $\frac{q}{a} > 0$.

42. Put $w = \tan\theta$, so that $dw = \sec^2\theta\,d\theta$. Thus, $\int \frac{dw}{(w^2 + 1)^k} = \int \frac{\sec^2\theta\,d\theta}{\sec^{2k}\theta} = \int \frac{1}{\sec^{2(k-1)}\theta}\,d\theta =$ $\int \cos^n\theta\,d\theta$, where $n = 2(k - 1)$.

43. $D_w\left[\frac{1}{2k - 2}\cdot\frac{w}{(w^2 + 1)^{k-1}} + \frac{2k - 3}{2k - 2}\int \frac{dw}{(w^2 + 1)^{k-1}}\right] =$

$\frac{1}{2k - 2}\left[\frac{(w^2 + 1)^{k-1} - w(k - 1)(2w)(w^2 + 1)^{k-2}}{(w^2 + 1)^{2k-2}} + (2k - 3)\frac{1}{(w^2 + 1)^{k-1}}\right] =$

$\frac{1}{2k - 2}\left[\frac{(w^2 + 1)^{k-2}(w^2 + 1 - 2w^2k + 2w^2)}{(w^2 + 1)^{2k-2}} + \frac{2k - 3}{(w^2 + 1)^{k-1}}\right] = \frac{1}{2k - 2}\left[\frac{3w^2 + 1 - 2w^2k + (2k - 2)(w^2 + 1)}{(w^2 + 1)^k}\right] =$

$\frac{1}{2k - 2}\left(\frac{3w^2 + 1 - 2w^2k + 2kw^2 - 3w^2 + 2k - 3}{(w^2 + 1)^k}\right) =$

$\frac{1}{2k - 2}\left[\frac{2k - 2}{(w^2 + 1)^k}\right] = \frac{1}{(w^2 + 1)^k}.$

44. This fact will be shown by induction on k. First, if $k = 1$, then $\int \frac{dx}{x^2 + 1} = \tan^{-1}x + C$. Now, assume

for $k - 1$ that $\int \frac{dx}{(x^2+1)^{k-1}}$ can be expressed in terms of rational functions and the inverse tangent function. Then, $\int \frac{dx}{(x^2+1)^k} = \frac{1}{2k-2} \cdot \frac{x}{(x^2+1)^{k-1}} + \frac{2k-3}{2k-2}\int \frac{dx}{(x^2+1)^{k-1}}$ is a sum of a rational function and an integral which, by the induction hypothesis, can be expressed in terms of rational functions and the inverse tangent function. Hence, we have shown what was desired.

45. (a) $\int \frac{dw}{(w^2+1)^2} = \frac{1}{2(2)-2} \cdot \frac{w}{(w^2+1)^{2-1}} + \frac{2(2)-3}{2(2)-2}\int \frac{dw}{(w^2+1)^{2-1}} = \frac{1}{2}\left(\frac{w}{w^2+1}\right) + \frac{1}{2}\int \frac{dw}{w^2+1} = \frac{w}{2(w^2+1)} + \frac{1}{2}\tan^{-1} w + C.$

(b) $\int \frac{dw}{(w^2+1)^3} = \frac{1}{2(3)-2} \cdot \frac{w}{(w^2+1)^{3-1}} + \frac{2(3)-3}{2(3)-2}\int \frac{dw}{(w^2+1)^{3-1}} = \frac{w}{4(w^2+1)^2} + \frac{3}{4}\int \frac{dw}{(w^2+1)^2} = \frac{w}{4(w^2+1)^2} + \frac{3}{4}\left[\frac{w}{2(w^2+1)} + \frac{1}{2}\tan^{-1} w\right] + C$ where the second integral is evaluated by using part (a).

46. Every rational function can be decomposed into partial fractions, the denominators of which are powers of either linear or quadratic factors. The integral of a fraction with a power of a linear factor in the denominator can be expressed in terms of rational functions and logarithms (of absolute value). By Problems 40, 44, and 45 above, the integral of an expression with a power of a quadratic factor in the denominator can be expressed in terms of rational functions, the inverse tangent function, and logarithms (of absolute value). Therefore, the integral of a rational function can be expressed in terms of rational functions, inverse tangents, and logarithms (of absolute values).

47. Put $z = \sqrt[3]{a - bx}$. Then $z^3 = a - bx$ and $dz = \frac{1}{3}(a - bx)^{-2/3}(-b)dx$. Thus, $\frac{dx}{(a-bx)^{2/3}(c-x)} = -\frac{3}{b}\frac{dz}{c - \left(\frac{a-z^3}{b}\right)} = -3\frac{dz}{bc - a + z^3}$. Call $q^3 = bc - a$. Then $\frac{dx}{(a-bx)^{2/3}(c-x)} = \frac{-3dz}{q^3+z^3}$. Now $\frac{-3}{q^3+z^3}$ is a rational function of z.

48. $\int \frac{-3\,dz}{q^3+z^3} = \int \frac{-3}{(q+z)(q^2-qz+z^2)}\,dz$. Now $\frac{-3}{(q+z)(q^2-qz+z^2)} = \frac{A}{q+z} + \frac{Bz+C}{q^2-qz+z^2}$. $\frac{-3}{3q^2} = -\frac{1}{q^2} = A$. Now $-3 = -\frac{1}{q^2}(q^2 - qz + z^2) + (Bz+C)(q+z)$. $0 = -\frac{1}{q^2} + B$, so $B = \frac{1}{q^2}$; $0 = \frac{1}{q} + Bq + C = \frac{2}{q} + C$, and so $C = -\frac{2}{q}$. Therefore, $\int \frac{-3}{(q+z)(q^2-qz+z^2)}\,dz = \int \frac{-\frac{1}{q^2}}{q+z}\,dz + \int \frac{\frac{z}{q^2} - \frac{2}{q}}{q^2-qz+z^2}\,dz = -\frac{1}{q^2}\ln|q+z| + \int \frac{\frac{z}{q^2} - \frac{2}{q}}{(z - \frac{q}{2})^2 + \frac{3}{4}q^2}\,dz$. Now put $u = z - \frac{q}{2}$, so that $du = dz$. Then $\int \frac{\frac{z}{q^2} - \frac{2}{q}}{(z-\frac{q}{2})^2 + \frac{3}{4}q^2}\,dz = \frac{1}{q^2}\int \frac{u + \frac{q}{2} - 2q}{u^2 + \frac{3}{4}q^2}\,du = \frac{1}{q^2}\left[\frac{1}{2}\ln\left(u^2 + \frac{3}{4}q^2\right) - \frac{3}{4}q\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\frac{2u}{\sqrt{3}q}\right] + C = \frac{1}{q^2}\left[\frac{1}{2}\ln(z^2 - qz + q^2) - \frac{\sqrt{3}}{2}q\tan^{-1}\left(\frac{2z-q}{\sqrt{3}q}\right)\right] + C$. Hence, $\int \frac{-3\,dz}{q^3+z^3} = -\frac{1}{q^2}\ln|q+z| + \frac{1}{2q^2}\ln(z^2 - qz + q^2) - \frac{\sqrt{3}}{2q}\tan^{-1}\left(\frac{2z-q}{\sqrt{3}q}\right) + C = \ln\left|\frac{(z^2-qz+q^2)^{\frac{1}{2q^2}}}{(q+z)^{1/q^2}}\right| - \frac{\sqrt{3}}{2q}\tan^{-1}\left(\frac{2z-q}{\sqrt{3}q}\right) + C.$

Problem Set 8.7, page 527

1. Put $z = \sqrt{x}$, so that $z^2 = x$ and $dx = 2z\,dz$. Thus, $\int \frac{dx}{1-\sqrt{x}} = \int \frac{2z\,dz}{1-z} = \int -2\,dz + \int \frac{2}{1-z}\,dz =$

$-2z - 2\ln|1 - z| + C = -2\sqrt{x} - \ln(1 - \sqrt{x})^2 + C.$

2. Put $z = \sqrt{x}$, so that $z^2 = x$ and $2z\,dz = dx$. Thus, $\int \frac{dx}{4 + \sqrt{x}} = \int \frac{2z\,dz}{4 + z} = \int 2\,dz - \int \frac{8}{4 + z}\,dz = 2z -$ $8\ln|4 + z| + C = 2\sqrt{x} - 8\ln(4 + \sqrt{x}) + C.$

3. Put $z = \sqrt[3]{x}$, so that $z^3 = x$ and $3z^2dz = dx$. Thus, $\int \frac{dx}{1 + \sqrt[3]{x}} = \int \frac{3z^2dz}{1 + z} = \int (3z - 3)dz + \int \frac{3}{1 + z}\,dz =$ $\frac{3}{2}z^2 - 3z + 3\ln|1 + z| + C = \frac{3}{2}x^{2/3} - 3x^{1/3} +$ $3\ln|1 + \sqrt[3]{x}| + C.$

4. Put $z = \sqrt[3]{x}$, so that $x = z^3$ and $dx = 3z^2dz$. Thus, $\int \frac{x\,dx}{1 - \sqrt[3]{x}} = \int \frac{z^3(3z^2dz)}{1 - z} = -3\int (z^4 + z^3 + z^2 + z + 1)dz +$ $\int \frac{3}{1 - z}\,dz = -3(\frac{z^5}{5} + \frac{z^4}{4} + \frac{z^3}{3} + \frac{z^2}{2} + z) -$ $3\ln|1 - z| + C = -3(\frac{x^{5/3}}{5} + \frac{x^{4/3}}{4} + \frac{x}{3} + \frac{x^{2/3}}{2} + x^{1/3}) -$ $3\ln|1 - \sqrt[3]{x}| + C.$

5. Put $z = \sqrt{x}$, so that $x = z^2$ and $dx = 2z\,dz$. Thus, $\int \frac{x\,dx}{2 + \sqrt{x}} = \int \frac{z^2(2z\,dz)}{2 + z} = \int (2z^2 - 4z + 8)dz - \int \frac{16}{z + 2}dz =$ $\frac{2}{3}z^3 - 2z^2 + 8z - 16\ln|z + 2| + C = \frac{2}{3}x^{3/2} - 2x +$ $8\sqrt{x} - 16\ln|\sqrt{x} + 2| + C.$

6. Put $z = \sqrt[6]{x}$, so that $x = z^6$ and $dx = 6z^5dz$. Thus, $\int \frac{2\sqrt{x}\,dx}{1 + \sqrt[3]{x}} = \int \frac{2z^3\cdot(6z^5dz)}{1 + z^2} =$ $\int (12z^6 - 12z^4 + 12z^2 - 12)dz + \int \frac{12}{1 + z^2}\,dz = \frac{12}{7}z^7 -$ $\frac{12}{5}z^5 + 4z^3 - 12z + 12\tan^{-1}z + C = \frac{12}{7}x^{7/6} -$ $\frac{12}{5}x^{5/6} + 4x^{1/2} - 12x^{1/6} + 12\tan^{-1}(x^{1/6}) + C.$

7. Put $z = \sqrt{2x^2 - 1}$, so that $z^2 = 2x^2 - 1$, $2z\,dz = 4x\,dx$, and $x\,dx = \frac{z}{2}\,dz$. Now $\int x^3\sqrt{2x^2 - 1}\,dx =$ $\int (\frac{z^2 + 1}{2})z(\frac{z}{2})dz = \frac{1}{4}\int (z^4 + z^2)dz = \frac{1}{4}(\frac{z^5}{5} + \frac{z^3}{3}) + C =$ $\frac{(2x^2 - 1)^{5/2}}{20} + \frac{(2x^2 - 1)^{3/2}}{12} + C.$

8. Put $z = \sqrt{5 - 2x^2}$, so that $x^2 = \frac{5 - z^2}{2}$ and $2z\,dz =$ $-4x\,dx$. Thus, $\int x^5\sqrt{5 - 2x^2}\,dx = \int x^4\sqrt{5 - 2x^2}(x\,dx) =$ $\int (\frac{5 - z^2}{2})^2 z(-\frac{z\,dz}{2}) = -\frac{1}{8}\int (z^6 - 10z^4 + 25z^2)dz =$ $-\frac{1}{8}(\frac{z^7}{7} - 2z^5 + \frac{25}{3}z^3) + C = -\frac{1}{8}[\frac{(5 - 2x^2)^{7/2}}{7} -$ $2(5 - 2x^2)^{5/2} + \frac{25}{3}(5 - 2x^2)^{3/2}] + C.$

9. Put $z = \sqrt[3]{3x + 1}$, so that $x = \frac{z^3 - 1}{3}$ and $dx = z^2dz$. Thus, $x^3\sqrt{3x + 1}\,dx = \int (\frac{z^3 - 1}{3})z(z^2dz) =$ $\frac{1}{3}\int (z^6 - z^3)dz = \frac{1}{3}(\frac{z^7}{7} - \frac{z^4}{4}) + C = \frac{(3x + 1)^{7/3}}{21} -$ $\frac{(3x + 1)^{4/3}}{12} + C.$

10. Put $z = \sqrt{1 + 2x^5}$, so that $x^5 = \frac{z^2 - 1}{2}$ and $5x^4dx =$ $z\,dz$. Thus, $\int x^9\sqrt{1 + 2x^5}\,dx = \int x^5\sqrt{1 + 2x^5}\,x^4dx =$ $\int (\frac{z^2 - 1}{2})\frac{z^2}{5}\,dz = \frac{1}{10}\int (z^4 - z^2)dz = \frac{1}{10}(\frac{z^5}{5} - \frac{z^3}{3}) + C$ $\frac{(1 + 2x^5)^{5/2}}{50} - \frac{(1 + 2x^5)^{3/2}}{30} + C.$

11. Put $z = (4x + 1)^{1/2}$, so that $x = \frac{z^2 - 1}{4}$ and $dx = \frac{z}{2}\,dz$ Thus, $\int x^2(4x + 1)^{3/2}dx = \int [\frac{(z^2 - 1)}{4}]^2(z^3)(\frac{z}{2}\,dz) =$ $\frac{1}{32}\int (z^8 - 2z^6 + z^4)dz = \frac{1}{32}(\frac{z^9}{9} - \frac{2}{7}z^7 + \frac{z^5}{5}) + C =$ $\frac{(4x + 1)^{9/2}}{288} - \frac{(4x + 1)^{7/2}}{112} + \frac{(4x + 1)^{5/2}}{160} + C.$

12. Put $z = (1 + x)^{1/3}$, so that $x = z^3 - 1$ and $dx =$ $3z^2dz$. Thus, $\int x(1 + x)^{2/3}dx = \int (z^3 - 1)(z^2)(3z^2dz)$ $\int (3z^7 - 3z^4)dz = \frac{3}{8}z^8 - \frac{3}{5}z^5 + C = \frac{3}{8}(1 + x)^{8/3} -$ $\frac{3}{5}(1 + x)^{5/3} + C.$

13. Put $z = \tan\frac{x}{2}$, so that $\sin x = \frac{2z}{1 + z^2}$ and $dx =$ $\frac{2\,dz}{1 + z^2}$. Thus, $\int \frac{dx}{3 + 5\sin x} = \int \frac{\frac{2}{1 + z^2}\,dz}{3 + \frac{10z}{1 + z^2}} =$

$\int \frac{2}{3z^2 + 10z + 3}\,dz = \int \frac{2}{(3z + 1)(z + 3)}\,dz =$

$\int \frac{(\frac{3}{4})}{3z + 1}\,dz = \int \frac{-(\frac{1}{4})}{z + 3}\,dz = \frac{1}{4}\ln|3z + 1| -$

$\frac{1}{4}\ln|z + 3| + C = \frac{1}{4}\ln\left|\frac{3z + 1}{z + 3}\right| + C =$

$\frac{1}{4}\ln\left|\frac{3\tan\frac{x}{2} + 1}{\tan\frac{x}{2} + 3}\right| + C.$

4. Put $z = \tan \frac{t}{2}$, so that $\sin t = \frac{2z}{1 + z^2}$, $\cos t = \frac{1 - z^2}{1 + z^2}$, and $dt = \frac{2\,dz}{1 + z^2}$. Thus, $\int \frac{\sin t}{1 + \cos t}\,dt =$

$$\int \frac{\frac{2z}{1 + z^2}}{1 + \frac{1 - z^2}{1 + z^2}} \left(\frac{2}{1 + z^2}\right)dz = \int \frac{2z}{1 + z^2}\,dz = \ln(1 + z^2) + C =$$

$\ln\,(1 + \tan^2 \frac{t}{2}) + C.$

5. Put $z = \tan \frac{x}{2}$, so that $\sin x = \frac{2z}{1 + z^2}$, $\cos x = \frac{1 - z^2}{1 + z^2}$, and $dx = \frac{2\,dz}{1 + z^2}$. Thus, $\int \frac{\cos x\,dx}{\sin x(\cos x + 1)} =$

$$\int \frac{\left(\frac{1 - z^2}{1 + z^2}\right)\left(\frac{2\,dz}{1 + z^2}\right)}{\left(\frac{2z}{1 + z^2}\right)\left(\frac{1 - z^2}{1 + z^2} + 1\right)} = \int \frac{2(1 - z^2)}{2z(2)}\,dz =$$

$$\frac{1}{2}\int \left(\frac{1}{z} - z\right)dz = \frac{1}{2}\left(\ln|z| - \frac{z^2}{2}\right) + C =$$

$$\frac{1}{2}\ln|\tan \tfrac{x}{2}| - \frac{1}{4}\tan^2 \tfrac{x}{2} + C.$$

. Put $z = \tan \frac{x}{2}$. $\int \frac{dx}{\sin x + \sqrt{3}\cos x} =$

$$\int \frac{\frac{2\,dz}{1 + z^2}}{\frac{2z}{1 + z^2} + \sqrt{3}\left(\frac{1 - z^2}{1 + z^2}\right)} = \int \frac{2\,dz}{2z + \sqrt{3} - \sqrt{3}\,z^2} =$$

$$\int \frac{-2\sqrt{3}\,dz}{3z^2 - 2\sqrt{3}\,z - 3} = \int \frac{-2\sqrt{3}\,dz}{(3z + \sqrt{3})(z - \sqrt{3})} =$$

$$\int \frac{3/2}{3z + \sqrt{3}}\,dz + \int \frac{-\frac{1}{2}\,dz}{z - \sqrt{3}} = \frac{1}{2}\ln|3z + \sqrt{3}| -$$

$$\frac{1}{2}\ln|z - \sqrt{3}| + C = \frac{1}{2}\ln|3\tan \tfrac{x}{2} + \sqrt{3}| -$$

$$\frac{1}{2}\ln|\tan \tfrac{x}{2} - \sqrt{3}| + C.$$

. Let $z = \tan \frac{\theta}{2}$, so that $\sin\theta = \frac{2z}{1 + z^2}$, $\cos\theta = \frac{1 - z^2}{1 + z^2}$, and $d\theta = \frac{2\,dz}{1 + z^2}$. Thus, $\int \frac{d\theta}{\tan\theta - \sin\theta} =$

$$\int \frac{2\,dz}{(1 + z^2)\left(\frac{2z}{1 - z^2} - \frac{2z}{1 + z^2}\right)} = \frac{1}{2}\int \frac{1 - z^2}{z^3}\,dz =$$

$$\frac{1}{2}\int z^{-3}dz - \frac{1}{2}\int \frac{1}{z}\,dz = -\frac{1}{4}z^{-2} - \frac{1}{2}\ln|z| + C =$$

$$\frac{-1}{4\tan^2 \frac{\theta}{2}} - \frac{1}{2}\ln|\tan \tfrac{\theta}{2}| + C.$$

18. Let $z = \tan(u/2)$. Then $\cos u = \frac{1 - z^2}{1 + z^2}$, $du = \frac{2\,dz}{1 + z^2}$, and $\int \frac{du}{(1 - \cos u)^2} = \frac{1}{2}\int \frac{1 + z^2}{z^4}\,dz =$

$$\frac{1}{2}\int z^{-4}dz + \frac{1}{2}\int z^{-2}dz = -\frac{1}{6}z^{-3} - \frac{1}{2}z^{-1} + C =$$

$$-\frac{1}{6\tan^3 \frac{u}{2}} - \frac{1}{2\tan \frac{u}{2}} + C.$$

19. $$\int \frac{dx}{x\sqrt{1 + x^2}} = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{1 + \frac{1}{t^2}}} = \int \frac{-dt}{\sqrt{t^2 + 1}} = \sinh^{-1}t =$$

$$-\sinh^{-1}\left(\tfrac{1}{x}\right) = -\operatorname{csch}^{-1}x + C.$$

20. $$\int \frac{dx}{x^2\sqrt{x^2 + 2x}} = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t^2}\sqrt{\frac{1}{t^2} + \frac{2}{t}}} = \int \frac{-t\,dt}{\sqrt{1 + 2t}}.$$ Now put

$u = \sqrt{1 + 2t}$, so that $u^2 = 1 + 2t$, $t = \frac{u^2 - 1}{2}$, and

$2u\,du = 2\,dt$. Thus, $\int \frac{-t\,dt}{\sqrt{1 + 2t}} = \int \frac{-\left(\frac{u^2 - 1}{2}\right)u\,du}{u} =$

$\frac{1}{2}u - \frac{u^3}{6} + C = \frac{\sqrt{1 + 2t}}{2} - \frac{(1 + 2t)^{3/2}}{6} + C$. Hence,

$$\int \frac{dx}{x^2\sqrt{x^2 + 2x}} = \frac{\sqrt{1 + \frac{2}{x}}}{2} - \frac{\left(1 - \frac{2}{x}\right)^{3/2}}{6} + C.$$

21. Let $u = \sqrt[4]{t}$ and $u^2 = \sqrt{t}$. Since $u^4 = t$, $4u^3du = dt$.

Then $\int \frac{1 - \sqrt{t}}{1 + \sqrt[4]{t}}\,dt = \int \frac{1 - u^2}{1 + u} \cdot 4u^3du = 4\int (1 - u)u^3du =$

$$u^4 - \frac{4u^5}{5} + C = t - \frac{4}{5}t^{5/4} + C.$$

22. Let $v = y^{1/5}$. Then $\int \frac{dy}{y - y^{3/5}} = \int \frac{5v^4dv}{v^5 - v^3} =$

$$\frac{5}{2}\int \frac{2v}{v^2 - 1}\,dv = \frac{5}{2}\ln|v^2 - 1| + C =$$

$$\frac{5}{2}\ln|y^{2/5} - 1| + C.$$

23. Put $z = \sqrt[4]{x}$, so that $x = z^4$ and $dx = 4z^3dz$. Thus,

$$\int \frac{dx}{\sqrt[4]{x} + \sqrt{x}} = \int \frac{4z^3dz}{z + z^2} = 4\int (z - 1)dz + 4\int \frac{dz}{1 + z} =$$

$$2z^2 - 4z + 4\ln(1 + z) + C =$$

$$2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(1 + \sqrt[4]{x}) + C.$$

24. Put $z = \sqrt{x + 1}$, so that $x = z^2 - 1$ and $dx = 2z\,dz$.

Thus, $\int \frac{dx}{1+\sqrt{x+1}} = \int \frac{2z\,dz}{1+z} = \int 2\,dz - \int \frac{2}{1+z}\,dz =$
$2z - 2\ln|1+z| + C = 2\sqrt{x+1} - 2\ln(1+\sqrt{x+1}) + C.$

25. Put $z = \sqrt[4]{1-x}$, so that $x = 1 - z^4$ and $dx = -4z^3dz$. Thus, $\int \frac{x\,dx}{\sqrt[4]{1-x}} = \int \frac{1-z^4}{z}(-4z^3)dz = -4\int (z^2 - z^6)dz =$ $-4(\frac{z^3}{3} - \frac{z^7}{7}) + C = -4[\frac{(1-x)^{3/4}}{3} - \frac{(1-x)^{7/4}}{7}] + C =$ $4[\frac{(1-x)^{7/4}}{7} - \frac{(1-x)^{3/4}}{3}] + C.$

26. Put $z = (2-3x^2)^{1/4}$, so that $x^2 = \frac{2-z^4}{3}$ and $2x\,dx =$ $\frac{4}{3}z^3dz$. Thus, $\int \frac{x^3dx}{(2-3x^2)^{3/4}} = \int \frac{x^2(x)dx}{(2-3x^2)^{3/4}} =$
$\int \frac{(\frac{2-z^4}{3})(\frac{-2z^3}{3})dz}{z^3} = -\frac{2}{9}\int(2-z^4)dz = -\frac{2}{9}(2z - \frac{z^5}{5}) + C =$
$-\frac{2}{9}[2\sqrt[4]{2-3x^2} - \frac{(2-3x^2)^{5/4}}{5}] + C = \frac{-4\sqrt[4]{2-3x^2}}{9} +$ $\frac{2(2-3x^2)^{5/4}}{45} + C.$

27. Put $z = x^{1/4}$, so that $x = z^4$ and $dx = 4z^3dz$. Thus, $\int \frac{dx}{x^{1/2} - x^{3/4}} = \int \frac{4z^3dz}{z^2 - z^3} = \int \frac{4z\,dz}{1-z} = \int -4\,dz + \int \frac{4}{1-z}dz =$ $-4z - 4\ln|1-z| + C = -4x^{1/4} - 4\ln|1-x^{1/4}| + C =$ $-4(x^{1/4} + \ln|1-x^{1/4}|) + C.$

28. Put $z = \sqrt{1-e^x}$, so that $z^2 = 1 - e^x$ and $2z\,dz =$ $-e^x\,dx$. Thus, $\int e^x\sqrt{1-e^x}\,dx = \int z(-2z\,dz) =$ $\int -2z^2dz = \frac{2}{3}z^3 + C = -\frac{2}{3}(1-e^x)^{3/2} + C.$

29. Put $z = \sqrt{1+e^x}$, so that $z^2 = 1 + e^x$ and $2z\,dz =$ e^xdx. Therefore, $\int e^{2x}\sqrt{1+e^x}\,dx =$ $\int (z^2-1)z(2z\,dz) = \int(2z^4 - 2z^2)dz = 2(\frac{z^5}{5} - \frac{z^3}{3}) + C =$ $2[\frac{(1+e^x)^{5/2}}{5} - \frac{(1+e^x)^{3/2}}{3}] + C.$

30. Put $z = \sqrt{1+\sin x}$, so that $z^2 = 1 + \sin x$ and $2z\,dz = \cos x\,dx$. Thus, $\int \sin x \cos x\sqrt{1+\sin x}\,dx =$ $\int (z^2-1)z(2z\,dz) = 2\int(z^4 - z^2)dz = 2(\frac{z^5}{5} - \frac{z^3}{3}) + C =$ $\frac{2(1+\sin x)^{5/2}}{5} - \frac{2(1+\sin x)^{3/2}}{3} + C.$

31. Put $z = \sqrt{\frac{1-x}{x}}$, so that $z^2 = \frac{1-x}{x}$ and $2z\,dz =$ $-\frac{1}{x^2}\,dx$. Now $\int\sqrt{\frac{1-x}{x}}\,dx = \int\sqrt{\frac{1-x}{x}}\,(\frac{x^2}{x^2})dx =$ $\int z(\frac{1}{z^2+1})^2(-2z\,dz) = -2\int\frac{z^2}{(z^2+1)^2}\,dz$. Now put $z = \tan\theta$, so $dz = \sec^2\theta\,d\theta$ and $-2\int\frac{z^2}{(z^2+1)^2}\,dz$ $= -2\int\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta =$
$-2\int\sin^2\theta\,d\theta = -\int(1-\cos 2\theta)d\theta = \frac{\sin 2\theta}{2} - \theta + C =$
$\sin\theta\cos\theta - \theta + C = \frac{z}{1+z^2} - \tan^{-1}z + C =$
$x\sqrt{\frac{1-x}{x}} - \tan^{-1}\sqrt{\frac{1-x}{x}} + C.$

32. Put $z = \sqrt[3]{\frac{1-x}{1+x}}$, so that $z^3 = \frac{1-x}{1+x}$ and $3z^2dz =$ $\frac{-2}{(1+x)^2}\,dx$. Thus, $\int\frac{1}{(1+x)^2}\sqrt[3]{\frac{1-x}{1+x}}\,dx =$
$\int -\frac{3}{2}z^2(z)dz = -\frac{3}{2}\int z^3dz = -\frac{3}{2}\cdot\frac{z^4}{4} + C = -\frac{3}{8}(\frac{1-x}{1+x})^{4/3} +$

33. Put $u = w + 32$, so that $\int\frac{w}{\sqrt[5]{w+32}} = u^{4/5} - 32u^{-1/5}$ and $dw = du$. Now $\int\frac{w\,dw}{\sqrt[5]{w+32}} = \int u^{4/5}du - 32\int u^{-1/5}du =$
$\frac{5}{9}u^{9/5} - 40u^{4/5} + C = \frac{5}{9}(w+32)^{9/5} -$ $40(w+32)^{4/5} + C.$

34. Put $z = \sqrt{\frac{1}{3x+1}}$, so that $z^2 = \frac{1}{3x+1}$, $2z\,dz =$ $-\frac{3}{(3x+1)^2}\,dx$, and $x = \frac{1-z^2}{3z^2}$. Thus,
$\int\frac{x}{(3x+1)^2}\sqrt{\frac{1}{3x+1}}\,dx = \int\frac{1-z^2}{3z^2}(z)(-\frac{2}{3}z\,dz) =$
$-\frac{2}{9}\int(1-z^2)dz = -\frac{2}{9}(z - \frac{z^3}{3}) + C =$
$-\frac{2}{9}\sqrt{\frac{1}{3x+1}}\,(1 - \frac{1}{3}\cdot\frac{1}{3x+1}) + C =$
$-\frac{2}{27}\sqrt{\frac{1}{3x+1}}\,\frac{9x+2}{3x+1} + C.$

35. Put $z = \tan\frac{x}{2}$. $\int\frac{dx}{1+\sin x+\cos x} =$
$\int\frac{\frac{2\,dz}{1+z^2}}{1 + \frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} = \int\frac{2\,dz}{2(z+1)} = \ln|z+1| + C =$
$\ln|1 + \tan\frac{x}{2}| + C.$

36. Put $z = \tan\frac{t}{2}$. $\int\frac{dt}{\sin t + \cos t} = \int\frac{\frac{2\,dz}{1+z^2}}{\frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} =$
$\int\frac{-2}{z^2 - 2z - 1}\,dz = \int\frac{-2}{(z-a)(z-b)}\,dz$, where

$a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2} = \int \frac{\left(\frac{-2}{a-b}\right)}{z-a}\,dz + \int \frac{\left(\frac{-2}{b-a}\right)}{z-b}\,dz = \int \frac{\left(-\frac{1}{\sqrt{2}}\right)}{z-a}\,dz + \int \frac{\left(\frac{1}{\sqrt{2}}\right)}{z-b}\,dz = -\frac{1}{\sqrt{2}} \ln|z-a| + \frac{1}{\sqrt{2}} \ln|z-b| + C = \frac{1}{\sqrt{2}} \ln\left|\frac{\tan\frac{t}{2} - 1 + \sqrt{2}}{\tan\frac{t}{2} - 1 - \sqrt{2}}\right| + C.$

7. Put $z = \tan\frac{t}{2}$. $\int \frac{\sec t}{1 + \sin t}\,dt = \int \frac{1}{\cos t(1+\sin t)}\,dt = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)\left(1 + \frac{2z}{1+z^2}\right)} = \int \frac{-2(1+z^2)}{(z-1)(z+1)^3}\,dz = 2\left[-\int \frac{(\frac{1}{4})}{z-1}\,dz + \int \frac{(\frac{1}{4})}{z+1}\,dz - \int \frac{(\frac{1}{2})}{(z+1)^2}\,dz + \int \frac{1}{(z+1)^3}\,dz\right] = 2\left(-\tfrac{1}{4}\ln|z-1| + \tfrac{1}{4}\ln|z+1| + \frac{1}{2(z+1)} - \frac{1}{2(z+1)^2}\right) + C = \ln\frac{\sqrt{|z+1|}}{\sqrt{|z-1|}} + \frac{1}{z+1} - \frac{1}{(z+1)^2} + C = \ln\frac{\sqrt{|\tan\frac{t}{2} + 1|}}{\sqrt{|\tan\frac{t}{2} - 1|}} + \frac{1}{\tan\frac{t}{2} + 1} - \frac{1}{(\tan\frac{t}{2} + 1)^2} + C.$

. Put $z = \tan\frac{u}{2}$. $\int \frac{du}{2\csc u - \sin u} = \int \frac{\frac{2\,dz}{1+z^2}}{2\left(\frac{1+z^2}{2z}\right) - \left(\frac{2z}{1+z^2}\right)} = \int \frac{2z}{z^4 + 1}\,dz = \int \frac{2z}{(z^2 + \sqrt{2}z + 1)(z^2 - \sqrt{2}z + 1)}\,dz = \int \frac{-\frac{\sqrt{2}}{2}}{z^2 + \sqrt{2}z + 1}\,dz + \int \frac{\frac{\sqrt{2}}{2}}{z^2 - \sqrt{2}z + 1}\,dz = \int \frac{-\frac{\sqrt{2}}{2}}{(z + \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}\,dz + \int \frac{\frac{\sqrt{2}}{2}}{(z - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}$. Now put $v = z + \frac{1}{\sqrt{2}}$, so that $dv = dz$. Then $\int \frac{-\frac{\sqrt{2}}{2}}{(z + \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}\,dz = \int \frac{-\frac{\sqrt{2}}{2}}{v^2 + \frac{1}{2}}\,dv = -\frac{\sqrt{2}}{2}(\sqrt{2})\tan^{-1}\sqrt{2}\,v = -\tan^{-1}\sqrt{2}\left(z + \frac{1}{\sqrt{2}}\right) = -\tan^{-1}(\sqrt{2}z + 1) + C$. Similarly, $\int \frac{\frac{\sqrt{2}}{2}}{(z - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}\,dz = \tan^{-1}(\sqrt{2}z - 1) + C$. Thus, $\int \frac{du}{2\csc u - \sin u} = -\tan^{-1}(\sqrt{2}\tan\frac{u}{2} + 1) + \tan^{-1}(\sqrt{2}\tan\frac{u}{2} - 1) + C.$

39. Put $z = \sqrt{x-1}$, so that $z^2 = x - 1$ and $2z\,dz = dx$. Thus, $\int_1^4 x\sqrt{x-1}\,dx = \int_0^{\sqrt{3}} (z^2+1)z(2z\,dz) = \int_0^{\sqrt{3}} (2z^4 + 2z^2)dz = 2\left(\frac{z^5}{5} + \frac{z^3}{3}\right)\Big|_0^{\sqrt{3}} = 2\left(\frac{9\sqrt{3}}{5} + \frac{3\sqrt{3}}{3}\right) = \frac{28\sqrt{3}}{5}$.

40. Put $z = \sqrt{2x+3}$, so that $z^2 = 2x + 3$ and $2z\,dz = 2\,dx$. Thus, $\int_3^{11} x\sqrt{2x+3}\,dx = \int_3^5 \left(\frac{z^2-3}{2}\right)z^2dz = \frac{1}{2}\left(\frac{z^5}{5} - z^3\right)\Big|_3^5 = \frac{1}{2}\left(625 - 125 - \frac{243}{5} + 27\right) = \frac{1196}{5}$.

41. Put $z = \sqrt{x}$, so that $z^2 = x$ and $2z\,dz = dx$. Thus, $\int_1^4 \frac{4 - \sqrt{x}}{1 + x}\,dx = \int_1^2 \frac{4 - z}{1 + z^2}(2z\,dz) = \int_1^2 \frac{8z - 2z^2}{1 + z^2}\,dz = \int_1^2 -2\,dz + \int_1^2 \frac{2 + 8z}{z^2 + 1}\,dz = -2z\Big|_1^2 + 2\tan^{-1}z\Big|_1^2 + 4\ln(z^2+1)\Big|_1^2 = -4 + 2 + 2\tan^{-1}2 - 2\tan^{-1}1 + 4\ln 5 - 4\ln 2 = 2\left(-1 + \tan^{-1}2 - \frac{\pi}{4} + 2\ln\frac{5}{2}\right)$.

42. Put $z = \sqrt{x}$, so that $z^2 = x$ and $2z\,dz = dx$. Thus, $\int_4^9 \frac{1 - \sqrt{x}}{1 + \sqrt{x}}\,dx = \int_2^3 \frac{1 - z}{1 + z}(2z\,dz) = \int_2^3 (-2z + 4)dz - \int_2^3 \frac{4}{1+z}\,dz = -z^2 + 4z\Big|_2^3 - 4\ln|1+z|\Big|_2^3 = -9 + 12 + 4 - 8 - 4\ln 4 + 4\ln 3 = 4\ln\left(\frac{3}{4}\right) - 1$.

43. Put $z = \sqrt{1-x}$, so that $z^2 = 1 - x$ and $2z\,dz = -dx$. Thus, $\int_{-3}^{-1} \frac{x^2dx}{\sqrt{1-x}} = \int_2^{\sqrt{2}} \frac{(1-z^2)^2}{z}(-2z\,dz) = -2\int_2^{\sqrt{2}} (1 - 2z^2 + z^4)dz = 2\left(-z + \frac{2}{3}z^3 - \frac{z^5}{5}\right)\Big|_2^{\sqrt{2}} = 2\left(-\sqrt{2} + \frac{2}{3}(2)\sqrt{2} - \frac{4\sqrt{2}}{5} + 2 - \frac{16}{3} + \frac{32}{5}\right) = \frac{92 - 14\sqrt{2}}{15}$.

44. Put $z = \sqrt{3x+2}$, so that $z^2 = 3x + 2$ and $2z\,dz = 3\,dx$. Thus, $\int_1^{7/3} \frac{1 - \sqrt{3x+2}}{1 + \sqrt{3x+2}}\,dx =$

$\int_{\sqrt5}^{3} (\frac{1 - z}{1 + z})(\frac{2}{3}z\, dz) = \frac{2}{3}[\int_{\sqrt5}^{3}(2 - z)dz + \int_{\sqrt5}^{3} - \frac{2}{1 + z}dz] =$

$\frac{2}{3}[(2z\ \frac{z^2}{2}\Big|_{\sqrt5}^{3} - 2\ \ln\ |1 + z|\Big|_{\sqrt5}^{3}] =$

$\frac{2}{3}[6 - \frac{9}{2} - 2\sqrt5 + \frac{5}{2} - 2\ \ln\ 4 + 2\ \ln\ (1 + \sqrt5)] =$

$\frac{4}{3}(2 - \sqrt5 + \ln \frac{1 + \sqrt5}{4})$.

45. Put $u = 2 + \sin x$, so that $du = \cos x\, dx$ and $\int_0^{\pi/2} \frac{\cos x\, dx}{2 + \sin x} = \int_2^3 \frac{du}{u} = \ln |u|\Big|_2^3 = \ln 3 - \ln 2 =$ $\ln \frac{3}{2}$.

46. Put $u = \sqrt{t^4 + 1}$, so that $u^2 = t^4 + 1$ and $2u\, du = 4t^3 dt$. Thus, $\frac{dt}{t} = \frac{u\, du}{2t^4} = \frac{u\, du}{2(u^2 - 1)}$, and

$\int_1^2 \frac{\sqrt{t^4 + 1}}{t}\, dt = \int_{\sqrt2}^{\sqrt{17}} \frac{u^2 du}{2(u^2 - 1)} = \frac{1}{2}\int_{\sqrt2}^{\sqrt{17}} du +$

$\frac{1}{2}\int_{\sqrt2}^{\sqrt{17}} \frac{du}{u^2 - 1} = \frac{1}{2}\, u\Big|_{\sqrt2}^{\sqrt{17}} + \frac{1}{4}\int_{\sqrt2}^{\sqrt{17}} \frac{du}{u - 1} -$

$\frac{1}{4}\int_{\sqrt2}^{\sqrt{17}} \frac{du}{u + 1} = \frac{\sqrt{17} - \sqrt2}{2} + \frac{1}{4}(\ln |u - 1| -$

$\ln |u + 1|)\Big|_{\sqrt2}^{\sqrt{17}} = \frac{\sqrt{17} - \sqrt2}{2} + \frac{1}{4}\ln \left|\frac{u - 1}{u + 1}\right|\Big|_{\sqrt2}^{\sqrt{17}} =$

$\frac{\sqrt{17} - \sqrt2}{2} + \frac{1}{4}\ln\left|\frac{\sqrt{17} - 1}{\sqrt{17} + 1}\right| - \frac{1}{4}\ln\left|\frac{\sqrt2 - 1}{\sqrt2 + 1}\right| = \frac{\sqrt{17} - \sqrt2}{2} +$

$\frac{1}{4}\ln \frac{(\sqrt{17} - 1)(\sqrt2 + 1)}{(\sqrt{17} + 1)(\sqrt2 - 1)}$.

47. Put $z = \tan \frac{x}{2}$. $\int_{\pi/3}^{\pi/2} \frac{dx}{\csc x - \cot x} =$

$\int_{\pi/3}^{\pi/2} \frac{\sin x}{1 - \cos x}\, dx = \int_{1/\sqrt3}^{1} \frac{\frac{2z}{1 + z^2}}{1 - \frac{1 - z^2}{1 + z^2}}\ \frac{2\, dz}{1 + z^2} =$

$\int_{1/\sqrt3}^{1} \frac{4z}{(1 + z^2)2z^2}\, dz = \int_{1/\sqrt3}^{1} \frac{2}{z(1 + z^2)}\, dz =$

$\int_{1/\sqrt3}^{1} \frac{2}{z}\, dz - \int_{1/\sqrt3}^{1} \frac{2z}{z^2 + 1}\, dz =$

$(2 \ln |z| - \ln (z^2 + 1))\Big|_{1/\sqrt3}^{1} = -\ln 2 - 2 \ln \frac{1}{\sqrt3} +$ $\ln \frac{4}{3} = -\ln 2 + \ln 3 + \ln 2^2 - \ln 3 = \ln 2$.

48. $\int_2^3 \frac{dx}{x\sqrt{3x^2 - 2x - 1}} = \int_{1/2}^{1/3} \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{3}{t^2} - \frac{2}{t} - 1}} =$

$\int_{1/2}^{1/3} \frac{-dt}{\sqrt{3 - 2t - t^2}} = \int_{1/2}^{1/3} \frac{-dt}{\sqrt{4 - (t + 1)^2}} =$

$-\sin^{-1}(\frac{t + 1}{2})\Big|_{1/2}^{1/3} = \sin^{-1}(\frac{3}{4}) - \sin^{-1}(\frac{2}{3})$.

49. (a) $\cosh x = 2 \cosh^2 \frac{x}{2} - 1 = \frac{2}{\operatorname{sech}^2 \frac{x}{2}} - 1 =$

$\frac{2}{1 - \tanh^2 x} - 1 = \frac{2}{1 - z^2} - 1 =$

$\frac{2 - 1 + z^2}{1 - z^2} = \frac{1 + z^2}{1 - z^2}$.

(b) $\sinh x = 2 \sinh x \cosh x = \frac{2 \sinh x}{\cosh x} \cosh^2 x =$

$\tanh x(\cosh x + 1) = z(\frac{1 + z^2}{1 - z^2} + 1) = \frac{2z}{1 - z^2}$.

(c) Since $z = \tanh \frac{x}{2}$, then $dz = \operatorname{sech}^2 \frac{x}{2}(\frac{dx}{2}) =$

$(1 - \tanh^2 \frac{x}{2})(\frac{dx}{2}) = \frac{1 - z^2}{2}\, dx$. Hence, $dx = \frac{2\, dz}{1 - z^2}$

50. Put $z = \tanh \frac{x}{2}$. $\int \frac{dx}{1 - \sinh x} = \int \frac{\frac{2\, dz}{1 - z^2}}{1 - \frac{2z}{1 - z^2}} =$

$\int \frac{-2\, dz}{z^2 + 2z - 1} = \int \frac{-2}{(z - a)(z - b)}\, dz$ where $a = -1 + \sqrt2$,

$b = -1 - \sqrt2$. So $\int \frac{-2}{(z-a)(z-b)} dz = \int \frac{-\frac{1}{\sqrt2}}{z-a} dz + \int \frac{\frac{1}{\sqrt2}}{z-b}\, dz$

$-\frac{1}{\sqrt2}\ln|z - (-1 + \sqrt2)| + \frac{1}{\sqrt2}\ln|z - (-1 - \sqrt2)| + C$

$\frac{1}{\sqrt2}\ln\left|\frac{\tanh \frac{x}{2} + 1 + \sqrt2}{\tanh \frac{x}{2} + 1 - \sqrt2}\right| + C$.

51. $\int \frac{dx}{\cosh x - \sinh x} = \int \frac{\frac{2\, dz}{1 - z^2}}{\frac{1 + z^2}{1 - z^2} - \frac{2z}{1 - z^2}} = \int \frac{2\, dz}{z^2 - 2z + 1} =$

$\int \frac{2\, dz}{(z - 1)^2} = -\frac{2}{z - 1} + C = \frac{-2}{\tanh \frac{x}{2} - 1} + C =$

$\frac{2}{1 - \tanh \frac{x}{2}} + C$.

52. $\int \frac{\tanh x}{1 + \cosh x}\, dx = \int \frac{\frac{\sinh x}{\cosh x}}{1 + \cosh x}\, dx =$

$\int \frac{\frac{2z}{1 - z^2}(\frac{1 - z^2}{1 + z^2})\frac{2\, dz}{1 - z^2}}{1 + \frac{1 + z^2}{1 - z^2}} = \int \frac{2z\, dz}{1 + z^2} = \ln(1 + z^2) + C =$

$\ln\ (1 + \tanh^2 \frac{x}{2}) + C.$

53. $A = \int_0^9 \frac{5x}{1 + \sqrt{x}}\, dx$. Put $z = \sqrt{x}$, so that $z^2 = x$ and $2z\, dz = dx$. Thus, $A = \int_0^3 \frac{5z^2}{1 + z}(2z\, dz) =$
$\int_0^3 (10z^2 - 10z + 10)dz + \int_0^3 \frac{-10}{1 + z}\, dz =$
$(\frac{10}{3} z^3 - 5z^2 + 10z)\Big|_0^3 - 10 \ln |1 + z|\Big|_0^3 =$
$90 - 45 + 30 - 10 \ln 4 = 75 - 10 \ln 4$ square units.

54. $V = \pi\int_0^8 (x + \sqrt{x + 1})^2 dx = \pi\int_0^8 (x + 2\sqrt{x+1} + x + 1)dx =$
$\pi\int_0^8 (2x + 2\sqrt{x + 1} + 1)dx = \pi(x^2 + x)\Big|_0^8 +$
$2\pi\int_0^8 \sqrt{x + 1}\, dx$. Put $z = \sqrt{x + 1}$, so that $z^2 = x + 1$ and $2z\, dz = dx$. Thus, $\int_0^8 \sqrt{x + 1}\, dx = \int_1^3 2z^2 dz =$
$\frac{2}{3} z^3\Big|_1^3 = 18 - \frac{2}{3} = \frac{52}{3}$. Hence, $V = \pi(64 + 8) +$
$2\pi(\frac{52}{3}) = (72 + \frac{104}{3})\pi = \frac{320}{3}\pi$ cubic units.

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1. By Formula 71, $\int \frac{u\, du}{(3 + 5u)^2} =$
$\frac{1}{25}\left[\frac{3}{3 + 5u} + \ln |3 + 5u|\right] + C.$

2. By Formula 73, $\int \frac{5x\, dx}{(2 - 3x)^3} = 5\int \frac{x\, dx}{(2 - 3x)^3} =$
$-\frac{5}{9}\left[\frac{1}{2 - 3x} - \frac{2}{2(2 - 3x)^2}\right] + C =$
$-\frac{5}{9}\left[\frac{1}{2 - 3x} - \frac{1}{(2 - 3x)^2}\right] + C.$

3. By Formula 67, $\int \frac{dx}{x^2\sqrt{5 + x^2}} = -\frac{\sqrt{5 + x^2}}{5x} + C.$

4. By Formula 65, $\int (7 + 3x^2)^{-3/2} dx =$
$3^{-3/2} \int (\frac{7}{3} + x^2)^{-3/2} dx = \frac{1}{3\sqrt{3}}\ \frac{x}{\frac{7}{3}\sqrt{\frac{7}{3} + x^2}} + C =$
$\frac{x}{7\sqrt{7 + 3x^2}} + C.$

5. By Formula 64, $\int \frac{3\, dy}{\sqrt{11 + 5y^2}} = \frac{3}{\sqrt{5}} \int \frac{dy}{\sqrt{\frac{11}{5} + y^2}} =$
$\frac{3}{\sqrt{5}} \ln |y + \sqrt{\frac{11}{5} + y^2}| + C_1 =$
$\frac{3}{\sqrt{5}} \ln |\sqrt{5}y + \sqrt{11 + 5y^2}| - \ln \sqrt{5} + C_1 =$
$\frac{3}{\sqrt{5}} \ln |\sqrt{5}y + \sqrt{11 + 5y^2}| + C.$

6. $\int \frac{\sqrt{2 + 7x^2}}{x}\, dx = \sqrt{7} \int \frac{\sqrt{\frac{2}{7} + x^2}}{x}\, dx = \sqrt{7}\sqrt{\frac{2}{7} + x^2} -$
$\sqrt{7}\sqrt{\frac{2}{7}} \ln\left|\frac{\sqrt{\frac{2}{7}} + \sqrt{\frac{2}{7} + x^2}}{x}\right| + C$ (Formula 63) $=$
$\sqrt{2 + 7x^2} - \sqrt{2} \ln\left|\frac{\sqrt{2} + \sqrt{2 + 7x^2}}{\sqrt{7}\, x}\right| + C.$

7. Here we use Formula 62. $t^2 \sqrt{13 + 8t^2}\, dt =$
$\sqrt{8} \int t^2\sqrt{\frac{13}{8} + t^2}\, dt = \frac{t}{\sqrt{8}} (\frac{13}{8} + 2t^2)\sqrt{\frac{13}{8} + t^2} -$
$\frac{169\sqrt{8}}{512} \ln |t + \sqrt{\frac{13}{8} + t^2}| + C_1 =$
$\frac{t}{64} (13 + 16t^2)\sqrt{13 + 8t^2} - \frac{169\sqrt{8}}{512} \ln|\sqrt{8}t + \sqrt{13 + 8t^2}| + C.$

8. $\int \frac{dw}{w^2\sqrt{5 - 2w^2}} = \frac{1}{\sqrt{2}} \int \frac{dw}{w^2\sqrt{\frac{5}{2} - w^2}} = -\frac{1}{\sqrt{2}} \frac{\sqrt{\frac{5}{2} - w^2}}{\frac{5}{2} w^2} + C =$
$-\frac{1}{5} \frac{\sqrt{5 - 2w^2}}{w^2} + C$, by Formula 50.

9. We use Formula 59. $\int \frac{\sqrt{3y^2 - 5}}{y^2}\, dy = \sqrt{3}\int \frac{\sqrt{y^2 - 5/3}}{y^2} dy =$
$-\sqrt{3}\, \frac{\sqrt{y^2 - \frac{5}{3}}}{y} + \sqrt{3} \ln |y + \sqrt{y^2 - \frac{5}{3}}| + C_1 =$
$-\frac{\sqrt{3y^2 - 5}}{y} + \sqrt{3} \ln|\sqrt{3}y + \sqrt{3y^2 - 5}| + C.$

10. By Formula 91, $\int \frac{\sqrt{3 + 5z}}{z^2}\, dz = -\frac{\sqrt{(3 + 5z)^3}}{3z} +$
$\frac{5}{6} \int \frac{\sqrt{3 + 5z}}{z}\, dz = -\frac{\sqrt{(3 + 5z)^3}}{3z} +$
$\frac{5}{6}(2\sqrt{3 + 5z} + 3 \int \frac{dz}{z\sqrt{3 + 5z}})$ (Formula 87) $=$
$-\frac{\sqrt{(3 + 5z)^3}}{3z} + \frac{5}{3} \sqrt{3 + 5z} +$
$\frac{5}{2} (\frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{3 + 5z} - \sqrt{3}}{\sqrt{3 + 5z} + \sqrt{3}}\right| + C)$ (Formula 86) $=$

$-\frac{\sqrt{(3 + 5z)^3}}{3z} + \frac{5\sqrt{3 + 5z}}{3} + \frac{5}{2\sqrt{3}} \ln\left|\frac{\sqrt{3 + 5z} - \sqrt{3}}{\sqrt{3 + 5z} + \sqrt{3}}\right| + C.$

11. By Formula 55, $\int t^2\sqrt{t^2 - 5}\, dt = \frac{t}{8}(2t^2 - 5)\sqrt{t^2 - 5} - \frac{25}{8} \ln |t + \sqrt{t^2 + 5}| + C.$

12. $\int \frac{\sqrt{2 - 3x^2}}{x^2}\, dx = \sqrt{3}\int \frac{\sqrt{\frac{2}{3} - x^2}}{x^2}\, dx = -\sqrt{3}\, \frac{\sqrt{\frac{2}{3} - x^2}}{x} - \sin^{-1}\left(\frac{x}{2/3}\right) + C$ (Formula 51) $= -\frac{3\sqrt{2 - 3x^2}}{x} - \sin^{-1}\frac{3x}{2} + C.$

13. We use Formula 55. $\int x^2\sqrt{5 - 7x^2}\, dx = \sqrt{7}\int x^2\sqrt{\frac{5}{7} - x^2}\, dx = \sqrt{7}\left[\frac{x(2x^2 - \frac{5}{7})\sqrt{\frac{5}{7} - x^2}}{8} + \frac{25}{392}\sin^{-1}\left(\sqrt{\frac{7}{5}}\, x\right)\right] + C$

$\frac{1}{7}\left[\frac{x(14x^2 - 5)\sqrt{5 - 7x^2}}{8} + \frac{25}{8\sqrt{7}}\sin^{-1}\left(\sqrt{\frac{7}{5}}\, x\right)\right] + C.$

14. $\int \frac{\sqrt{2 - 3x^2}}{x^2}\, dx = \sqrt{3}\int \frac{\sqrt{\frac{2}{3} - x^2}}{x^2}\, dx = -\sqrt{3}\, \frac{\sqrt{\frac{2}{3} - x^2}}{x} - \sin^{-1}\left(\sqrt{\frac{3}{2}}\, x\right) + C$ (Formula 51) $= -\frac{\sqrt{2 - 3x^2}}{x} - \sin^{-1}\left(\sqrt{\frac{3}{2}}\, x\right) + C.$

15. $\int \frac{dt}{2 + 3t + 4t^2} = \frac{2}{\sqrt{32 - 9}} \tan^{-1} \frac{8t + 3}{\sqrt{32 - 9}} + C = \frac{2}{\sqrt{23}} \tan^{-1} \frac{8t + 3}{\sqrt{23}} + C$ by Formula 78.

16. By Formula 94, $\int \frac{dx}{\sqrt{2 + 3x + 4x^2}} = \frac{1}{2} \ln\left(\sqrt{2 + 3x + 4x^2} + 2x + \frac{3}{4}\right) + C.$

17. By Formula 92, $\int \frac{dx}{x\sqrt{5 - 4x + 2x^2}} = -\frac{1}{\sqrt{5}} \ln\left(\frac{\sqrt{5 - 4x + 2x^2} + \sqrt{5}}{x} - \frac{4}{2\sqrt{5}}\right) + C = -\frac{1}{\sqrt{5}} \ln\left(\frac{\sqrt{5 - 4x + 2x^2} + \sqrt{5}}{x} - \frac{2}{\sqrt{5}}\right) + C.$

18. By Formula 93, $\int \frac{\sqrt{16t^2 - 5t + 7}}{t}\, dt = \sqrt{16t^2 - 5t + 7} - \frac{5}{2}\int \frac{dt}{\sqrt{16t^2 - 5t + 7}} + 7\int \frac{dt}{t\sqrt{16t^2 - 5t + 7}} = \sqrt{16t^2 - 5t + 7} - \frac{5}{8} \ln\left(\sqrt{16t^2 - 5t + 7} + 4t - \frac{5}{8}\right) - \sqrt{7} \ln\left(\frac{\sqrt{16t^2 - 5t + 7} + \sqrt{7}}{t} - \frac{5}{2\sqrt{7}}\right) + C$, by Formulas 94 and 92.

19. By Formula 37, $\int \sin^{-1}(3y + 2)dy = \frac{1}{3}\int \sin^{-1}u\, du = \frac{1}{3} u \sin^{-1}u + \frac{1}{3}\sqrt{1 - u^2} + C = \frac{1}{3}(3y + 2)\sin^{-1}(3y + 2) + \frac{1}{3}\sqrt{1 - (3y + 2)^2} + C.$

20. By Formula 39, $\int \tan^{-1}(2t + 1)dt = \frac{1}{2}\int \tan^{-1}(u)du = \frac{1}{2} u \tan^{-1}u - \frac{1}{4} \ln (1 + u^2) + C = \frac{1}{2}(2t + 1)\tan^{-1}(2t + 1) - \frac{1}{4} \ln (4t^2 + 4t + 1) + C.$

21. Put $u = 5x$, so that $du = 5\, dx$ and $\int x \cos^{-1}5x\, dx = \int \frac{u}{5} \cos^{-1}u\, \frac{du}{5} = \frac{1}{25}\int u \cos^{-1}u\, du = \frac{1}{25}\left(\frac{2u^2 - 1}{4} \cos^{-1}u - \frac{u\sqrt{1 - u^2}}{4}\right) + C = \frac{1}{25}\left(\frac{50x^2 - 1}{4} \cos^{-1}5x - \frac{5x\sqrt{1 - 25x^2}}{4}\right) + C$ by Formula 41.

22. $\int w \sin^{-1}(3w - 1)dw = \frac{1}{3}\int (u + 1)\sin^{-1}u\, \frac{du}{3} = \frac{1}{9}\left[\int u \sin^{-1}u\, du + \int \sin^{-1}u\, du\right] = \frac{1}{9}\left[\frac{u^2}{2} \sin^{-1}u - \frac{1}{4} \sin^{-1}u + \frac{u}{4}\sqrt{1 - u^2} + u \sin^{-1}u + \sqrt{1 - u^2}\right] + C$ (Formulas 40 and 37) $= \frac{1}{18}(3w - 1)^2\sin^{-1}(3w - 1) - \frac{1}{36} \sin^{-1}(3w - 1) + \frac{(3w - 1)}{36}\sqrt{1 - (3w - 1)^2} + \frac{1}{9}(3w - 1) \sin^{-1}(3w - 1)$

$\frac{1}{9}\sqrt{1 - (3w - 1)^2} + C = \frac{1}{12}\left[(6w^2 - 1) \sin^{-1}(3w - 1) + (w + 1)\sqrt{6w - 9w^2}\right] + C.$

23. $\int \frac{dx}{(x^2 + 1)^2} = \frac{1}{2}\, \frac{x}{x^2 + 1} + \frac{1}{2}\int \frac{dx}{x^2 + 1} = \frac{1}{2}\, \frac{x}{x^2 + 1} + \frac{1}{2} \tan^{-1}x + C$ by Formula 77.

24. By Formula 77, $\int \frac{dx}{(x^2 + 1)^3} = \frac{1}{4}\, \frac{x}{(x^2 + 1)^2} + \frac{3}{4}\int \frac{dx}{(x^2 + 1)^2} = \frac{1}{4}\, \frac{x}{(x^2 + 1)^2} + \frac{3}{4}\left[\frac{1}{2}\, \frac{x}{x^2 + 1} + \frac{1}{4} \tan^{-1}x\right] + C.$

25. By using Formula 85 twice and then Formula 84, we have $\int \frac{4v^4\, dv}{\sqrt{3 + 2v}} = 4\left[\frac{2v^4\sqrt{3 + 2v}}{18} - \frac{24}{18}\int \frac{v^3\, dv}{\sqrt{3 + 2v}}\right] =$

$$4\left[\frac{v^4\sqrt{3+2v}}{9} - \frac{4}{3}\left(\frac{2v^3\sqrt{3+2v}}{14} - \frac{18}{14}\int\frac{v^2\,dv}{\sqrt{3+2v}}\right)\right] =$$

$$\frac{4}{9}v^4\sqrt{3+2v} - \frac{16}{21}v^3\sqrt{3+2v} +$$

$$\frac{48}{7}\left[\frac{2(12v^2 - 24v + 72)}{120}\right]\sqrt{3+2v} + C =$$

$$[\tfrac{4}{9}v^4 - \tfrac{16}{21}v^3 + \tfrac{4}{35}(12v^2 - 24v + 72)]\sqrt{3+2v} + C.$$

26. $\int\frac{dy}{(y^2+2y+2)^4} = \int\frac{dy}{[(y+1)^2+1]^2} =$

$$\frac{y+1}{8(y^2+2y+2)^3} + \frac{5}{6}\int\frac{dy}{(y^2+2y+2)^3} \text{ (Formula 77) } =$$

$$\frac{y+1}{8(y^2+2y+2)^3} + \frac{5}{6}\left[\frac{y+1}{4(y^2+2y+2)^2} + \right.$$

$$\left.\frac{3}{4}\int\frac{dy}{(y^2+2y+2)^2}\right] = \frac{y+1}{8(y^2+2y+2)^3} +$$

$$\frac{5(y+1)}{24(y^2+2y+2)^2} + \frac{5}{8}\left[\frac{y+1}{2(y^2+2y+2)} + \frac{1}{2}\int\frac{dy}{y^2+2y+2}\right] =$$

$$(y+1)\left[\frac{1}{8(y^2+2y+2)^3} + \frac{5}{24(y^2+2y+2)^2} + \right.$$

$$\left.\frac{5}{16(y^2+2y+2)}\right] + \frac{5}{16}\tan^{-1}\left(\frac{2y+2}{2}\right) + C \text{ (Formula 78).}$$

27. By Formula 34, $\int\csc^5 3x\,dx =$

$$\frac{1}{3}\left[\frac{-1}{4}\csc^3 3x\cot 3x + \frac{3}{4}\int\csc^3 3x\,dx\right] =$$

$$-\frac{1}{12}\csc^3 3x\cot 3x + \frac{1}{4}\left[-\frac{1}{2}\cot 3x\csc 3x + \right.$$

$$\left.\frac{1}{2}\int\csc 3x\right] + C = -\frac{1}{12}\csc^3 3x\cot 3x -$$

$$\frac{1}{8}\cot 3x\csc 3x + \frac{1}{8}\ln|\csc 3x - \cot 3x| + C$$

(Formula 12).

28. $\int\cot^5(2x-1)dx = \frac{1}{2}\int\cot^5 u\,du = -\frac{1}{8}\cot^4(2x-1) -$

$$\frac{1}{2}\int\cot^3 u\,du \text{ (Formula 32) } = -\frac{1}{8}\cot^4(2x-1) +$$

$$\frac{1}{4}\cot^2(2x-1) + \frac{1}{2}\ln\sin|2x-1| + C \text{ (Formula 10).}$$

29. By Formula 31, $\int\tan^5 7x\,dx = \frac{1}{28}\tan^4 7x - \int\tan^3 7x\,dx =$

$$\frac{1}{28}\tan^4 7x - \frac{1}{14}\tan^2 7x - \frac{1}{7}\ln|\cos 7x| + C.$$

30. $\int\sec^7\left(\frac{t}{2}\right)dt = 2\int\sec^7 u\,du = \frac{1}{3}\sec^5 u\tan u +$

$$\frac{5}{3}\int\sec^5 u\,du \text{ (Formula 33) } = \frac{1}{3}\sec^5 u\tan u +$$

$$\frac{5}{3}\left(\frac{1}{4}\sec^3 u\tan u + \frac{3}{4}\int\sec^3 u\,du\right) = \left(\frac{1}{3}\right)\sec^5 u\tan u +$$

$$\frac{5}{12}\sec^3 u\tan u + \left(\frac{5}{4}\right)\left(\frac{1}{2}\right)[\tan\left(\frac{t}{2}\right)\sec\left(\frac{t}{2}\right) +$$

$$\frac{1}{2}\ln\left|\sec\frac{t}{2} + \tan\frac{t}{2}\right|] = \frac{1}{3}\sec^5\frac{t}{2}\tan\frac{t}{2} +$$

$$\frac{5}{12}\sec^3\frac{t}{2}\tan u + \frac{5}{8}\tan\frac{t}{2}\sec\frac{t}{2} + \frac{5}{16}\ln|.$$

31. $\int\sin^n ax\,dx = \int\sin^{n-1}ax\cdot\sin ax\,dx =$

$$-\frac{\sin^{n-1}ax\cos ax}{a} + \frac{1}{a}\int\cos ax\frac{d}{dx}(\sin^{n-1}ax)\,dx =$$

$$-\frac{\sin^{n-1}ax\cos ax}{a} + (n-1)\int\cos^2 ax\sin^{n-2}ax\,dx =$$

$$-\frac{\sin^{n-1}ax\cos ax}{a} + (n-1)\int(1-\sin^2 ax)\sin^{n-2}ax\,dx.$$

Hence, $n\int\sin^n ax\,dx = -\frac{\sin^{n-1}ax\cos ax}{a} +$

$(n-1)\int\sin^{n-2}ax\,dx$. Thus, $\int\sin^n ax\,dx =$

$$-\frac{\sin^{n-1}ax\cos ax}{na} + \left(\frac{n-1}{n}\right)\int\sin^{n-2}ax\,dx.$$

32. $\int\tan^n u\,du = \int\tan^{n-2}u\tan^2 u\,du =$

$$\int\tan^{n-2}u(\sec^2 u - 1)\,du = \int\tan^{n-2}u\frac{d}{du}(\tan u)\,du -$$

$$\int\tan^{n-2}u\,du = \frac{1}{n-1}\tan^{n-1}u - \int\tan^{n-2}u\,du.$$

33. (a) $\int\sin^2 ax\,dx = -\frac{\sin ax\cos ax}{2a} + \frac{x}{2} + C.$

(b) $\int\sin^3 ax\,dx = -\frac{\sin^2 ax\cos ax}{3a} + \frac{2}{3}\int\sin ax\,dx =$

$$-\frac{\sin^2 ax\cos ax}{3a} - \frac{2}{3a}\cos ax + C.$$

(c) $\int\sin^4 ax\,dx = -\frac{\sin^3 ax\cos ax}{4a} + \frac{3}{4}\int\sin^2 ax\,dx =$

$$-\frac{\sin^3 ax\cos ax}{4a} + \frac{3}{4}\cdot\frac{1}{2}\int(1 - \cos 2ax)\,dx =$$

$$-\frac{\sin^3 ax\cos ax}{4a} + \frac{3}{8}\left(x - \frac{\sin 2ax}{2a}\right) + C =$$

$$-\frac{\sin^3 ax\cos ax}{4a} - \frac{3}{8a}\sin ax\cos ax + \frac{3}{8}x + C.$$

34. $\int\sec^n u\,du = \int\sec^{n-2}u\frac{d}{du}(\tan u)\,du = \sec^{n-2}u\tan u -$

$$\int\tan u\cdot(n-2)\sec^{n-3}u\sec u\tan u\,du =$$

$$\sec^{n-2}u\tan u - (n-2)\int\sec^{n-2}u\tan^2 u\,du =$$

$$\sec^{n-2}u\tan u - (n-2)\int\sec^{n-2}u(\sec^2 u - 1)du.$$

Hence, $(n-1)\int\sec^n u\,du = \sec^{n-2}u\tan u +$

$(n-2)\int\sec^{n-2}u\,du$ and $\int\sec^n u\,du = \frac{\sec^{n-2}u\tan u}{n-1} +$

$$\left(\frac{n-2}{n-1}\right)\int\sec^{n-2}u\,du.$$

35. (a) $I_1 = \int_0^{\pi/2}\sin x\,dx = -\cos x\Big|_0^{\pi/2} = -0 + 1 = 1,$

$$I_2 = \int_0^{\pi/2}\sin^2 x\,dx = \left(\frac{x}{2} - \frac{\sin x\cos x}{2}\right)\Big|_0^{\pi/2} = \frac{\pi}{4},$$

$$I_3 = \int_0^{\pi/2}\sin^3 x\,dx = \left(-\frac{\sin^2 x\cos x}{3} - \frac{2\cos x}{3}\right)\Big|_0^{\pi/2} = \frac{2}{3},$$

$I_4 = \int_0^{\pi/2} \sin^4 x\, dx = \left(-\frac{\sin^3 x \cos x}{4} + \frac{3x}{8} - \frac{3 \sin x \cos x}{8}\right)\Big|_0^{\pi/2} = \frac{3\pi}{16}.$

(b) $I_n = \int_0^{\pi/2} \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n}\Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx = 0 + \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} I_{n-2}.$

36. (a) We prove $I_{2k} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2k-1)}{2 \cdot 4 \cdot 6 \cdot \ldots (2k)} \cdot \frac{\pi}{2}$ by induction on k. It is true for k = 1 by Problem 35, part (a). Assume true for a given value of k. By Problem 35, part (b), $I_{2(k+1)} = \frac{2(k+1)-1}{2(k+1)} I_{2(k+1)-2} = I_{2k} \frac{2(k+1)-1}{2(k+1)} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2k-1)}{2 \cdot 4 \cdot 6 \cdot \ldots (2k)} \cdot \frac{2(k+1)-1}{2(k+1)}$; hence, it is true for k + 1.

(b) We prove $I_{2k+1} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)}{3 \cdot 5 \cdot 7 \cdot 9 \cdot \ldots (2k+1)}$ by induction on k. It is true for k = 1 by Problem 35, part (a). Assume true for a given value of k. By Problem 35, part (b), $I_{2(k+1)+1} = \frac{[2(k+1)+1]-1}{2(k+1)+1} I_{[2(k+1)+1]-2} = I_{2k+1}\frac{2(k+1)}{2(k+1)+1} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)}{3 \cdot 5 \cdot 7 \cdot 9 \cdot \ldots (2k+1)} \cdot \frac{2(k+1)}{2(k+1)+1}$; hence, it is true for k + 1.

37. $I_{2k+1}I_{2k} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)}{3 \cdot 5 \cdot 7 \cdot 9 \cdot \ldots (2k+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots (2k-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)} \cdot \frac{\pi}{2} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots (2k-1)}{3 \cdot 5 \cdot 7 \cdot 9 \cdot \ldots (2k-1)(2k+1)} \cdot \frac{\pi}{2} = \frac{1}{2k+1} \cdot \frac{\pi}{2}$; hence, $(2k+1) I_{2k+1} I_{2k} = \frac{\pi}{2}$.

38. By part (b) of Problem 35, $I_{2k+1} = \frac{2k}{2k+1} I_{2k-1}$; hence, by Problem 37, $\frac{\pi}{2} = (2k+1) I_{2k+1} I_{2k} = (2k+1)\frac{2k}{2k+1} I_{2k-1} I_{2k} = 2k\, I_{2k-1} I_{2k}$.

39. Suppose $1 \le k \le n$. Then, since $0 \le \sin x \le 1$, it follows that $0 \le \sin^n x \le \sin^k x$; hence, $\int_0^{\pi/2} \sin^n x\, dx \le \int_0^{\pi/2} \sin^k x\, dx$. Therefore, $I_n \le I_k$.

40. By Problem 39, $I_{2k+1} \le I_{2k} \le I_{2k-1}$. By Problem 37, $I_{2k+1} = \frac{1}{2k+1} \cdot \frac{\pi}{2 I_{2k}}$, and by Problem 38, $I_{2k-1} = \frac{1}{2k} \cdot \frac{\pi}{2 I_{2k}}$. Hence, $\frac{1}{2k+1} \cdot \frac{\pi}{2 I_{2k}} \le I_{2k} \le \frac{1}{2k} \cdot \frac{\pi}{2 I_{2k}}$

41. By Problem 40, $\frac{1}{2k+1} \cdot \frac{\pi}{2} \le (I_{2k})^2 \le \frac{1}{2k} \cdot \frac{\pi}{2}$. Using part (a) of Problem 36, we obtain $\frac{1}{2k+1} \cdot \frac{\pi}{2}$ $\left[\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots (2k-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)} \cdot \frac{\pi}{2}\right]^2 \le \frac{1}{2k} \cdot \frac{\pi}{2}$, or

$$\frac{1}{2k+1} \cdot \frac{\pi}{2} \le \left[\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots (2k-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)}\right] \cdot \left(\frac{\pi^2}{4}\right) \le \frac{1}{2k} \cdot \frac{\pi}{2}.$$

Multiplying the latter inequality by $\frac{4}{\pi^2}$, we obtain

$$\frac{2}{\pi(2k+1)} \le \left[\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots (2k-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)}\right]^2 \le \frac{1}{\pi k}.$$

42. Taking reciprocals and reversing inequalities in Problem 41, we have $\frac{(2k+1)\pi}{2} \ge \left[\frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots (2k)}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots (2k-1)}\right]^2 \ge k\pi$. Thus, $k\pi \le \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot \ldots (2k)(2k)}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot \ldots (2k-1)(2k-1)} \le \frac{(2k+1)\pi}{2}$. Multiplying the latter inequality by $\frac{1}{2(2k+1)}$, we obtain $\frac{2k}{2k+1} \cdot \frac{\pi}{4} \le \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot \ldots (2k)(2k)}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot \ldots (2k-1)(2k+1)} \le \frac{\pi}{4}$. As $k \to \infty$, $\frac{2k}{2k+1} \to 1$; hence,

$$\lim_{k \to \infty} \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot \ldots (2k)(2k)}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot \ldots (2k-1)(2k+1)} = \frac{\pi}{4}.$$

Review Problem Set, Chapter 8, page 531

1. $\int \cos^3 2x\, dx = \int (1 - \sin^2 2x) \cos 2x\, dx$. Put $u = \sin 2x$, so that $du = 2 \cos 2x\, dx$. Thus, $\int \cos^3 2x\, dx = \int \frac{1-u^2}{2}\, du = \frac{u}{2} - \frac{u^3}{6} + C = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + C$.

$\int \sin^3 4x \cos^2 4x\, dx = \int (1 - \cos^2 4x)\cos^2 4x \sin 4x\, dx.$ Put $u = \cos 4x$, so that $du = -4 \sin 4x\, dx$. Thus, $\int \sin^3 4x \cos^2 4x\, dx = \int (1 - u^2)u^2(-\frac{du}{4}) = -\frac{1}{4}(\frac{u^3}{3} - \frac{u^5}{5}) + C = -\frac{1}{4}(\frac{\cos^3 4x}{3} - \frac{\cos^5 4x}{5}) + C.$

$\int \sin^3 3x \cos^3 3x\, dx = \int \sin^3 3x(1 - \sin^2 3x)\cos 3x\, dx.$ Put $u = \sin 3x$, so that $du = 3 \cos 3x\, dx$. Thus, $\int \sin^3 3x \cos^3 3x\, dx = \int (u^3 - u^5)\frac{du}{3} = \frac{1}{3}(\frac{u^4}{4} - \frac{u^6}{6}) + C = \frac{1}{3}(\frac{\sin^4 3x}{4} - \frac{\sin^6 3x}{6}) + C.$

$\int \sqrt{\cos x}\, \sin^5 x\, dx = \int \sqrt{\cos x}\, \sin^4 x \sin x\, dx = \int [\sqrt{\cos x}\,(1 - \cos^2 x)^2 \sin x]dx.$ Put $u = \cos x$, $du = -\sin x\, dx$. Thus, $\int \sqrt{\cos x}\, \sin^5 x\, dx = \int u^{\frac{1}{2}}(1 - 2u^2 + u^4)(-du) = -\frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2} - \frac{2}{11}u^{11/2} + C = -\frac{2}{3}\cos^{3/2}x + \frac{2}{7}\cos^{7/2}x - \frac{2}{11}\cos^{11/2}x + C.$

$\int \sin^3(1 - 2x)dx = \int \sin^2(1 - 2x)\sin(1 - 2x)dx = \int [1 - \cos^2(1 - 2x)]\sin(1 - 2x)dx.$ Put $u = \cos(1 - 2x)$, so that $du = 2 \sin(1 - 2x)dx$. Thus, $\int \sin^3(1 - 2x)dx = \int (1 - u^2)\frac{du}{2} = \frac{1}{2}(u - \frac{u^3}{3}) + C = \frac{1}{2}[\cos(1 - 2x) - \frac{\cos^3(1 - 2x)}{3}] + C.$

$\int \sin^3 \frac{x}{2} \cos^{3/2} \frac{x}{2}\, dx = \int \sin^2 \frac{x}{2} \cos^{3/2} \frac{x}{2} \sin \frac{x}{2}\, dx = \int (1 - \cos^2 \frac{x}{2})\cos^{3/2} \frac{x}{2} \sin \frac{x}{2}\, dx.$ Put $u = \cos \frac{x}{2}$, so that $du = -\frac{1}{2} \sin \frac{x}{2}\, dx$. Thus, $\int \sin^3 \frac{x}{2} \cos^{3/2} \frac{x}{2}\, dx = \int (1 - u^2)u^{3/2}(-2\, du) = 2(\frac{2}{9}u^{9/2} - \frac{2}{5}u^{5/2}) + C = \frac{4}{9}\cos^{9/2}(\frac{x}{2}) - \frac{4}{5}\cos^{5/2}(\frac{x}{2}) + C.$

$\int \sin^{-2/3}5x \cos^3 5x\, dx = \int \sin^{-2/3}5x(1 - \sin^2 5x)\cos 5x\, dx.$ Put $u = \sin 5x$, so that $du = 5 \cos 5x\, dx$. Therefore, $\int \sin^{-2/3}5x \cos^3 5x\, dx = \int u^{-2/3}(1 - u^2)(\frac{du}{5}) = \frac{1}{5}(3u^{1/3} - \frac{3}{7}u^{7/3}) + C = \frac{3}{5}\sin^{1/3}5x - \frac{3}{35}\sin^{7/3}5x + C.$

$\int \sin^4 \frac{2x}{5} \cos^3 \frac{2x}{5}\, dx = \int \sin^4 \frac{2x}{5}(1 - \sin^2 \frac{2x}{5}) \cos \frac{2x}{5}\, dx.$ Put $u = \sin \frac{2x}{5}$, $du = \frac{2}{5}\cos \frac{2x}{5}\, dx.$ $\int \sin^4 \frac{2x}{5} \cos^3 \frac{2x}{5}\, dx = \int u^4(1 - u^2)\frac{5}{2}\, du = \frac{5}{2}\left[\frac{u^5}{5} - \frac{u^7}{7}\right] + C = \frac{1}{2}(\sin^5 \frac{2x}{5} - \frac{5}{7}\sin^7 \frac{2x}{5}) + C.$

9. Put $u = 2 - 3x$, so that $du = -3\, dx$. Thus, $\int \sin^2(2 - 3x)dx = -\frac{1}{3}\int \sin^2 u\, du = -\frac{1}{3}\int \frac{1 - \cos 2u}{2}\, du = \frac{1}{6}\frac{\sin 2u}{2} - \frac{1}{6}u + C_1 = \frac{1}{12}\sin(4 - 6x) - \frac{1}{6}(2 - 3x) + C_1 = \frac{1}{12}\sin(4 - 6x) + \frac{x}{2} + C.$

10. $\int (4 + \cos x)(3 - \cos x)dx = \int (12 - \cos x - \cos^2 x)dx = 12x - \sin x - \int \frac{1 + \cos 2x}{2}\, dx = \frac{23}{2}x - \sin x - \frac{1}{4}\sin 2x + C.$

11. $\int (\sin x - \cos x)^2 dx = \int (\sin^2 x - 2 \sin x \cos x + \cos^2 x)\, dx = \int (1 - 2 \sin x \cos x)dx = x - \sin^2 x + C.$

12. $\int \sin^2(1 - 2x)\cos^2(1 - 2x)dx = \int (\frac{1 - \cos 2u}{2})(\frac{1 + \cos 2u}{2})(\frac{-du}{2}).$ Put $u = 1 - 2x$, so that $du = -2\, dx$. Substituting gives $-\frac{1}{8}\int (1 - \cos^2 2u)du = -\frac{1}{8}\int [1 - (\frac{1 + \cos 4u}{2})]du = -\frac{1}{8}(\frac{u}{2} - \frac{\sin 4u}{8}) + C_1 = \frac{1}{8}[\frac{\sin(4 - 8x)}{8} - \frac{1 - 2x}{2}] + C_1 = \frac{1}{8}[\frac{\sin(4 - 8x)}{8} + x] + C.$

13. $\int \sin^2 6x \cos^2 6x\, dx = \int (\frac{1 - \cos 12x}{2})(\frac{1 + \cos 12x}{2})dx = \frac{1}{4}\int (1 - \cos^2 12x)dx = \frac{1}{4}\int \sin^2 12x\, dx = \frac{1}{4}\int \frac{1 - \cos 24x}{2}\, dx = \frac{1}{8}(x - \frac{\sin 24x}{24}) + C.$

14. $\int \sin^4 4x \cos^2 4x\, dx = \int (\frac{1 - \cos 8x}{2})^2(\frac{1 + \cos 8x}{2})dx = \frac{1}{8}\int (1 - \cos 8x - \cos^2 8x + \cos^3 8x)dx = \frac{1}{8}(x - \frac{\sin 8x}{8}) - \frac{1}{8}\int \frac{1 + \cos 16x}{2}\, dx + \frac{1}{8}\int \cos^2 8x \cos 8x\, dx = \frac{1}{8}x - \frac{\sin 8x}{64} - \frac{x}{16} - \frac{\sin 16x}{256} + \frac{1}{8}\int (1 - \sin^2 8x) \cos 8x\, dx.$ Put $u = \sin 8x$, so that $du = 8 \cos 8x\, dx$. Thus, $\int (1 - \sin^2 8x)\cos 8x\, dx = \int (1 - u^2)\frac{du}{8} = \frac{u}{8} - \frac{u^3}{24} + C.$ Hence, $\int \sin^4 4x \cos^2 4x\, dx = \frac{x}{16} - \frac{\sin 8x}{64} - \frac{\sin 16x}{256} + \frac{\sin 8x}{64} - \frac{\sin^3 8x}{192} + C =$

$\frac{x}{16} - \frac{\sin 16x}{256} - \frac{\sin^3 8x}{192} + C.$

15. Put $u = \sin \frac{3t}{2}$, so that $du = \frac{3}{2} \cos \frac{3}{2}t\, dt$. Thus,

$$\int \frac{\cos^3 \frac{3t}{2}}{\sqrt[3]{\sin \frac{3t}{2}}} dt = \int \frac{(1 - \sin^2 \frac{3t}{2})}{\sqrt[3]{\sin \frac{3t}{2}}} (\cos \frac{3t}{2}\, dt) =$$

$$\frac{2}{3} \int \frac{1 - u^2}{u^{1/3}} du = \frac{2}{3} (\tfrac{3}{2}u^{2/3} - \tfrac{3}{8}u^{8/3}) + C = \sin^{2/3} \frac{3t}{2} - \frac{1}{4} \sin^{8/3} \frac{3t}{2} + C.$$

16. $\int \frac{\cos x}{\sin^4 x} dx = \int \sin^{-4} x \frac{d}{dx} (\sin x) dx = -\frac{1}{3} \sin^{-3} x + C.$

17. $\int \sin 8x \sin 3x\, dx = \frac{1}{2} \int (\cos 5x - \cos 11x) dx = \frac{\sin 5x}{10} - \frac{\sin 11x}{22} + C.$

18. $\int \cos 13x \cos 2x\, dx = \frac{1}{2} \int (\cos 11x + \cos 15x) dx = \frac{1}{22} \sin 11x + \frac{1}{30} \sin 15x + C.$

19. $\int \sin x \sin 2x \sin 3x\, dx = \int (\sin 2x \sin x) \sin 3x\, dx = \int \frac{1}{2}(\cos x - \cos 3x) \sin 3x\, dx = \int \frac{1}{2} \cos x \sin 3x\, dx - \frac{1}{2} \int \cos 3x \sin 3x\, dx = \frac{1}{4} \int \sin 4x\, dx + \int \frac{1}{4} \sin 2x\, dx - \frac{1}{4} \int \sin 6x\, dx - \frac{1}{16} \cos 4x - \frac{1}{8} \cos 2x + \frac{1}{24} \cos 6x + C.$

20. $\int \cos 3x \cos 5x \cos 9x\, dx = \int \cos 5x \cos 3x \cos 9x\, dx = \int [\frac{1}{2} \cos 2x + \frac{1}{2} \cos 8x] \cos 9x\, dx = \int \frac{1}{2} \cos 9x \cos 2x\, dx + \frac{1}{2} \int \cos 9x \cos 8x\, dx = \int (\frac{1}{4} \cos 7x + \frac{1}{4} \cos 11x + \frac{1}{4} \cos x + \frac{1}{4} \cos 17x) dx = \frac{1}{28} \sin 7x + \frac{1}{44} \sin 11x + \frac{1}{4} \sin x + \frac{1}{68} \sin 17x + C.$

21. $\int \tan^4 (2x - 1) dx = \int \tan^2 (2x - 1)[\sec^2 (2x - 1) - 1] dx = \int \tan^2 (2x - 1) \cdot \sec^2 (2x - 1) dx - \int \tan^2 (2x - 1) dx = \frac{1}{2} \frac{\tan^3 (2x - 1)}{3} - \int [\sec^2 (2x - 1) - 1] dx = \frac{1}{6} \tan^3 (2x - 1) - \frac{1}{2} \tan (2x - 1) + x + C$, where the first integral was evaluated by letting $u = \tan (2x - 1)$.

22. $\int \cot^4 (2 - 3x) dx = \int \cot^2 (2 - 3x)[\csc^2 (2 - 3x) - 1] dx = \int \cot^2 (2 - 3x) \csc^2 (2 - 3x) dx - \int \cot^2 (2 - 3x) dx = \frac{1}{3} \frac{\cot^3 (2 - 3x)}{3} - \int [\csc^2 (2 - 3x) - 1] dx = \frac{1}{9} \cot^3 (2 - 3x) - \frac{\cot (2 - 3x)}{3} + x + C$, where the first integral can be evaluated by putting $u = \cot (2 - 3x)$.

23. $\int x \tan^3 5x^2 dx$. Put $u = 5x^2$, so that $du = 10x\, dx$. Thus, $\int x \tan^3 5x^2 dx = \frac{1}{10} \int \tan^3 u\, du = \frac{1}{10} \int \tan u (\sec^2 u - 1) du = \frac{1}{10} \frac{\tan^2 u}{2} - \frac{1}{10} \ln |\sec u| + C$ $\frac{1}{20} \tan^2 5x^2 - \frac{1}{10} \ln |\sec (5x^2)| + C.$

24. Put $u = 5 - x^3$, so that $du = -3x^2 dx$. Thus, $\int x^2 \cot^3 (5 - x^3) dx = -\frac{1}{3} \int \cot^3 u\, du = -\frac{1}{3} \int \cot u (\csc^2 u - 1) du = -\frac{1}{3} \int \cot u \csc^2 u\, du + \frac{1}{3} \int \cot u\, du = \frac{1}{3} \frac{\cot^2 u}{2} + \frac{1}{3} \ln |\sin u| + C = \frac{1}{6} \cot^2 (5 - x^3) + \frac{1}{3} \ln |\sin (5 - x^3)| + C.$

25. $\int (\sec t - \tan t)^2 dt = \int (\sec^2 t - 2 \sec t \tan t + \tan^2 t) dt = \int (2 \sec^2 t - 1 - \frac{2 \sin t}{\cos^2 t}) dt = 2 \tan t - t - 2 \sec t + C$, where $\int \frac{-2 \sin t}{\cos^2 t} dt$ is evaluated by putting $u = \cos t$.

26. Put $u = \tan x$, so that $du = \sec^2 x\, dx$. Thus, $\int \frac{\cos (\tan x)}{\cos^2 x} dx = \int \cos (\tan x) \sec^2 x\, dx = \int \cos u\, du = \sin u + C = \sin (\tan x) + C.$

27. $\int \frac{dx}{(1 - \sin x)^2} = \int \frac{(1 + \sin x)^2}{[(1 - \sin x)(1 + \sin x)]^2} dx = \int \frac{1 + 2 \sin x + \sin^2 x}{\cos^4 x} dx = \int (\sec^4 x + 2 \tan x \sec^3 x + \tan^2 x \sec^2 x) dx = \int [\sec^2 x (\tan^2 x + 1) + 2 \tan x \sec x \sec^2 x + \tan^2 x \sec^2 x] dx = \int (2 \tan^2 x \sec^2 x + \sec^2 x + 2 \tan x \sec x \sec^2 x) dx = \frac{2}{3} \tan^3 x + \tan x + \frac{2}{3} \sec^3 x + C.$

28. $\int \sqrt{1 + \cos x}\, dx = \int \frac{\sqrt{1 + \cos x} \sqrt{1 - \cos x}}{\sqrt{1 - \cos x}} dx = \int \frac{\sin x}{\sqrt{1 - \cos x}} dx$. Now put $u = 1 - \cos x$, so that $du = \sin x\, dx$. Thus, $\int \sqrt{1 + \cos x}\, dx = \int -\frac{du}{u^{\frac{1}{2}}} =$

$2u^{\frac{1}{2}} + C = 2\sqrt{1 - \cos x} + C.$

29. $\int \sec^4(1 + 2x)dx = \int \sec^2(1 + 2x)[\tan^2(1+2x)+1]dx.$ Now put $u = \tan(1 + 2x)$, so that $du = 2\sec^2(1 + 2x)dx$. Thus, $\int \sec^4(1 + 2x)dx = \frac{1}{2}\int (u^2 + 1)du = \frac{1}{2}(\frac{u^3}{3} + u) + C = [\frac{\tan^3(1 + 2x)}{3} + \tan(1 + 2x)] + C.$

30. $\int \csc^4(3 - 2x)dx = \int \csc^2(3 - 2x)[\cot^2(3-2x)+1]dx.$ Put $u = \cot(3 - 2x)$, so that $du = 2\csc^2(3 - 2x)dx$. Thus, $\int \csc^4(3 - 2x)dx = \frac{1}{2}\int (u^2 + 1)du = \frac{1}{2}(\frac{u^3}{3} + u) + C = \frac{1}{2}[\frac{\cot^3(3 - 2x)}{3} + \cot(3 - 2x)] + C.$

31. $\int \tan^3(2 + 3x)\sec^4(2 + 3x)dx = \int \tan^3(2 + 3x)[\tan^2(2 + 3x) + 1]\cdot\sec^2(2 + 3x)dx.$ Now put $u = \tan(2 + 3x)$, so that $du = 3\sec^2(2 + 3x)dx$. Therefore, $\int \tan^3(2 + 3x)\sec^4(2 + 3x)dx = \int (u^5 + u^3)\frac{du}{3} = \frac{1}{3}(\frac{u^6}{6} + \frac{u^4}{4}) + C = \frac{1}{3}[\frac{\tan^6(2 + 3x)}{6} + \frac{\tan^4(2 + 3x)}{4}] + C.$

32. $\int \cot^3(1 - x)\csc^4(1 - x)dx = \int \cot^3(1 - x)[\cot^2(1 - x) + 1]\csc^2(1 - x)dx.$ Now put $u = \cot(1 - x)$, so that $du = \csc^2(1 - x)dx$. Thus, $\int \cot^3(1 - x)\csc^4(1 - x)dx = \int (u^5 + u^3)du = \frac{u^6}{6} + \frac{u^4}{4} + C = \frac{\cot^6(1 - x)}{6} + \frac{\cot^4(1 - x)}{4} + C.$

33. $\int \frac{dx}{\sqrt{x^2 + 64}} = \sinh^{-1}\frac{x}{8} + C.$

34. Put $x = 9\sin\theta$, so that $dx = 9\cos\theta\, d\theta$. Thus, $\int \frac{dx}{x^2\sqrt{81 - x^2}} = \int \frac{9\cos\theta\, d\theta}{81\sin^2\theta(9\cos\theta)} = \frac{1}{81}\int \csc^2\theta\, d\theta = -\frac{1}{81}\cot\theta + C = \frac{-\sqrt{81 - x^2}}{81x} + C.$

35. Put $x = \sin\theta$, so that $dx = \cos\theta\, d\theta$. Thus, $\int \frac{dx}{(\sqrt{1 - x^2})^5} = \int \frac{\cos\theta\, d\theta}{\cos^5\theta} = \int \sec^4\theta\, d\theta = \frac{\tan^3\theta}{3} + \tan\theta + C = \frac{1}{3}(\frac{x}{\sqrt{1 - x^2}})^3 + \frac{x}{\sqrt{1 - x^2}} + C.$ (For the last integral, see Problem 29.)

36. Put $x^3 = u$, $3x^2dx = du$. $\int \frac{4\, dx}{x\sqrt{x^6 - 16}} = \int \frac{4\cdot 3x^2dx}{3x^2\cdot x\sqrt{x^6 - 16}} = \int \frac{4\, du}{3u\sqrt{u^2 - 16}} = \int \frac{du}{3u\sqrt{(\frac{u}{4})^2 - 1}}.$ Put $z = \frac{u}{4}$, $dz = \frac{1}{4}du$; so the last integral becomes $\int \frac{4\, dz}{3\cdot 4z\sqrt{z^2 - 1}} = \frac{1}{3}\sec^{-1}z + C = \frac{1}{3}\sec^{-1}\frac{u}{4} + C = \frac{1}{3}\sec^{-1}\frac{x^3}{4} + C.$

37. Put $u = x^2 - 4$, so that $du = 2x\, dx$. Thus, $\int x\sqrt{x^2 - 4}\, dx = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2 - 4)^{3/2} + C.$

38. Put $u = t - 4$, then $\int \frac{dt}{(t - 4)\sqrt{t^2 - 8t + 41}} = \int \frac{du}{u\sqrt{u^2 + 25}} = \int \frac{5\sec^2\theta\, d\theta}{5\tan\theta\sqrt{25\tan^2\theta + 25}}$ (where $u = 5\tan\theta$) $= \int \frac{1}{5}\csc\theta\, d\theta = \frac{1}{5}\ln|\csc\theta - \cot\theta| + C = \frac{1}{5}\ln\left|\frac{\sqrt{u^2 + 25}}{u} - \frac{5}{u}\right| + C = \frac{1}{5}\ln\left|\frac{\sqrt{t^2 - 8t + 41} - 5}{t - 4}\right| + C.$

39. $\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{-(x - 1)^2 + 1}}.$ Now put $u = x - 1$, so that $du = dx$. Thus, $\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1}u + C = \sin^{-1}(x - 1) + C.$

40. $\int \frac{dt}{\sqrt{1 + 2t - 2t^2}} = \int \frac{dt}{\sqrt{-2(t - \frac{1}{2})^2 + \frac{3}{2}}}.$ Now put $u = t - \frac{1}{2}$, so $du = dt$. Thus, $\int \frac{dt}{\sqrt{1 + 2t - 2t^2}} = \int \frac{du}{\sqrt{\frac{3}{2} - 2u^2}} = \sqrt{2}\int \frac{du}{\sqrt{3 - 4u^2}} = \frac{\sqrt{2}}{2}\sin^{-1}\frac{2u}{\sqrt{3}} + C = \frac{\sqrt{2}}{2}\sin^{-1}\frac{(2t - 1)}{\sqrt{3}} + C.$

41. $\int \frac{dx}{\sqrt{x^2 + 6x + 13}} = \int \frac{dx}{\sqrt{(x + 3)^2 + 4}}.$ Put $u = x + 3$, so that $du = dx$. Thus, $\int \frac{dx}{\sqrt{x^2 + 6x + 13}} = \int \frac{du}{\sqrt{u^2 + 4}} = \sinh^{-1}\frac{u}{2} + C = \sinh^{-1}\frac{x + 3}{2} + C.$

42. $\int \frac{dx}{\sqrt{8 + 4x - 4x^2}} = \int \frac{dx}{\sqrt{-4(x - \frac{1}{2})^2 + 9}}$. Now put $u = x - \frac{1}{2}$, so that $du = dx$. Thus, $\int \frac{dx}{\sqrt{8 + 4x - 4x^2}} = \int \frac{du}{\sqrt{9 - 4u^2}} = \frac{1}{2} \sin^{-1} \frac{2u}{3} + C = \frac{1}{2} \sin^{-1}(\frac{2x - 1}{3}) + C.$

43. By the tabular method of integration by parts:

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ x^2 & e^{-7x} & \\ 2x & -\frac{1}{7}e^{-7x} & \xrightarrow{+} + x^2(-\frac{1}{7}e^{-7x}) \\ 2 & \frac{1}{49}e^{-7x} & \xrightarrow{-} - 2x(\frac{1}{49}e^{-7x}) \\ 0 & -\frac{1}{343}e^{-7x} & \xrightarrow{+} + 2(-\frac{1}{343}e^{-7x}) \end{array}$$

Thus, $\int x^2 e^{-7x} = -\frac{x^2}{7} e^{-7x} - \frac{2x}{49} e^{-7x} - \frac{2}{343} e^{-7x} + C.$

44. Put $u = \ln 2x$ and $dv = \sqrt{x}\, dx$. So $du = \frac{1}{x}\, dx$ and $v = \frac{2}{3}x^{3/2}$. Thus, $\int \sqrt{x} \ln 2x\, dx = uv - \int v\, du = (\ln 2x)\, \frac{2}{3}x^{3/2} - \int \frac{2}{3}x^{3/2} \frac{dx}{x} = \frac{2}{3}x^{3/2} \ln 2x - \frac{4}{9}x^{3/2} + C.$

45. Put $u = \sin^{-1}2t$ and $dv = t^2dt$, so that $du = \frac{2}{\sqrt{1 - 4t^2}}\, dt$ and $v = \frac{t^3}{3}$. Thus, $\int t^2\sin^{-1}2t\, dt = uv - \int v\, du = \frac{t^3}{3} \sin^{-1}2t - \int \frac{t^3(2)}{3\sqrt{1 - 4t^2}}\, dt$. Now let $z = \sqrt{1 - 4t^2}$, so that $z^2 = 1 - 4t^2$ and $2z\, dz = -8t\, dt$. Thus, $\int \frac{t^3\, dt}{\sqrt{1 - 4t^2}} = \int \frac{(\frac{1 - z^2}{4})(-\frac{1}{4}z\, dz)}{z} = \frac{1}{16} \int (z^2 - 1)dz = \frac{1}{16} (\frac{z^3}{3} - z) + C_1 = \frac{1}{16}[\frac{(1 - 4t^2)^{3/2}}{3} - \sqrt{1 - 4t^2}] + C_1$. Thus, $\int t^2\sin^{-1}(2t)dt = \frac{t^3}{3} \sin^{-1}2t - \frac{1}{24}[\frac{(1 - 4t^2)^{3/2}}{3} - (1 - 4t^2)^{\frac{1}{2}}] + C.$

46. Put $u = \ln (x^2 + 16)$ and $dv = dx$, so that $du = \frac{2x}{x^2 + 16}\, dx$ and $v = x$. Thus, $\int \ln(x^2 + 16)dx = uv - \int v\, du = x \ln (x^2 + 16) - \int \frac{2x^2}{x^2 + 16}\, dx = x \ln (x^2 + 16) - \int 2\, dx + \int \frac{32}{x^2 + 16}\, dx = x \ln (x^2 + 16) - 2x + 8 \tan^{-1} \frac{x}{4} + C.$

47. We use the tabular method:

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ x + 2 & e^{3x} & \\ 1 & \frac{1}{3}e^{3x} & \xrightarrow{+} + (x + 2)(\frac{1}{3})e^{3x} \\ 0 & \frac{1}{9}e^{3x} & \xrightarrow{-} - (\frac{1}{9}e^{3x}) \end{array}$$

Thus, $\int (x + 2)e^{3x}dx = \frac{1}{3}(x + 2)e^{3x} - \frac{1}{9}e^{3x} + C.$

48. Let $u = \ln (x + \sqrt{x^2 + 4})$, $dv = dx$. Then $du = \frac{dx}{\sqrt{x^2 + 4}}$, $v = x$, and $\int \ln (x + \sqrt{x^2 + 4})dx = x \ln (x + \sqrt{x^2 + 4}) - \int \frac{x\, dx}{\sqrt{x^2 + 4}} = x \ln (x + \sqrt{x^2 + 4}) - \frac{1}{2} \frac{\sqrt{x^2 + 4}}{1/2} + C = x \ln (x + \sqrt{x^2 + 4}) - \sqrt{x^2 + 4} + C.$

49.
$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ t^3 & \cos 3t & \\ 3t^2 & \frac{1}{3} \sin 3t & \xrightarrow{+} t^3(\frac{1}{3} \sin 3t) \\ 6t & -\frac{1}{9} \cos 3t & \xrightarrow{-} -3t^2(-\frac{1}{9} \cos 3t) \\ 6 & -\frac{1}{27} \sin 3t & \xrightarrow{+} +6t(-\frac{1}{27} \sin 3t) \\ 0 & +\frac{1}{81} \cos 3t & \xrightarrow{-} -6(\frac{1}{81} \cos 3t) \end{array}$$

Thus, $\int t^3 \cos 3t\, dt = \frac{1}{3} t^3 \sin 3t + \frac{1}{3} t^2 \cos 3t - \frac{2t}{9} \sin 3t - \frac{2}{27} \cos 3t + C.$

50. Put $u = \sin(\ln x)$ and $dv = dx$, so that $du = \frac{\cos(\ln x)}{x}\, dx$ and $v = x$. Thus, $\int \sin(\ln x)dx = uv - \int v\, du = x \sin(\ln x) - \int \cos(\ln x)dx$. Now put $u_1 = \cos(\ln x)$ and $dv_1 = dx$, so that $du_1 = \frac{-\sin(\ln x)}{x}\, dx$ and $v_1 = x$. So, $\int \cos(\ln x)dx = u_1v_1 - \int v_1du_1 = x \cos(\ln x) + \int \sin(\ln x)dx$. Hence, $2\int \sin(\ln x)dx = x \sin (\ln x) - x \cos(\ln x) + C_1$ and $\int \sin(\ln x)dx = \frac{1}{2}[x \sin(\ln x) - x \cos(\ln x)] + C.$

51. Put $u = \tan^{-1} \sqrt{x}$ and $dv = dx$, so that $du = \frac{1}{1 + x} \cdot \frac{dx}{2\sqrt{x}}$ and $v = x$. Therefore, $\int \tan^{-1} \sqrt{x}\, dx = uv - \int v\, du = x \tan^{-1} \sqrt{x} - \frac{1}{2} \int \frac{\sqrt{x}}{1 + x}\, dx$. Now put $w = \sqrt{x}$, so that $w^2 = x$ and $2w\, dw = dx$. Thus, $\frac{1}{2} \int \frac{\sqrt{x}}{1 + x}\, dx = \frac{1}{2} \int \frac{w}{1 + w^2} (2w\, dw) = \int \frac{w^2dw}{1 + w^2} = \int (1 - \frac{1}{1 + w^2})dw =$

$w - \tan^{-1}w + C_1 = \sqrt{x} - \tan^{-1}\sqrt{x} + C_1$. Hence, $\int \tan^{-1}\sqrt{x}\,dx = x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$.

52. $\int e^{\sin t}\left(\frac{t\cos^3 t - \sin t}{\cos^2 t}\right)dt = \int t\cos t\, e^{\sin t}dt - \int \frac{\sin t\, e^{\sin t}}{\cos^2 t}\,dt$. Now to evaluate the first integral, put $u = t$ and $dv = \cos t\, e^{\sin t}\,dt$, so that $du = dt$ and $v = e^{\sin t}$. Thus, $\int t\cos t\, e^{\sin t}\,dt = te^{\sin t} - \int e^{\sin t}\,dt$. To evaluate the second integral, put $u_1 = e^{\sin t}$ and $dv_1 = \frac{\sin t}{\cos^2 t}\,dt$, so that $du_1 = \cos t\, e^{\sin t}\,dt$ and $v_1 = \frac{1}{\cos t}$. Thus, $\int \frac{\sin t}{\cos^2 t} e^{\sin t}\,dt = \frac{e^{\sin t}}{\cos t} - \int e^{\sin t}\,dt$. Hence, $\int e^{\sin t}\left(\frac{t\cos^3 t - \sin t}{\cos^2 t}\right)dt = te^{\sin t} - \int e^{\sin t}\,dt - \frac{e^{\sin t}}{\cos t} + \int e^{\sin t}dt = te^{\sin t} - \frac{e^{\sin t}}{\cos t} + C$.

53. $\int \frac{\tan^{-1}x}{x^2}\,dx = -\frac{\tan^{-1}x}{x} + \int \frac{dx}{x(1 + x^2)} = -\frac{\tan^{-1}x}{x} + \int\frac{dx}{x} - \int\frac{x\,dx}{1 + x^2} = -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln(1 + x^2) + C = -\frac{\tan^{-1}x}{x} + \ln\frac{|x|}{\sqrt{1 + x^2}} + C$.

54. $\int e^{2x}\sin^{2x}dx = \int e^{2x}\left(\frac{1 - \cos 2x}{2}\right)dx = \frac{1}{2}\int e^{2x}dx - \frac{1}{2}\int e^{2x}\cos 2x\,dx = \frac{1}{4}e^{2x} - \frac{1}{2}\int e^{2x}\cos 2x\,dx$. Now put $u = \cos 2x$ and $dv = e^{2x}dx$, such that $du = -2\sin 2x\,dx$ and $v = \frac{1}{2}e^{2x}$. So $\int e^{2x}\cos 2x\,dx = \frac{1}{2}e^{2x}\cos 2x + \int e^{2x}\sin 2x\,dx$. Now put $u_1 = \sin 2x$ and $dv_1 = e^{2x}dx$, so that $du_1 = 2\cos 2x\,dx$ and $v_1 = \frac{1}{2}e^{2x}dx$. Thus, $\int e^{2x}\sin 2x\,dx = \frac{1}{2}e^{2x}\sin 2x - \int e^{2x}\cos 2x\,dx$. Thus, $2\int e^{2x}\cos 2x\,dx = \frac{1}{2}e^{2x}\cos 2x + \frac{1}{2}e^{2x}\sin 2x + C_1$. Therefore, $\int e^{2x}\sin^2 x\,dx = \frac{1}{4}e^{2x} - \frac{1}{2}\left[\frac{1}{4}e^{2x}\cos 2x + \frac{1}{4}e^{2x}\sin 2x\right] + C$.

55. $\int e^{3x}\cos^2 x\,dx = \int e^{3x}\left(\frac{1 + \cos 2x}{2}\right)dx = \frac{1}{2}\int e^{3x}dx + \frac{1}{2}\int e^{3x}\cos 2x\,dx = \frac{1}{6}e^{3x} + \frac{1}{2}\int e^{3x}\cos 2x\,dx$. Put $u = \cos 2x$ and $dv = e^{3x}dx$, so that $du = -2\sin 2x\,dx$ and $v = \frac{1}{3}e^{3x}$. Thus, $\int e^{3x}\cos 2x\,dx = \frac{1}{3}e^{3x}\cos 2x + \int\frac{2}{3}\sin 2x\, e^{3x}dx$. Now let $u_1 = \sin 2x$ and $dv = e^{3x}dx$, such that $du_1 = 2\cos 2x\,dx$ and $v_1 = \frac{1}{3}e^{3x}$. Hence, $\int\frac{2}{3}\sin 2x\, e^{3x}dx = \frac{2}{3}\left(\frac{1}{3}\sin 2x\, e^{3x} - \int\frac{2}{3}\cos 2x\, e^{3x}dx\right)$. Thus, $\int e^{3x}\cos 2x\,dx = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}\sin 2x\, e^{3x} - \frac{4}{9}\cos 2x\, e^{3x} + C_1$ and $\int e^{3x}\cos 2x\,dx = \frac{9}{13}\left(\frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}\sin 2x\, e^{3x}\right) + C_1$. Hence, $\int e^{3x}\cos^2 x\,dx = \frac{1}{6}e^{3x} + \frac{3}{26}e^{3x}\cos 2x + \frac{1}{13}\sin 2x\, e^{3x} + C = e^{3x}\left(\frac{1}{6} + \frac{3}{26}\cos 2x + \frac{1}{13}\sin 2x\right) + C$.

56. $\int \sec^3 5x\,dx = \frac{1}{5}\left[\frac{1}{2}\sec 5x\tan 5x + \frac{1}{2}\ln|\sec 5x + \tan 5x|\right] + C$.

57. Put $z = -x^4$, so that $dz = -4x^3dx$. Thus, $\int x^{11}e^{-x^4}dx = \int z^2e^z\left(-\frac{dz}{4}\right) = -\frac{1}{4}\int z^2e^zdz$. Now we use the tabular method:

u	v'		
z^2	e^z		
$2z$	e^z	$+$	z^2e^z
2	e^z	$-$	$-2ze^z$
0	e^z	$+$	$+2e^z$

Hence, $\int x^{11}e^{-x^4}dx = -\frac{1}{4}(z^2e^z - 2ze^z + 2e^z) + C = -\frac{e^{-x^4}}{4}(x^8 + 2x^4 + 2) + C$.

58. Put $y = x^2$, so that $dy = 2x\,dx$. Now $\int x^5\sin x^2dx = \frac{1}{2}\int y^2(\sin y)dy$.

u	v'		
y^2	$\sin y$		
$2y$	$-\cos y$	$+$	$-y^2\cos y$
2	$-\sin y$	$-$	$+2y\sin y$
0	$\cos y$	$+$	$+2\cos y$

Hence, $\int x^5\sin x^2dx = \frac{1}{2}[-x^4\cos^2 x + 2x^2\sin x^2 + 2\cos x^2] + C$.

59. Put $y = -3x^2$, so that $dy = -6x\,dx$. Thus, $\int x^3\cos(-3x^2)dx = -\frac{1}{6}\int -\frac{y}{3}\cos y\,dy = \frac{1}{18}\int y\cos y\,dy$.

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ y & \cos y & \\ 1 & \sin y & \xrightarrow{+} +y\sin y \\ 0 & -\cos y & \xrightarrow{-} -(-\cos y) \end{array}$$

Hence, $\int x^3\cos(-3x^2)dx = \frac{-x^2\sin(-3x^2)}{6} + \frac{\cos(-3x^2)}{18} + C = \frac{x^2\sin 3x^2}{6} + \frac{\cos 3x^2}{18} + C.$

60. Put $y = x^6$, so that $dy = 6x^5dx$. Thus, $\int x^{17}\cos x^6 dx = \frac{1}{6}\int y^2\cos y\, dy.$

$$\begin{array}{lll} \underline{u} & \underline{v'} & \\ y^2 & \cos y & \\ 2y & \sin y & \xrightarrow{+} +y^2\sin y \\ 2 & -\cos y & \xrightarrow{-} -(2y)(-\cos y) \\ 0 & -\sin y & \xrightarrow{+} +(2)(-\sin y) \end{array}$$

Therefore, $\int x^{17}\cos x^6 dx = \frac{1}{6}(x^{12}\sin x^6 + 2x^6\cos x^6 - 2\sin x^6) + C.$

61. $\frac{3y^2 - y + 1}{y(y-1)(y+1)} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}$. $A = \frac{1}{-1}$; $B = \frac{3}{2}$; $C = \frac{5}{2}$. Hence, $\int \frac{3y^2 - y + 1}{(y^2 - y)(y+1)}dy = \int -\frac{1}{y}dy + \int\frac{(3/2)}{y-1}dy + \int\frac{(5/2)}{y+1}dy = -\ln|y| + \frac{3}{2}\ln|y-1| + \frac{5}{2}\ln|y+1| + C = \ln\frac{|y-1|^{3/2}|y+1|^{5/2}}{|y|} + C.$

62. $\frac{2x+1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$. $A = \frac{1}{2}$; $B = \frac{-1}{-1} = 1$; $C = -\frac{3}{2}$. Hence, $\int\frac{2x+1}{x(x+1)(x+2)}dx = \int\frac{\frac{1}{2}}{x}dx + \int\frac{1}{x+1}dx + \int\frac{-3/2}{x+2}dx = \frac{1}{2}\ln|x| + \ln|x+1| - \frac{3}{2}\ln|x+2| + C = \ln\frac{|x|^{1/2}|x+1|}{|x+2|^{3/2}} + C.$

63. $\int\frac{3x^2 - x + 1}{x^3 - x^2}dx = \int\frac{3x^2 - x + 1}{x^2(x-1)}dx$. $\frac{3x^2 - x + 1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$. $\frac{1}{-1} = B$; $\frac{3}{1} = C$. $3x^2 - x + 1 = Ax(x-1) + B(x-1) + Cx^2$. $3 = A + C$, so $A = 0$. Thus, $\int\frac{3x^2 - x + 1}{x^3 - x^2}dx = \int -\frac{1}{x^2}dx + \int\frac{3}{x-1}dx = \frac{1}{x} + 3\ln|x-1| + C.$

64. $\frac{1}{x^3(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{1+x}$. $C = 1$; $D = -1$; and $1 = Ax^2(1+x) + Bx(1+x) + (1+x) - x^3$. Thus, $0 = A - 1$ and so $A = 1$. $0 = A + B$, so $B = -1$ Hence, $\int\frac{1}{x^3(1+x)}dx = \int\frac{1}{x}dx - \int\frac{1}{x^2}dx + \int\frac{1}{x^3}dx - \int\frac{1}{x+1}dx = \ln|x| + \frac{1}{x} - \frac{1}{2x^2} - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + \frac{2x-1}{2x^2} + C.$

65. $\frac{t^2 + 6t + 4}{t^4 + 5t^2 + 4} = \frac{t^2 + 6t + 4}{(t^2+4)(t^2+1)} = \frac{At+B}{t^2+4} + \frac{Ct+D}{t^2+1}$. So $t^2 + 6t + 4 = (At+B)(t^2+1) + (Ct+D)(t^2+4) = At^3 + Bt^2 + At + Bt + Ct^3 + Dt^2 + 4Ct + 4D$. So $0 = A + C$ and $1 = B + D$ and $6 = A + 4C$. Hence, $6 = 3C$ and $C = 2$. Thus, $A = -2$. Also, $4 = B + 4D$, so $4 = 1 - D + 4D$ and $D = 1$, so $B = 0$. Thus, $\int\frac{t^2 + 6t + 4}{t^4 + 5t^2 + 4}dt = \int\frac{-2t}{t^2+4}dt + \int\frac{2t+1}{t^2+1}dt = -\ln(t^2+4) + \ln(t^2+1) + \tan^{-1}t + C = \ln(\frac{t^2+1}{t^2+4}) + \tan^{-1}t + C.$

66. $\frac{x^2 - 4x - 4}{(x-2)(x^2+9)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+9}$. $A = \frac{-8}{13}$. $x^2 - 4x - 4 = \frac{-8}{13}x^2 - \frac{72}{13} + Bx^2 - 2Bx + Cx - 2C$. $1 = \frac{-8}{13} + B$, so $B = \frac{21}{13}$; $-4 = -2B + C$, so $C = -\frac{10}{13}$. Hence, $\int\frac{x^2 - 4x - 4}{(x-2)(x^2+9)}dx = \int\frac{-\frac{8}{3}}{x-2}dx + \int\frac{\frac{21}{13}x - \frac{10}{13}}{x^2+9}dx = -\frac{8}{3}\ln|x-2| + \frac{21}{13}(\frac{1}{2})\ln(x^2+9) - \frac{10}{13}(\frac{1}{3})\tan^{-1}\frac{x}{3} + C = \ln\left|\frac{(x^2+9)^{21/26}}{(x-2)^{8/3}}\right| - \frac{10}{39}\tan^{-1}\frac{x}{3} + C.$

67. $\int\frac{t^4 + 4t^3 + 6t^2 + 4t - 3}{t^4 - 1}dt = \int dt + \int\frac{4t^3 + 6t^2 + 4t - 2}{(t^2+1)(t-1)(t+1)}dt.$ Now

$$\frac{4t^3 + 6t^2 + 4t - 2}{(t^2+1)(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1},$$

where $A = 3$, $B = 1$ by short substitution, and

$C = 0$, $D = 4$ by equating coefficients. Now $\int \frac{3}{t - 1}\,dt + \int \frac{1}{t + 1}\,dt + \int \frac{4}{t^2 + 1}\,dt =$ $3 \ln|t - 1| + \ln|t + 1| + 4 \tan^{-1}t + C$. Thus, $\int \frac{t^4 + 4t^3 + 6t^2 + 4t - 3}{t^4 - 1}\,dt =$ $t + 3 \ln|t - 1| + \ln|t + 1| + 4 \tan^{-1} + C$.

68. Let $u = \cos t$, then $\int \frac{\sin t\,dt}{\cos^3 t + \cos t} = -\int \frac{du}{u^3 + u} =$ $-\int \frac{du}{u(u^2 + 1)} = -\int \frac{du}{u} + \int \frac{u\,du}{u^2 + 1} = -\ln|u| +$ $\frac{1}{2} \ln(u^2 + 1) + C = -\ln|\cos t| +$ $\frac{1}{2} \ln(1 + \cos^2 t) + C$.

69. Put $z = (1 + 2x)^{1/4}$, so $z^4 = 1 + 2x$ and $4z^3 dz = 2\,dx$. Thus, $\int \frac{x\,dx}{\sqrt[4]{1 + 2x}} = \int \frac{\frac{z^4 - 1}{2}}{z}(2z^3 dz) = \int (z^6 - z^3)dz =$ $\frac{z^7}{7} - \frac{z^3}{3} + C = \frac{(1 + 2x)^{7/4}}{7} - \frac{(1 + 2x)^{3/4}}{3} + C$.

70. Put $z = \sqrt[4]{t}$, $z^4 = t$ and $4z^3 dz = dt$. Thus, $\int \frac{dt}{\sqrt[4]{t} + 3} =$ $\int \frac{4z^3 dz}{z + 3} = \int (4z^2 - 12z + 36)dz - \int \frac{108}{z + 3}\,dz = \frac{4}{3}t^{3/4} -$ $6\sqrt{t} + 36\sqrt[4]{t} - 108 \ln(\sqrt[4]{t} + 3) + C$.

71. $\int \frac{\sqrt[5]{x^3} + \sqrt[6]{x}}{\sqrt{x}}\,dx = \int (x^{3/5 - 1/2} + x^{1/6 - 1/2})dx =$ $\int (x^{1/10} + x^{-1/3})dx = \frac{10}{11}x^{11/10} + \frac{3}{2}x^{2/3} + C$.

72. Put $u = y^{1/8}$, so that $u^8 = y$ and $8u^7 du = dy$. Thus, $\int \frac{dy}{\sqrt{y} + y^{3/4}} = \int \frac{8u^7 du}{u^4 + u^6} = \int \frac{8u^3 du}{1 + u^2} = \int 8u\,du -$ $\int \frac{8u}{1 + u^2}\,du = 4u^2 - 4 \ln(1 + u^2) + C =$ $4y^{1/4} - 4 \ln (1 + y^{1/4}) + C$.

73. Put $z = \sqrt{e^t + 1}$, so that $z^2 = e^t + 1$, $2z\,dz = e^t dt$, and $e^t = z^2 - 1$. Thus, $\int \frac{dt}{\sqrt{e^t + 1}} = \int \frac{e^t dt}{e^t\sqrt{e^t + 1}} =$ $\int \frac{2z\,dz}{(z^2 - 1)z} = \int \frac{2}{z^2 - 1}\,dz = \int \frac{-1}{z + 1}\,dz + \int \frac{1}{z - 1}\,dz =$ $-\ln|z + 1| + \ln|z - 1| + C = \ln\left|\frac{\sqrt{e^t + 1} - 1}{\sqrt{e^t + 1} + 1}\right| + C$.

74. Put $z = \sqrt{x}$, so that $z^2 = x$ and $2z\,dz = dx$. Thus, $\int \frac{\sqrt{x} + 1}{\sqrt{x} - 1}\,dx = \int \frac{(z + 1)}{z - 1}(2z\,dz) = \int (2z + 4)dz +$ $\int \frac{4}{z - 1}\,dz = z^2 + 4z + 4 \ln|z - 1| + C =$ $x + 4\sqrt{x} + 4 \ln|\sqrt{x} - 1| + C$.

75. Put $z = \sqrt{x + 1}$, so that $z^2 = x + 1$ and $2z\,dz = dx$. Thus, $\int \frac{dx}{\sqrt{4 + \sqrt{x + 1}}} = \int \frac{2z\,dz}{\sqrt{4 + z}}$. Now put $u = \sqrt{4 + z}$, so that $u^2 = 4 + z$ and $2u\,du = dz$. Hence, $\int \frac{2z\,dz}{\sqrt{4 + z}} = 2\int \frac{u^2 - 4}{u}(2u\,du) = \frac{4u^3}{3} - 16u + C =$ $\frac{4}{3}(4 + z)^{3/2} - 16\sqrt{4 + z} + C = \frac{4}{3}(4 + \sqrt{x + 1})^{3/2} -$ $16\sqrt{4 + \sqrt{x + 1}} + C$.

76. Put $u = \ln y$, so that $du = \frac{1}{y}\,dy$. Thus, $\int \frac{dy}{y \ln y (\ln y + 5)} = \int \frac{du}{u(u + 5)} = \int \frac{1/5}{u}\,du +$ $\int \frac{-1/5}{u + 5}\,du = \frac{1}{5} \ln|u| - \frac{1}{5} \ln|u + 5| + C =$ $\ln\left[\frac{\ln y}{\ln(\ln y + 5)}\right]^{1/5} + C$.

77. Put $u = \ln\sqrt{x^2 + 3} = \frac{1}{2} \ln(x^2 + 3)$ and $dv = dx$, so that $du = \frac{x\,dx}{x^2 + 3}$ and $v = x$. Thus, $\int \ln\sqrt{x^2 + 3}\,dx =$ $x \ln\sqrt{x^2 + 3} - \int x\left(\frac{x}{x^2 + 3}\right)dx = x \ln\sqrt{x^2 + 3} - \int dx +$ $\int \frac{3}{x^2 + 3}\,dx = x \ln\sqrt{x^2 + 3} - x + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + C$.

78. Put $u = \ln\sqrt[3]{5y + 2}$ and $dv = y\,dy$, so that $du = \frac{1}{3}\frac{5}{(5y + 2)}\,dy$ and $v = \frac{y^2}{2}$. Therefore, $\int y \ln\sqrt[3]{5y + 2}\,dy = \frac{y^2}{2} \ln\sqrt[3]{5y + 2} - \int \frac{5}{6}\left(\frac{y^2}{5y + 2}\right)dy$. Now $\int \frac{y^2}{5y + 2}\,dy = \int \left(\frac{1}{5}y - \frac{2}{25}\right)dy + \frac{4}{25}\int \frac{1}{5y + 2}\,dy =$ $\frac{y^2}{10} - \frac{2y}{25} + \frac{4}{125} \ln|5y + 2| + C_1$. Thus, $\int y \ln\sqrt[3]{5y + 2}\,dy = \frac{y^2}{2} \ln\sqrt[3]{5y + 2} - \frac{1}{12}y^2 +$ $\frac{1}{15}y - \frac{2}{75} \ln|5y + 2| + C$.

79. Put $u = x^{2/3}$, so that $u^3 = x^2$ and $3u^2 du = 2x\,dx$. Hence, $\int x\sqrt{1 - x^{2/3}}\,dx = \int \frac{3}{2}u^2 \sqrt{1 - u}\,du$. Now let

$z = \sqrt{1 - u}$, so that $z^2 = 1 - u$ and $2z\,dz = -du$.

Thus, $\int \frac{3}{2}u^2 \sqrt{1 - u}\,du = -\int \frac{3}{2}(1 - z^2)^2 z(2z\,dz) =$

$-3\int (z^6 - 2z^4 + z^2)dz = -\frac{3}{7}z^7 + \frac{6}{5}z^5 - z^3 + C =$

$-\frac{3}{7}(1 - u)^{7/2} + \frac{6}{5}(1 - u)^{5/2} - (1 - u)^{3/2} + C =$

$-\frac{3}{7}(1 - x^{2/3})^{7/2} + \frac{6}{5}(1 - x^{2/3})^{5/2} - (1 - x^{2/3})^{3/2} + C.$

80. Put $z = \sqrt{x}$, so that $z^2 = x$ and $2z\,dz = dx$. Hence,

$\int \sqrt{\frac{1 + \sqrt{x}}{x}}\,dx = \int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{\sqrt{1 + z}}{z}(2z\,dz) =$

$2\int \sqrt{1 + z}\,dz$. Now let $u = 1 + z$, so that $du = dz$.

Thus, $2\int \sqrt{1 + z}\,dz = 2\int u^{1/2}du = \frac{4}{3}u^{3/2} + C =$

$\frac{4}{3}(1 + z)^{3/2} + C$. Therefore, $\int \sqrt{\frac{1 + \sqrt{x}}{x}}\,dx =$

$\frac{4}{3}(1 + \sqrt{x})^{3/2} + C.$

81. Put $z = \sqrt[3]{x}$, so that $z^3 = x$ and $3z^2dz = dx$. Thus,

$\int \cos\sqrt[3]{x}\,dx = \int 3z^2 \cos z\,dz$. Now we use the tabular method:

u	v'	
$3z^2$	$\cos z$	
$6z$	$\sin z$	$\xrightarrow{+}$ $+\,3z^2(\sin z)$
6	$-\cos z$	$\xrightarrow{-}$ $-\,6z(-\cos z)$
0	$-\sin z$	$\xrightarrow{+}$ $+\,6(-\sin z)$

So $\int \cos\sqrt[3]{x}\,dx = \int 3z^2 \cos z\,dz = 3z^2 \sin z +$

$6z \cos z - 6 \sin z + C =$

$3x^{2/3}\sin\sqrt[3]{x} + 6\sqrt[3]{x}\cos\sqrt[3]{x} - 6\sin\sqrt[3]{x} + C.$

82. Put $z = \sqrt[4]{e^y + 3}$, so that $z^4 = e^y + 3$ and $4z^3dz = e^y dy$. Thus, $\int \frac{e^y e^y\,dy}{\sqrt[4]{e^y + 3}} = \int \frac{(z^4 - 3)4z^3dz}{z} =$

$4\int (z^6 - 3z^2)dz = \frac{4}{7}z^7 - 4z^3 + C =$

$\frac{4}{7}(e^y + 3)^{7/4} - 4(e^y + 3)^{3/4} + C.$

83. Put $z = \tan\frac{x}{2}$, so that $dz = \frac{2\,dz}{1 + z^2}$, $\cos x = \frac{1 - z^2}{1 + z^2}$,

and $\int \frac{dx}{10 + 11\cos x} = \int \frac{\left(\frac{2\,dz}{1 + z^2}\right)}{10 + 11\frac{1 - z^2}{1 + z^2}} = \int \frac{2\,dz}{21 - z^2} =$

$\frac{1}{\sqrt{21}}\int \frac{dz}{\sqrt{21} - z} + \frac{1}{\sqrt{21}}\int \frac{dz}{\sqrt{21} + z} =$

$-\frac{1}{\sqrt{21}}\ln|\sqrt{21} - z| + \frac{1}{\sqrt{21}}\ln|\sqrt{21} + z| + C =$

$\frac{1}{\sqrt{21}}\ln\left|\frac{\sqrt{21} + z}{\sqrt{21} - z}\right| + C = \frac{1}{\sqrt{21}}\ln\left|\frac{\sqrt{21} + \tan\frac{x}{2}}{\sqrt{21} - \tan\frac{x}{2}}\right| + C.$

84. Put $z = \tan\frac{x}{2}$. Then, $\int \frac{\sin x}{8 + \cos x}\,dx =$

$\int \frac{\frac{2z}{z^2 + 1}\left(\frac{2\,dz}{1 + z^2}\right)}{8 + \left(\frac{1 - z^2}{1 + z^2}\right)}\,dz = \int \frac{4z\,dz}{8(1 + z^2)^2 + (1 - z^2)(1 + z^2)} =$

$\int \frac{4z\,dz}{(7z^2 + 9)(z^2 + 1)} = \int \frac{-14}{7z^2 + 9}\,dz + \int \frac{2\,dz}{z^2 + 1} =$

$\int \frac{-2}{z^2 + \frac{9}{7}}\,dz + 2\tan^{-1}z + C = -2\frac{\sqrt{7}}{3}\tan^{-1}\frac{\sqrt{7}z}{3} +$

$2\tan^{-1}z + C = -2\frac{\sqrt{7}}{3}\tan^{-1}\left(\frac{\sqrt{7}}{3}\tan\frac{x}{2}\right) +$

$2\tan^{-1}\left(\tan\frac{x}{2}\right) + C = -\frac{2\sqrt{7}}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}\tan\frac{x}{2}\right) + x + C.$

85. Put $z = \tan\frac{y}{2}$. $\int \frac{dy}{3 + 2\sin y + \cos y} =$

$\frac{\frac{2\,dz}{1 + z^2}}{3 + 2\left(\frac{2z}{z^2 + 1}\right) + \left(\frac{1 - z^2}{1 + z^2}\right)} = \int \frac{2\,dz}{2z^2 + 4z + 4} =$

$\int \frac{dz}{z^2 + 2z + 2} = \int \frac{dz}{(z + 1)^2 + 1} = \tan^{-1}(z + 1) + C =$

$\tan^{-1}\left(1 + \tan\frac{y}{2}\right) + C.$

86. $\int \frac{\cot x}{\cot x + \csc x}\,dx = \int \frac{\cos x}{\cos x + 1}\,dx =$

$\int \frac{\frac{1 - z^2}{1 + z^2}}{\frac{1 - z^2}{1 + z^2} + 1}\left(\frac{2\,dz}{1 + z^2}\right) = \int \frac{2(1 - z^2)\,dz}{(1 - z^2)(1 + z^2) + (1 + z^2)^2}$

$\int \frac{1 - z^2}{1 + z^2}\,dz = \int -1\,dz + \int \frac{2}{1 + z^2}\,dz = -z + 2\tan^{-1}z + C$

$-\tan\frac{x}{2} + 2\tan^{-1}\left(\tan\frac{x}{2}\right) + C = -\tan\frac{x}{2} + x + C.$

87. Put $z = \tan\frac{x}{2}$. $\int \frac{\sec x}{1 + \sin x}\,dx = \int \frac{dx}{\cos x(1 + \sin x)}$

$\int \frac{\frac{2\,dz}{1 + z^2}}{\left(\frac{1 - z^2}{1 + z^2}\right)\left(1 + \frac{2z}{z^2 + 1}\right)} =$

$$\int \frac{2(1+z^2)}{(1-z^2(1+z^2)+(1-z^2)2z} dz =$$

$$\int \frac{2(1+z^2)}{(1-z)(1+z)^3} dz = \int \frac{(\frac{1}{2})}{1+z} dz + \int \frac{(-1)}{(1+z)^2} dz +$$

$$\int \frac{2}{(1+z)^3} dz + \int \frac{(\frac{1}{2})}{1-z} dz = \frac{1}{2} \ln|1+z| + \frac{1}{1+z} -$$

$$\frac{1}{(1+z)^2} - \frac{1}{2} \ln|1-z| + C = \frac{1}{2} \ln\left|\frac{1+z}{1-z}\right| +$$

$$\frac{z}{(1+z)^2} + C = \frac{1}{2} \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + \frac{\tan\frac{x}{2}}{(1+\tan\frac{x}{2})^2} + C.$$

88. $$\int \frac{dx}{3-\cos x + 2\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{3-(\frac{1-z^2}{1+z^2}) + 2(\frac{2z}{z^2+1})} =$$

$$\int \frac{2\,dz}{4z^2+4z+2} = \int \frac{2\,dz}{4(z+\frac{1}{2})^2+1} = \int \frac{\frac{1}{2}}{(z+\frac{1}{2})^2+\frac{1}{4}} dz =$$

$$\tfrac{1}{2}(2)\tan^{-1}\frac{(z+\frac{1}{2})}{\frac{1}{2}} + C = \tan^{-1}(2z+1) + C =$$

$$\tan^{-1}(2\tan\frac{x}{2}+1) + C.$$

89. Put $u = \sqrt[4]{e^{2x}+1}$, so that $u^4 = e^{2x}+1$ and $4u^3du = 2e^{2x}dx$. Thus, $\int \frac{e^{4x}}{\sqrt[4]{e^{2x}+1}} dx = \int \frac{(u^4-1)(2u^3du)}{u} =$

$$2\int(u^6-u^2)du = \frac{2}{7}u^7 - \frac{2}{3}u^3 + C = \frac{2}{7}(e^{2x}+1)^{7/4} -$$

$$\frac{2}{3}(e^{2x}+1)^{3/4} + C.$$

90. Put $z = \tan\frac{x}{2}$, so that $\int \frac{dx}{a^2\cos x + b^2\sin x} =$

$$\int \frac{\frac{2\,dz}{1+z^2}}{a^2(\frac{1-z^2}{1+z^2}) + b^2(\frac{2z}{z^2+1})} = \int \frac{2\,dz}{a^2 - a^2z^2 + 2b^2z} =$$

$$\int \frac{-2\,dz}{a^2z^2 - 2b^2z - a^2} = \int \frac{-2\,dz}{a^2(z-\frac{b^2}{a^2})^2 - (\frac{a^4+b^4}{a^2})}.$$

Now put $u = z - \frac{b^2}{a^2}$, so that $du = dz$. Thus,

$$\int \frac{-2\,dz}{a^2(z-\frac{b^2}{a^2})^2 - (\frac{a^4+b^4}{a^2})} = \int \frac{-2\,du}{a^2u^2 - (\frac{a^4+b^4}{a^2})}.$$

Call $\frac{a^4+b^4}{a^2} = A^2$. So $\int \frac{-2\,du}{a^2u^2 - A^2} =$

$$\int \frac{-2\,du}{(au+A)(au-A)} = \int \frac{(\frac{1}{A})}{au+A} dt + \int \frac{(-\frac{1}{A})}{au-A} dt =$$

$$\frac{1}{aA}\ln|au+A| - \frac{1}{aA}\ln|au-A| + C =$$

$$\frac{1}{\sqrt{a^4+b^4}} \ln\left|\frac{a^2z - b^2 + \sqrt{a^4+b^4}}{a^2z - b^2 - \sqrt{a^4+b^4}}\right| =$$

$$\frac{1}{\sqrt{a^4+b^4}} \ln\left|\frac{a^2\tan\frac{x}{2} + \sqrt{a^4+b^4} - b^2}{a^2\tan\frac{x}{2} - \sqrt{a^4+b^4} - b^2}\right| + C.$$

91. $$\int_0^{\pi/4} \cos x \cos 5x\, dx = \int_0^{\pi/4} \tfrac{1}{2}\cos(-4x)dx +$$

$$\int_0^{\pi/4} \tfrac{1}{2}\cos(6x)dx = \frac{1}{8}\sin 4x\Big|_0^{\pi/4} + \frac{1}{12}\sin 6x\Big|_0^{\pi/4} =$$

$$\frac{1}{12}\sin\frac{3\pi}{2} = -\frac{1}{12}.$$

92. $$\int_0^{\pi/4} \sin^3 2t\cos^3 2t\, dt =$$

$\int_0^{\pi/4} \sin 2t(1-\cos^2 2t)\cos^3 2t\, dt$. Put $u = \cos 2t$, so that $du = -2\sin 2t\, dt$. Hence,

$$\int_0^{\pi/4} \sin^3 2t\cos^3 2t\, dt = \int_1^0 -\frac{1}{2}(u^3-u^5)du =$$

$$(\frac{u^6}{12} - \frac{u^4}{8})\Big|_1^0 = -(\frac{1}{12} - \frac{1}{8}) = \frac{1}{24}.$$

93. $$\int_{\pi/12}^{\pi/8} \tan^3 2x\, dx = \int_{\pi/12}^{\pi/8} \tan 2x(\sec^2 2x - 1)dx =$$

$$\int_{\pi/12}^{\pi/8} \tan 2x\sec^2 2x\, dx - \int_{\pi/12}^{\pi/8} \tan 2x\, dx =$$

$$\frac{1}{4}\tan^2 2x\Big|_{\pi/12}^{\pi/8} + \frac{1}{2}\ln|\cos 2x|\Big|_{\pi/12}^{\pi/8} = \frac{1}{4}(1-\frac{1}{3}) +$$

$$\frac{1}{2}(\ln\frac{\sqrt{2}}{2} - \ln\frac{\sqrt{3}}{2}) = \frac{1}{6} + \frac{1}{2}\ln(\sqrt{\frac{2}{3}}) = \frac{1}{6} + \frac{1}{4}\ln\frac{2}{3}.$$

94. Put $u = \tan^{-1}x$ and $dv = x\,dx$, so that $du = \frac{1}{1+x^2}dx$ and $v = x^2$. Thus, $\int_0^1 x\tan^{-1}x\, dx = x^2\tan^{-1}x\Big|_0^1 -$

$$\int_0^1 \frac{x^2}{1+x^2}dx = (x^2\tan^{-1}x)\Big|_0^1 - \int_0^1 dx + \int_0^1 \frac{1}{1+x^2}dx =$$

$$(x^2\tan^{-1}x)\Big|_0^1 - x\Big|_0^1 + \tan^{-1}x\Big|_0^1 = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1.$$

95. Put $u = (\ln t)^2$ and $dv = dt$, so that $du = \frac{2\ln t}{t}dt$ and $v = t$. Thus, $\int_1^2 (\ln t)^2 dt = t(\ln t)^2\Big|_1^2 -$

$\int_1^2 2 \ln t\, dt$. Now let $u_1 = \ln t$ and $dv_1 = dt$, so that $du_1 = \frac{1}{t}\, dt$ and $v_1 = t$. Hence, $\int_1^2 2 \ln t\, dt = 2t(\ln t)\Big|_1^2 - \int_1^2 2\, dt = 4 \ln 2 - 2t\Big|_1^2 = 4 \ln 2 - 2$.

Therefore, $\int_1^2 (\ln t)^2 dt = t(\ln t)^2\Big|_1^2 - [4 \ln 2 - 2] = 2(\ln 2)^2 - 4 \ln 2 + 2$.

96.

u	v'		
x^2	$\sin 2x$		
$2x$	$-\frac{1}{2}\cos 2x$	$+$	$-\frac{1}{2}x^2 \cos 2x$
2	$-\frac{1}{4}\sin 2x$	$-$	$\frac{1}{2}x \sin 2x$
0	$\frac{1}{8}\cos 2x$	$+$	$\frac{1}{4}\cos 2x$

Hence, $\int_0^{\pi/4} x^2 \sin 2x\, dx = -\frac{1}{2}x^2 \cos 2x\Big|_0^{\pi/4} + \frac{1}{2}x \sin 2x\Big|_0^{\pi/4} + \frac{1}{4}\cos 2x\Big|_0^{\pi/4} = 0 + \frac{\pi}{8} + 0 = \frac{\pi}{8}$.

97. Put $u = \ln t$ and $dv = t^3 dt$, so that $du = \frac{1}{t}\, dt$ and $v = \frac{t^4}{4}$. Thus, $\int_1^2 t^3 \ln t\, dt = \frac{t^4}{4} \ln t\Big|_1^2 - \int_1^2 \frac{t^3}{4}\, dt = 4 \ln 2 - \frac{t^4}{16}\Big|_1^2 = 4 \ln 2 - 1 + \frac{1}{16} = 4 \ln 2 - \frac{15}{16}$.

98. $\int_0^{\pi} \sin^3 x\, dx = \int_0^{\pi} (1 - \cos^2 x)\sin x\, dx = (-\cos x + \frac{\cos^3 x}{3})\Big|_0^{\pi} = -\cos \pi + \frac{\cos^3 \pi}{3} + \cos 0 - \frac{\cos^3 0}{3} = 1 - \frac{1}{3} + 1 - \frac{1}{3} = \frac{4}{3}$.

99. $\int_{-\pi/8}^{\pi/8} |\tan^3 2x|\, dx = 2\int_0^{\pi/8} \tan^3 2x\, dx = 2\int_0^{\pi/8} \tan 2x\, (\sec^2 2x - 1)dx = 2(\frac{1}{4}\tan^2 2x - \frac{1}{2}\ln|\sec 2x|)\Big|_0^{\pi/8} = 2(\frac{1}{4} - \frac{1}{2}\ln \sqrt{2}) = \frac{1}{2} - \ln \sqrt{2}$.

100. $\int_0^1 \cosh^4 x\, dx = \int_0^1 (\frac{1 + \cosh 2x}{2})^2 dx = \frac{1}{4}\int_0^1 (1 + 2\cosh 2x + \cosh^2 2x)dx = \frac{1}{4}(x + \sinh 2x)\Big|_0^1 + \frac{1}{4}\int_0^1 \frac{1 + \cosh 4x}{2}\, dx = \frac{1}{4}(1 + \sinh 2) + \frac{1}{8}(x + \frac{1}{4}\sinh 4x)\Big|_0^1 = \frac{1}{4}(1 + \sinh 2) + \frac{1}{8}(1 + \frac{\sinh 4}{4}) = \frac{3}{8} + \frac{\sinh 2}{4} + \frac{\sinh 4}{32}$.

101. Put $x = 3 \sin \theta$, so that $dx = 3 \cos \theta\, d\theta$. Thus, $\int_3^{3/2} \frac{(9 - x^2)^{3/2}}{x^2}\, dx = \int_{\pi/2}^{\pi/6} \frac{27 \cos^3\theta (3 \cos \theta)d\theta}{9 \sin^2\theta} = 9\int_{\pi/2}^{\pi/6} \frac{(1 - \sin^2\theta)^2}{\sin^2\theta}\, d\theta = 9\int_{\pi/2}^{\pi/6} (\csc^2\theta - 2 + \sin^2\theta)d\theta = 9(-\cot \theta - 2\theta)\Big|_{\pi/2}^{\pi/6} + 9(\int_{\pi/2}^{\pi/6} \frac{1 - \cos 2\theta}{2}\, d\theta) = -9\,(\sqrt{3} + \frac{\pi}{3} - 0 - \pi) + 9(\frac{\theta}{2} - \frac{\sin 2\theta}{4})\Big|_{\pi/2}^{\pi/6} = -9\sqrt{3} + 6\pi + \frac{3\pi}{4} - \frac{9\sqrt{3}}{8} - \frac{9\pi}{4} + 0 = \frac{9\pi}{2} - \frac{81\sqrt{3}}{8}$.

102. Put $u = 3t^2$, so that $du = 6t\, dt$. Thus, $\int_0^{1/3} \frac{t\, dt}{\sqrt{1 - 9t^4}} = \int_0^{1/3} \frac{\frac{1}{6}\, du}{\sqrt{1 - u^2}} = \frac{1}{6}\sin^{-1} u\Big|_0^{1/3} = \frac{1}{6}\sin^{-1}\frac{1}{3} - \frac{1}{6}\sin^{-1} 0 = \frac{1}{6}\sin^{-1}\frac{1}{3}$.

103. Put $t = 5 \sec \theta$, so that $dt = 5 \sec \theta \tan \theta\, d\theta$ and $\int_5^{10} \frac{\sqrt{t^2 - 25}}{t}\, dt = \int_0^{\pi/3} \frac{5 \tan \theta\, (5 \sec \theta \tan \theta\, d\theta)}{5 \sec \theta} = 5\int_0^{\pi/3} (\sec^2\theta - 1)d\theta = 5(\tan \theta - \theta)\Big|_0^{\pi/3} = 5(\sqrt{3} - \frac{\pi}{3})$.

104. Put $x = a \sin \theta$, so that $dx = a \cos \theta\, d\theta$. Thus, $\int_0^a x^2\sqrt{a^2 - x^2}\, dx = \int_0^{\pi/2} a^2 \sin^2\theta (a \cos \theta) a \cos \theta\, d\theta = \frac{a^4}{4}\int_0^{\pi/2} (1 - \cos 2\theta)(1 + \cos 2\theta)d\theta = \frac{a^4}{4}\int_0^{\pi/2} (1 - \cos^2 2\theta)d\theta = \frac{a^4\theta}{4}\Big|_0^{\pi/2} - \frac{a^4}{4}\int_0^{\pi/2} (\frac{1 + \cos 4\theta}{2})d\theta = \frac{a^4\pi}{8} - \frac{a^4}{8}(\theta + \frac{\sin 4\theta}{4})\Big|_0^{\pi/2} = \frac{a^4\pi}{16}$.

105. Put $u = \frac{x}{3}$, so that $du = \frac{1}{3}\, dx$. Thus, $\int_0^{\pi} \sqrt{1 + \cos \frac{x}{3}}\, dx = \int_0^{\pi/3} 3\sqrt{1 + \cos u}\, du$. Now let $z = \tan \frac{u}{2}$. Then $\int_0^{\pi/3} 3\sqrt{1 + \cos u}\, du =$

$$\int_0^{\sqrt{3}/3} 3\sqrt{1+\frac{1-z^2}{1+z^2}}\cdot\frac{2\,dz}{1+z^2} = \int_0^{\sqrt{3}/3}\frac{3(2\sqrt{2})}{(1+z^2)^{3/2}}\,dz.$$

Now put $z = \tan\theta$, so that $\int_0^{\sqrt{3}/3}\frac{6\sqrt{2}}{(1+z^2)^{3/2}}\,dz =$

$$\int_0^{\pi/6}\frac{6\sqrt{2}\sec^2\theta\,d\theta}{\sec^3\theta} = 6\sqrt{2}\int_0^{\pi/6}\cos\theta\,d\theta =$$

$6\sqrt{2}\sin\theta\Big|_0^{\pi/6} = 3\sqrt{2}$. Hence, $\int_0^{\pi}\sqrt{1+\cos\frac{x}{3}}\,dx = 3\sqrt{2}$.

106. $$\int_{\pi/4}^{\pi/2}\frac{\cot x\,dx}{1-\cos x} = \int_{\pi/4}^{\pi/2}\frac{\cos x}{\sin x(1-\cos x)}\,dx =$$

$$\int_{\tan\pi/8}^{1}\frac{\frac{1-z^2}{1+z^2}\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{z^2+1}\right)\left(1-\frac{1-z^2}{1+z^2}\right)} =$$

$$\int_{\tan\pi/8}^{1}\frac{2(1-z^2)dz}{4z^3} = \left(-\frac{1}{4z^2}-\frac{1}{2}\ln|z|\right)\Big|_{\tan\pi/8}^{1} =$$

$$-\frac{1}{4}+\frac{1}{4(\tan^2\frac{\pi}{8})}+\frac{1}{2}\ln(\tan\tfrac{\pi}{8}).$$

107. Put $u = t - 2$, so that $du = dt$. Thus,

$$\int_3^5\frac{t^2-1}{(t-2)^2}\,dt = \int_1^3\frac{u^2+4u+3}{u^2}\,du =$$

$$\left(u+4\ln|u|-\frac{3}{u}\right)\Big|_1^3 = 3+4\ln 3-1-1-0+3 =$$

$4 + 4\ln 3$.

108. $$\int_1^2\frac{5x^2-3x+18}{x(9-x^2)}\,dx = \int_1^2\frac{2}{x}\,dx + \int_1^2\frac{-4}{3+x}\,dx +$$

$$\int_1^2\frac{3}{3-x}\,dx = 2\ln(x)\Big|_1^2 - 4\ln(3+x)\Big|_1^2 -$$

$$3\ln(3-x)\Big|_1^2 = 2\ln 2 - 4\ln 5 + 4\ln 4 + 3\ln 2 =$$

$5\ln 2 - 4\ln 5 + 4\ln 4$.

109. $$\int_0^1\frac{x^2+3x+1}{x^4+2x^2+1}\,dx = \int_0^1\frac{x^2+3x+1}{(x^2+1)^2}\,dx =$$

$$\int_0^1\frac{1}{x^2+1}\,dx + \int_0^1\frac{3x}{(x^2+1)^2}\,dx = \tan^{-1}x\Big|_0^1 -$$

$$\frac{3}{2(x^2+1)}\Big|_0^1 = \frac{\pi}{4}-\left(\frac{3}{4}-\frac{3}{2}\right) = \frac{\pi+3}{4}.$$

110. Put $u = t^2 + 1$, so that $du = 2t\,dt$. Thus,

$$\int_0^1\frac{t^5\,dt}{(t^2+1)^2} = \int_1^2\frac{(u-1)^2}{u^2}\left(\frac{du}{2}\right) =$$

$$\frac{1}{2}\int_1^2\left(1-\frac{2}{u}+\frac{1}{u^2}\right)du = \frac{1}{2}\left(u-2\ln|u|-\frac{1}{u}\right)\Big|_1^2 =$$

$\frac{1}{2}(2 - 2\ln 2 - \frac{1}{2} - 1 + 1) = \frac{3}{4} - \ln 2$.

111. Put $t = \frac{1}{x}$, so that $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2}\,dt$. Thus,

$$\int_{1/2}^2\frac{dx}{x\sqrt{5x^2+4x-1}} = \int_2^{1/2}\frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{5}{t^2}+\frac{4}{t}-1}} = \int_{1/2}^2\frac{dt}{\frac{5+4t-t^2}{t^2}} =$$

$$\int_{1/2}^2\frac{dt}{\sqrt{9-(t-2)^2}} = \int_{-3/2}^0\frac{du}{\sqrt{9-u^2}} = \sin^{-1}\frac{u}{3}\Big|_{-3/2}^0 =$$

$\sin^{-1}0 + \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ where we made the substitution $u = t - 2$.

112. $$\int_0^{1/5}(2x-x^2)^{3/2}dx = \int_0^{1/5}[1-(x-1)^2]^{3/2}dx.$$ Now put $u = x - 1$, so that $du = dx$. Then

$$\int_0^{1/5}[1-(x-1)^2]^{3/2}dx = \int_{-1}^{-4/5}(1-u^2)^{3/2}du.$$ Now let $u = \sin\theta$, so that $du = \cos\theta\,d\theta$. Then

$$\int_{-1}^{-4/5}(1-u^2)^{3/2}du = \int_{-\pi/2}^{\sin^{-1}4/5}\cos^3\theta(\cos\theta\,d\theta) =$$

$$\frac{1}{4}\left[x+\sin 2x+\frac{x}{2}+\frac{\sin 4x}{8}\right]\Big|_{-\pi/2}^{\sin^{-1}4/5}$$ by a method similar to that of Problem 100. Thus,

$$\int_0^{1/5}[2x-x^2]^{3/2}dx = \frac{1}{4}\left[\frac{3x}{2}+2\sin x\cos x+\frac{1}{8}(4\sin x\cos x)(1-\sin^2 x)\right]\Big|_{-\pi/2}^{\sin^{-1}4/5} =$$

$$\frac{1}{4}\left[\frac{3}{2}\sin^{-1}\frac{4}{5}+2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)+\frac{1}{2}\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(1-\frac{16}{25}\right)\right] - \left(-\frac{3\pi}{2}\right) = \frac{1}{4}\left(\frac{3}{2}\sin^{-1}\frac{4}{5}+\frac{654}{625}+\frac{3\pi}{2}\right).$$

113. Put $z = \sqrt[3]{x}$, so that $z^3 = x$ and $3z^2dz = dx$. Then

$$\int_1^8\frac{dx}{x+\sqrt[3]{x}} = \int_1^2\frac{3z^2dz}{z^3+z} = \int_1^2\frac{3z\,dz}{z^2+1} =$$

$$\frac{3}{2}\ln(z^2+1)\Big|_1^2 = \frac{3}{2}(\ln 5 - \ln 2) = \frac{3}{2}\ln\left(\frac{5}{2}\right).$$

114. Put $u = \sqrt{x}$, so that $u^2 = x$ and $2u\,du = dx$. Thus,

$$\int_1^4\frac{\sqrt{x}+1}{\sqrt{x}(x+1)}\,dx = \int_1^2\frac{(u+1)}{u(u^2+1)}(2u\,du) =$$

$$2\int_1^2\frac{u}{u^2+1}\,du + 2\int_1^2\frac{du}{u^2+1} = \ln(u^2+1)\Big|_1^2 +$$

$2 \tan^{-1}u\Big|_1^2 = \ln(\frac{5}{2}) + 2\tan^{-1}2 - \frac{\pi}{2}$.

115. Put $u = \sqrt{t-1}$, so that $u^2 = t - 1$, $t = u^2 + 1$, and $dt = 2u\,du$. Then $\int_2^5 \frac{t\,dt}{(t-1)^{3/2}} =$
$\int_1^2 \frac{(u^2+1)(2u\,du)}{u^3} = \int_1^2 (2 + \frac{2}{u^2})du = (2u - \frac{2}{u})\Big|_1^2 =$
$(4 - 1) - (2 - 2) = 3$.

116. Put $u = \sqrt{1+x}$, so that $u^2 = 1 + x$ and $2u\,du = dx$. Then $\int_{-1}^8 \frac{dx}{\sqrt{1+\sqrt{1+x}}} = \int_0^3 \frac{2u\,du}{\sqrt{1+u}}$. Now let $y = 1 + u$, so that $dy = du$. Thus, $\int_0^3 \frac{2u\,du}{\sqrt{1+u}} =$
$\int_1^4 \frac{2y-2}{\sqrt{y}}dy = (\frac{4}{3}y^{3/2} - 4y^{\frac{1}{2}})\Big|_1^4 = \frac{4}{3}(8) - 8 - \frac{4}{3} + 4 =$
$\frac{16}{3}$.

117. $\int_{1/4}^{5/4} \frac{dt}{\sqrt{t+1}-\sqrt{t}} = \int_{1/4}^{5/4} \frac{\sqrt{t+1}+\sqrt{t}}{1}dt =$
$\int_{1/4}^{5/4}\sqrt{t+1}\,dt + \int_{1/4}^{5/4}\sqrt{t}\,dt = \frac{2}{3}(t+1)^{3/2}\Big|_{1/4}^{5/4} +$
$\frac{2}{3}t^{3/2}\Big|_{1/4}^{5/4} = \frac{2}{3}[(\frac{9}{4})^{3/2} - (\frac{5}{4})^{3/2}] +$
$\frac{2}{3}[(\frac{5}{4})^{3/2} - (\frac{1}{4})^{3/2}] = \frac{2}{3}(\frac{27}{8} - \frac{1}{8}) = \frac{13}{6}$.

118. Put $u = \sqrt{y}$, so that $u^2 = y$ and $2u\,du = dy$. Thus,
$\int_{16}^{25} \frac{dy}{y - 2\sqrt{y} - 3} = \int_4^5 \frac{2u\,du}{u^2 - 2u - 3} =$
$\int_4^5 \frac{2u\,du}{(u-3)(u+1)} = \int_4^5 \frac{3/2}{u-3}du + \int_4^5 \frac{\frac{1}{2}}{u+1}du =$
$\frac{3}{2}\ln(u-3)\Big|_4^5 + \frac{1}{2}\ln(u+1)\Big|_4^5 = \frac{3}{2}\ln 2 + \frac{1}{2}\ln\frac{6}{5}$.

119. $\int_0^{\ln 4} \frac{dx}{\sqrt{e^{-2x} + 2e^{-x}}} = \int_0^{\ln 4} \frac{e^{-x}\,dx}{e^{-x}\sqrt{(e^{-x})^2 + 2e^{-x}}}$. Put $u = e^{-x}$, so that $du = -e^{-x}dx$. Then
$\int_0^{\ln 4} \frac{e^{-x}\,dx}{e^{-x}\sqrt{(e^{-x})^2 + 2e^{-x}}} = \int_1^{\frac{1}{4}} \frac{-du}{u\sqrt{u^2 + 2u}}$. Now let $u = \frac{1}{t}$, so that $du = -\frac{1}{t^2}dt$. Then $\int_1^4 \frac{\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2} + \frac{2}{t}}} =$
$\int_1^4 \frac{dt}{\sqrt{1+2t}} = \sqrt{1+2t}\Big|_1^4 = 3 - \sqrt{3}$, where we evaluate $\int_1^4 \frac{dt}{\sqrt{1+2t}}$ by putting $z = 1 + 2t$. Thus,
$\int_0^{\ln 4} \frac{dx}{\sqrt{e^{-2x} + 2e^{-x}}} = 3 - \sqrt{3}$.

120. $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x + 2\cos x} =$
$\int_0^{\tan\frac{\pi}{8}} \frac{\frac{2\,dz}{1+z^2}}{1 - \frac{2z}{1+z^2} + \frac{2(1-z^2)}{1+z^2}} =$
$\int_0^{\tan\frac{\pi}{8}} \frac{-2\,dz}{(z+3)(z-1)} = \int_0^{\tan\frac{\pi}{8}} \left(\frac{\frac{1}{2}}{z+3} - \frac{\frac{1}{2}}{z-1}\right)dz =$
$(\frac{1}{2}\ln|z+3| - \frac{1}{2}\ln|z-1|)\Big|_0^{\tan\frac{\pi}{8}} =$
$\frac{1}{2}\ln|3 + \tan\frac{\pi}{8}| - \frac{1}{2}\ln|-1 + \tan\frac{\pi}{8}| - \frac{1}{2}\ln 3 =$
$\frac{1}{2}\left(\ln\left|\frac{3 + \tan\frac{\pi}{8}}{3(-1 + \tan\frac{\pi}{8})}\right|\right)$.

121. $\int_{\pi/4}^{\pi/8} \frac{dx}{\sin x + \tan x} = \int_{\pi/4}^{\pi/8} \frac{\cos x\,dx}{\sin x\cos x + \sin x} =$
$\int_{\tan\pi/8}^{\tan\pi/16} \frac{(\frac{1-z^2}{1+z^2})(\frac{2\,dz}{1+z^2})}{\frac{2z}{z^2+1}(\frac{1-z^2}{1+z^2} + 1)} =$
$\int_{\tan\pi/8}^{\tan\pi/16} \frac{2(1-z^2)dz}{2z(1 - z^2 + 1 + z^2)} =$
$\int_{\tan\pi/8}^{\tan\pi/16} \frac{(2 - 2z^2)}{4z}dz = (\frac{1}{2}\ln|z| - \frac{z^2}{4})\Big|_{\tan\pi/8}^{\tan\pi/16} =$
$\frac{1}{2}\ln(\tan\frac{\pi}{16}) - \frac{\tan^2\frac{\pi}{16}}{4} - \frac{1}{2}\ln(\tan\frac{\pi}{8}) + \frac{\tan^2\frac{\pi}{8}}{4} =$
$\frac{1}{2}\ln(\frac{\tan\frac{\pi}{16}}{\tan\frac{\pi}{8}}) - \frac{1}{4}(\tan^2\frac{\pi}{16} - \tan^2\frac{\pi}{8})$.

122. Put $u = \sqrt[3]{x}$ and $u^3 = x$, so that $3u^2du = dx$. Thus,
$\int_{1/8}^1 \frac{x\,dx}{x + \sqrt[3]{x}} = \int_{\frac{1}{2}}^1 \frac{u^3(3u^2du)}{u^3 + u} = \int_{\frac{1}{2}}^1 \frac{3u^4}{u^2+1}du =$

$\int_{1/2}^{1} (3u^2 - 3)du + \int_{1/2}^{1} \frac{3}{u^2 + 1} du = (u^3 - 3u)\Big|_{1/2}^{1} +$

$3\tan^{-1}u\Big|_{1/2}^{1} = (1 - 3) - (\frac{1}{8} - \frac{3}{2}) + 3\tan^{-1}1 -$

$3\tan^{-1}\frac{1}{2} = -\frac{5}{8} + \frac{3\pi}{4} - 3\tan^{-1}\frac{1}{2}$.

123. $\frac{c \sin\theta + d\cos\theta}{e\sin\theta + f\cos\theta} =$

$\frac{Ae\sin\theta + Af\cos\theta + Be\cos\theta - Bf\sin\theta}{e\sin\theta + f\cos\theta}$.

Hence, $\begin{cases} c = Ae - Bf \\ d = Af + Be \end{cases}$ and so $\begin{cases} cf = Aef - Bf^2 \\ de = Aef + Be^2 \end{cases}$.

Thus, $cf - de = -B(f^2 + e^2)$ and $B = \frac{de - cf}{e^2 + f^2}$. Also,

$\begin{cases} ce = Ae^2 - Bef \\ df = Af^2 + Bef \end{cases}$ and so $ce + df = A(e^2 + f^2)$.

Thus, $A = \frac{ce + df}{e^2 + f^2}$. Now $\int \frac{c\sin\theta + d\cos\theta}{e\sin\theta + f\cos\theta} d\theta =$

$\int A\, d\theta + \int B\,\frac{e\cos\theta - f\sin\theta}{e\sin\theta + f\cos\theta} d\theta = A\theta + \int B\,\frac{du}{u}$

where $u = e\sin\theta + f\cos\theta$ and $du =$

$(e\cos\theta - f\sin\theta)d\theta) = A\theta + B\ln|u| + C =$

$A\theta + B\ln|e\sin\theta + f\cos\theta| + C = (\frac{ce + df}{e^2 + f^2})\theta +$

$(\frac{de - cf}{e^2 + f^2})\ln|e\sin\theta + f\cos\theta| + C$.

124. For $0 < t < 1$, $t^3 < t^2$, so that $-t^3 > -t^2$ and $1 - t^3 > 1 - t^2$; thus, $\sqrt{1 - t^3} > \sqrt{1 - t^2}$. Hence,

$\frac{1}{\sqrt{1 - t^3}} < \frac{1}{\sqrt{1 - t^2}}$. Now $\int_0^x \frac{1}{\sqrt{1 - t^3}} dt < \int_0^x \frac{1}{\sqrt{1 - t^2}}$,

so that $\int_0^x \frac{1}{\sqrt{1 - t^3}} dt < \sin^{-1}t\Big|_0^x = \sin^{-1}x$.

125. Put $\sqrt{x^2 - 2x + 5} = z - x$, so that $x^2 - 2x + 5 =$

$z^2 - 2zx + x^2$, $x = \frac{z^2 - 5}{2z - 2}$, $-2\,dx = 2z\,dz -$

$2z\,dx - 2x\,dz$, and $dx = \frac{(z - x)}{z - 1} dz$. Thus,

$\int \frac{dx}{\sqrt{x^2 - 2x + 5}} = \int \frac{(\frac{z - x}{z - 1})dz}{\frac{z^2 - 5}{2(z - 1)}(z - x)} = \int \frac{2}{z^2 - 5} dz =$

$\int \frac{-\frac{1}{\sqrt{5}}}{z + \sqrt{5}} dz + \int \frac{\frac{1}{\sqrt{5}}}{z - \sqrt{5}} dz = -\frac{1}{\sqrt{5}}\ln|z + \sqrt{5}| +$

$\frac{1}{\sqrt{5}}\ln|z - \sqrt{5}| + C = \frac{1}{\sqrt{5}}\ln\left|\frac{z - \sqrt{5}}{z + \sqrt{5}}\right| + C =$

$\frac{1}{\sqrt{5}}\ln\left|\frac{x + \sqrt{x^2 - 2x + 5} - \sqrt{5}}{x + \sqrt{x^2 - 2x + 5} + \sqrt{5}}\right| + C$.

126. $\frac{1}{t + \sqrt{t^2}} < \frac{1}{t + \sqrt{t^2 - 1}} \le \frac{1}{t}$, $t \ge 1$, so

$\int_1^x \frac{1}{t + \sqrt{t^2}} dt \le \int_1^x \frac{dt}{t + \sqrt{t^2 - 1}} \le \int_1^x \frac{dt}{t}$ for $x \ge 1$.

Thus $\frac{1}{2}\int_1^x \frac{1}{t} dt \le f(x) \le \ln t\Big|_1^x$ and so

$\frac{1}{2}\ln x \le f(x) \le \ln x$.

127.
$$\begin{array}{lllll} u & v' & & \\ y^2 & e^{-y} & & e^{-y} \\ 2y & -e^{-y} & \xrightarrow{+} & -y^2e^{-y} \\ 2 & e^{-t} & \xrightarrow{-} & -2ye^{-y} \\ 0 & -e^{-y} & \xrightarrow{+} & -2e^{-y} \end{array}$$

So $\int e^{-y}y^2 dy = e^{-y}(-2 - 2y - y^2) + C =$

$2e^{-y}(-1 - y - \frac{y^2}{2}) + C$. Hence, $\int_0^x e^{-y}y^2 dy =$

$2e^{-x}(-1 - x - \frac{x^2}{2}) - 2(-1) = 2e^{-x}(e^x - 1 - x - \frac{x^2}{2})$.

128. Put $u = \frac{1}{t}$, so that $du = -\frac{1}{t^2} dt$ and $-t\,du = \frac{dt}{t}$.

Thus, $g(x) = \int_1^x f(t + \frac{1}{t})\frac{dt}{t} = \int_1^{1/x} f(\frac{1}{u} + u)(-t\,du) =$

$-\int_1^{1/x} f(u + \frac{1}{u})\frac{du}{u} = -g(\frac{1}{x})$. Hence, $g(\frac{1}{x}) = -g(x)$.

129. Put $u = (\ln x)^n$ and $dv = x^m dx$, so that $du =$

$\frac{n(\ln x)^{n-1}}{x} dx$ and $v = \frac{x^{m+1}}{m + 1}$, $m \ne -1$. Thus,

$\int x^m(\ln x)^n dx = uv - \int v\,du = \frac{x^{m+1}(\ln x)^n}{m + 1} -$

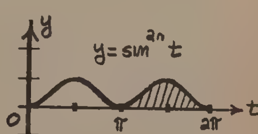
$\frac{n}{m + 1}\int (\ln x)^{n-1}x^m dx$, $m \ne -1$.

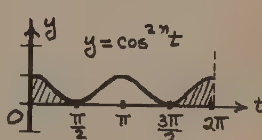
130. The areas represented by

$\int_0^{2\pi} \sin^{2n}t\,dt$ and

$\int_0^{2\pi} \cos^{2n}t\,dt$ are

the same. As indicated in the adjacent figure, the area of the shaded region under the





first graph corresponds to the two shaded regions under the second graph. Similarly, the unshaded regions correspond. Hence, the total areas are equivalent.

131. $A = \int_1^4 (\frac{1}{2}x^2 - \frac{1}{2}\ln x)dx = \frac{x^3}{12}\Big|_1^4 - \frac{1}{2}\int_1^4 \ln x\,dx$. Now put $u = \ln x$ and $dv = dx$, so that $du = \frac{1}{x}dx$ and $v = x$. Thus, $\int_1^4 \ln x\,dx = x\ln x\Big|_1^4 - \int_1^4 dx = x\ln x\Big|_1^4 - x\Big|_1^4$. Hence, $A = (\frac{4^3}{12} - \frac{1}{12}) - \frac{1}{2}[4\ln 4 - (4 - 1)]$. $A = \frac{27}{4} - 2\ln 4$ square units.

132. $A = \int_0^1 x^2e^{-x}dx$. We evaluate $\int x^2e^{-x}dx$ by using Problem 130. Hence, $A = 2!e^{-x}(e^x - 1 - x - \frac{x^2}{2!})\Big|_0^1 = 2(\frac{1}{e})(e - 1 - 1 - \frac{1}{2}) - 2(1 - 1) = 2 - \frac{5}{e}$ square unit.

133. $A = \int_0^\pi \sin^3 x\,dx = \int_0^\pi (1 - \cos^2 x)\sin x\,dx$. Put $u = \cos x$ so that $du = -\sin x\,dx$. Hence, $A = \int_1^{-1}(u^2 - 1)du = (\frac{u^3}{3} - u)\Big|_1^{-1} = -2(\frac{1}{3} - 1) = \frac{4}{3}$ square units.

134. $V = \pi\int_1^4 (x\ln x)^2dx = \pi\int_1^4 x^2(\ln x)^2dx$. Put $u = (\ln x)^2$ and $dV = x^2dx$, so that $du = \frac{2\ln x}{x}dx$ and $V = \frac{x^3}{3}$. Hence, $V = \pi[\frac{x^3(\ln x)^2}{3}\Big|_1^4 - \int_1^4 \frac{2}{3}x^2\ln x\,dx]$. Now put $u_1 = \ln x$ and $dV_1 = x^2dx$, so that $du_1 = \frac{dx}{x}$ and $V_1 = \frac{x^3}{3}$. Thus $\int_1^4 \frac{2}{3}x^2\ln x\,dx = \frac{2}{3}(\frac{x^3\ln x}{3})\Big|_1^4 - \frac{2}{3}\int_1^4 \frac{x^2}{3}dx = \frac{2}{9}x^3\ln x\Big|_1^4 - \frac{2}{27}x^3\Big|_1^4$. Therefore, $V = \pi[\frac{x^3(\ln x)^2}{3}\Big|_1^4 - \frac{2}{9}x^3\ln x\Big|_1^4 + \frac{2}{27}x^3\Big|_1^4] = \pi[\frac{64(\ln 4)^2}{3} - \frac{128}{9}\ln 4 + \frac{14}{3}]$ cubic units.

135. $V = \int_0^\pi \pi(y + 2)^2dx = \int_0^\pi \pi(\sin x + 2)^2dx = \pi\int_0^\pi (\sin^2 x + 4\sin x + 4)dx = \pi\int_0^\pi (\frac{1}{2} - \frac{\cos 2x}{2} + 4\sin x + 4)dx = \pi[\frac{x}{2} - \frac{\sin 2x}{4} - 4\cos x + 4x]\Big|_0^\pi = \pi(\frac{9\pi}{2} + 4 + 4) = \frac{\pi}{2}(9\pi + 16)$ cubic units.

136. $V = \frac{t + 3}{t^3 + t}$, so that $s = \int_1^3 \frac{t + 3}{t(t^2 + 1)}dt = \int_1^3 \frac{3}{t}dt + \int_1^3 \frac{-3t + 1}{t^2 + 1}dt = 3\ln t\Big|_1^3 - \frac{3}{2}\ln(t^2 + 1)\Big|_1^3 + \tan^{-1}t\Big|_1^3 = 3\ln 3 - \frac{3}{2}\ln 10 + \frac{3}{2}\ln 2 + \tan^{-1}3 - \tan^{-1}1 = \ln(\frac{3\sqrt{2}}{\sqrt{10}})^3 - \frac{\pi}{4} + \tan^{-1}3$ meters.

137. $\int_0^{\pi/3}\sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/3}\sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/3} = \ln(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}) - \ln 1 = \ln(2 + \sqrt{3})$.

138. $s = \int_1^{\sqrt{3}}\sqrt{1 + \frac{1}{x^2}}\,dx = \int_1^{\sqrt{3}}\frac{\sqrt{1 + x^2}}{x}dx$. Put $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Thus, $\int_1^{\sqrt{3}}\frac{\sqrt{1 + x^2}}{x}dx = \int_{\pi/4}^{\pi/3}\frac{\sec\theta\sec^2\theta\,d\theta}{\tan\theta} = \int_{\pi/4}^{\pi/3}\frac{\sec^2\theta\sec\theta\tan\theta}{\tan^2\theta}d\theta = \int_{\pi/4}^{\pi/3}\frac{\sec^2\theta(\sec\theta\tan\theta)}{(\sec^2\theta - 1)}d\theta$. Now let $u = \sec\theta$, so that $du = \sec\theta\tan\theta\,d\theta$. Hence, $\int_{\pi/4}^{\pi/3}\frac{\sec^2\theta(\sec\theta\tan\theta)d\theta}{(\sec^2\theta - 1)} = \int_{\sqrt{2}}^2\frac{u^2du}{u^2 - 1} = \int_{\sqrt{2}}^2 1\,du + \int_{\sqrt{2}}^2\frac{-\frac{1}{2}}{u + 1}du + \int_{\sqrt{2}}^2\frac{\frac{1}{2}}{u - 1}du = u\Big|_{\sqrt{2}}^a - \frac{1}{2}\ln(u + 1)\Big|_{\sqrt{2}}^2 + \frac{1}{2}\ln(u - 1)\Big|_{\sqrt{2}}^2 = 2 - \sqrt{2} - \frac{1}{2}[\ln 3 - \ln(\sqrt{2} + 1) + \ln(\sqrt{2} - 1)] = 2 - \sqrt{2} - \frac{1}{2}\ln[\frac{3(\sqrt{2} - 1)}{\sqrt{2} + 1}]$. Hence, $s = 2 - \sqrt{2} - \frac{1}{2}\ln[\frac{3(\sqrt{2} - 1)}{\sqrt{2} + 1}]$ units.

139. $s = \int_{\pi/6}^{\pi/2}\sqrt{1 + (\frac{-\csc x\cot x}{\csc x})^2}\,dx = \int_{\pi/6}^{\pi/2}\sqrt{1 + \cot^2 x}\,dx = \int_{\pi/6}^{\pi/2}\csc x\,dx = \ln|\csc x - \cot x|\Big|_{\pi/6}^{\pi/2} = \ln(1 - 0) - \ln(2 - \sqrt{3}) =$

$\ln(\frac{1}{2 - \sqrt{3}}) = -\ln(2 - \sqrt{3})$ units.

140. $S = 2\pi\int_0^1 e^x\sqrt{1 + (e^x)^2}\,dx$. Put $u = e^x$, so that

$du = e^x dx$. Thus, $S = 2\pi\int_1^e \sqrt{1 + u^2}\,du =$

$2\pi\int_{\pi/4}^{\tan^{-1}e} \sqrt{1 + \tan^2\theta}\cdot\sec^2\theta\,d\theta$, where $u = \tan\theta$.

Thus, $S = 2\pi\int_{\pi/4}^{\tan^{-1}e} \sec^3\theta\,d\theta =$

$2\pi(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta|)\Big|_{\pi/4}^{\tan^{-1}e} =$

$\pi[\sqrt{1 + e^2}\cdot(e) + \ln(\sqrt{1 + e^2} + e) - \sqrt{2} - \ln(\sqrt{2} + 1)] =$

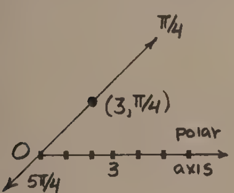
$\pi[e\sqrt{1 + e^2} - \sqrt{2} + \ln(\frac{e + \sqrt{1 + e^2}}{\sqrt{2} + 1})]$ square units.

9

POLAR COORDINATES AND ANALYTIC GEOMETRY

Problem Set 9.1, page 540

1.

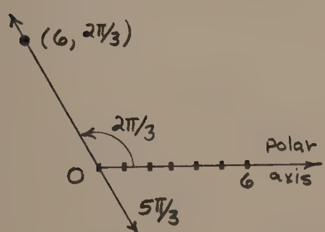


(a) $(-3, \frac{\pi}{4} + \pi) = (-3, \frac{5\pi}{4})$.

(b) $(3, \frac{\pi}{4} - 2\pi) = (3, -\frac{7\pi}{4})$.

(c) $(-3, \frac{5\pi}{4} - 2\pi) = (-3, -\frac{3\pi}{4})$.

2.

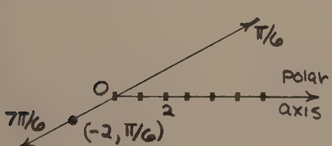


(a) $(-6, \frac{2\pi}{3} + \pi) = (-6, \frac{5\pi}{3})$.

(b) $(6, \frac{2\pi}{3} - \pi) = (6, -\frac{4\pi}{3})$.

(c) $(-6, \frac{5\pi}{3} - 2\pi) = (-6, -\frac{\pi}{3})$.

3.

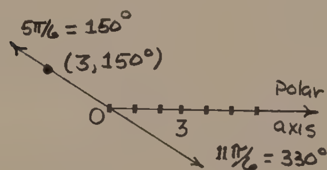


(a) Already in this form.

(b) $(2, \frac{\pi}{6} - \pi) = (2, -\frac{5\pi}{6})$.

(c) $(-2, \frac{\pi}{6} - 2\pi) = (-2, -\frac{11\pi}{6})$.

4.



(a) $(-3, 150° + 180°) = (-3, 330°)$.

(b) $(3, 150° - 360°) = (3, -210°)$.

(c) $(-3, 330° - 360°) = (-3, -30°)$.

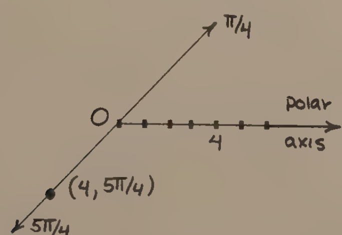
5.

(a) $(-4, 180° - 180°) = (-4, 0°)$.

(b) $(4, 180° - 360°) = (4, -180°)$.

(c) $(-4, 360° - 360°) = (-4, 0°)$.

6.



(a) $(-4, \frac{5\pi}{4} - \pi) = (-4, \frac{\pi}{4})$.

(b) $(4, \frac{5\pi}{4} - 2\pi) = (4, -\frac{3\pi}{4})$.

(c) $(-4, \frac{\pi}{4} - 2\pi) = (-4, -\frac{7\pi}{4})$.

7. $x = 7 \cos \frac{\pi}{3} = 7(\frac{1}{2}) = \frac{7}{2}$, $y = 7 \sin \frac{\pi}{3} = 7(\frac{\sqrt{3}}{2})$.

8. $x = 0$, $y = 0$.

9. $x = (-2) \cos \frac{\pi}{4} = -2(\frac{\sqrt{2}}{2}) = -\sqrt{2}$, $y =$
$-2 \sin \frac{\pi}{4} = -2(\frac{\sqrt{2}}{2}) = -\sqrt{2}$.

10. $x = 6 \cos \frac{13\pi}{6} = 6(\frac{\sqrt{3}}{2}) = 3\sqrt{3}$, $y =$
$6 \sin \frac{13\pi}{6} = 6(-\frac{1}{2}) = -3$.

11. $x = 1 \cdot \cos (-\frac{\pi}{3}) = \frac{1}{2}$, $y = 1 \cdot \sin (-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$.

12. $x = (-5) \cos 150° = (-5)(-\frac{\sqrt{3}}{2}) = \frac{5\sqrt{3}}{2}$,
$y = (-5) \sin 150° = (-5)(\frac{1}{2}) = -\frac{5}{2}$.

13. $r = \sqrt{x^2 + y^2} = \sqrt{49 + 49} = 7\sqrt{2}$, $\theta =$
$\tan^{-1} \frac{7}{7} = \tan^{-1} 1 = \frac{\pi}{4}$.

14. $r = \sqrt{1 + 3} = 2$, $\theta = \tan^{-1} - \frac{\sqrt{3}}{1} = -\frac{\pi}{3}$.

15. $r = \sqrt{9 + 27} = 6$, $\theta = \tan^{-1}(\frac{-3\sqrt{3}}{-3}) - \pi =$
$\tan^{-1}\sqrt{3} - \pi = \frac{-2\pi}{3}$.

16. $r = \sqrt{25 + 25} = 5\sqrt{2}$, $\theta = \tan^{-1} - \frac{5}{5} + \pi =$
$\tan^{-1}(-1) + \pi = \frac{3\pi}{4}$.

17. $r = \sqrt{0 + 49} = 7$, $\theta = \frac{\pi}{2}$.

18. $r = \sqrt{4 + 0} = 2$, $\theta = \tan^{-1} - \frac{0}{2} + \pi = \pi$.

19. (13) Same.

(14) $r = 2$, $= \frac{5\pi}{3}$.

(15) $r = 6$, $= \frac{4\pi}{3}$.

(16) Same.

(17) Same.

(18) Same.

20. (13) $r = -7\sqrt{2}$, $= \frac{5\pi}{4}$.

(14) $r = -2$, $\theta = \frac{2\pi}{3}$.

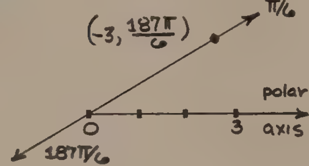
(15) $r = -6$, $\theta = \frac{\pi}{3}$.

(16) $r = -5\sqrt{2}$, $\theta = \frac{7\pi}{4}$.

(17) $r = -7$, $\theta = \frac{3\pi}{2}$.

(18) $r = -2$, $\theta = 0$.

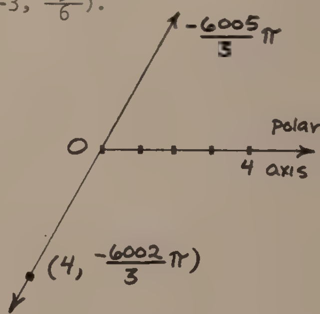
21.



Note that $\frac{187\pi}{6} = 31\pi + \frac{\pi}{6} = 30\pi + \frac{7\pi}{6}$.

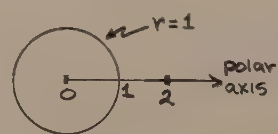
Thus, $(-3, \frac{7\pi}{6})$ or $(3, \frac{\pi}{6})$ or $(3, -\frac{11\pi}{6})$ or $(3, \frac{13\pi}{6})$ or $(-3, \frac{-5\pi}{6})$.

22.

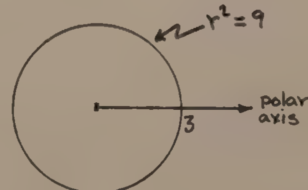


$(4, -\frac{2\pi}{3})$ or $(-4, \frac{\pi}{3})$ or $(4, \frac{4\pi}{3})$ or $(4, -\frac{8\pi}{3})$ or $(-4, -\frac{6005\pi}{3})$.

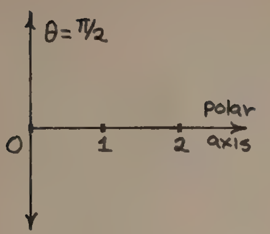
23.



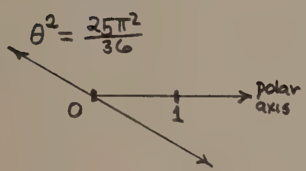
24.



25.

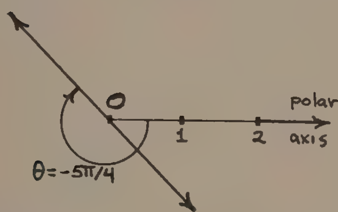


26.

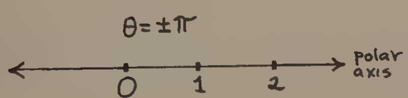


27. Same as Problem 25.

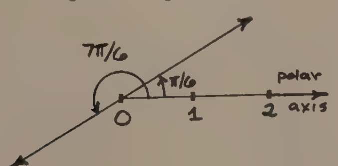
28.



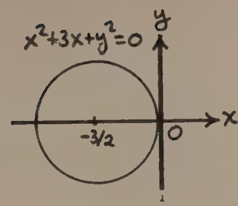
29. $|\theta| = \pi$. Thus, $\theta = \pm\pi$.



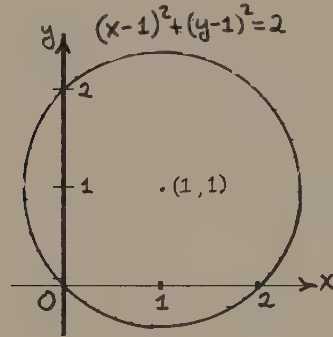
30. Factor $\theta^2 - \frac{4\pi}{3}\theta + \frac{7\pi^2}{36}$ and obtain $(\theta - \frac{\pi}{6})(\theta - \frac{7\pi}{6}) = 0$; so $\theta = \frac{\pi}{6}$ and $\theta = \frac{7\pi}{6}$.



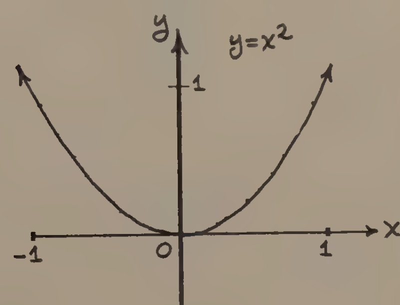
31. $r = -3 \cos \theta$. Multiply both sides by r and obtain $r^2 = 3r \cos \theta$. Then $x^2 + y^2 = -3x$; that is, $x^2 + 3x + y^2 = 0$ or $(x + \frac{3}{2})^2 + y^2 = \frac{9}{4}$ is the Cartesian equation, which is a circle of radius $\frac{3}{2}$ centered at $(-\frac{3}{2}, 0)$.



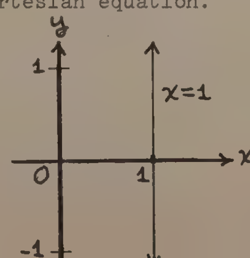
32. Multiply both sides by r and obtain $r^2 = 2r \cos \theta + 2r \sin \theta$. Then $x^2 + y^2 = 2x + 2y$ or $(x - 1)^2 + (y - 1)^2 = 2$, which is a circle of radius $\sqrt{2}$ centered at $(1, 1)$.



33. $r \cos^2 \theta = \sin \theta$. Multiply both sides by r and obtain $r^2 \cos^2\theta = r \sin \theta$. Thus, $x^2 = y$. Note: $\theta \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

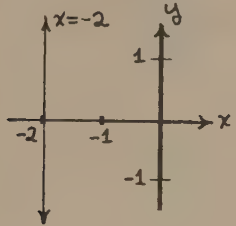


34. $r = \frac{1}{\cos \theta}$, or $r \cos \theta = 1$. Thus, $x = 1$ is the Cartesian equation.

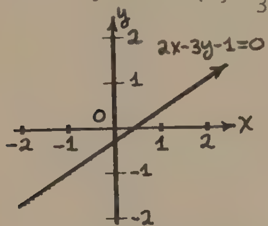


35. $r \cos \theta = -2$, so that $x = -2$.

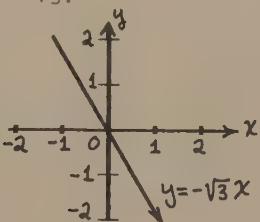
Note: $\theta \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$.



36. $2r \cos \theta - 3r \sin \theta = 1$, so that $2x - 3y = 1$, which is a straight line whose intercepts are $(0, -\frac{1}{3})$ and $(\frac{1}{2}, 0)$.



37. $\theta = -\frac{\pi}{3}$. The slope of the ray $\theta = -\frac{\pi}{3}$, which contains $(0, -\frac{\pi}{3}) = (0, 0)$, is $\tan^{-1}(-\frac{\pi}{3}) = -\sqrt{3}$.



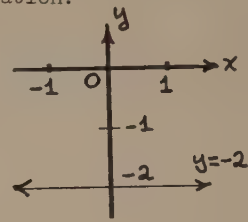
38. $r^2\ 2 \sin \theta \cos \theta = 2$, or $(r \sin \theta)\cdot(r \cos \theta) = 2$. Thus, $xy = 2$.

39. $r^2 = \cos 2\theta = 1 - 2 \sin^2\theta$, so that $r^4 = r^2 - 2r^2\sin^2\theta$. Thus, $(x^2 + y^2)^2 = x^2 + y^2 - 2y^2$, or $(x^2 + y^2)^2 = x^2 - y^2$.

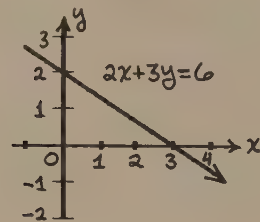
40. $\frac{r}{5} = \theta$, so that $\tan \frac{r}{5} = \tan \theta = \frac{y}{x}$. Thus, $x \tan \frac{r}{5} = y$, or $x \tan (\pm\frac{\sqrt{x^2 + y^2}}{5}) = y$, or $\pm x \tan (\frac{\sqrt{x^2 + y^2}}{5}) = y$.

41. $x^2 + y^2 = 25$ becomes $r^2 \cos^2 \theta + r^2 \sin^2 \theta = 25$, or $r^2 = 25$, or $r = 5$.

42. $y = -2$, so $r \sin \theta = -2$ or $r = -2 \csc$ is the polar equation.



43. $2x + 3y = 6$, so that $2r \sin \theta + 3r \cos \theta =$ 6 or $r\ (2 \sin \theta + 3 \cos \theta) = 6$.

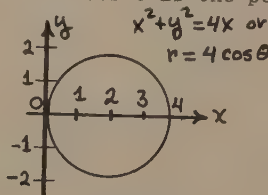


44. $x^2 + y^2 = 4x$.

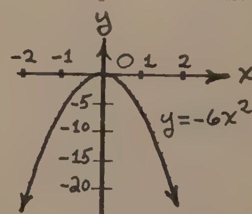
$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 4r \cos \theta$.

$r^2(\cos^2 \theta + \sin^2 \theta) = 4r \cos \theta$.

Thus, $r^2 = 4r \cos \theta$. If $r \neq 0$, then r $4 \cos \theta$. Since $(0, \frac{\pi}{2})$ is a point of th graph, $r = 4 \cos \theta$ is the polar equatio



45. $y = -6x^2$. Therefore, $r \sin \theta = -6r^2 \cos^2 \theta$. If $r \neq 0$, then $-\sin \theta = 6r \cos^2 \theta$. Note: $(0, 0°)$ satisfies the polar equation $-\sin \theta = 6r \cos^2 \theta$ as well.



46. $(r \sin\theta)(r\cos\theta) = 1$, or $r^2 \sin\theta\cos\theta = 1$. Thus, $r^2(2\sin\theta\cos\theta) = 2$, and $r^2 \sin 2\theta = 2$ is the polar equation.

47. $\frac{x^2}{4} + y^2 = 1$ becomes $\frac{r^2\cos^2\theta}{4} + r^2\sin^2\theta = 1$, or $r^2\cos^2\theta + 4r^2\sin^2\theta = 4$; that is, $r^2(\cos^2\theta + 4\sin^2\theta) = 4$ or $r^2(1 + 3\sin^2\theta) = 4$.

48. $x^4 + 2x^2y^2 y^4 = (x^2 + y^2)^2 = 4xy$. Thus, $r^4 = 4(r\cos\theta)(r\sin\theta)$ or $r^2 = 2(2\cos\theta\cos\theta)$, and then $r^2 = 2\sin 2\theta$.

49. $r^2 = 4r\cos\theta$, or $r(r - 4\cos\theta) = 0$. Thus, points of $r = 4\cos\theta$ satisfy $r^2 = 4r\cos\theta$. Since $(0, \frac{\pi}{2})$ satisfies equation $r = 4\cos\theta$, the points on the graph of $r^2 = 4r\cos\theta$ satisfy the equation $r = 4\cos\theta$. Thus, both graphs are the same. (That is, we do not introduce the origin as a new point - it was already on the first graph.)

50. When the graph of the first equation already contains the origin.

51. $r^2 + 8r\sin\theta = 0$, or $r(r + 8\sin\theta) = 0$. Thus, points of $r + 8\sin\theta$ satisfy $r^2 + 8r\sin\theta$. Since $(0, 0°)$ satisfies $r + 8\sin\theta = 0$, the points of $r^2 + 8r\sin\theta = 0$ satisfy $r + 8\sin\Theta = 0$. Therefore, both graphs are the same. (That is, we do not lose the origin - it is still on the graph of $r + 8\sin\theta = 0$.)

52. The rules are as follows:

(i) $0 = (0, \theta_1) = (0, \theta_2)$ for all values of θ_1 and θ_2.

(ii) If $(r_1, \theta_1) = (r_2, \theta_2)$ the $|r_1| = |r_2|$.

(iii) If $r_1, r_2 \neq 0$, then $(r_1, \theta_1) = (r_2, \theta_2)$ if and only if there is an integer n such that either $r_1 = r_2$ and $\theta_1 - \theta_2 = 2n\pi$ or else $r_1 = -r_2$ and $\theta_1 - \theta_2 = (2n + 1)\pi$.

Condition (i) follows immediately from the observation that if $r_1 = r_2 = 0$, then $r_1\cos\theta_1 = r_2\cos\theta_2$ and $r_1\sin\theta_1 = r_2\sin\theta_2$. To prove (ii), assume that $r_1\cos\theta_1 = r_2\cos\theta_2$ and $r_1\sin\theta_1 = r_2\sin\theta_2$. Squaring, we have $r_1^2\cos^2\theta_1 = r_2^2\cos^2\theta_2$ and $r_1^2\sin^2\theta_1 = r_2^2\sin^2\theta_2$. Adding the latter two equations, we obtain $r_1^2(\cos^2\theta_1 + \sin^2\theta_1) = r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$, so that $r_1^2 = r_2^2$; hence, taking square roots, $|r_1| = |r_2|$.

To prove (iii), assume that $r_1, r_2 \neq 0$, and suppose that $r_1\cos\theta_1 = r_2\cos\theta_2$ and $r_1\sin\theta_1 = r_2\sin\theta_2$. If $r_1 = r_2$, then $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$; hence, $0 = \cos\theta_1 - \cos\theta_2 = -2\sin\frac{1}{2}(\theta_1 + \theta_2)\cdot\sin\frac{1}{2}(\theta_1 - \theta_2)$ and $0 = \sin\theta_1 - \sin\theta_2 = 2\cos\frac{1}{2}(\theta_1 + \theta_2)\cdot\sin\frac{1}{2}(\theta_1 - \theta_2)$. Therefore, $\sin\frac{1}{2}(\theta_1 + \theta_2) = 0$ or $\sin\frac{1}{2}(\theta_1 - \theta_2) = 0$ and $\cos\frac{1}{2}(\theta_1 + \theta_2) = 0$ or $\sin\frac{1}{2}(\theta_1 - \theta_2) = 0$. It follows that $\sin\frac{1}{2}(\theta_1 - \theta_2) = 0$ or else $\sin\frac{1}{2}(\theta_1 + \theta_2) = 0$ and $\cos\frac{1}{2}(\theta_1 + \theta_2) = 0$. But there is no value of $\frac{1}{2}(\theta_1 + \theta_2)$ for which both sine and cosine are zero; hence, we must have $\sin\frac{1}{2}(\theta_1 - \theta_2) = 0$, that is, $\frac{1}{2}(\theta_1 - \theta_2) = n$, or $\theta_1 - \theta_2 = 2n\pi$. On the other hand, if $r_1 = -r_2$, then $\cos\theta_1 = -\cos\theta_2$ and $\sin\theta_1 = -\sin\theta_2$; hence, $0 = \cos\theta_1 + \cos\theta_2 = 2\cos\frac{1}{2}(\theta_1 + \theta_2)\cdot\cos\frac{1}{2}(\theta_1 - \theta_2)$ and $0 = \sin\theta_1 + \sin\theta_2 = 2\sin\frac{1}{2}(\theta_1 + \theta_2)\cdot\cos\frac{1}{2}(\theta_1 - \theta_2)$. Arguing as above, we must have $\cos\frac{1}{2}(\theta_1 - \theta_2) = 0$, that is, $\frac{1}{2}(\theta_1 - \theta_2) = (2n + 1)\frac{\pi}{2}$, or $\theta_1 - \theta_2 = (2n + 1)\pi$. This establishes the "only if" part of (iii). The "if" part is obvious.

53. (a) $r = \pm 2a\cos\theta$, so that $r^2 = \pm 2ar\cos\theta$. Thus, $x^2 + y^2 = \pm 2ax$. Completing the square: $x^2 \pm 2ax + a^2 + y^2 = a^2$ or $(x \pm a)^2 + y^2 = a^2$. This is the equation of a circle with center $(h, k) = (\pm a, 0)$ of radius a.

(b) $r = \pm 2a \sin\theta$, so that $r^2 = \pm 2ar \sin\theta$. Thus, $x^2 + y^2 = \pm 2ay$. Completing the square: $x^2 + (y \pm a)^2 = a^2$. This is a circle with center $(h, k) = (0, \pm a)$ and radius a.

54. (a) Suppose the Cartesian coordinates of (r_1, θ_1) and (r_2, θ_2) are (x_1, y_1) and (x_2, y_2), respectively. Then the distance between (r_1, θ_1) and (r_2, θ_2) is given in Cartesian coordinates by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} =$$
$$\sqrt{x_1^2 + y_1^2 - 2x_1x_2 - 2y_1y_2 + x_2^2 + y_2^2} =$$
$$\sqrt{r_1^2 - 2r_1\cos\theta_1 r_2\cos\theta_2 - 2r_1\sin\theta_1 r_2\sin\theta_2 + r_2^2} =$$
$$\sqrt{r_1^2 - 2r_1r_2(\cos{}_1\cos{}_2 + \sin{}_1\sin{}_2) + r_2^2} =$$
$$\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2},$$

where $x_1^2 + y_1^2 = r_1^2$, $x_2^2 + y_2^2 = r_2^2$, $x_1 = r_1\cos\theta_1$, $x_2 = r_2\cos\theta_2$, $y_1 = r_1\sin\theta_1$, and $y_2 = r_2\sin\theta_2$.

(b) A point (r, θ) on a circle with center (r_o, θ_o) and radius a satisfies the equation $\sqrt{r^2 - 2rr_o\cos(\theta - \theta_o) + r_o^2} = a$ by part (a). Thus, $r^2 - 2rr_o\cos(\theta - \theta_o) + r_o^2 = a^2$ is the equation of the circle.

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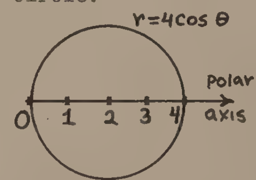
1. (c) Circle.
2. (f) Four-leaved rose.
3. (d) $r = a + b\sin\theta$, $0 < a < b$.
4. (h) Lemniscate.
5. (g) Archimedean spiral.
6. (a) Circle.
7. (e) $r = a + b\sin\theta$, $0 < b \le \frac{a}{2}$.
8. (b) Cardioid.
9. (a) Replace θ by $-\theta$: $r = 4\cos(-\theta) = 4\cos\theta$, which is equivalent to $r = 4\cos\theta$. The graph is symmetric about the polar axis.

(b) Replace θ by $\pi - \theta$: $r = 4\cos(\pi - \theta) = 4\cos\pi\cos\theta + 4\sin\pi\sin\theta = -4\cos\theta$, which is not equivalent to the original. Replace θ by $-\theta$ and r by $-r$: $-r = 4\cos(-\theta)$ or $-r = 4\cos\theta$, which is not equivalent to the original.

(c) Replace θ by $\theta + \pi$: $r = 4\cos(\theta + \pi) = 4\cos\theta\cos\pi - 4\sin\theta\sin\pi = -4\cos\theta$, which is not equivalent to the original equation. Replace r by $-r$: $-r = 4\cos\theta$, which is not equivalent to the given equation. The graph is a circle.

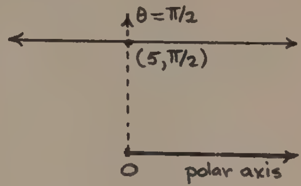


10. (a) Replace θ by $-\theta$: $r\sin(-\theta) = 5$ or $-4\sin\theta = 5$ is not equivalent to the original. Replace θ by $\pi - \theta$ and r by $-r$: $-r\sin(\pi - \theta) = 5$ or $-r(\sin\pi\cos\theta - \cos\pi\sin\theta) = 5$ or $-r\sin\theta = 5$ is not equivalent to the given equation.

(b) Replace θ by $\pi - \theta$: $r\sin(\pi - \theta) = 5$ or $r(\sin\pi\cos\theta - \cos\pi\sin\theta) = 5$ or $r\sin\theta = 5$, which is equivalent to the original, so there is symmetry about the $\theta = \pm\frac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r\sin(\theta + \pi) = r\sin\theta\cos\pi + r\cos\theta\sin\pi = -r\sin\theta = 5$ is not equivalent to the original. Replace r by $-r$: $-r\sin\theta = 5$ is not equivalent to the original.

The graph is a straight line($y = 5$ in Cartesian coordnates).

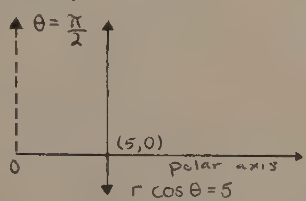


11. (a) Replace θ by $-\theta$: $r\cos(-\theta) = r\cos\theta = 5$ is equivalent to the original. So there is symmetry about the polar axis.

(b) Replace θ by $\pi - \theta$: $r\cos(\pi - \theta) = r\cos\pi\cos\theta + r\sin\pi\sin\theta = -r\cos\theta = 5$ is not equivalent to the original. Replace θ by $-\theta$ and r by $-r$: $-r\cos(-\theta) = -r\cos\theta = 5$ is not equivalent to the given equation.

(c) Replace θ by $\theta + \pi$: $r\cos(\theta + \pi) = r\cos\theta\cos\pi - r\sin\theta\sin\pi = -r\cos\theta = 5$ which is not equivalent to the original. Replace r by $-r$: $-r\cos\theta = 5$ is not equivalent to the original equation.

The graph is a straight line ($x = 5$ in Cartesian coordinates)

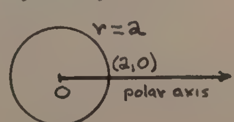


12. (a) Replace θ by $-\theta$: we get the same result, so there is symmetry about the polar axis.

(b) Replace θ by $\pi - \theta$: we get the same result, so there is symmetry about the line $\theta = \frac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: we get the same result, so there is symmetry about the pole.

The graph is a circle.

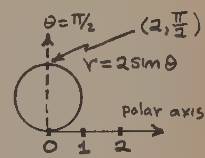


13. (a) Replace θ by $-\theta$: $r = 2\sin(-\theta) = -2\sin\theta$. Replace θ by $\pi - \theta$ and r by $-r$: $-r = 2\sin(\pi-\theta)$ or $-r = 2\sin\pi\cos\theta - 2\cos\pi\sin\theta = 2\sin\theta$, which is not equivalent to the given equation.

(b) Replace θ by $\pi - \theta$: $r = 2\sin(\pi - \theta) = 2\sin\pi\cos\theta - 2\cos\pi\sin\theta = 2\sin\theta$. We have symmetry about the line $\theta = \frac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r = 2\sin(\theta + \pi) = 2\sin\theta\cos\pi + 2\sin\theta\cos\pi = -2\sin\theta$, which is not equivalent to the given equation. Replace r by $-r$: $-r = 2\sin\theta$ is not equivalent to the original.

The graph is a circle.

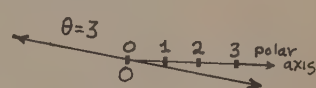


14. (a) Replace θ by $-\theta$: $-\theta = 3$ is not equivalent to the original equation. Replace θ by $\pi - \theta$ and r by $-r$: $(-r, \pi - \theta)$ is not the same point.

(b) Replace θ by $\pi - \theta$: $\pi - \theta = 3$ does not yield the same equation. Replace θ by $-\theta$ and r by $-r$: $(-r, -3)$ is not the same point $(r, 3)$.

(c) Replace θ bye $\theta + \pi$ $(r, \pi + \theta)$ is not the same point.

The graph is a line.

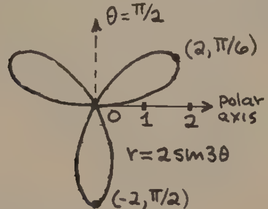


15. (a) Replace θ by $-\theta$: $r = 2\sin 3(-\theta) = -2\sin 3\theta$, which is not equivalent to the given equation. Replace θ by $\pi - \theta$ and r by $-r$: $-r = 2\sin(3\pi - 3\theta) = 2(\sin 3\pi\cos 3\theta - \cos 3\pi\sin 3\theta) = 2\sin 3\theta$, which is not equivalent to the original equation.

(b) Replace θ by $\pi - \theta$: $r = 2\sin 3(\pi - \theta) =$

$2 \sin(3\pi - 3\theta) = 2\sin 3\pi \cos 3\theta - 2\cos 3\pi \sin 3\theta = 2 \sin 3\theta$. Hence, there is symmetry about the line $\theta = \frac{\pi}{2}$.

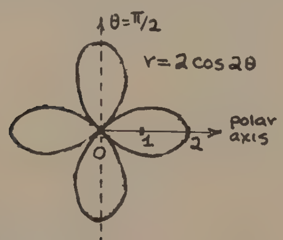
(c) Replace θ by $\theta + \pi$: $r = 2 \sin(3\theta + 3\pi) = 2\sin 3\theta \cos 3\pi + 2\cos 3\theta \sin 3\pi = -2\sin 3\theta$, which is not equivalent to the original equation. Replace r by −r: $-r = 2\sin 3\theta$, which is not equivalent to the original equation.



16. (a) Replace θ by $-\theta$: $r = 2\cos 2(-\theta) = 2\cos 2\theta$. Thus we have symmetry about the polar axis.

(b) Replace θ by $\pi - \theta$: $r = 2\cos 2(\pi - \theta) = 2\cos(2\pi - 2\theta) = 2\cos 2\pi \cos 2\theta + 2\sin 2\pi \sin 2\theta = 2 \cos 2\theta$. Thus, there is symmetry about the line $\theta = \frac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r = 2 \cos 2(\theta + \pi) = 2\cos 2\theta \cos 2\pi - 2\sin 2\theta \sin 2\pi = 2\cos 2\theta$. Hence, there is symmetry about the pole.

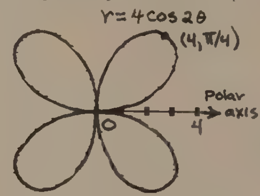


17. (a) Replace θ by $-\theta$: $r = 4\sin 2(-\theta) = -4\sin 2\theta$. Replace θ by $\pi - \theta$ and r by −r: $-r = 4 \sin 2(\pi - \theta) = 4[\sin 2\pi \cos 2\theta - \cos 2\pi \sin 2\theta] = -4\sin 2\theta$, so we have an equivalent equation. Thus, there is symmetry about the polar axis.

(b) Replace θ by $\pi - \theta$: $r = 4\sin 2(\pi - \theta) = 4\sin(2\pi - 2\theta) = 4[\sin 2\pi \cos 2\theta - \cos 2\pi \sin 2\theta] = -4\sin 2\theta$.

Replace θ by $-\theta$ and r by −r: $-r = 4\sin 2(-\theta)$ o $-r = -4\sin 2\theta$, which is equivalent to the given equation, so we have symmetry about $\theta = \frac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r = 4\sin 2(\theta + \pi) = 4[\sin 2\theta \cos 2\pi + \cos 2\theta \sin 2\pi] = 4\sin 2\theta$. So there is symmetry about the pole.

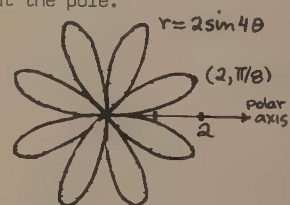


18. (a) Replace θ by $-\theta$: $r = 2\sin 4(-\theta) = -2\sin 4\theta$. Replace θ by $\pi - \theta$ and r by −r: $-r = 2\sin 4(\pi - \theta) = 2\sin 4\pi \cos 4\theta - 2\cos 4\pi \sin 4\theta = -2\sin 4\theta$ is equivalent to the given equation. Thus, we have symmetry about the polar axis.

(b) Replace θ by $\pi - \theta$: $r = 2\sin 4(\pi - \theta) = 2\sin(4\pi - 4\theta) = 2\sin 4\pi \cos 4\theta - 2\cos 4\pi \sin 4\theta = -2\sin 4\theta$.

Replace θ by $-\theta$ and r by −r: $-r = 2\sin 4(-\theta) = 2\sin 4\theta$. Thus, there is symmetry about $\theta = \frac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r = 2\sin 4(\theta + \pi) = 2\sin 4\theta \cos 4\pi + 2\cos 4\theta \sin 4\pi = 2\sin 4\theta$ is equivalent to the original so there is symmetry about the pole.

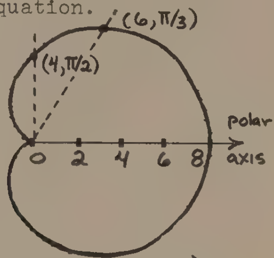


19. (a) Replace θ by $-\theta$: $r = 4(1 + \cos(-\theta)) = 4(1 + \cos\theta)$. Thus, there is symmetry about the polar axis.

(b) Replace θ by $\pi - \theta$: $r = 4(1 + \cos(\pi - \theta) = 4(1 + \cos\pi \cos\theta + \sin\pi \sin\theta) = 4(1 - \cos\theta)$, which is not equivalent. Replace θ by $-\theta$ and r by −r: $-r = 4(1 + \cos(-\theta)) = -r =$

$4(1 + \cos\theta)$, which is not equivalent to the original equation.

(c) Replace θ by $\theta + \pi$: $r = 4(1 + \cos\theta\cos\pi - \sin\theta\sin\pi)$, or $r = 4(1 - \cos\theta)$, which is not equivalent to the original equation. Replace r by $-r$: $-r = 4(1 + \cos\theta)$, which is not equivalent to the original equation.

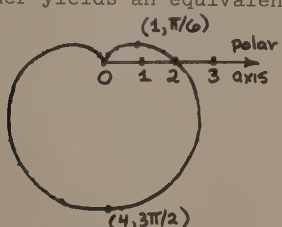


$r = 4(1 + \cos\theta)$

20. (a) Replace θ by $-\theta$: $r = 2(1 - \sin(-\theta)) = 2(1 + \sin\theta)$, which is not equivalent to the original equation. Replace θ by $\pi - \theta$ and r by $-r$: $-r = 2(1 - \sin(\pi - \theta))$ or $-r = 2(1 - \sin\pi\cos\theta + \cos\pi\sin\theta) = 2(1 - \sin\theta)$, which is not equivalent to the given equation.

(b) Replace θ by $\pi - \theta$: $r = 2(1 - \sin\pi\cos\theta + \cos\pi\sin\theta)$ or $r = 2(1 - \sin\theta)$. Thus, there is symmetry about the line $\theta = \dfrac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r = 2(1 - \sin\theta\cos\pi - \cos\theta\sin\pi) = 2(1 + \sin\theta)$. Replace r by $-r$: $-r = 2(1 - \sin\theta)$. Neither yields an equivalent equation.

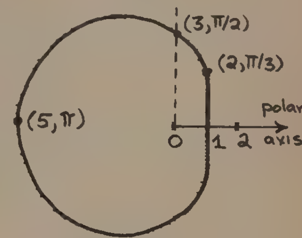


$r = 2(1 - \sin\theta)$

21. (a) Replace θ by $-\theta$: $r = 3 - 2\cos(-\theta) = 3 - 2\cos\theta$. Thus, there is symmetry about the axis.

(b) Replace θ by $\pi - \theta$: $r = 3 - 2(\cos\pi\cos\theta + \sin\pi\sin\theta) = 3 + 2\cos\theta$. Replace θ by $-\theta$ and r by $-r$: $-r = 3 - 2\cos(-\theta)$. Replace θ by $-\theta$ and r by $-r$: $-r = 3 - 2\cos(-\theta) = 3 - 2\cos\theta$. Neither is equivalent to the original equation.

(c) Replace θ by $\theta + \pi$: $r = 3 - 2(\cos\theta\cos\pi - \sin\theta\sin\pi)$ or $r = 3 + 2\cos\theta$. Replace r by $-r$: $-r = 3 - 2\cos\theta$. Neither is equivalent to the given equation.

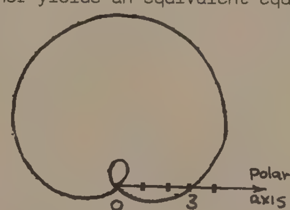


$r = 3 - 2\cos\theta$

22. (a) Replace θ by $-\theta$: $r = 3 + 4\sin(-\theta) = 3 - 4\sin\theta$. Replace θ by $\pi - \theta$ and r by $-r$: $-r = 3 + 4(\sin\pi\cos\theta - \cos\pi\sin\theta)$ or $-r = 3 + 4\sin\theta$. Neither yields an equivalent equation.

(b) Replace θ by $\pi - \theta$: $r = 3 + 4(\sin\pi\cos\theta - \cos\pi\sin\theta)$ or $r = 3 + 4\sin\theta$. Thus, there is symmetry about $\theta = \dfrac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r = 3 + 4\sin(\theta + \pi) = 3 + 4\sin\theta\cos\pi + 4\cos\theta\sin\pi = 3 - 4\sin\theta$. Replace r by $-r$: $-r = 3 + 4\sin\theta$. Neither yields an equivalent equation.



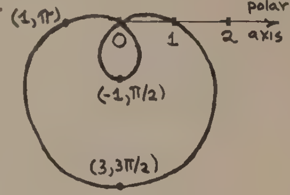
$r = 3 + 4\sin\theta$

23. (a) Replace θ by $-\theta$: $r = 1 - 2\sin(-\theta) = 1 + 2\sin\theta$. Replace θ by $\pi - \theta$ and r by $-r$: $-r = 1 - 2\sin(\pi - \theta) = 1 - 2\sin\pi\cos\theta + 2\cos\pi\sin\theta = 1 - 2\sin\theta$. Neither yields an equivalent equation.

(b) Replace θ by $\pi - \theta$: $r = 1 - 2\sin(\pi - \theta) = 1 - 2\sin\pi\cos\theta + 2\cos\pi\sin\theta = 1 - 2\sin\theta$. Thus, there is symmetry about the line $\theta = \dfrac{\pi}{2}$.

(c) Replace θ by $\theta + \pi$: $r = 1 - 2\sin(\theta + \pi) =$

$1-2\sin\theta\cos\pi - 2\cos\theta\sin\pi = 1 + 2\sin\theta$: replace r by $-r$: $-r = 1-2\sin\theta$. Neither yields an equivalent equation.

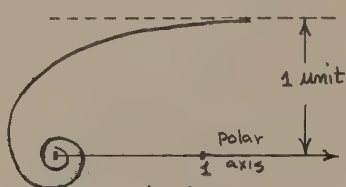


$r = 1-2\sin\theta$

24. (a) Replace θ by $-\theta$: $r = \dfrac{-1}{\theta}$. Replace θ by $\pi-\theta$ and r by $-r$: $-r = \dfrac{1}{\pi-\theta}$. Neither yields an equivalent equation.

(b) Replace θ by $\pi-\theta$: $r = \dfrac{1}{\pi-\theta}$. Replace θ by $-\theta$ and r by $-r$: $-r = \dfrac{1}{-\theta}$ is equivalent to $r = \dfrac{1}{\theta}$, so that there is symmetry about the line $\theta = \dfrac{\pi}{2}$.

(c) Replace θ by $\theta+\pi$: $r = \dfrac{1}{\pi+\theta}$. Replace r by $-r$: $-r = \dfrac{1}{\theta}$. Neither yields an equivalent equation.



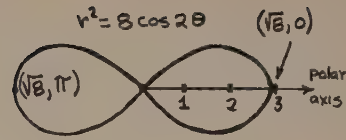
$r = 1/\theta,\ \theta > 0$

The line $y = 1$ in Cartesian coordinates is an asymptote: Consider $(x, y) = (r\cos\theta, r\sin\theta)$ on the graph. Then $\lim\limits_{\theta\to 0} r\sin\theta = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1.$

25. (a) Replace θ by $-\theta$: $r^2 = 8\cos 2(-\theta) = 8\cos 2\theta$
There is symmetry about the polar axis.

(b) Replace θ by $\pi-\theta$: $r^2 = 8\cos(2\pi-2\theta) = 8\cos 2\pi\cos 2\theta + \sin 2\pi \sin 2\theta = 8\cos 2\theta$. There is symmetry about the line $\theta = \dfrac{\pi}{2}$.

(c) Replace θ by $\theta+\pi$: $r^2 = 8\cos(2\theta + 2\pi) = 8\cos 2\theta\cos 2\pi - \sin 2\theta\sin 2\pi = 8\cos 2\theta$. There is symmetry about the pole.



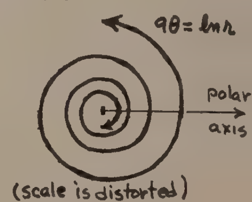
Note: $\dfrac{3\pi}{4} \le \theta \le \dfrac{5\pi}{4}$ and $-\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{4}$.

26. (a) Replace θ by $-\theta$: $-9\theta = \ln r$. This is not an equivalent equation. We cannot replace θ by $\pi-\theta$ and r by $-r$, since r cannot be negative.

(b) Replace θ by $\pi-\theta$: $9(\pi-\theta) = \ln r$. This is not an equivalent equation.

(c) Replace θ by $\theta+\pi$: $9(\theta+\pi) = \ln r$. This is not an equivalent equation.

We write $r = e^{9\theta}$.



27.
$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\cdot\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cdot\cos\theta - r\sin\theta}$$
$$= \frac{(-3\sin\theta)\sin\theta + 3(1+\cos\theta)\cos\theta}{(-3\sin\theta)\cos\theta - 3(1+\cos\theta)\sin\theta}.$$

For $\theta = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = \dfrac{-3}{-3} = 1.$

28.
$$\frac{dy}{ex} = \frac{\dfrac{dr}{d\theta}\cdot\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cdot\cos\theta - r\sin\theta} = \frac{(-2\cos\theta)\sin\theta + r\ \text{[illegible]}}{(-2\cos\theta)\cos\theta - r\ \text{[illegible]}}$$

$$\text{For } \theta = \frac{\pi}{6},\ \frac{dy}{dx} = \frac{-2(\frac{\sqrt{3}}{2})(\frac{1}{2}) + (1)(\frac{\sqrt{3}}{2})}{-2(\frac{\sqrt{3}}{2})^2 - (1)(\frac{1}{2})} = \frac{0}{-2} = 0$$

29.
$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\cdot\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cdot\cos\theta - r\sin\theta} = \frac{16\cos 2\theta\cdot\sin\theta + r\cos\theta}{16\cos 2\theta\cdot\cos\theta - r\sin\theta}$$
$$= \frac{0}{16} = 0.$$

30.
$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\cdot\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cdot\cos\theta - r\sin\theta} = \frac{2\sec^2\theta\cdot\tan\theta\cdot\sin\theta + r\cos\theta}{2\sec^2\theta\cdot\tan\theta\cdot\cos\theta - r\sin\theta}$$

$$= \frac{8\sqrt{3}\cdot\frac{\sqrt{3}}{2} + 4(\frac{1}{2})}{8\sqrt{3}(\frac{1}{2})\ \frac{-4\sqrt{3}}{2}} = \frac{7\sqrt{3}}{3}.$$

1. (a) If $\frac{dr}{d\theta}\sin\theta + r\cos\theta = 0$ and $\frac{dr}{d\theta}\cos\theta - r\sin\theta \neq 0$ at (r, θ), then $\frac{dy}{dx} = 0$; so the tangent line is horizontal at this point.

(b) If $\frac{dr}{d\theta}\cos\theta - r\sin\theta = 0$ and $\frac{dr}{d\theta}\sin\theta + r\cos\theta \neq 0$ for a point (r, θ), then $\frac{dy}{dx}$ is undefined; so the tangent line at this point is vertical. If both the numerator and denominator of dy/dx are 0 at a point (r, θ), then dy/dx does not exist at this point.

2. $\tan\psi = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$

$$= \frac{\dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} - \dfrac{\sin\theta}{\cos\theta}}{1 + \left[\dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\text{dps}\,\theta - r\sin\theta}\right]\dfrac{\sin\theta}{\cos\theta}}$$

$$\frac{(\frac{dr}{d\theta}\sin\theta + r\cos\theta)\cos\theta - (\sin\theta)(\frac{dr}{d\theta}\cos\theta - r\sin\theta)}{(\cos\theta)(\frac{dr}{d\theta}\cos\theta - r\sin\theta) + \sin\theta(\frac{dr}{d\theta}\sin\theta + r\cos\theta)} = \frac{r}{\frac{dr}{d\theta}}.$$

. $\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{1/2}{\cos\theta} = \frac{1/2}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}.$

. $\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{0}{-2\sin 2\theta} = \frac{0}{-1} = 0.$

. $\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{e^2}{e^\theta} = \frac{e^2}{e^2} = 1.$

. $r^2 = \csc 2\theta$ at $\left(1, \frac{\pi}{4}\right)$.

$\tan\psi = \frac{r}{\frac{dr}{d\theta}}$. To find $\frac{dr}{d\theta}$, differentiate implicitly;

so $2r\frac{dr}{d\theta} = -2\csc 2\theta\cot 2\theta$ and $\frac{dr}{d\theta} = \frac{-\csc 2\theta\cot 2\theta}{r}$.

Thus, $\tan\psi = \frac{r}{\frac{-\csc 2\theta\cot 2\theta}{r}} = \frac{r^2}{-\csc 2\theta\cot 2\theta} = \frac{\csc 2\theta}{-\csc 2\theta\cot 2\theta}$

or $\tan\psi = -\tan 2\theta$. When $\theta = \frac{\pi}{4}$, $\tan\psi$ is undefined.

37. $\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{4}{4\sec\theta\tan\theta}$, which is undefined when $\theta = 0$.

38. If $r = 0$, then $\sin\theta = -\frac{a}{b}$.

If $r = 0$, then $\cos\theta = \frac{\pm\sqrt{b^2 - a^2}}{b}$. Now $\frac{dr}{d\theta} = b\cos\theta$.

Therefore, $\frac{dy}{dx} = \frac{b\cos\theta\sin\theta + r\cos\theta}{b\cos\theta\cos\theta - r\sin\theta}$; so $\frac{dy}{dx}$

$$= \frac{b\left(\frac{\pm\sqrt{b^2 - a^2}}{b}\right)\left(\frac{-a}{b}\right)}{b\left(\frac{\sqrt{b^2 - a^2}}{b}\right)^2} = \pm\frac{a}{\sqrt{b^2 - a^2}}.$$ Note: $\frac{dy}{dx} = \tan\theta$

if $\theta = \sin^{-1}\left(\frac{-a}{b}\right)$ or $\theta = \pi - \sin^{-1}\left(\frac{-a}{b}\right)$.

39.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\cdot\sin\theta + (4 + 3\sin\theta)\cos\theta}{\frac{dr}{d\theta}\cdot\cos\theta - (4 + 3\sin\theta)\sin\theta} =$$

$$\frac{3\cos\theta\sin\theta + 4\cos\theta + 3\sin\theta\cdot\cos\theta}{3\cos^2\theta - 4\sin\theta - 3\sin^2\theta}$$

The tangent line is horizontal provided

$6\cos\theta\sin\theta + 4\cos\theta = 0$.

Now $2\cos\theta(3\sin\theta + 2) = 0$ provided $\cos\theta = 0$ or $\sin\theta = -\frac{2}{3}$; so $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, or $\theta = \sin^{-1}\left(-\frac{2}{3}\right)$ or $\theta = \pi - \sin^{-1}\left(-\frac{2}{3}\right)$. (These values do not make the denominator 0.) Now, $3\cos^2\theta - 4\sin\theta - 3\sin^2\theta = 0$ yields values where the tangent line is vertical. Thus, $3(1 - \sin^2\theta) - 3\sin^2\theta - 4\sin\theta = 0$ or $-6\sin^2\theta - 4\sin\theta + 3 = 0$ or $6\sin^2\theta + 4\sin\theta - 3 = 0$, and so $\sin\theta = \frac{-4 \pm \sqrt{88}}{12}$; that is, $\sin\theta = \frac{-2 \pm\sqrt{22}}{6}$.

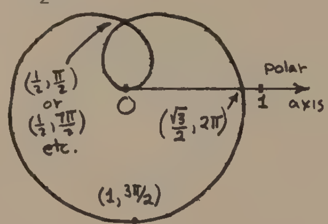
But $\sin\theta$ cannot equal $\frac{-2 - \sqrt{22}}{6}$. Hence, $\theta = \sin^{-1}\left(\frac{\sqrt{22} - 2}{6}\right)$ and $\theta = \pi - \sin^{-1}\left(\frac{\sqrt{22} - 2}{6}\right)$.

These values do not make the numerator 0. Therefore, the points where the tangent to the graph is horizontal are $(7, \frac{\pi}{2})$, $(1, \frac{3\pi}{2})$, $(2, \sin^{-1}(-\frac{2}{3}))$ and $(2, \pi - \sin^{-1}(-\frac{2}{3}))$. The points where the tangent to the graph is vertical are $\left(\frac{18 + 3\sqrt{22}}{6}, \sin^{-1}\left(\frac{\sqrt{22} - 2}{6}\right)\right)$ and $\left(\frac{18 + 3\sqrt{22}}{6}, \pi - \sin^{-1}\left(\frac{\sqrt{22} - 2}{6}\right)\right)$.

40. r takes on its maximum value of 2 when $\cos k\theta = 1$; that is, when $k\theta = 2n\pi$ or $\theta = \frac{2n\pi}{k}$, where $n = 0, 1, 2, \ldots, k - 1$. Thus, the coordinates of the tips of the leaves are $(2, \frac{2n\pi}{k})$, where $n = 0, 1, 2, \ldots, k - 1$.

41. The minimum occurs when $\sin \theta = -1$. Since $a > b$, then $r = a - b$ is the minimum value of r.

42. The symmetry test fails when we substitute $-r$ for r and $-\theta$ for θ, but $(-r, -\theta + 6\pi)$ is another representation for $(-r, -\theta)$. Now, $-r = \sin(\frac{6\pi - \theta}{3})$ $= \sin(2\pi - \frac{\theta}{3}) = \sin 2\pi \cos\frac{\theta}{3} - \cos 2\pi \sin\frac{\theta}{3} = -\sin\frac{\theta}{3}$, which is equivalent to $r = \sin\frac{\theta}{3}$. Hence, there is symmetry about $\theta = \frac{\pm\pi}{2}$.



43. (a) Replace θ by $(-\theta)$: $r = f(-\theta) = f(\theta)$ if f is an even function. Thus, the graph is symmetric about $\theta = 0$.

(b) Replace θ by $-\theta$ and r by $-r$: $-r = f(-\theta) = -f(\theta)$, since f is odd. This equation is equivalent to $r = f(\theta)$. Thus the graph is symmetric about the line $\theta = \frac{\pm\pi}{2}$.

44. 1. The graph is symmetric with respect to the polar axis and its extension if and only if at least one of the two following conditions holds:

(a) There exists an integer n such that when θ is replaced by $-\theta + 2n\pi$ an equivalent equation is obtained.

(b) There exists an integer n such that when r is replaced by $-r$ and θ is replaced by $\pi - \theta + 2n\pi$, an equivalent equation is obtained.

2. The graph is symmetric with respect to the line $\theta = \frac{\pm\pi}{2}$ if and only if at least one of the two following conditions holds:

(a) There exists an integer n such that when θ is replaced by $\pi - \theta + 2n\pi$, an equivalent equation is obtained.

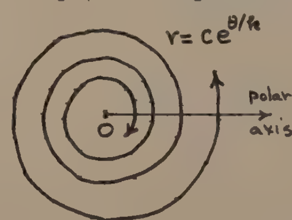
(b) There exists an integer n such that when r is replaced by $-r$ and θ is replaced by $-\theta + 2n\pi$, an equivalent equation is obtained.

3. The graph is symmetric with respect to the pole if and only if at least one of the two following conditions holds:

(a) There exists an integer n such that when θ is replaced by $\pi + \theta + 2n\pi$, an equivalent equation is obtained.

(b) There exists an integer n such that when r is replaced by $-r$ and θ by $\theta + 2n\pi$, an equivalent equation is obtained.

45. We require that $k = \tan\psi = \frac{r}{\frac{dr}{d\theta}}$, that is, $\frac{dr}{d\theta} = \frac{1}{k}r$. The solution of this differential equation is $r = ce^{\theta/k}$, where c is a constant. The graph is a logarithmic spiral.



46. We will first determine the values of θ

for which $\frac{dy}{dx} = 0$ for the polar equation $r =$ $a + b \sin \theta$: $\frac{dy}{dx} = \frac{2b \sin \theta \cos \theta + a \cos \theta}{b^2\cos^2\theta - a \sin \theta - b \sin^2\theta}$

Using this formula, $\frac{dy}{dx} = 0$ if $2b \sin \theta \cos \theta + a \cos \theta = 0$ and $b^2\cos^2\theta - a \sin \theta - b \sin^2\theta \neq 0$. Solving $2b \sin \theta \cos \theta + a \cos \theta = 0$, we find $\cos \theta\ (2b \sin \theta + a) = 0$, or $\cos \theta = 0$ and $2b \sin \theta + a = 0$. Therefore, $\frac{dy}{dx} = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ and when $\sin \theta = \frac{-a}{2b}$. $b^2\cos^2\theta - a \sin \theta - b \sin^2\theta \neq 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. If $\sin \theta = \frac{-a}{2b}$ then $\cos \theta = \frac{\pm\sqrt{4b^2 - a^2}}{2b}$. Substituting into $b^2\cos^2\theta - a \sin \theta - b \sin^2\theta$, we obtain

$$\frac{4b^2 - a^2}{4b} + \frac{a^2}{2b} - \frac{a^2}{4b} = \frac{4b^2}{4b} \neq 0.$$

(a) Assume $0 < a/2 < b < a$; then $a < 2b$. Therefore, $|\sin \theta| = \left|\frac{-a}{2b}\right| < 1$, so values of θ exist and are the numbers: $\theta_1 = \pi - \sin^{-1}\left(\frac{-a}{2b}\right)$ and $\theta_2 = \sin^{-1}\left(\frac{-a}{2b}\right)$. Then $r = f(\theta_1) = f(\theta_2) = a + b\left(\frac{-a}{2b}\right) = \frac{a}{2}$. Consequently, the y coordinate of the points on the polar graph at (r, θ_1) and (r, θ_2) is $y = r \sin \theta = \frac{a}{2}\left(\frac{-a}{2b}\right) = -\frac{a^2}{4b}$. To see if there is an indentation at $\theta = \frac{3\pi}{2}$, compare the y coordinates at the points where $\frac{dy}{dx} = 0$. If $\theta = \frac{3\pi}{2}$, the y coordinate is $a - b$. The difference is: $\left|\frac{-a^2}{4b}\right| - |a - b| = \frac{a^2}{4b} - (a - b) = \frac{a^2 - 4ab + 4b^2}{4b} = \frac{(a - 2b)^2}{4b} > 0$. Thus, $\left|-\frac{a^2}{4b}\right| > |a - b|$. This indicates that on either side of the point $\left(\frac{a}{2}, \frac{3\pi}{2}\right)$, there are points whose distance from the x axis is greater than that of $\left(\frac{a}{2}, \frac{3\pi}{2}\right)$.

Hence, there is an indentation in the graph of $r = a + b \sin \theta$ when $0 < \frac{a}{2} < b < a$.

(b) Assume $0 < b \le \frac{a}{2}$. Since $\sin \theta = \frac{-a}{2b}$, we see that $\left|\frac{-a}{2b}\right| \ge 1$ since $2b \le a$. If $2b = a$, then $\sin \theta = -1$ and $\theta = \frac{3\pi}{2}$, which is a point where $\frac{dy}{dx} = 0$. If $2b < a$, then $\left|\frac{-a}{2b}\right| > 1$ and $\sin \theta$ does not exist. Thus, there are no points on either side of $\left(\frac{a}{2}, \frac{3\pi}{\theta}\right)$ or $\left(a + b, \frac{\pi}{2}\right)$ where $\frac{dy}{dx} = 0$. Thus, there is no indentation in the graph of $r = a + b \sin\theta$.

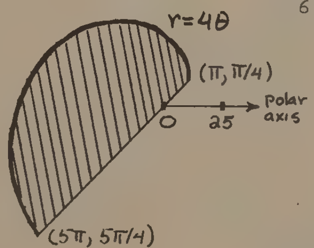
Problem Set 9.3, page 554

1.

$$A = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (4\theta)^2\, d\theta$$

$$= \frac{8}{3}\theta^3\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{8}{3}\left[\frac{125\pi^3}{64} - \frac{\pi^3}{64}\right]$$

$$= \frac{31}{6}\pi^3 \text{ square units.}$$



2.

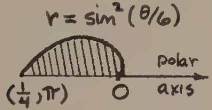
$$A = \frac{1}{2}\int_0^{\pi} \left(\sin^2\frac{\theta}{6}\right)^2 d\theta = \frac{1}{2}\int_0^{\pi}\left[\frac{1 - \cos\frac{\theta}{3}}{2}\right]^2 d\theta$$

$$= \frac{1}{8}\int_0^{\pi} \left(1 - 2\cos\frac{\theta}{3} + \cos^2\frac{\theta}{3}\right) d\theta$$

$$= \frac{1}{8}\left[\theta - 6\sin\frac{\theta}{3}\right]_0^{\pi} + \frac{1}{8}\int_0^{\pi} \cos^2\frac{\theta}{3}\, d\theta$$

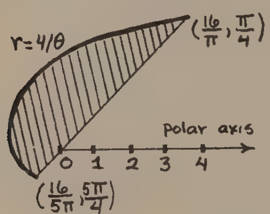
$$= \frac{1}{8}(\pi - 3\sqrt{3}) + \frac{1}{8}\int_0^{\pi} \frac{\left(1 + \cos\frac{2\theta}{3}\right)}{2} d\theta$$

$$= \frac{1}{8}(\pi - 3\sqrt{3}) + \frac{1}{8}\left(\frac{\theta}{2} + \frac{3\sin\frac{2\theta}{3}}{4}\right)\Bigg|_0^{\pi} = \frac{3\pi}{16} - \frac{21\sqrt{3}}{64} \text{ square units.}$$



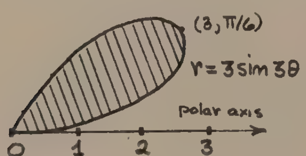
3.
$$A = \frac{1}{2}\int_{\pi/4}^{5\pi/4}\left(\frac{4}{\theta}\right)^2 d\theta = 8\int_{\pi/4}^{5\pi/4}\theta^{-2}\,d\theta$$

$$= -8\left(\frac{1}{\theta}\right)\Bigg|_{\pi/4}^{5\pi/4} = -8\left(\frac{4}{5\pi} - \frac{4}{\pi}\right) = \frac{128}{5\pi} \text{ square units.}$$



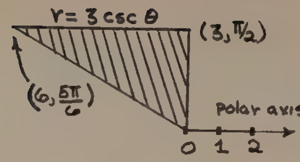
4.
$$A = \frac{1}{2}\int_0^{\pi/3}(3\sin 3\theta)^2\,d\theta = \frac{9}{2}\int_0^{\pi/3}\frac{1-\cos 6\theta}{2}\,d\theta$$

$$= \frac{9}{4}\left(\theta - \frac{\sin 6\theta}{6}\right)\Bigg|_0^{\pi/3} = \frac{9}{4}\left(\frac{\pi}{3}\right) = \frac{3\pi}{4} \text{ square units.}$$



5.
$$A = \frac{1}{2}\int_{\pi/2}^{5\pi/6}(3\csc\theta)^2 d\theta = \frac{9}{2}\int_{\pi/2}^{5\pi/6}\csc^2\theta\,d\theta$$

$$= \frac{9}{2}(-\cot\theta)\Bigg|_{\pi/2}^{5\pi/6} = \frac{9}{2}\left(-\cot\frac{5\pi}{6} + \cot\frac{\pi}{2}\right) = \frac{9\sqrt{3}}{2} \text{ square units.}$$

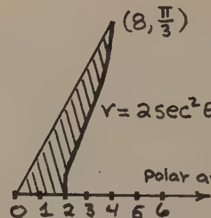


6.
$$A = \frac{1}{2}\int_0^{\pi/3}(2\sec^2\theta)^2 d\theta = 2\int_0^{\pi/3}(\sec^2\theta)(\sec^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/3}(\sec^2\theta + \sec^2\theta\tan^2\theta)\,d\theta$$

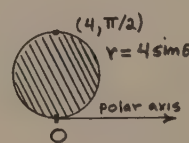
$$= 2(\tan\theta)\Bigg|_0^{\pi/3} + 2\int_0^{\pi/3}\sec^2\theta\tan^2\theta\,d\theta$$

$$= 2\sqrt{3} + \frac{2(\tan^3\theta)}{3}\Bigg|_0^{\pi/3} = 2\sqrt{3} + \frac{2}{3}\left((\sqrt{3})^3 - 0\right) = 4\sqrt{3} \text{ square units.}$$



7.
$$A = 2\left(\frac{1}{2}\int_0^{\pi/2}(4\sin\theta)^2 d\theta\right) = 16\int_0^{\pi/2}\sin^2\theta\,d\theta$$

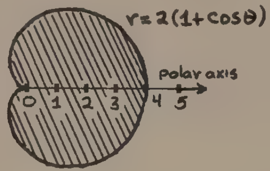
$$= -8\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/2} = 8\left(\frac{\pi}{2}\right) = 4\pi \text{ square units.}$$



8.
$$A = 2\left(\frac{1}{2}\int_0^{\pi}[2(1+\cos\theta)]^2\,d\theta\right) = 4\int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 4(\theta + 2\sin\theta)\Bigg|_0^{\pi} + 4\int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta$$

$= 4\pi + 2(\theta + \frac{1}{2}\sin2\theta)\Big|_0^{\pi} = 4\pi + 2\pi = 6\pi$ square units.

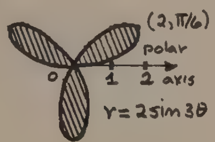


.

$A = 3\left(\frac{1}{2}\int_0^{\pi/3}(2\sin3\theta)^2 d\theta\right) = 6\int_0^{\pi/3}\sin^2 3\theta\, d\theta$

$= 6\int_0^{\pi/3}\frac{1-\cos6\theta}{2}\, d\theta = 3(\theta - \frac{1}{6}\sin6\theta)\Big|_0^{\pi/3}$

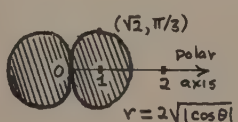
$= 3(\frac{\pi}{3} - \frac{1}{6}\sin2\pi) = \pi$ square units.



0.

$= 4\left(\frac{1}{2}\int_0^{\pi/2}(2\sqrt{|\cos\theta|})^2\, d\theta\right)$

$8\int_0^{\pi/2}\cos\theta\, d\theta = 8(\sin\theta)\Big|_0^{\pi/2} = 8$ square units.

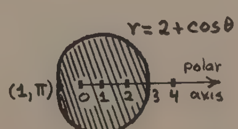


1.

$= 2\left(\frac{1}{2}\int_0^{\pi}[2+\cos\theta]^2\, d\theta\right) = \int_0^{\pi}(4 + 4\cos\theta + \cos^2\theta)\, d\theta$

$[4\theta + 4\sin\theta]_0^{\pi} + \int_0^{\pi}\frac{1+\cos2\theta}{2}\, d\theta = 4\pi + \left[\frac{\theta}{2} + \sin2\theta\right]_0^{\pi}$

$4\pi + \frac{\pi}{2} = \frac{9\pi}{2}$ square units.



12.

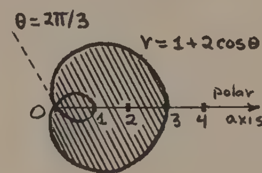
$A = 2\left(\frac{1}{2}\int_0^{2\pi/3}[1 + 2\cos\theta]^2 d\theta\right)$

$= \int_0^{2\pi/3}(1 + 4\cos\theta + 4\cos^2\theta)\, d\theta$

$= (\theta + 4\sin\theta)\Big|_0^{2\pi/3} + \int_0^{2\pi/3}(2 + 2\cos2\theta)\, d\theta$

$= \frac{2\pi}{3} + 4\left(\frac{\sqrt{3}}{2}\right) + (2\theta + \sin2\theta)\Big|_0^{2\pi/3}$

$= \frac{2\pi}{3} + 2\sqrt{3} + \frac{4\pi}{3} + \left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi + 3\sqrt{3}}{2}$ square units.

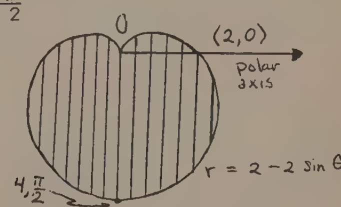


13.

$A = \frac{1}{2}\left(2\int_{-\pi/2}^{\pi/2}(2 - 2\sin\theta)^2 d\theta\right) = \int_{-\pi/2}^{\pi/2}(4 - 8\sin\theta + 4\sin^2\theta)d\theta$

$= (4\theta + 8\cos\theta)\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}2(1 - \cos2\theta)\, d\theta$

$= 4\pi + (2\theta - \sin2\theta)\Big|_{-\pi/2}^{\pi/2} = 4\pi + 2\pi + 6\pi$ square units.



14.

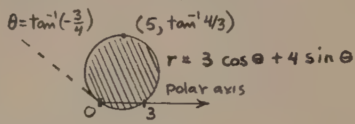
$A = \frac{1}{2}\int_0^{\pi}(3\cos\theta + 4\sin\theta)^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi}\left[9\cos^2\theta + 24\sin\theta\cos\theta + 16\sin^2\theta\right] d\theta$

$= \frac{1}{2}\int_0^{\pi}\left[\frac{9}{2} + \frac{9\cos2\theta}{2} + 12\sin2\theta + 8 - 8\cos2\theta\right] d\theta$

$= \frac{1}{2}\left(\frac{9\theta}{2} + \frac{9}{4}\sin 2\theta - 6\cos 2\theta + 8\theta - 4\sin 2\theta\right)\Big|_0^{\pi} = \frac{25\pi}{4}$ square units.

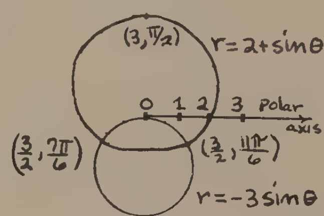
The graph is a circle.



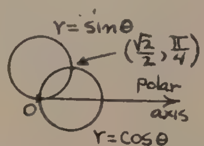
15\. (1) 0 is not a point of intersection.

(2) Now we solve the simultaneous equations $r = -3\sin(\theta + 2n\pi)$ and $r = 2 + \sin\theta$. So $r = -3\sin\theta = 2 + \sin\theta$ and we have $4\sin\theta = -2$ and $\sin\theta = -\frac{1}{2}$. Hence, $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$. The points of intersection obtained are $(\frac{3}{2}, \frac{7\pi}{6})$ and $(\frac{3}{2}, \frac{11\pi}{6})$.

(3) Now we consider $-r = -3\sin(\theta + (2n + 1)\pi)$ and $r = 2 + \sin\theta$ or $-r = 3\sin\theta$ and $r = 2 + \sin\theta$. Thus, $4\sin\theta$ and $-2 = \sin\theta = -\frac{1}{2}$. Thus, $\theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$.



16\. $r = 0$ is a point of intersection; solving $\sin\theta = \cos\theta$; we find that $\tan\theta = 1$ or $\theta = \frac{\pi}{4}$. Using the fact that the graphs are circles, we see the only points of intersection are (0, 0) and $(\frac{\sqrt{2}}{2}, \frac{\pi}{4})$.



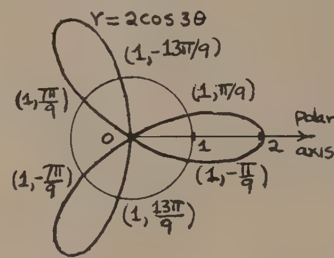
17\. (1) 0 is not a point of intersection.

(2) We solve the simultaneous equations $r = 1$ and $r = 2\cos 3(\theta + 2n\pi) = 2\cos 3\theta$. Thus, $1 = 2\cos 3\theta$ when $\cos 3\theta = \frac{1}{2}$ or when $3\theta = \pm\frac{\pi}{3}$, $3\theta = \pm\frac{7\pi}{3}$, $3\theta = \pm\frac{13\pi}{9}$.

Thus, $\theta = \pm\frac{\pi}{9}$, $\theta = \pm\frac{7\pi}{9}$, $\theta = \pm\frac{13\pi}{9}$. Therefore, the points of intersection are

$$\left(1, \pm\frac{\pi}{9}\right), \left(1, \pm\frac{7\pi}{9}\right), \left(1, \pm\frac{13\pi}{9}\right)$$

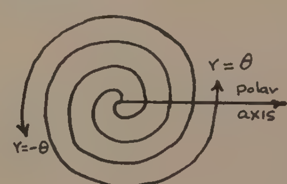
(3) Now we solve $r = 1$ together with $r = -2\cos 3(\theta + (2n + 1)\pi))$ or $r = 1$ with $r = 2\cos 3\theta$, so that $\frac{1}{2} = \cos 3\theta$, and we obtain the same solutions as above.



18\. (1) 0 is a point of intersection.

(2) We solve $r = \theta + 2n\pi$ and $r = -\theta$ simultaneously. Thus, $-\theta = \theta + 2n\pi$ when $2\theta = -2n\pi$ or $\theta = -n\pi$. But the domains of the two functions are such that $\theta = -n\pi \geq 0$, and $\theta = -n\pi + 2n\pi \geq 0$, so that $n\pi \geq 0$. Thus, $n = 0$. The point of intersection is 0.

(3) Now we solve $-r = \theta + (2n + 1)\pi$ and $r = -\theta$ simultaneously. Thus, $\theta = \theta + (2n + 1)\pi$ and so $(2n + 1)\pi = 0$, so that $n = -\frac{1}{2}$, which is not possible.

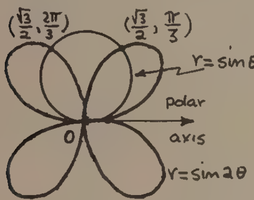


19\. (1) 0 is a point of intersection.

(2) We solve simultaneously $r = \sin(\theta + 2n\pi)$ and $r = \sin 2\theta$. Thus, $r = \sin\theta = \sin 2\theta$ or $\sin\theta = 2\sin\theta\cos\theta$, and so $\sin\theta(2\cos\theta - 1) = 0$. Hence, $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. Thus, $\theta = 0, \pi$

2π or $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$. The points of intersection are $(0,0)$, $(\frac{\sqrt{3}}{2},\frac{\pi}{3})$, $(\frac{\sqrt{3}}{2},\frac{2\pi}{3})$.

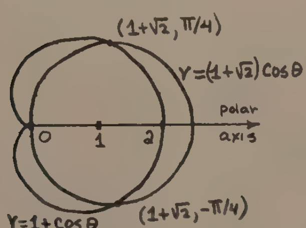
(3) We solve simultaneously $r = -\sin[\theta+(2n+1)\pi]$ and $r = \sin 2\theta$. Thus, $r=\sin 2\theta=-\sin(\theta+\pi) = \sin\theta$. Thus, $2\sin\theta\cos\theta-\sin\theta=0$ yields the same solutions as above.



20. (1) Since $0=1+\cos\pi$ and $0=(1+\sqrt{2})\cos\frac{\pi}{2}$, then 0 is a point of intersection.

(2) We solve $r=1+\cos(\theta+2n\pi)$ and $r=(1+\sqrt{2})\cos\theta$ simultaneously. Thus, $1+\cos\theta = \cos\theta + \sqrt{2}\cos\theta$, and so $\cos\theta = \frac{1}{\sqrt{2}}$. $\theta = \frac{\pi}{4}$ or $\theta = -\frac{\pi}{4}$. The points of intersection are $(1+\frac{\sqrt{2}}{2}, \frac{\pi}{4})$ and $(1+\frac{\sqrt{2}}{2}, -\frac{\pi}{4})$.

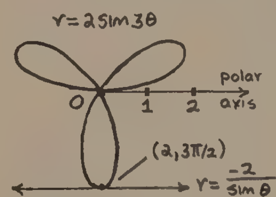
(3) Now we solve $r = -(1+\cos[\theta+(2n+1)\pi])$ and $r = (1+\sqrt{2})\cos\theta$ simultaneously. Thus, $\cos\theta - 1 = (1+\sqrt{2})\cos\theta$ and $\cos\theta = -\frac{1}{\sqrt{2}}$, so that $\theta = \frac{3\pi}{4}$ and $\theta = \frac{5\pi}{4}$. The points of intersection are $(\frac{-(1+\sqrt{2})}{\sqrt{2}}, \frac{3\pi}{4})$ and $(\frac{-(1+\sqrt{2})}{\sqrt{2}}, \frac{5\pi}{4})$, which are those same points obtained in part (2).



21. (1) 0 is not a point of intersection.

(2) We solve simultaneously $r = 2\sin 3\theta$ and $r = \frac{-2}{\sin(\theta+2n\pi)} = \frac{-2}{\sin\theta}$. Thus $2\sin 3\theta = -\frac{2}{\sin\theta}$. We solve $\sin(2\theta+\theta)(\sin\theta) = -1$, that is, $\left[\sin 2\theta\cos\theta+\cos 2\theta\sin\theta\right]\sin\theta = -1$. Multiplying out and using the facts that $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1-2\sin^2\theta$, and $\sin^2\theta + \cos^2\theta = 1$, we get $4\sin^4\theta - 3\sin^2\theta - 1 = 0$, so that $(4\sin^2\theta + 1)(\sin^2\theta - 1) = 0$. Hence, $\sin^2\theta = 1$ or $\sin\theta = 1$ or $\sin\theta = -1$. Thus, $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. The points of intersection are $(-2,\frac{\pi}{2})$ and $(2,\frac{3\pi}{2})$, but these two are representations of the same point. The point of intersection is $(2,\frac{3\pi}{2})$.

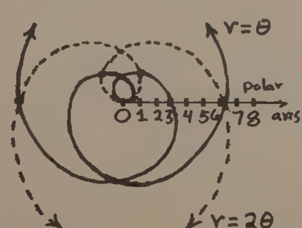
(3) We solve simultaneously $r = 2\sin 3\theta$ and $-r = \frac{-2}{\sin[\theta+(2n+1)\pi]}$. Thus, $2\sin 3\theta = r = \frac{2}{-\sin\theta}$; this is equivalent to the equation considered in part (2).



22. (1) 0 is a point of intersection.

(2) We solve $r = \theta+2n\pi$ and $r = 2\theta$ simultaneously. Thus, $\theta+2n\pi = 2\theta$ when $\theta=2n\pi$, $n=0,\pm1,\pm2,\ldots$ The points of intersection are $(4n\pi,2n\pi)$; that is, $(4n\pi,0)$, $n = 0, \pm1,\pm2,\ldots$

(3) Now we solve $r=-(\theta+2n\pi+\pi)$ and $r=2\theta$ simultaneously. Hence, $2\theta=-\theta-2n\pi-\pi$ when $\theta=\frac{-(2n+1)\pi}{3}$, $n=0,\pm1,\pm2,\ldots$ The points of intersection are $(\frac{-(4n+2)\pi}{3},\frac{-(2n+1)\pi}{3})$.



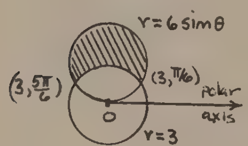
23. Solving $r = 6\sin(\theta+2n\pi)$ and $r = 3$ simultaneously, we have $6\sin\theta = 3$, so that $\sin\theta = \frac{1}{2}$; thus, $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$. The points of intersection are $(3,\frac{\pi}{6})$ and $(3,\frac{5\pi}{6})$.

$$A = 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (6\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3)^2 d\theta\right)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (36\sin^2\theta-9)d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[9-18\cos 2\theta\right]d\theta$$

$$= (9\theta-9\sin 2\theta)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{9\pi}{2} - \frac{9\pi}{6} + \frac{9\sqrt{3}}{2}$$

$$= \frac{6\pi + 9\sqrt{3}}{2} \text{ square units.}$$



24. We solve $r = 1$ and $r = 1+\cos(\theta+2n\pi)$ simultaneously. Thus, $1 = 1+\cos\theta$ when $\cos\theta = 0$; thus, $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. The points of intersection are $(1,\frac{\pi}{2})$ and $(1,\frac{3\pi}{2})$. Now,

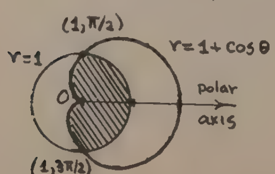
$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (1)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1+\cos\theta)^2 d\theta\right)$$

$$= \theta\Big|_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} (1+2\cos\theta + \cos^2\theta)d\theta$$

$$= \frac{\pi}{2} + (\theta+2\sin\theta)\Big|_{\frac{\pi}{2}}^{\pi} + \int_{\frac{\pi}{2}}^{\pi} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - 2 + \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \pi - 2 + \frac{\pi}{4} = \frac{5\pi-8}{4} \text{ square units.}$$



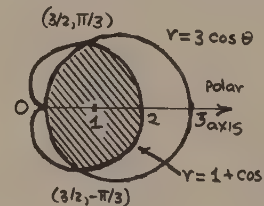
25. We solve $r = 3\cos(\theta+2n\pi)$ and $r = 1+\cos\theta$ simultaneously. Thus, $3\cos\theta = 1+\cos\theta$, so that $2\cos\theta = 1$ and $\cos\theta = \frac{1}{2}$ for $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$. The points of intersection are $(\frac{3}{2},\frac{\pi}{3})$ and $(\frac{3}{2},-\frac{\pi}{3})$. Now,

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (3\cos\theta)^2 d\theta\right)$$

$$= \int_0^{\frac{\pi}{3}} (1+2\cos\theta+\cos^2\theta)d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9\cos^2\theta\, d\theta$$

$$= \left(\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_0^{\frac{\pi}{3}} + 9\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \frac{9\sqrt{3}}{8} + 9\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) = \frac{5\pi}{4} \text{square units}$$

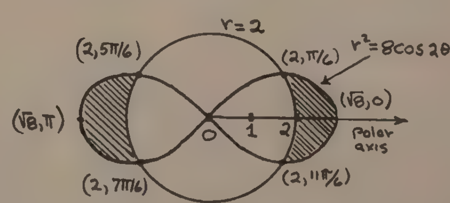


26. We solve $r^2 = 8\cos(2\theta+4n\pi)$ and $r = 2$ simultaneously. Thus, $8\cos 2\theta = 4$, $\cos 2\theta = \frac{1}{2}$, $2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$ or $\frac{11\pi}{3}$. The points of intersection are $(2,\frac{\pi}{6})(2,\frac{5\pi}{6})$, $(2,\frac{\pi}{6})$, $(2,\frac{11\pi}{6})$.

$$A = 4\left(\frac{1}{2}\int_0^{\pi/6} 8\cos 2\theta\, d\theta - \frac{1}{2}\int_0^{\pi/6} 4d\theta\right)$$

$$= 2(4\sin 2\theta-2\theta)\Big|_0^{\frac{\pi}{6}} = 4\sqrt{3}-\frac{2\pi}{3}$$

$$= \frac{12\sqrt{3}-2\pi}{3} \text{ square units.}$$

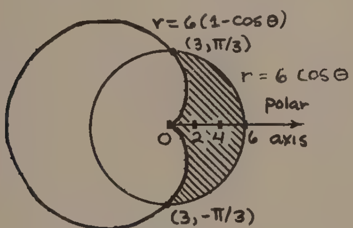


27. We solve $r = 6\cos(\theta+2n\pi)$ and $r = 6(1-\cos\theta)$ simultaneously. Thus, $6\cos\theta = 6-6\cos\theta$ or $\cos\theta = \frac{1}{2}$; $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$. The points of intersection are $(3,\frac{\pi}{3})$ and $(3,-\frac{\pi}{3})$.

$$A = 2\left[\frac{1}{2}\int_0^{\frac{\pi}{3}} (6\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}} 6(1-\cos\theta)^2 d\theta\right]$$

$$= \int_0^{\frac{\pi}{3}} (72\cos\theta - 36)d\theta = (72\sin\theta - 36\theta)\Big|_0^{\frac{\pi}{3}}$$

$$= 36\sqrt{3} - 12\pi = 12(3\sqrt{3}-\pi) \text{ square units.}$$



28. $r = a + b\sin\theta = 0$ when $\sin\theta = -\frac{a}{b}$, that is, when $\theta = \pi + \sin^{-1}\frac{a}{b}$ or $\theta = 2\pi - \sin^{-1}\frac{a}{b}$. The tip of the inner loop is $(a-b,\frac{3\pi}{2}) = (b-a,\frac{\pi}{2})$.

$$A = 2\left[\frac{1}{2}\int_{\pi+\sin^{-1}\frac{a}{b}}^{\frac{3\pi}{2}} (a+b\sin\theta)^2 d\theta\right]$$

$$= \int_{\pi+\sin^{-1}\frac{a}{b}}^{\frac{3\pi}{2}} (a^2+2ab\sin\theta+b^2\sin^2\theta)d\theta$$

$$= \left(a^2\theta - 2ab\cos\theta + \frac{b^2\theta}{2} - \frac{b^2\sin 2\theta}{4}\right)\Bigg|_{\pi+\sin^{-1}\frac{a}{b}}^{\frac{3\pi}{2}}$$

$$= \left(a^2\left(\frac{3\pi}{2}\right) + \frac{b^2 3\pi}{4} - a^2\left(\pi+\sin^{-1}\frac{a}{b}\right) + 2a\sqrt{b^2-a^2} - \frac{b^2}{2}\left(\pi+\sin^{-1}\frac{a}{b}\right) + \frac{b^2}{4}\left(\frac{a}{b}\right)\frac{\sqrt{b^2-a^2}}{b}\right.$$

$$= (2a^2+b^2)\left(\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{a}{b}\right) + \frac{9}{4}a\sqrt{b^2-a^2} \text{ square units.}$$

We used the fact that $\sin 2\theta = 2\sin\theta\cos\theta$ in evaluating the integral.

29. The points of intersection are $(\frac{\pi}{2},\frac{\pi}{2})$ and $(\frac{3\pi}{2},\frac{3\pi}{2})$.

$$A = 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \theta^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta^2 d\theta\right)$$

$$= \frac{\theta^3}{3}\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \frac{\theta^3}{3}\Bigg|_0^{\frac{\pi}{2}} = \frac{25}{24}\pi^3 \text{ square units.}$$

30. $$A = 2\left[\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2-4\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} (-6\cos\theta)^2 d\theta\right]$$

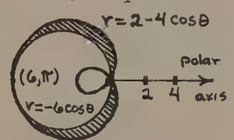
$$= \int_{\frac{\pi}{2}}^{\pi}\left[4-16\cos\theta+16\cos^2\theta\right]dt - 36\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)dt$$

$$= (4\theta - 16\sin\theta)\Big|_{\frac{\pi}{2}}^{\pi} + 16\int_{\frac{\pi}{2}}^{\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - 18\left[\theta + \frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{2}}\right]$$

$$= 4\pi - 2\pi + 16 + 8\left[\theta + \frac{\sin 2\theta}{2}\Big|_{\frac{\pi}{2}}^{\pi}\right] - 18\left[\frac{\pi}{2}\right]$$

$$= 2\pi + 16 + 8\left[\pi - \frac{\pi}{2}\right] - 9\pi$$

$$= 16 - 3\pi \text{ square units.}$$



31. $$s = \int_0^{2\pi} \sqrt{36\cos^2\theta + 36\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} 6d\theta = 6\theta\Big|_0^{2\pi} = 6\pi \text{ units.}$$

32. $$s = \int_0^{2\pi} \sqrt{0^2+(-2)^2}\, d\theta = \int_0^{2\pi} 2d\theta$$

$$= 2\theta\Big|_0^{2\pi} = 4\pi \text{ units.}$$

33. $$s = \int_0^{3/2} \sqrt{(8\theta)^2 + (4\theta^2)^2}\, d\theta$$

$$= \int_0^{3/2} 4\theta\sqrt{\theta^2+4}\, d\theta$$

$$= \int_4^{25/4} 2\sqrt{u}\, du = \frac{4u^{3/2}}{3}\Bigg|_4^{25/4}$$

$$= \frac{4}{3}\left(\frac{25}{4}\right)^{3/2} - \frac{4}{3}(4)^{3/2} = \frac{61}{6} \text{ units.}$$

34. $$s = 2\cdot \int_0^{\pi}\sqrt{\left[2(-\sin\theta)\right]^2+\left[2(1+\cos\theta)\right]^2}\ d\theta$$
$$= 2\cdot \int_0^{\pi}\sqrt{8(1+\cos\theta)}\ d\theta = 2\cdot \int_0^{\pi} 8(2\cos^2\tfrac{\theta}{2})d\theta$$
$$= 2\cdot \int_0^{\pi} 4\cos\tfrac{\theta}{2}\ d\theta = 16\ \sin\tfrac{\theta}{2}\ \Big|_0^{\pi}$$
$= 16$ units.

35. $$s = \int_0^{4\pi}\sqrt{(e^{\theta})^2 + (e^{\theta})^2}\ d\theta$$
$$= \int_0^{4\pi}\sqrt{2}\ e^{\theta}\ d\theta = \sqrt{2}\ e^{\theta}\ \Big|_0^{4\pi}$$
$= \sqrt{2}e^{4\pi} - \sqrt{2} = \sqrt{2}(e^{4\pi}-1)$ units.

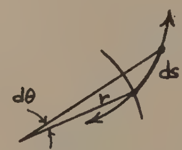
36. $$s = \int_0^{3\pi}\sqrt{(2\sin^2\tfrac{\theta}{3}\cos\tfrac{\theta}{3})^2+(2\sin^3\tfrac{\theta}{3})^2}\ d\theta$$
$$= \int_0^{3\pi}\sqrt{4\sin^4\tfrac{\theta}{3}\cos^2\tfrac{\theta}{3}+4\sin^6\tfrac{\theta}{3}}\ d\theta$$
$$= \int_0^{3\pi} 2\sin^2\tfrac{\theta}{3}\sqrt{\cos^2\tfrac{\theta}{3}+\sin^2\tfrac{\theta}{3}}\ d\theta$$
$$= \int_0^{3\pi} 2\sin^2\tfrac{\theta}{3}\ d\theta = \int_0^{3\pi}(1-\cos\tfrac{2\theta}{3})\ d\theta$$
$= (\theta-\frac{3}{2}\sin\frac{2\theta}{3})\ \Big|_0^{3\pi} = 3\pi$ units.

37. $$s = \int_0^{\pi/2}\sqrt{(-3\sin\theta+4\cos\theta)^2+(3\cos\theta+4\sin\theta)^2}d\theta$$
$$= \int_0^{\pi/2}\sqrt{9(\sin^2\theta+\cos^2\theta)+16(\sin^2\theta+\cos^2\theta)}d\theta$$
$= \int_0^{\pi/2} 5d\theta = 5\theta\ \Big|_0^{\pi/2} = \frac{5\pi}{2}$ units.

38. $$s = \int_{\pi/4}^{3\pi/4}\sqrt{(7\ \csc\ \theta\ \cot\ \theta)^2+(-7\ \csc\theta)^2}\ d\theta$$
$$= \int_{\pi/4}^{3\pi/4}\sqrt{49\ \csc^2\theta(1+\cot^2\theta)}\ d\theta$$
$$= \int_{\pi/4}^{3\pi/4} 7\csc^2\theta\ d\theta = -7\ \cot\ \theta\ \Big|_{\pi/4}^{3\pi/4}$$
$= -7(-1-1) = 14$ units.

39. The limits of integration are in error since, when θ goes from π to 2π, the circumference is counted for the second time. The correct integral is
$$s = \int_0^{\pi}\sqrt{(\tfrac{dr}{d\theta})^2 + r^2}\ d\theta = 4\pi.$$

40. In fact, ds is not the arc length of the portion of the circumference of a circle of radius $|r|$ cut off by the angle $d\theta$ radians, as can be seen in the accompanying figure. The portion of the circumference of the circle of radius $|r|$ cut off by $d\theta$ here is much smaller than ds.



41. In Cartesian coordinates,
$A = 2\pi\int_{x=a}^{x=b} y\ ds$. Thus, in polar coordinates,
$A = 2\pi\int_{\theta=\alpha}^{\theta=\beta} r\sin\theta\ \sqrt{(\frac{dr}{d\theta})^2+r^2}\ d\theta$, provided that the arc length is not counted twice between $\theta = \alpha$ and $\theta = \beta$.

42. In Cartesian coordinates,
$A = 2\pi\int_{x=a}^{x=b} x ds$, so that
$$A = 2\pi\int_{\alpha}^{\beta} r\cos\theta\ \sqrt{(\tfrac{dr}{d\theta})^2+r^2}\ d\theta$$
in polar coordinates, provided that the arc length is not counted twice between $\theta = \alpha$ and $\theta = \beta$.

43. $$A = 2\pi\int_0^{\pi} 2\ \sin\ \theta\ \sqrt{0^2+4}\ d\theta$$
$$= 2\pi\int_0^{\pi} 4\ \sin\ \theta\ d\theta = -8\pi\ \cos\ \theta\ \Big|_0^{\pi}$$
$= 8\pi - (-8\pi) = 16\pi$ square units.

44. $$A = 2\pi\int_{-\pi/2}^{\pi/2} 4\cos\theta\ \sqrt{0^2+16}\ d\theta$$
$$= 32\pi\int_{-\pi/2}^{\pi/2}\cos\theta\ d\theta = 32\pi(\sin\theta)\Big|_{-\pi/2}^{\pi/2}$$
$= 32\pi + 32\pi = 64\pi$ square units.

45. $A = 2\pi\int_0^{\pi/2}(5\cos\theta)\sin\theta\sqrt{(-5\sin\theta)^2 + 5\cos\theta)^2}\,d\theta$

$= 50\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$

$= 25(\sin^2\theta)\Big|_0^{\pi/2} = 25\pi$ square units.

46. $A = 2\pi\int_{-\pi/4}^{\pi/4} 2\sqrt{\cos 2\theta}\cdot\cos\theta\sqrt{\left(\frac{-2\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2 + 4\cos 2\theta}\,d\theta$

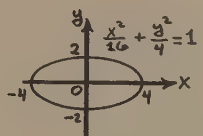
$= 4\pi\int_{-\pi/4}^{\pi/4}\sqrt{\cos 2\theta}\cdot\cos\theta\cdot\sqrt{\frac{4\sin^2 2\theta + 4\cos^2 2\theta}{\cos 2\theta}}\,d\theta$

$= 4\pi\int_{-\pi/4}^{\pi/4} 2\cos\theta\,d\theta = 8\pi(\sin\theta)\Big|_{-\pi/4}^{\pi/4}$

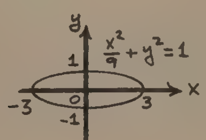
$= 8\pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 8\pi\sqrt{2}$ square units.

Problem Set 9.4, page 562

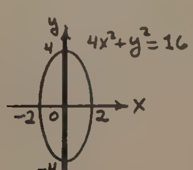
1. $a = 4$, $b = 2$, $c = \sqrt{12}$. Foci at $(-\sqrt{12},0)$, $(\sqrt{12},0)$. Vertices at $(-4,0)$, $(4,0)$, $(0,2)$, $(0,-2)$.



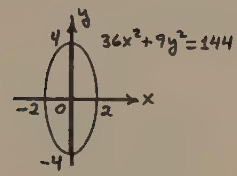
2. $a = 3$, $b = 1$, $c = \sqrt{8}$. Foci at $(-\sqrt{8},0)$, $(\sqrt{8},0)$. Vertices at $(-3,0)$, $(3,0)$, $(0,1)$, $(0,-1)$.



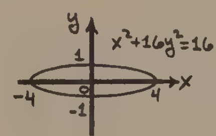
3. $\frac{x^2}{4} + \frac{y^2}{16} = 1$. $a = 4$, $b = 2$, $c = \sqrt{12}$. Foci at $(0,-\sqrt{12})$, $(0,\sqrt{12})$. Vertices at $(-2,0)$, $(2,0)$, $(0,4)$, $(0,-4)$.



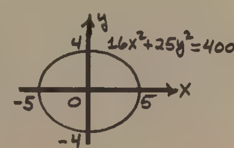
4. $\frac{x^2}{4} + \frac{y^2}{16} = 1$. $a = 4$, $b = 2$, $c = \sqrt{12}$. Same as 3 above.



5. $\frac{x^2}{16} + \frac{y^2}{1} = 1$. $a = 4$, $b = 1$, $c = \sqrt{15}$. Foci at $(-\sqrt{15},0)$, $(\sqrt{15},0)$. Vertices at $(-4,0)$, $(4,0)$, $(0,1)$, $(0,-1)$.

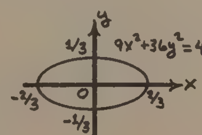


6. $\frac{x^2}{25} + \frac{y^2}{16} = 1$. $a = 5$, $b = 4$, $c = 3$. Foci at $(-3,0)$, $(3,0)$. Vertices at $(-5,0)$, $(5,0)$, $(0,4)$, $(0,-4)$.



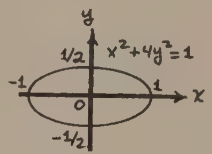
7. $\frac{x^2}{\left(\frac{4}{9}\right)} + \frac{y^2}{\left(\frac{4}{36}\right)} = 1$. $a = \frac{2}{3}$, $b = \frac{2}{6} = \frac{1}{3}$, $c = \frac{\sqrt{3}}{3}$.

Foci at $(-\frac{\sqrt{3}}{3},0)$, $(\frac{\sqrt{3}}{3},0)$. Vertices at $(-\frac{2}{3},0)$, $(\frac{2}{3},0)$, $(0,\frac{1}{3})$, $(0,-\frac{1}{3})$.



8. $\frac{x^2}{1} + \frac{y^2}{(\frac{1}{4})} = 1$. $a = 1$, $b = \frac{1}{2}$, $c = \frac{\sqrt{3}}{2}$.

Foci at $(-\frac{\sqrt{3}}{2},0)$, $(\frac{\sqrt{3}}{2},0)$. Vertices at $(-1,0)$, $(1,0)$, $(0,\frac{1}{2})$, $(0,-\frac{1}{2})$.



9. $c = 4$, $a = 5$, $b = 3$; $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

10. $a = 5$; $c = 3$; $b^2 = a^2 - c^2 = 25 - 9 = 16$. Thus, $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

11. $c = 12$; $a = 13$; $b^2 = a^2 - c^2 = 169-144=25$. Thus, $\frac{x^2}{25} + \frac{y^2}{169} = 1$.

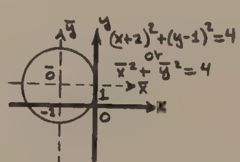
12. $b = 6$; $c = 8$; $a^2 = b^2 + c^2 = 36 + 64 = 100$. Thus, $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

13. $\bar{x} = x - h$, $\bar{y} = y - k$, $h = -1$, $k = 2$. Thus, $\bar{x} = x + 1$ and $\bar{y} = y - 2$.

(a) $(1,-2)$ (b) $(-1,-1)$ (c) $(4,-5)$

(d) $(-2,-4)$ (e) $(6,3)$ (f) $(7,-2)$

14. $x^2+y^2+4x-2y+1 = 0$ can be written as $(x+2)^2+(y-1)^2 = 4$. Thus, $\bar{x} = x + 2$ and $\bar{y} = y - 1$ gives $\bar{x}^2 + \bar{y}^2 = 4$, so $r = 2$.



15. Let (h,k) be the "old" xy coordinates of $\bar{O}$. Then $\bar{x} = x-h$, $\bar{y} = y-k$. Since O has "old" coordinates $x = 0$, $y = 0$ and "new" coordinates $\bar{x} = -3$, $\bar{y} = 2$, then $-3 = 0-h$, $2 = 0-k$ and $h = 3$, $k = -2$. Thus, $x = \bar{x} + h = \bar{x} + 3$ while $y = \bar{y} + k = \bar{y} - 2$ Consequently, the "old" coordinates of the given points are:

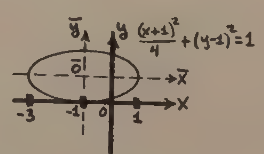
(a) $(3,-2)$ (b) $(6,0)$ (c) $(0,2)$

(d) $(3+\sqrt{2},-4)$ (e) $(3,-2-\pi)$ (f) $(0,0)$

16. $x = \bar{x} - \frac{b}{3}$, $y = \bar{y} - \frac{cb}{3} + d + \frac{2b^3}{27}$. Therefore, $\bar{y} - \frac{cb}{3} + d - \frac{2b^3}{27} = (\bar{x} - \frac{b}{3})^3 + b(\bar{x} - \frac{b}{3})^2 + c(\bar{x} - \frac{b}{3}) + d$. Now simplify: $\bar{y} - \frac{cb}{3} + d - \frac{2b^3}{27} = \bar{x}^3 - 3\bar{x}^2(\frac{b}{3}) + 3\bar{x}(\frac{b}{3})^2 - (\frac{b}{3})^3 + b(\bar{x}^2 - \frac{2\bar{x}b}{3} + \frac{b^2}{3}) + c\bar{x} - \frac{cb}{3} + d$ or $\bar{y} - \frac{2b^3}{27} = \bar{x}^3 - b\bar{x}^2 + \frac{b^2\bar{x}}{3} - \frac{b^3}{27} + b\bar{x}^2 - \frac{2b^2}{3}\bar{x} + \frac{b^3}{3} + c\bar{x}$. Thus, $\bar{y} = \bar{x}^3 + (c - \frac{b^2}{3})x$ or $\bar{y} = x^3 + px$, where $p = c - \frac{b^2}{3}$.

17. Complete the square to determine (h,k). $(x^2+2x+1)+4(y^2-2y) = 0$ or $(x+1)^2 + 4(y^2-2y+1) = 4$. Thus, $\frac{(x+1)^2}{4} + (y-1)^2 = 1$. $(h,k) = (-1,1)$. Therefore, $\bar{x} = x+1$ and $\bar{y} = y-1$ are the translation equations and $\frac{\bar{x}^2}{4} + \bar{y}^2 = 1$.

Center: $(-1,1)$ Foci: $(-\sqrt{3}-1,0)$, $(\sqrt{3}-1,0)$ Vertices: $(-3,1)$, $(1,1)$, $(-1,2)$, $(-1,0)$.

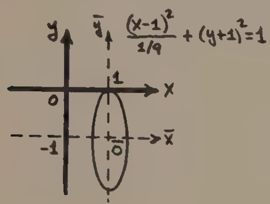


18. Complete the squares to determine (h,k). $9(x^2 - 2x) + y^2 + 2y = -9$ or $9(x^2 - 2x + 1) + (y^2 + 2y + 1)^2 = -9+9+1$ Thus, $9(x - 1)^2 + (y + 1)^2 = 1$.

Therefore, $\bar{x} = x - 1$ and $\bar{y} = y + 1$, and $\frac{\bar{x}^2}{\frac{1}{9}} + \frac{\bar{y}^2}{1} = 1$. Center: $(1,-1)$

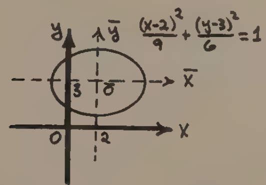
Foci: $(1,-1+\frac{2\sqrt{2}}{3})$ $(1,-1-\frac{2\sqrt{2}}{3})$

Vertices: $(1,0)$, $(1,-2)$, $(\frac{2}{3},-1)$, $(\frac{4}{3},-1)$



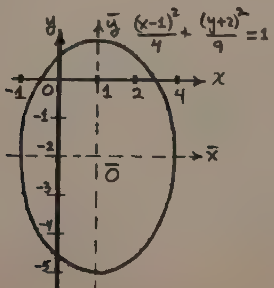
19. Complete the squares to determine h and k.
$6(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -51+24+81$.
Thus, $6(x-2)^2 + 9(y-3)^2 = 54$. Therefore, $\bar{x} = x - 2$ and $\bar{y} = y - 3$ and $\frac{\bar{x}^2}{9} + \frac{\bar{y}^2}{6} = 1$.

Center: $(2,3)$ Foci: $(-\sqrt{3}+2,3)$, $(\sqrt{3}+2,3)$
Vertices: $(-1,3)$, $(5,3)$, $(2,3+\sqrt{6})$, $(2,3-\sqrt{6})$



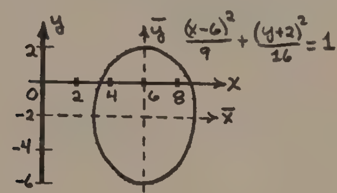
20. Complete the squares to determine h and k.
$9(x^2-2x+1) + 4(y^2+4y+4) = 11 + 9 + 16$.
Thus, $9(x-1)^2 + 4(y+2)^2 = 36$. Therefore, $\bar{x} = x - 1$ and $\bar{y} = y + 2$ and $\frac{\bar{x}^2}{4} + \frac{\bar{y}^2}{9} = 1$.

Center: $(1,-2)$ Foci: $(1,-2+\sqrt{5})$, $(1,-2-\sqrt{5})$
Vertices: $(3,-2)$, $(-1,-2)$, $(1,-5)$, $(1,1)$



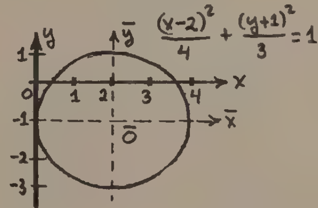
21. Complete the squares to determine h and k.
$16(x^2-12x+36)+9(y^2+4x+4) = -468+576+36$.
Therefore, $16(x-6)^2+9(y+2)^2 = 144$ so $\bar{x} = x - 6$ and $\bar{y} = y + 2$ and $\frac{\bar{x}^2}{9} + \frac{\bar{y}^2}{16} = 1$.

Center: $(6,-2)$ Foci: $(6,\sqrt{7}-2)$, $(6,-\sqrt{7}-2)$
Vertices: $(6,2)$, $(6,-6)$, $(3,-2)$, $(9,-2)$

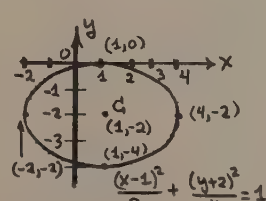


22. Complete the squares to determine h and k.
$3(x^2-4x+4)+4(y^2+2y+1) = -4+12+4$
Thus, $3(x-2)^2+4(y+1)^2 = 12$, so $\bar{x} = x-2$ and $\bar{y} = y+1$ and $\frac{\bar{x}^2}{4} + \frac{\bar{y}^2}{3} = 1$.

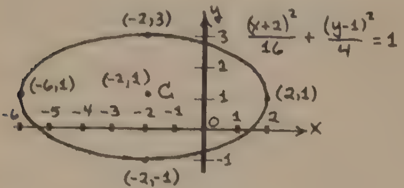
Center: $(2,-1)$ Foci: $(1,-1)(3,-1)$
Vertices: $(0,-1)$, $(4,-1)$, $(2,-1+\sqrt{3})$, $(2,-1-\sqrt{3})$



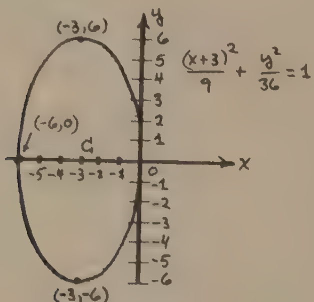
23. $(h,k) = (1,-2)$ $a = 3$, $b = 2$, $c = \sqrt{5}$.
Foci at $(1-\sqrt{5},-2)$, $(1+\sqrt{5},-2)$. Vertices at $(-2,-2)$, $(4,-2)$, $(1,0)$, $(1,-4)$.



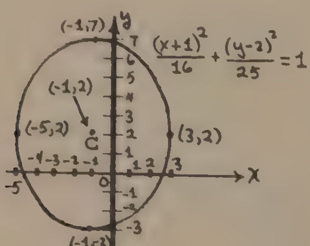
24. $(h,k) = (-2,1)$. $a = 4$, $b = 2$, $c = \sqrt{12}$. Foci at $(-2-\sqrt{12},1)$, $(-2+\sqrt{12},1)$. Vertices at $(-6,1)$, $(2,1)$, $(-2,3)$, $(-2,-1)$.



25. $\dfrac{(x+3)^2}{9} + \dfrac{y^2}{36} = 1$. $(h,k) = (-3,0)$.
$a = 6$, $b = 3$, $c = \sqrt{27}$. Foci at $(-3,-\sqrt{27})$, $(-3,\sqrt{27})$. Vertices at $(-3,-6)$, $(-3,6)$, $(-6,0)$, $(0,0)$.

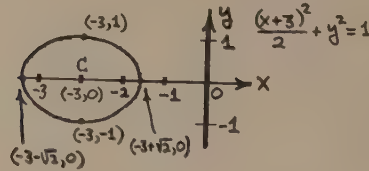


26. $\dfrac{(x+1)^2}{16} + \dfrac{(y-2)^2}{25} = 1$. $(h,k) = (-1,2)$.
$a = 5$, $b = 4$, $c = 3$. Foci at $(-1,-1)$, $(-1,5)$. Vertices at $(-1,-3)$, $(-1,7)$, $(-5,2)$, $(3,2)$.

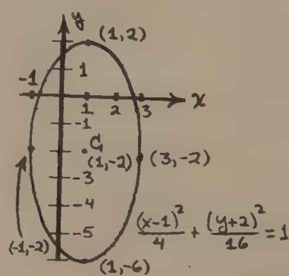


27. $x^2 + 6x + 9 + 2y^2 + 7 = 9$.
$(x+3)^2 + 2y^2 = 2$. $\dfrac{(x+3)^2}{2} + \dfrac{y^2}{1} = 1$.
$(h,k) = (-3,0)$. $a = \sqrt{2}$, $b = 1$, $c = 1$.

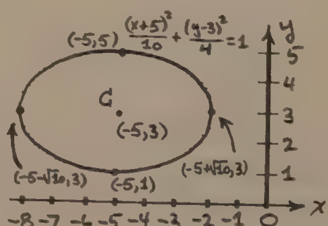
Foci at $(-4,0)$, $(-2,0)$. Vertices at $(-3-\sqrt{2},0)$, $(-3+\sqrt{2},0)$, $(-3,1)$, $(-3,-1)$.



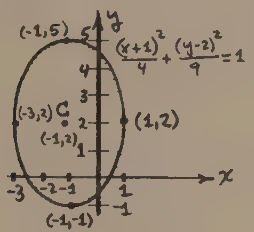
28. $4(x^2-2x+1) + (y^2+4y+4) - 8 = 4 + 4$.
$4(x-1)^2 + (y+2)^2 = 16$.
$\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{16} = 1$.
$(h,k) = (1,-2)$. $a = 4$, $b = 2$, $c = \sqrt{12}$.
Foci at $(1,-2-\sqrt{12})$, $(1,-2+\sqrt{12})$.
Vertices at $(1,-6)$, $(1,2)$, $(-1,-2)$, $(3,-2)$



29. $2(x^2+10x+25) + 5(y^2-6y+9) + 75 = 50 + 45$.
$2(x+5)^2 + 5(y-3)^2 = 20$.
$\dfrac{(x+5)^2}{10} + \dfrac{(y-3)^2}{4} = 1$. $(h,k) = (-5,3)$.
$a = \sqrt{10}$, $b = 2$, $c = \sqrt{6}$. Foci at $(-5-\sqrt{6},3)$, $(-5+\sqrt{6},3)$. Vertices at $(-5-\sqrt{10},3)$, $(-5+\sqrt{10},3)$, $(-5,5)$, $(-5,1)$.



30. $9(x^2+2x+1) + 4(y^2-4y+4) - 11 = 9 + 16.$
$9(x + 1)^2 + 4(y - 2)^2 = 36.$
$\frac{(x + 1)^2}{4} + \frac{(y - 2)^2}{9} = 1.$
$(h,k) = (-1,2).$ $a = 3,$ $b = 2,$ $c = \sqrt{5}.$
Foci at $(-1,2-\sqrt{5})$, $(-1,2+\sqrt{5})$. Vertices at $(-1,-1)$, $(-1,5)$, $(-3,2)$, $(1,2)$.



31. $c = 4$ and $a = 5$. Thus, $b^2 = a^2 - c^2 = 25-16$ or $b^2 = 9$. $(h,k) = (0,1)$, so that
$\frac{x^2}{25} + \frac{(y-1)^2}{9} = 1.$

32. For this ellipse, $(h,k) = (1,0)$, $a = 4$, and $c = 2$. Thus, $b^2 = a^2 - c^2 = 12$. So
$\frac{(x-1)^2}{12} + \frac{y^2}{16} = 1.$

33. Equation has the form $\frac{x^2}{b^2} + \frac{y^2}{64} = 1$. Since $x = 6$, $y = 0$ satisfies the equation,
$\frac{36}{b^2} + 0 = 1$, $b^2 = 36$. Equation is
$\frac{x^2}{36} + \frac{y^2}{64} = 1.$

34. $\frac{x^2}{b^2} + \frac{y^2}{9} = 1.$ $\frac{4}{9b^2} + \frac{8}{9} = 1$, $b^2 = 4$, so
$\frac{x^2}{4} + \frac{y^2}{9} = 1.$

35. $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1.$ $\frac{4^2}{p^2} + 0 = 1$, so $p^2 = 16.$
$\frac{x^2}{16} + \frac{y^2}{q^2} = 1$, $\frac{9}{16} + \frac{4}{q^2} = 1$, $q^2 = \frac{64}{7}$,
$\frac{x^2}{16} + \frac{7y^2}{64} = 1.$

36. $a = 4$, $b = 2\sqrt{3}$, $\frac{x^2}{12} + \frac{y^2}{16} = 1.$

37. $a = \frac{3+7}{2} = 5$, $b = \frac{5+3}{2} = 4$, $(h,k) = (-2,1)$,
$\frac{(x+2)^2}{25} + \frac{(y-1)^2}{16} = 1.$

38. $(h,k) = (\frac{1+5}{2},3) = (3,3)$, $c = 3-1 = 2$,
$a = \frac{10}{2} = 5$, $b = \sqrt{a^2-c^2} = \sqrt{21}$,
$\frac{(x-3)^2}{25} + \frac{(y-3)^2}{21} = 1.$

39. $a = 3$, $b = 2$, $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1.$

40. $a = \frac{5+3}{2} = 4$, $b = 2$, $(h,k) = (5-4,2)=(1,2)$,
$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{4} = 1.$

41. $a = \frac{5+3}{2} = 4$ and $c = \frac{4+2}{2} = 3$ so
$b^2 = a^2-c^2 = 16 - 9 = 7$. $(h,k) = (2,1)$;
thus, $\frac{(x-2)^2}{7} + \frac{(y-1)^2}{16} = 1.$

42. $k = 5$ and $h = \frac{-4+2}{2} = -1$; thus, the center is $(-1,5)$. $c = \frac{4-(-2)}{2} = 3$ and $b = 3$.
Now $a^2 = b^2+c^2 = 18$. So the equation is $\frac{(x+1)^2}{18} + \frac{(y-5)^2}{9} = 1.$

43. $2x + 18y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{x}{9y}$. When $x = 9$ and $y = 4$, $\frac{dy}{dx} = -\frac{1}{4}$.
Tangent line: $y-4 = -\frac{1}{4}(x-9)$ or $y=-\frac{x}{4}+\frac{25}{4}$.
Normal line: $y-4 = 4(x-9)$ or $y=4x-32$.

44. $8x+18y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{4x}{9y}$. When $x = 3$ and $y = 1$, $\frac{dy}{dx} = -\frac{4}{3}$.
Tangent line: $y-1 = -\frac{4}{3}(x-3)$ or $y=-\frac{4}{3}x+5$.
Normal line: $y-1=\frac{3}{4}(x-3)$ or $y=\frac{3}{4}x-\frac{5}{4}$.

45. $2x+8y\frac{dy}{dx}-2+8\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{1-x}{4(y+1)}$. When $x = 3$ and $y = 2$, $\frac{dy}{dx} = -\frac{1}{6}$.
Tangent line: $y - 2 = -\frac{1}{6}(x - 3)$ or
$x + 6y = 15.$

Normal line: $y - 2 = 6(x - 3)$ or

$y = 6x - 16$.

46. $18x + 50y\frac{dy}{dx} - 50\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{9x}{25(1 - y)}$.

When $y = 1$, $\frac{dy}{dx}$ is undefined; thus, $(5,1)$ is a vertex of the ellipse at which the tangent line is vertical. Therefore, the equation of the tangent line is $x = 5$ and the equation of the normal line is $y = 1$.

47. (a) Let q denote the length of the latus rectum. Then, the point $(c,\frac{q}{2})$ belongs to the ellipse, so $b^2c^2 + \frac{a^2q^2}{4} = a^2b^2$, $b^2(a^2-b^2) + \frac{a^2q^2}{4} = a^2b^2$, $\frac{a^2q^2}{4} = b^4$, $q = \frac{2b^2}{a}$.

(b) $\frac{x^2}{16} + \frac{y^2}{9} = 1$, $a = 4$, $b = 3$; hence $q = \frac{2b^2}{a} = \frac{18}{4} = \frac{9}{2}$ units.

48. $0 < b < a$, $c = \sqrt{a^2-b^2}$ or $c^2 = a^2-b^2$, and $y = \pm \frac{b}{a}\sqrt{a^2-x^2}$ or $x = \pm \frac{a}{b}\sqrt{b^2-y^2}$. From the above relationships we have: $0 < c < a$ as well as $-a < x < a$ and $-b < y < b$, which will be used in part (a) or (b).

To show (a): $|x| \leq a$ or $c|x| \leq ca < a^2$ since $c < a$.

To show (b): $|\overline{PF_2}| = \sqrt{(x-c)^2+y^2} < \sqrt{(a+c)^2+b^2}$ because $x-c < a-c < a+c$ since $c > 0$; also $y < b$. Consequently, $(x-c)^2 < (a+c)^2$ and $y^2 < b^2$. Thus, $\sqrt{(x-c)^2+y^2} < \sqrt{a^2+2ac+c^2+b^2} = \sqrt{2a^2+2ac} < \sqrt{4a^2} = 2a$.

To show (c): We will show that $4a^2 = |\overline{PF_1}|^2+2|\overline{PF_1}|\,|\overline{PF_2}|+|\overline{PF_2}|^2$. If this is true, then $(2a)^2 = (|\overline{PF_1}|+|\overline{PF_2}|)^2$ and it will follow that $2a = |\overline{PF_1}|+|\overline{PF_2}|$.

First, $|\overline{PF_1}| = \sqrt{(x+c)^2+\frac{b^2}{a^2}(a^2-x^2)} = \frac{\sqrt{x^2c^2+2ca^2x+a^4}}{a}$ and $|\overline{PF_2}| = \sqrt{(x-c)^2+\frac{b^2}{a^2}(a^2-x$

$= \frac{\sqrt{x^2c^2-2ca^2x+a^4}}{a}$. Now,

$|\overline{PF_1}|^2 + 2|\overline{PF_1}|\,|\overline{PF_2}| + |\overline{PF_2}|^2$

$= \frac{x^2c^2+2ca^2x+a^4}{a^2} + \frac{2\sqrt{x^2c^2+2ca^2x+a^4}\sqrt{x^2c^2-2ca}}{a^2}$

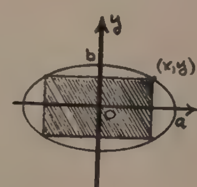
$+ \frac{x^2c^2-2ca^2x+a^4}{a^2}$

$= \frac{2x^2c^2+2a^4+2\sqrt{x^4c^4-2c^2a^4x^2+a^8}}{a^2}$

$= \frac{2x^2c^2+2a^4+2(a^4-x^2c^2)}{a^2}$ since $a^2 > c|x|$ or $a^4 > c^2x^2$.

$= \frac{4a^4}{a^2} = 4a^2$. Hence, $2a = |\overline{PF_1}| + |\overline{PF_2}|$.

49. From the accompanying figure, the area is $A = (2x)(2y) = 4xy$. Since $y = b\sqrt{1 - \frac{x^2}{a^2}}$, then $A = 4b\sqrt{x^2 - \frac{x^4}{a^2}}$. Let f be the function defined by the equation $f(x) = x^2 - \frac{x^4}{a^2}$ and notice that A is maximum when f takes on its maximum value. Since $f'(x) = 2x - \frac{4x^3}{a^2}$, the critical value $x = \frac{a}{\sqrt{2}}$ gives the desired maximum. The maximum area is $A = 4b\sqrt{(\frac{a}{\sqrt{2}})^2 - \frac{1}{a^2}(\frac{a}{\sqrt{2}})^4}$ $= 2ab$ square units.



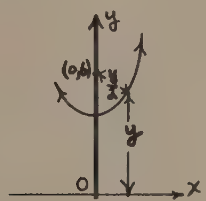
50. $\frac{y}{2} = \sqrt{x^2+(y-6)^2}$, $\frac{y^2}{4} = x^2+y^2-12y+36$,

$4x^2+3y^2-48y+144 = 0$,

$4x^2+3(y^2-16y+64) + 144 = 3(64)$,

$4x^2 + 3(y-8)^2 = 48$, $\frac{x^2}{12} + \frac{(y-8)^2}{16} = 1$.

The curve is an ellipse with center at (0,8), vertical major axis, $a = 4$, $b = 2\sqrt{3}$, $c = \sqrt{16-12} = 2$, foci at (0,6), (0,10). Thus, the university is at the lower focus of the ellipse.



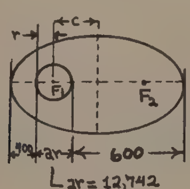
51. $a = \frac{60}{2} = 30$, $b = \frac{20}{2} = 10$, so $c = \sqrt{a^2-b^2} = \sqrt{800} = 20\sqrt{2}$. The required length of string is $\ell = 2a+2c = 60 + 40\sqrt{2}$ feet. The two stakes should be $2c = 40\sqrt{2}$ feet apart.

52. $2a = 400 + 12{,}742 + 600$.

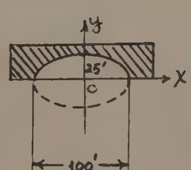
$a = 6871$. $a - c = 400 + r$, so

$c = a - r - 400$, $c = 100$. Therefore,

$b = \sqrt{a^2-c^2} = \sqrt{(6871)^2-(100)^2} \approx 6870.27$ km.



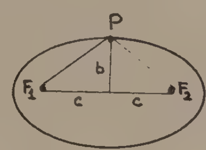
53. $\frac{x^2}{50^2} + \frac{y^2}{25^2} = 1$. $a = 50$, $b = 25$.



54. $|\overline{PF_1}| + |\overline{PF_2}| + |\overline{F_1F_2}| = \ell$.

If $|\overline{PF_1}| = |\overline{PF_2}|$, then P is directly above the midpoint of $\overline{F_1F_2}$ at a distance b, which is the semimajor axis. Therefore, $2c+2|\overline{PF_1}| = \ell$ or $|\overline{PF_1}| = \frac{\ell-2c}{2}$. Thus, $(\frac{\ell-2c}{2})^2 = c^2+b^2$ by the Pythagorean theorem; so $b^2 = (\frac{\ell-2c}{2})^2-c^2 = \frac{\ell^2-4\ell c+4c^2-4c^2}{4} = \frac{\ell^2-4\ell c}{4}$. Hence,

$b = \frac{1}{2}\sqrt{\ell^2-4\ell c}$.



55. (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so $y = \frac{b}{a}\sqrt{a^2-x^2}$ for $0 \le x \le a$. $A = 4\int_0^a y\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$.

(b) $A = \frac{4b}{a}\int_0^{\pi/2} a^2\cos^2\theta\,d\theta$ using the trig. substitution $x = a \sin\theta$, $dx = 9 \cos\theta\,d\theta$. Thus, $A = 4ab\left[\frac{1}{2}(\theta+\frac{\sin 2\theta}{2})\Big|_0^{\pi/2}\right] = 4ab(\frac{\pi}{4}) = ab\pi$ square units.

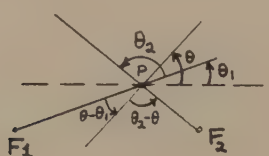
56. From the adjacent figure, $m_1 = \tan\theta_1$, $m = \tan\theta$, $m_2 = \tan\theta_2$, and we must prove that $\theta - \theta_1 = \theta_2 - \theta$. It will be enough to prove that $\tan(\theta-\theta_1) = \tan(\theta_2 - \theta)$. By the trigonometric identity for the tangent of the difference between two angles, the latter condition is equivalent to

$$\frac{\tan\theta - \tan\theta_1}{1+\tan\theta\tan\theta_1} = \frac{\tan\theta_2 - \tan\theta}{1+\tan\theta_2\tan\theta};$$

that is, $\frac{m - m_1}{1+mm_1} = \frac{m_2 - m}{1+m_2m}$. By cross multiplication, the latter equation is equivalent to $(m_1+m_2)m^2 + 2(1-m_1m_2)m = m_1 + m_2$. We have $m_1+m_2 = \frac{y}{x+c} + \frac{y}{x-c} = \frac{2xy}{x^2-c^2}$, $m^2 = (\frac{a^2y}{b^2x})^2 = \frac{a^4y^2}{b^4x^2}$, so the desired equation is equivalent to

$$(\frac{2xy}{x^2-c^2})\,\frac{a^4y^2}{b^4x^2} + 2(1 - \frac{y^2}{x^2-c^2})\frac{a^2y}{b^2x} = \frac{2xy}{x^2-c^2}.$$

If the latter equation is multiplied on both sides by $\frac{b^4x(x^2-c^2)}{2y}$, it is seen to be equivalent to $a^4y^2+a^2b^2(x^2-c^2-y^2) = b^4x^2$; that is, $(a^2-b^2)b^2x^2+(a^2-b^2)a^2y^2 = a^2b^2c^2$. Since $a^2-b^2= c^2$, the desired equation is equivalent to $b^2x^2 + a^2y^2 = a^2b^2$; that is, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which holds because $P = (x,y)$ belongs to the ellipse.

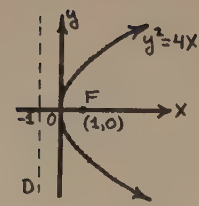


57. By the result in Problem 56 and the fact that the angle of incidence equals the angle of reflection, the path of the ray from focus to focus is clear from Figure 19 in Section 9.4.

Problem Set 9.5, page 568

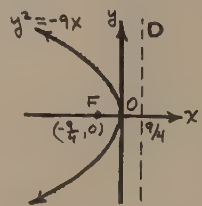
1. $x = \frac{1}{4} y^2$, $p = 1$, $V = (0,0)$, $F = (1,0)$,

   D: $x = -1$, length of focal chord = 4.



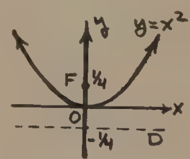
2. $x = -\frac{1}{9}y^2$, $p = \frac{9}{4}$, $V = (0,0)$, $F = (-\frac{9}{4},0)$,

   D: $x = \frac{9}{4}$, length of focal chord = 9.



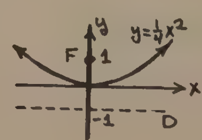
3. $y = x^2$, $p = \frac{1}{4}$, $V = (0,0)$, $F = (0,\frac{1}{4})$,

   D: $y = -\frac{1}{4}$, length of focal chord = 1.



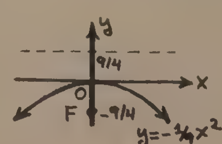
4. $y = \frac{1}{4} x^2$, $p = 1$, $V = (0,0)$, $F = (0,1)$,

   D: $y = -1$, length of focal chord = 4.



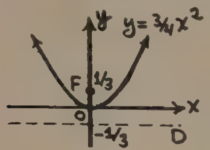
5. $y = -\frac{1}{9} x^2$, $p = \frac{9}{4}$, $V = (0,0)$, $F = (0,-\frac{9}{4})$,

   D: $y = \frac{9}{4}$, length of focal chord = 9.



6. $y = \frac{3}{4}x^2$, $p = \frac{1}{3}$, $V = (0,0)$, $F = (0,\frac{1}{3})$,

D: $y = -\frac{1}{3}$, length of focal chord $= \frac{4}{3}$.

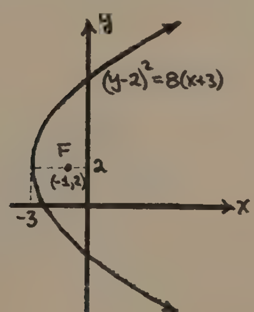


7. $V = (0,0)$, $p = 3$, $y = \frac{1}{12}x^2$.

8. $\frac{dy}{dx} = 2Ax + B$; $\frac{dy}{dx} = 0$ when $x = -\frac{B}{2A}$. When $x = -\frac{B}{2A}$, $y = A(-\frac{B}{2A})^2 + B(-\frac{B}{2A}) + C = \frac{4AC - B^2}{4A}$. Therefore, the vertex is at $(-\frac{B}{2A}, \frac{4AC - B^2}{4A})$.

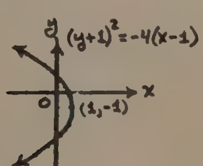
9. $V = (-3,2)$, $p = 2$, $F = (-1,2)$,

D: $x = -5$, length of focal chord $= 8$.



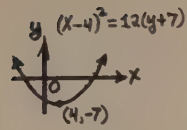
10. $V = (1,-1)$, $p = 1$, $F = (0,-1)$,

D: $x = 2$, length of focal chord $= 4$.



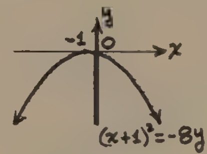
11. $V = (4,-7)$, $p = 3$, $F = (4,-4)$,

D: $y = -10$, length of focal chord $= 12$.



12. $V = (-1,0)$, $p = 2$, $F = (-1,-2)$,

D: $y = 2$, length of focal chord $= 8$.

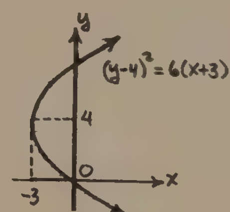


13. $(y^2 - 8y + 16) - 6x - 2 = 16$,

$(y - 4)^2 = 6(x + 3)$, $V = (-3,4)$,

$p = \frac{3}{2}$, $F = (-\frac{3}{2},4)$, D: $x = -\frac{9}{2}$,

length of focal chord $= 6$.

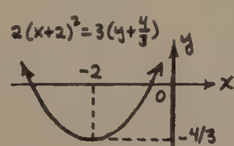


14. $2(x^2 + 4x + 4) - 3y + 4 = 8$.

$2(x + 2)^2 = 3(y + \frac{4}{3})$, $V = (-2,-\frac{4}{3})$,

$p = \frac{3}{8}$, $F = (-2,-\frac{23}{24})$, D: $y = -\frac{41}{24}$,

length of focal chord $= \frac{3}{2}$.

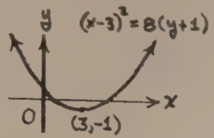


15. $(x^2 - 6x + 9) - 8y + 1 = 9$.

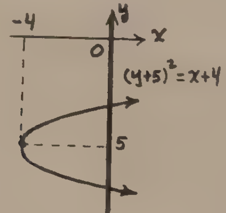
$(x - 3)^2 = 8(y + 1)$, $V = (3,-1)$,

$p = 2$, $F = (3,1)$, D: $y = -3$,

length of focal chord = 8.



16. $y^2 + 10y + 25 - x + 21 = 25.$
$(y + 5)^2 = x + 4$, $V = (-4,-5)$, $p = \frac{1}{4}$,
$F = (-\frac{15}{4},-5)$, D: $x = -\frac{17}{4}$,
length of focal chord = 1.



17. $x - 5 = -\frac{1}{4}(y - 2)^2.$

18. $y - 2 = -\frac{1}{12}(x - 3)^2.$

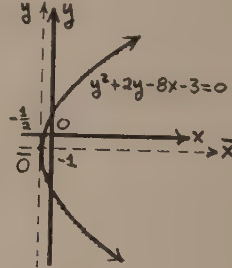
19. $x + 6 = \frac{1}{32}(y + 5)^2.$

20. $(x - 2) = \frac{1}{40}(y + 3)^2.$

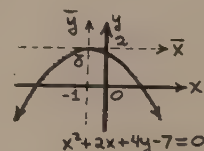
21. The equation has the form $x + \frac{1}{2} = \frac{1}{4p}(y + 1)^2$. Put $x = \frac{5}{8}$, $y = 2$ and solve for p to get $p = 2$. The equation is $x + \frac{1}{2} = \frac{1}{8}(y + 1)^2$.

22. The equation has the form $y - k = \frac{\pm 1}{4p} x^2$. Since (2,3) and (-1,-2) belong to the parabola, $3 - k = \frac{\pm 1}{4p} 4$ and $-2 - k = \frac{\pm 1}{4p} 1$ must hold. Thus, $3 - k = \frac{\pm 1}{p}$ and $-8 - 4k = \frac{\pm 1}{p}$, so $3 - k = -8 - 4k$, $3k = -11$, $k = -\frac{11}{3}$. Now $\frac{\pm 1}{p} = 3 - k = 3 + \frac{11}{3} = \frac{20}{3}$; since $k = -\frac{11}{3}$ and the axis is $x = 0$, then $p > 0$. Thus, $p = \frac{3}{20}$. The equation is $y + \frac{11}{3} = \frac{5}{3} x^2$.

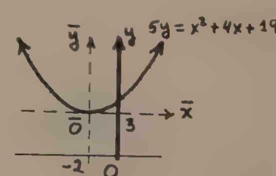
23. $y^2 + 2y - 8x - 3 = 0$ can be written as $(y + 1)^2 = 8x + 4$ or $(y + 1)^2 = 8(x + \frac{1}{2})$ by completing the square. Thus, $\bar{x} = x + \frac{1}{2}$ and $\bar{y} = y + 1$, and so $\bar{y}^2 = 8\bar{x}$ with $p = 2$.



24. $x^2 + 2x + 4y - 7 = 0$ can be written as $(x + 1)^2 = -4y + 8$ or $(x + 1)^2 = -4(y - 2$
Thus, $\bar{x} = x + 1$ and $\bar{y} = y - 2$, and so $\bar{x}^2 = -4\bar{y}$ with $p = 1$.



25. $5y = x^2 + 4x + 19$ can be written as $(x + 2)^2 = 5y - 19 + 4$ or $(x + 2)^2 = 5(y - 3)$. Thus, $\bar{x} = x + 2$ and $\bar{y} = y - 3$, and so $\bar{x}^2 = 5\bar{y}$ with $p = \frac{5}{4}$.



26. $y - C = A(x^2 + \frac{B}{A} x)$. Complete the square by adding $\frac{B^2}{4A}$ to both sides, $y + \frac{B^2}{4A} - C =$

$A(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}), y + \frac{B^2 - 4AC}{4A} =$ $A(x + \frac{B}{2A})^2$. This is the equation of a parabola with vertex at $(-\frac{B}{2A}, -\frac{B^2 - 4AC}{4A})$. The parabola has a vertical axis and (since $A > 0$) it opens upward. Since $A = \frac{1}{4p}$, then $p = \frac{1}{4A}$. The focus is therefore $F = (-\frac{B}{2A}, \frac{1 - B^2 + 4AC}{4A})$ and the length of the latus rectum is $4p = \frac{1}{A}$. Since the graph opens upward from the vertex which is $-\frac{B^2 - 4AC}{4A}$ units high, it will be entirely above the x axis unless $-\frac{B^2 - 4AC}{4A} < 0$; hence, the graph will intersect the x axis if and only if $B^2 - 4AC > 0$.

27. (a) $2y \frac{dy}{dx} = 8$. At $(2,-4)$, $\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{-4} = 1$.

Tangent line: $y+4 = -(x-2)$ or $y=-x-2$.

Normal line: $y+4 = x-2$ or $y = x-6$.

(b) $4y \frac{dy}{dx} = 9$. At $(2,-3)$, $\frac{dy}{dx} = \frac{9}{4y} =$ $-\frac{9}{12} = -\frac{3}{4}$.

Tangent line: $y+3 = -\frac{3}{4}(x - 2)$ or $y = -\frac{3}{4}x - \frac{3}{2}$ or $3x+4y = -6$.

Normal line: $y + 3 = \frac{4}{3}(x-2)$ or $y = \frac{4}{3}x - \frac{17}{3}$ or $4x - 3y = 17$.

(c) $2x = -12 \frac{dy}{dx}$. At $(-6,-3)$, $\frac{dy}{dx} = -\frac{x}{6} = 1$.

Tangent line: $y+3=x+6$ or $y = x+3$.

Normal line: $y+3 = -(x+6)$ or $y = -x-9$.

(d) $2x + 8 \frac{dy}{dx} + 4 = 0$. At $(1,\frac{15}{8})$, $\frac{dy}{dx} =$ $-\frac{(x+2)}{4} = -\frac{3}{4}$.

Tangent line: $y - \frac{15}{8} = -\frac{3}{4}(x - 1)$ or $y = -\frac{3}{4}x + \frac{21}{8}$ or $8y + 6x = 21$.

Normal line: $y - \frac{15}{8} = \frac{4}{3}(x - 1)$ or $y = \frac{4}{3}x + \frac{13}{24}$ or $24y - 32x = 13$.

(e) $2y \frac{dy}{dx} - 2 \frac{dy}{dx} + 10 = 0$. At $(\frac{9}{2},1)$, $\frac{dy}{dx} = \frac{5}{1-y}$ is undefined. Therefore,

Tangent line: $x = \frac{9}{2}$.

Normal line: $y = 1$.

28. The equation involves only terms of first degree in y and second degree or first degree in x; hence, its graph is a parabola with vertical axis. It opens upward because the coefficient of y on the left-hand side and the coefficient of x^2 on the right have the same algebraic sign.

29. (a) $\frac{dy}{dx} = 2x - 2$, so $\frac{dy}{dx} = 0$ when $x = 1$. When $x = 1$, $y = x^2 - 2x + 6 =$ $(1)^2 - 2(1) + 6 = 5$. The parabola has a vertical axis and it has a horizontal tangent at $(1,5)$. Therefore, its vertex is at $(1,5)$.

(b) $8x + 24 - 3 \frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{8x + 24}{3}$, so $\frac{dy}{dx} = 0$ when $x = -3$. When $x = -3$, $4(-3)^2 + 24(-3) + 39 - 3y = 0$, $y = 1$. The parabola has a vertical axis and its vertex is at $(-3,1)$.

(c) $2y - 10 = 4 \frac{dx}{dy}$. (Note that the parabola has a horizontal axis, so this time we want to set $\frac{dx}{dy} = 0$). We have $\frac{dx}{dy} = 0$ when $y = 5$. When $y = 5$, $(5)^2 - 10(5) = 4x - 21$, $x = -1$. The parabola with horizontal axis has a horizontal normal; hence, its vertex is at $(-1,5)$.

(d) $3\frac{dx}{dy} = 14 - 2y$, $\frac{dx}{dy} = 0$ when $y = 7$.

When $y = 7$, $3x = 14(7) - (7)^2 - 43$, $x = 2$. Vertex at $(2,7)$.

30. $ds = \sqrt{1+(\frac{dy}{dx})^2}dx$ where $\frac{dy}{dx} = \frac{2x}{4p}$.

$s = \int_0^a \sqrt{1 + \frac{4x^2}{16p^2}}\, dx = \int_0^a \sqrt{\frac{16p^2+4x^2}{16p^2}}\, dx =$

$\frac{1}{2p}\int_0^a \sqrt{4p^2+x^2}\, dx =$

$\frac{1}{2p}\int_0^{\tan^{-1}\frac{a}{2p}} \sqrt{4p^2(1+\tan^2\theta)}\ 2p \sec^2\theta\, d\,\theta$,

where $x = 2p\tan\theta$ and $dx = 2p\sec^2\theta d\theta$.

So $s = 2p\int_0^{\tan^{-1}\frac{a}{2p}} \sec^3\theta d\theta$. By

Formulas 33 and 11, $\int \sec^3\theta\, d\,\theta =$

$\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C$.

Hence, $s = p\left[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right|_0^{\tan^{-1}\frac{a}{2p}}\Big]$

$= p\left[\left(\frac{\sqrt{4p^2+a^2}}{2p}\right)\left(\frac{a}{2p}\right) + \ln\left|\frac{\sqrt{4p^2+a^2}}{2p} + \frac{a}{2p}\right|\right]$

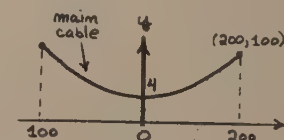
$= \frac{a\sqrt{4p^2+a^2}}{4p} + p\ln\left(\frac{\sqrt{4p^2+a^2}+a}{2p}\right)$

Note: $\tan\theta = \frac{a}{2p}$, so $\sec\theta = \frac{\sqrt{4p^2+a^2}}{2p}$.

31. The area of the rectangle is $A = (2x)y = 2x(12 - x^2)$. Therefore, $A = 24x - 2x^3$ and $\frac{dA}{dx} = 24 - 6x^2$. $\frac{dA}{dx} = 0$ for $x = \pm 2$. The critical number $x = 2$ gives the desired maximum. When $x = 2$, $y = 12 - 4 = 8$. The desired rectangle is 8 units high and its base is 4 units long.

32. Let the equation of the parabola be $y = \frac{1}{4p}x^2$, so that the focus is at the point $(0,p)$. If (x,y) is an endpoint of the focal chord, then $y = p$ and $y = \frac{1}{4p}x^2$; hence, $p = \frac{1}{4p}x^2$, $4p^2 = x^2$, and $x = \pm 2p$. Therefore, the endpoints of the focal chord are $(-2p,p)$ and $(2p,p)$. Its length is $4p$ units.

33. Set up an xy coordinate system so that the vertex of the parabola is 4 meters above the origin. The equation of the parabola is $y - 4 = \frac{1}{4p}(x - 0)^2$. When $x = 200$, then $y = 100$, and so $96 = \frac{200^2}{4p}$, $4p = \frac{200^2}{96} = \frac{1250}{3}$. Thus, the equation of the parabola is $y = 4 + \frac{3x^2}{1250}$. The vertical cables have x coordinates -150, -100, -50, 0, 50, 100, 150, and their lengths can be found directly from $y = 4 + \frac{3x^2}{1250}$ by substitution. The lengths are 58, 28, 10, 4, 10, 28, and 58 meters, respectively.



(not drawn to scale)

34. Set up a coordinate system as shown. The equation of the parabola is $y = -\frac{1}{4p}x^2$. Since $(30,-1)$ belongs to the parabola, $-1 = -\frac{1}{4p}30^2$, $4p = 900$, $y = -\frac{1}{900}x^2$. When $x = 15$, $y = -\frac{15^2}{900} = -0.25$. Thus, the roadway is $\frac{1}{4}$ foot higher than the ends at a point 15 feet from an end.

(not drawn to scale)

35. The y coordinate of the point whose x coordinate is a is $\frac{a^2}{4p}$ and the slope of the tangent line to the parabola at

$(a, \frac{a^2}{4p})$ is $\frac{dy}{dx} = \frac{2x}{4p} = \frac{a}{2p}$. Thus, the equation of the normal line at $(a, \frac{a^2}{4p})$ is $y - \frac{a^2}{4p} = -\frac{2p}{a}(x - a)$. This normal line will intersect the y axis at the center of the circle, so we obtain f(a) by solving the latter equation for y when $x = 0$. The result is:

(a) $f(a) = \frac{a^2}{4p} + 2p$. Thus,

(b) $\lim_{a \to 0} f(a) = \lim_{a \to 0} (\frac{a^2}{4p} + 2p) = 2p$.

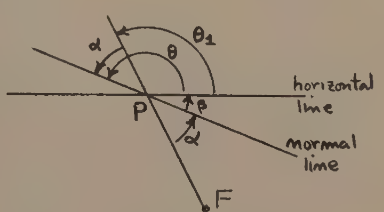
36. From the work in Problem 35, the desired circle has its center at $(0,2p)$ and has radius $r = 2p$.

37. In the accompanying figure, $\alpha = \theta - \theta_1$ and $\beta + \theta = \pi$ radians, so $\beta = \pi - \theta$. Thus, $\tan \alpha = \tan(\theta - \theta_1) =$

$$\frac{\tan\theta - \tan\theta_1}{1 + \tan\theta \tan\theta_1} = \frac{m - m_1}{1 + mm_1} \text{ and } \tan\beta =$$

$$\tan(\pi - \theta) = \frac{\tan\pi - \tan\theta}{1 + \tan\pi\tan\theta} =$$

$$\frac{0 - m}{1 + (0)m} = -m.$$

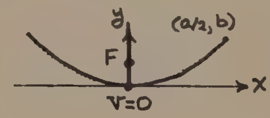


38. We must show that $\frac{m - m_1}{1 + mm_1} = -m$, that is, $m^2 m_1 + 2m - m_1 = 0$. Since $m = -\frac{y}{2p}$ and $m_1 = \frac{y}{x - p}$, then $m^2 m_1 + 2m - m_1 =$

$$\frac{y^3}{4p^2(x-p)} - \frac{y}{p} - \frac{y}{x-p} = \frac{y^3 - 4p(x-p)y - 4p^2 y}{4p^2(x-p)} =$$

$$\frac{y}{4p^2(x-p)}(y^2 - 4px) = \frac{y}{4p^2(x-p)}(0) = 0 \text{ as desired.}$$

39. Place the vertex at $(0,0)$; $(\frac{a}{2}, b)$ is a point on the graph of the parabola whose equation is $y = \frac{1}{4p}x^2$. To find p, let $x = \frac{a}{2}$ and $y = b$; then $b = \frac{1}{4p}\frac{a^2}{4}$ or $p = \frac{a^2}{16b}$. Thus, the focus is the point $(0, \frac{a^2}{16b})$.



40. Fix p; $F_1 = (0,p)$ and the center is $(0,k)$. Thus, $a = k$ and $c = k - p$. Also $b^2 = k^2 - (k - p)^2 = 2kp - p^2$. So the equation $\frac{x^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ becomes $\frac{x^2}{2pk - p^2} + \frac{(y-k)^2}{k^2} = 1$. Solving this equation for y, we obtain $(y - k)^2 = k^2(1 - \frac{x^2}{2pk - p^2})$, which can be written as $y^2 - 2ky = -\frac{k^2x^2}{2pk - p^2}$. Dividing both sides by $k > 0$, we have $\frac{y^2}{k} - 2y = -\frac{kx^2}{2pk - p^2}$ and then $\frac{y^2}{k} - 2y = \frac{-x^2}{2p - (p/k^2)}$. Taking the limit of both sides as $k \to \infty$, we obtain $2y = \frac{x^2}{2p}$ or $y = \frac{1}{4p}x^2$. Hence, the ellipse has this parabola as a limiting curve.

41. (a) Suppose you have two parabolas, one having equation $y_1 = \frac{1}{4p_1}x^2$ and the other $y_2 = \frac{1}{4p_2}x^2$. Fix x; then $\frac{y_1}{y_2} = \frac{\frac{x^2}{4p_1}}{\frac{x^2}{4p_2}} = \frac{p_2}{p_1}$.

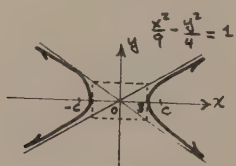
Thus, $y_1 = \frac{p_2}{p_1} y_2$. So y_1 is a multiple

of y_2 since $\frac{p_2}{p_1}$ is a constant; y_1 is called a "magnification" of y_2 and vice versa. So any two parabolas are "similar" in the sense that one is a magnification of the other.

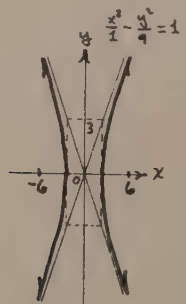
(b) This is not the case for two ellipses. Suppose $y_1^2 = \frac{b_1^2}{a_1^2}(a_1^2 - x^2)$ and $y_2^2 = \frac{b_2^2}{a_2^2}(a_2^2 - x^2)$ are their equations and suppose that x is in the domain of both. Then $\frac{y_1^2}{y_2^2} = \frac{a_2^2 b_1^2 (a_1^2 - x^2)}{a_1^2 b_2^2 (a_2^2 - x^2)} \neq k$ for some constant value.

Problem Set 9.6, page 575

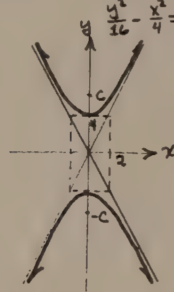
1. Vertices are (3,0) and (-3,0). $a^2 + b^2 = c^2$, $9 + 4 = 13$, $c = \pm\sqrt{13}$. Foci are $(\sqrt{13},0)$ and $(-\sqrt{13},0)$. Asymptotes: $y = \pm \frac{2}{3}x$.



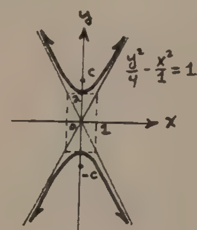
2. Vertices: (1,0), (-1,0). $1 + 9 = c^2$, $c = \pm\sqrt{10}$. Foci: $(\sqrt{10},0)$, $(-\sqrt{10},0)$. Asymptotes: $y = 3x$, $y = -3x$.



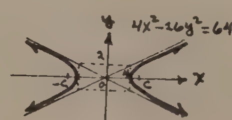
3. Vertices: (0,4), (0,-4). $a^2 + b^2 = c^2$, $4 + 16 = c^2$, $c = \pm\sqrt{20}$. Foci: $(0,\sqrt{20})$, $(0,-\sqrt{20})$. Asymptotes: $y = 2x$, $y = -2x$.



4. Vertices are (0,2) and (0,-2). $a^2 + b^2 = 1 + 4 = 5 = c^2$. Foci are $(0,\sqrt{5})$ and $(0,-\sqrt{5})$. Asymptotes: $y = 2x$ and $y = -2x$.

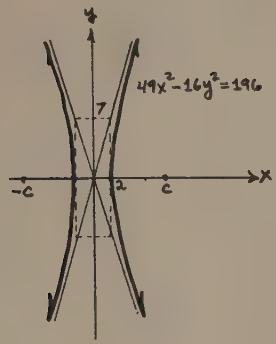


5. $\frac{x^2}{16} - \frac{y^2}{4} = 1$. $a^2 + b^2 = c^2$, $\sqrt{20} = c$. Vertices: (4,0), (-4,0). Foci: $(2\sqrt{5},0)$, $(-2\sqrt{5},0)$. Asymptotes: $y = \frac{1}{2}x$, $y = -\frac{1}{2}x$.



6. $\frac{x^2}{4} - \frac{y^2}{49} = 1$. $4 + 49 = c^2$, $\pm\sqrt{53} = c$. Vertices: (2,0), (-2,0). Foci: $(\sqrt{53},0)$, $(-\sqrt{53},0)$.

Asymptotes: $y = \pm \frac{7}{2}x$.

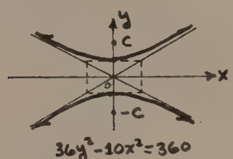


7. $\frac{y^2}{10} - \frac{x^2}{36} = 1$. $a^2 + b^2 = c^2$, $36 + 10 = c^2$,

$c^2 = 46$. Vertices: $(0,\sqrt{10})$, $(0,-\sqrt{10})$.

Foci: $(0,\sqrt{46})$, $(0,-\sqrt{46})$.

Asymptotes: $y = \pm \frac{\sqrt{10}}{6}x$.

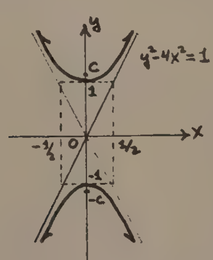


8. $\frac{y^2}{1} - \frac{x^2}{\frac{1}{4}} = 1$. $a^2 + b^2 + c^2$, $\frac{1}{4} + 1 = c^2$,

$\frac{5}{4} = c^2$. Vertices: $(0,1)$, $(0,-1)$.

Foci: $(0,\frac{\sqrt{5}}{2})$, $(0,-\frac{\sqrt{5}}{2})$.

Asymptotes: $y = 2x$, $y = -2x$.



9. $a^2 + b^2 = c^2$, $16 + b^2 = 36$, $b^2 = 20$.

$\frac{x^2}{16} - \frac{y^2}{20} = 1$.

10. $a^2 + b^2 = c^2$, $b^2 + \frac{1}{4} = 1$, $b^2 = \frac{3}{4}$.

$\frac{y^2}{\frac{1}{4}} - \frac{x^2}{\frac{3}{4}} = 1$, $4y^2 - \frac{4x^2}{3} = 1$,

$12y^2 - 4x^2 = 3$.

11. $a^2 = 16$, $b^2 = 25$. $\frac{x^2}{16} - \frac{y^2}{25} = 1$.

12. (a) $\left.\begin{aligned} 4b^2 - 25a^2 &= a^2b^2 \\ 9b^2 - 100a^2 &= a^2b^2 \end{aligned}\right\}$, so

$\left.\begin{aligned} 16b^2 - 100a^2 &= 4a^2b^2 \\ 9b^2 - 100a^2 &= a^2b^2 \end{aligned}\right\}$. Subtracting, we

get $7b^2 = 3a^2b^2$, $a^2 = \frac{7}{3}$ and so

$9b^2 - \frac{700}{3} = \frac{7}{3}b^2$ and $b^2 = 35$.

(b) $\left.\begin{aligned} 16b^2 - 9a^2 &= a^2b^2 \\ 49b^2 - 36a^2 &= a^2b^2 \end{aligned}\right\}$, so

$\left.\begin{aligned} 64b^2 - 36a^2 &= 4a^2b^2 \\ 49b^2 = 36a^2 &= a^2b^2 \end{aligned}\right\}$

Subtracting, we find that $15b^2 = 3a^2b^2$,

so that $5 = a^2$. Thus, substituting

$5 = a^2$ above, we have $16b^2 - 45 = 5b^2$,

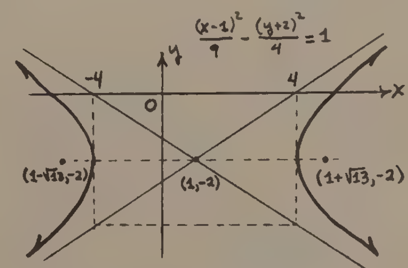
$11b^2 = 45$, $b^2 = \frac{45}{11}$.

13. Center: $(1,-2)$. Vertices: $(4,-2)$, $(-2,-2)$

Foci: $(1+\sqrt{13},-2)$, $(1-\sqrt{13},-2)$

Asymptotes: $y+2 = \frac{2}{3}(x-1)$ or $3y-2x+8 = 0$;

$y+2 = -\frac{2}{3}(x-1)$ or $3y+2x+4 = 0$.

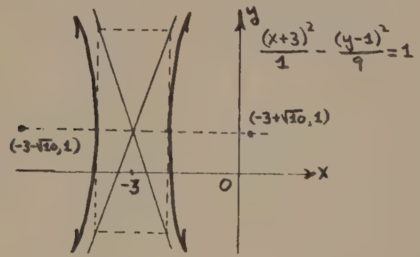


14. Center: $(-3,1)$.

Vertices: $(-2,1)$, $(-4,1)$.

Foci: $(-3+\sqrt{10},1)$, $(-3-\sqrt{10},1)$.

Asymptotes: $y - 1 = 3(x+3)$ or $y - 3x - 10 = 0$; $y - 1 = -3(x+3)$ or $y + 3x + 8 = 0$.



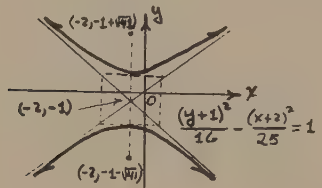
15. Center: $(-2,-1)$.

Vertices: $(-2,3)$, $(-2,-5)$.

Foci: $(-2,-1+\sqrt{41})$, $(-2,-1-\sqrt{41})$.

Asymptotes: $y + 1 = \frac{4}{5}(x+2)$ or $5y - 4x - 3 = 0$; $y + 1 = -\frac{4}{5}(x+2)$ or

$5y + 4x + 13 = 0$.



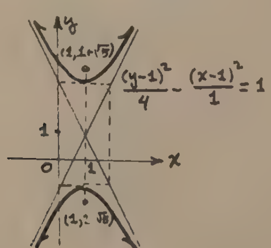
16. $4(x-1)^2 - (y-1)^2 = -4$.

$$\frac{(y-1)^2}{4} - \frac{(x-1)^2}{1} = 1.$$

Center: $(1,1)$. Vertices: $(1,3)$, $(1,-1)$.

Foci: $(1,1+\sqrt{5})$, $(1,1-\sqrt{5})$.

Asymptotes: $y-1 = 2(x-1)$ or $y-2x+1 = 0$; $y-1 = -2(x-1)$ or $y+2x-3 = 0$.



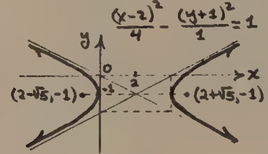
17. $(x-2)^2 - 4(y+1)^2 = 4$.

$$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{1} = 1.$$

Center: $(2,-1)$. Vertices: $(4,-1)$, $(0,-1)$.

Foci: $(2+\sqrt{5},-1)$, $(2-\sqrt{5},-1)$.

Asymptotes: $y+1 = \frac{1}{2}(x-2)$ or $2y-x+4 = 0$; $y+1 = -\frac{1}{2}(x-2)$ or $2y + x = 0$.



18. $9(y - 10)^2 - 16x^2 = 288$.

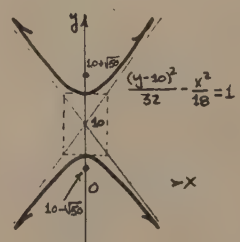
$$\frac{(y-10)^2}{32} - \frac{x^2}{18} = 1.$$

Center: $(0,10)$.

Vertices: $(0,10+\sqrt{32})$, $(0,10-\sqrt{32})$.

Foci: $(0,10+\sqrt{50})$, $(0,10-\sqrt{50})$.

Asymptotes: $y-10 = \frac{16}{9}(x)$ or $9y-16x-90 = 0$; $y-10 = -\frac{16}{9}x$ or $9y+16x-90 = 0$.



19. $25(y + 2)^2 - 9(x + 4)^2 = 225$.

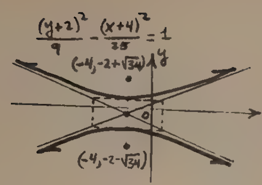
$$\frac{(y+2)^2}{9} - \frac{(x+4)^2}{25} = 1.$$

Center: $(-4,-2)$.

Vertices: $(-4,1)$, $(-4,-5)$.

Foci: $(-4,-2+\sqrt{34})$, $(-4,-2-\sqrt{34})$.

Asymptotes: $y+2 = \frac{3}{5}(x+4)$ or $5y-3x-2 = 0$; $y+2 = -\frac{3}{5}(x+4)$ or $5y+3x+22 = 0$.



20. $16(y+8)^2 - 9(x-5)^2 = 576.$

$\frac{(y+8)^2}{36} - \frac{(x-5)^2}{64} = 1.$

Center: $(5,-8)$.

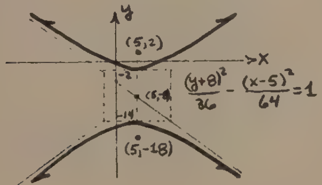
Vertices: $(5,-2), (5,-14)$.

Foci: $(5,2), (5,-18)$.

Asymptotes: $y+8 = \frac{3}{4}(x-5)$ or

$4y-3x+47 = 0$; $y+8 = -\frac{3}{4}(x-5)$ or

$4y + 3x + 17 = 0.$



21. (a) $2a = 2$, $a = 1$; $2c = 6$, $c = 3$.

Center: $(\frac{1+7}{2},-1) = (4,-1)$.

$$\frac{(x-4)^2}{1} - \frac{(y+1)^2}{8} = 1.$$

(b) Center: $(-2,3)$. So $c = 2\frac{1}{2} = \frac{5}{2}$,

$a = 2.$ $a^2 + b^2 = c^2$, $4 + b^2 = \frac{25}{4}$,

$b^2 = \frac{9}{4}.$ $\frac{(x+2)^2}{4} - \frac{(y-3)^2}{9/4} = 1,$

$\frac{(x+2)^2}{4} - \frac{4(y-3)^2}{9} = 1$, or

$9(x+2)^2 - 16(y-3)^2 = 36.$

(c) $b = 8 - 3 = 5$, $c = 3 - (-3) = 6.$

So $a^2 + b^2 = c^2$, $a^2 + 25 = 36$,

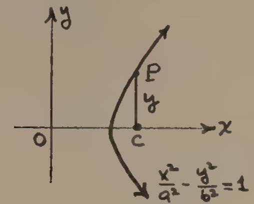
$a^2 = 11.$ $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{11} = 1.$

22. Find y when $x = c$, where $(c,0)$ is a focus.

$y^2 = b^2(\frac{c^2}{a^2} - 1) = \frac{b^2(c^2-a^2)}{a^2} = \frac{b^4}{a^2}$ since

$b^2 = c^2 - a^2.$ Thus, $y = \pm \frac{b^2}{a}.$

The focal chord has length $2y = \frac{2b^2}{a}.$



23. $\frac{x^2}{16} - \frac{y^2}{2} = 1.$ $a = 4$, $b^2 = 2.$ The length of a focal chord is $\frac{2 \cdot 2}{4} = 1.$ (See Problem 22.)

24. (a) $2x - 2y \frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{x}{y}.$ So $m = -\frac{5}{4}$ for tangent line.

Tangent line: $y - 4 = -\frac{5}{4}(x+5)$ or

$$4y + 5x + 9 = 0.$$

Normal line: $y - 4 = \frac{4}{5}(x+5)$ or

$$5y - 4x - 40 = 0.$$

(b) $8y \frac{dy}{dx} - 2x = 0$, $\frac{dy}{dx} = \frac{x}{4y}.$ So $m = -\frac{3}{8}$ for tangent line.

Tangent line: $y + 2 = -\frac{3}{8}(x-3)$ or

$$8y + 3x + 7 = 0.$$

Normal line: $y + 2 = \frac{8}{3}(x-3)$ or

$$3y - 8x + 30 = 0.$$

(c) $2x-4-2y \frac{dy}{dx} - 2 \frac{dy}{dx} = 0,$

$\frac{dy}{dx} = \frac{4-2x}{-(2y+2)}$, $\frac{dy}{dx} = \frac{x-2}{y+1}$, so $m = -2$ for tangent line.

Tangent line: $y = -2x.$

Normal line: $y = \frac{1}{2}x.$

(d) $18(x+1) - 32(y-2) \frac{dy}{dx} = 0,$

$\frac{dy}{dx} = \frac{9(x+1)}{16(y-2)}.$ Slope of tangent line is infinite.

Tangent line: $x = 3.$

Normal line: $y = 2.$

25. (a) Center is (0,0). $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\frac{b}{a} = \pm 2$. So $b = \pm 2a$. We have $\frac{x^2}{a^2} - \frac{y^2}{4a^2} = 1$,

$4x^2 - y^2 = 4a^2$. Since (1,1) is on the hyperbola, $4 - 1 = 4a^2$, $\frac{3}{4} = a^2$. The equation is $\frac{4x^2}{3} - \frac{y^2}{3} = 1$ or $4x^2 - y^2 = 3$.

(b) The center is obtained by solving $\begin{cases} y = -2x + 3 \\ y = 2x + 1 \end{cases}$ simultaneously. Hence, $2y = 4$, $y = 2$, and so $x = \frac{1}{2}$. Now,

$\frac{(y-2)^2}{b^2} - \frac{(x-\frac{1}{2})^2}{a^2} = 1$ and $b = \pm 2a$. So,

since (1,4) is on the hyperbola,

$\frac{4}{4a^2} - \frac{\frac{1}{4}}{a^2} = 1$, $\frac{3}{4} = a^2$. The equation is

$\frac{(y-2)^2}{3} - \frac{4(x-\frac{1}{2})^2}{3} = 1$.

(c) Find center: adding equations of asymptotes we get $2y = 8$, $y = 4$, so $x = -1$.

Now $\frac{(x+1)^2}{a^2} - \frac{(y-4)^2}{b^2} = 1$. Now $\frac{b}{a} = \pm 1$,

so $b = \pm a$. $\frac{(x+1)^2}{b^2} - \frac{(y-4)^2}{b^2} = 1$. Since

(2,4) is on the curve, $\frac{9}{b^2} - 0 = 1$ and

$b^2 = 9$. Hence, the equation is

$\frac{(x+1)^2}{9} - \frac{(y-4)^2}{9} = 1$.

26. $y = \pm\sqrt{\frac{b^2x^2 - a^2b^2}{a^2}} = \pm\frac{b}{a}\sqrt{x^2-a^2}$

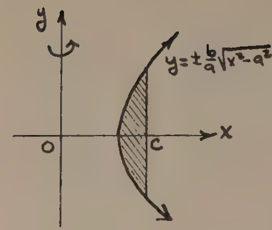
Using cylindrical shells, we have

$$V = 2\left[2\pi \int_a^c x\,\frac{b}{a}(x^2-a^2)^{\frac{1}{2}}\,dx\right]$$

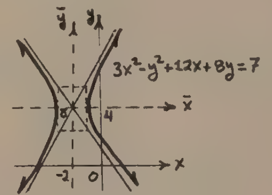
$$= \frac{4b\pi}{a\cdot 2}\int_a^c 2x(x^2-a^2)^{\frac{1}{2}}\,dx$$

$$= \frac{2b\pi}{a}\left[\frac{2}{3}(x^2-a^2)^{3/2}\Big|_a^c\right] = \frac{4b\pi}{3a}\left[(c^2-a^2)^{3/2}\right]$$

$$= \frac{4b^4\pi}{3a} \text{ cubic units.}$$

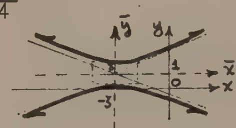


27. $3(x^2+4x+4) - (y^2-8x+16) = 7+12-16$ or $3(x+2)^2 - (y-4)^2 = 3$. Put $\bar{x} = x+2$ and $\bar{y} = y-4$, so $\bar{x}^2 - \frac{\bar{y}^2}{3} = 1$.

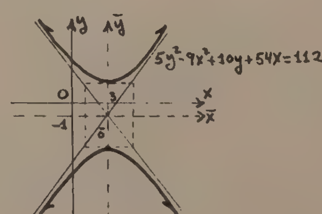


28. $4(x^2+6x+9)-25(y^2-2y+1) = -22+36-25$ or $25(y-1)^2-4(x+3)^2 = 11$. Put $\bar{y} = y-1$ and $\bar{x} = x+3$, so $25\bar{y}^2-4\bar{x}^2 = 11$. Hence,

$\frac{\bar{y}^2}{\frac{11}{25}} - \frac{\bar{x}^2}{\frac{11}{4}} = 1$.

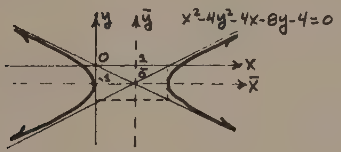


29. $5(y^2+2y+1)-9(x^2-6x+9) = 112+5-81$, so $5(y+1)^2-9(x-3)^2 = 36$. Put $\bar{x} = x-3$ and $\bar{y} = y+1$. Hence, $\frac{\bar{y}^2}{\frac{36}{5}} - \frac{\bar{x}^2}{4} = 1$ is the equation.

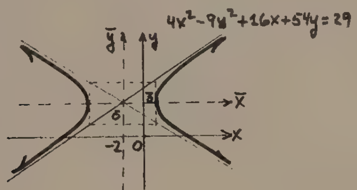


30. $(x^2-4x+4) - 4(y^2+2y+1) = 4 + 4 - 4$ or $(x-2)^2 - 4(y+1)^2 = 4$. Thus, $\frac{(x-2)^2}{4} - \frac{(y+1)^2}{1} = 1$. Put $\bar{x} = x-2$ and and $\bar{y} = y+1$. Then the equation is

$$\frac{\bar{x}^2}{4} - \bar{y}^2 = 1.$$



31. $4(x^2+4x+4)-9(y^2-6y+9) = 29+16-81$; $4(x+2)^2-9(y-3)^2 = 36$. Put $\bar{x} = x+2$ and $\bar{y} = y-3$, so $\frac{\bar{x}^2}{9} - \frac{\bar{y}^2}{4} = 1$.



32. Distance from P to a point on the circle= distance from P to (2,0). So there is a point such that $\sqrt{(x+2)^2 + y^2} - 3 = \sqrt{(x-2)^2 + y^2}$,

$\sqrt{(x+2)^2 + y^2} = 3 + \sqrt{(x-2)^2 + y^2}$,

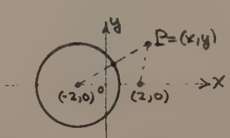
$x^2 + 4x + 4 + y^2 = 9 + 6\sqrt{(x^2-4x+4+y^2)} + x^2-4x+4+y^2$,

$8x-9 = 6\sqrt{x^2 -4x + 4 + y^2}$,

$64x^2-144x+81 = 36x^2-144x+144+36y^2$,

$28x^2 - 36y^2 = 63$,

$\frac{x^2}{\frac{63}{28}} - \frac{y^2}{\frac{63}{36}} = 1$ or $\frac{x^2}{\frac{9}{4}} - \frac{y^2}{\frac{7}{4}} = 1$. The path is an hyperbola with center (0,0), vertices $(\frac{3}{2},0)$ and $(-\frac{3}{2},0)$, and foci (2,0) and (-2,0).



33. $\frac{dx}{dt} = 6$ units/sec. We want $\frac{dy}{dt}$ at (3,1).

$8x\,\frac{dx}{dt} - 18y\,\frac{dy}{dt} = 0$, $24\cdot6-18\,\frac{dy}{dt} = 0$,

$\frac{dy}{dt} = \frac{144}{18} = 8$ units/sec.

34. Given that $m_1\cdot m_2 = 9$. $\left(\frac{y-b}{x-a}\right)\left(\frac{y-d}{x-c}\right) = 9$.

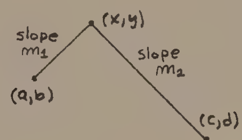
$y^2-y(b+d)+bd = 9x^2-9x(a+c)+9ac$.

$y^2-y(b+d)-9\left[x^2-(a+c)x\right] = 9ac-bd$.

$\left[y - \frac{(b+d)}{2}\right]^2-9\left[x - \frac{(a+c)}{2}\right]^2$

$= 9ac-bd + \left(\frac{b+d}{2}\right)^2 - 9\left(\frac{a+c}{2}\right)^2 = k$.

The path is an hyperbola with vertical transverse axis if $k > 0$, or an hyperbola with horizontal transverse axis if $k < 0$; its center is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.



35. The distance D from (3,0) to the hyperbola $y^2-x^2 = 18$ is $\sqrt{y^2+(x-3)^2}$. So $D(x) = \sqrt{18+x^2+x^2-6x+9} = \sqrt{2x^2-6x+27}$.

$D'(x) = \frac{2x-3}{\sqrt{2x^2-6x+27}} = 0$ for $x = \frac{3}{2}$.

So $y = \sqrt{\frac{81}{4}} = \frac{9}{2}$. The shortest distance is $\sqrt{\frac{81}{4} + \frac{9}{4}} = \sqrt{\frac{90}{4}} = \frac{3\sqrt{10}}{2}$. Note: If $x > \frac{3}{2}$, $D'(x) > 0$; and if $x < \frac{3}{2}$, $D'(x) < 0$. Therefore, $x = \frac{3}{2}$ yields the maximum value of D.

36. A(rectangle) $= 2\ell w$. (See figure.)

$A(y) = 2(8 - \sqrt{16 + y^2})y$

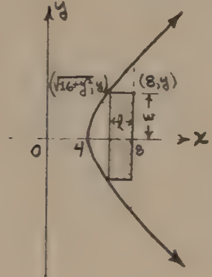
$A(y) = 16y - 2y\sqrt{16 + y^2}$.

$A'(y) = 16-2\sqrt{16+y^2}-2y^2(16+y^2)^{-\frac{1}{2}} = 0$.

So $4\sqrt{16 + y^2} = 8 + y^2$,

$256 + 16y^2 = 64 + 16y^2 + y^4$,

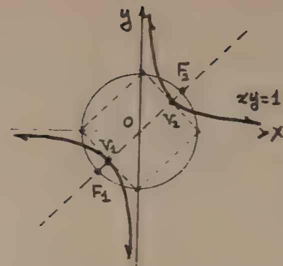
$0 = y^4 - 192$, $y^2 = \sqrt{192}$, so
$y = \sqrt[4]{192} \approx 3.72$. The dimensions are
$2w \approx 7.44$, $\ell \approx 2.54$.



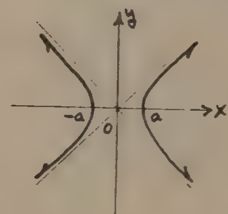
37. Solve the equation of the hyperbola for y to get $y = \pm \frac{b}{a}\sqrt{x^2-a^2}$. Since $x_0 > a$ and $y_0 > 0$, then (x_0,y_0) is on the portion of the hyperbola whose equation is $y = \frac{b}{a}\sqrt{x^2-a^2}$, $x > a$. Then $\frac{dy}{dx} = \frac{b}{a}\frac{x}{\sqrt{x^2-a^2}} = \frac{b}{a}\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}} = \frac{b}{a}\frac{1}{\sqrt{1-\frac{a^2}{x^2}}}$, so $m = \frac{b}{a}\frac{1}{\sqrt{1-\frac{a^2}{x_0^2}}}$.

It follows that $\lim\limits_{x_0 \to +\infty} m = \frac{b}{a}$. This limit is the slope of the asymptote whose equation is $y = \frac{b}{a}x$.

38. The transverse axis lies along the line whose equation is $y = x$ and the vertices are $V_1 = (-1,-1)$ and $V_2 = (1,1)$. The triangle OV_2A in the accompanying figure is a right triangle and $\overline{OV_2} = \sqrt{2}$, $\overline{V_2A} = \sqrt{2}$; hence, $\overline{OA} = 2$. The circle of radius $\overline{OA} = 2$ and center O intersects the line $y = x$ at the desired foci F_1 and F_2, so $F_1 = (-\sqrt{2},-\sqrt{2})$ and $F_2 = (\sqrt{2},\sqrt{2})$.

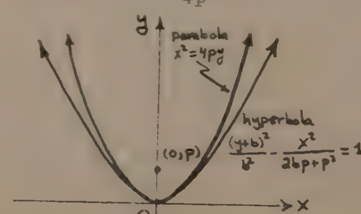


39. Since the asymptotes are perpendicular, their slopes are 1 and -1. The equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $\pm\frac{b}{a} = \pm 1$, $\frac{b}{a} = 1$, $b = a$. Therefore, the equation is $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$.



40. The sketch below shows the parabola $x^2 = 4py$ and the upper branch of the hyperbola $\frac{(y+b)^2}{b^2} - \frac{x^2}{2bp+p^2} = 1$. The vertex of the hyperbola is at $(0,0)$, its focus is at $(0,p)$ and its center is at $(0,-b)$. The equation of the hyperbola can be rewritten as $\frac{y^2}{b^2} + \frac{2y}{b} + 1 = 1 + \frac{x^2}{2bp+p^2}$ or $y = \frac{1}{4p + \frac{2p^2}{b}}x^2 - \frac{1}{2b}y^2$.

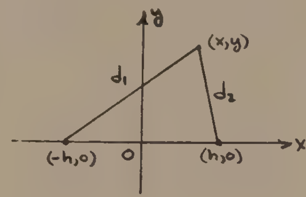
Letting $b \to +\infty$ (while holding p constant), we see that the hyperbola approaches the parabola $y = \frac{1}{4p}x^2$.



41. The hyperbola begins to look more and more like the intersecting asymptotes.

42. We show that if $P = (x,y)$ is a point on the hyperbola with foci $F_1 = (-c,0)$, $F_2 = (c,0)$ and vertices $V_1 = (-a,0)$, $V_2 = (a,0)$, then $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b = \sqrt{c^2-a^2}$. By reversing the argument, it can be seen that a point (x,y) satisfying the equation lies on the hyperbola. We have already seen that if $P = (x,y)$ belongs to the hyperbola, then $\left||\overline{PF_1}| - |\overline{PF_2}|\right| = 2a$; that is, $\left|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}\right| = 2a$. Thus, $\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$, so that $\sqrt{(x+c)^2+y^2} = \sqrt{(x-c)^2+y^2} \pm 2a$. Square both sides of the latter equation to get $(x+c)^2+y^2 = (x-c)^2+y^2 \pm 4a\sqrt{(x-c)^2+y^2}+4a^2$; that is $x^2 + 2cx + c^2 + y^2 = x^2-2cx+c^2+y^2 \pm 4a\sqrt{(x-c)^2+y^2}+4a^2$, or $4cx-4a^2 = \pm 4a\sqrt{(x-c)^2+y^2}$. Divide both sides of the latter equation by 4 and square to get $c^2x^2-2a^2cx+a^4 = a^2\left[(x-c)^2+y^2\right]$, or, $(c^2-a^2)x^2-a^2y^2 = a^2c^2-a^4$. Since $b^2 = c^2-a^2$, the latter equation can be rewritten as $b^2x^2-a^2y^2 = a^2(c^2-a^2) = a^2b^2$, or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

43. The boom of the gun requires $\frac{d_1}{s}$ seconds to reach (x,y). The bullet requires $\frac{2h}{b}$ seconds to reach $(h,0)$; then the ping of the bullet requires $\frac{d_2}{s}$ more seconds. to reach (x,y). Hence, $\frac{d_1}{s} = \frac{2h}{b} + \frac{d_2}{s}$ or $d_1-d_2 = \frac{2hs}{b}$. Thus, the point (x,y) lies on a hyperbola with foci at $(-h,0)$ and $(h,0)$. The vertices of the hyperbola are $\frac{hs}{b}$ units from the center. The equation of the hyperbola is $\frac{b^2x^2}{h^2s^2} - \frac{b^2y^2}{h^2(b^2-s^2)} = 1$. The boom of the gun and the ping of the bullet hitting the target can be heard simultaneously at any point on the right-hand branch of this hyperbola (except at the vertex, where the bullet will pass).



Problem Set 9.7, page 583

We use $r = \bar{r}$ and $\theta = \bar{\theta} + \phi$ in Problems 1 to 10.

1. $\bar{r} = 5$.

2. $\bar{\theta} + \phi = \frac{\pi}{3}$ and $\phi = -\frac{\pi}{6}$, so that $\bar{\theta} - \frac{\pi}{6} = \frac{\pi}{3}$ and $\bar{\theta} = \frac{\pi}{2}$.

3. $\bar{r} = 4\cos(\bar{\theta} + \phi) = 4\cos(\bar{\theta} - \frac{\pi}{2}) = 4 \sin \bar{\theta}$, that is, $\bar{r} = 4 \sin \theta$.

4. $\bar{r} = \frac{1}{1-\cos(\bar{\theta}+\phi)} = \frac{1}{1-\cos(\bar{\theta}+\pi)}$
$= \frac{1}{1+\cos\bar{\theta}}$. Thus, $\bar{r} = \frac{1}{1+\cos\bar{\theta}}$.

5. $\bar{r} = 3 + 5\sin(\bar{\theta}+\phi) = 3 + 5\sin(\bar{\theta}+\pi)$
$= 3 - 5\sin\bar{\theta}$. Hence, $\bar{r} = 3 - 5\sin\bar{\theta}$.

6. $\bar{r} = \bar{\theta} + \phi = \bar{\theta} + \frac{\pi}{3}$. Thus, $\bar{r} = \bar{\theta} + \frac{\pi}{3}$.

7. $\bar{r} = 3-5\sin(\bar{\theta}+\phi) = 3-5\sin(\bar{\theta} - \frac{\pi}{2}) =$

$3+5\cos\bar{\theta}$. Thus, $\bar{r} = 3+5\cos\bar{\theta}$.

8. $r = \dfrac{ed}{1+e\cos(\bar{\theta}+\phi)} = \dfrac{ed}{1+e\cos(\bar{\theta}+\frac{\pi}{2})}$

$= \dfrac{ed}{1-e\sin\bar{\theta}}$. Thus, $r = \dfrac{ed}{1-e\sin\bar{\theta}}$.

9. $\bar{r}^2 = 25\cos 2(\bar{\theta}+\phi) = 25\cos(2\bar{\theta}+2\phi)$

$= 25\cos(2\bar{\theta}+\pi) = -25\cos 2\bar{\theta}$. Thus

$r^2 = -25\cos 2\bar{\theta}$.

10. $\bar{r} = \dfrac{-C}{A\cos(\bar{\theta}+\emptyset) + B\sin(\bar{\theta}+\emptyset)} =$

$\dfrac{-C}{A\cos\bar{\theta}\cos\emptyset - A\sin\bar{\theta}\sin\emptyset + B\sin\bar{\theta}\cos\emptyset + B\cos\bar{\theta}\sin\emptyset}$

$= \dfrac{-C}{\left[A\cos\emptyset+B\sin\emptyset\right]\cos\bar{\theta} + \left[B\cos\emptyset - A\sin\emptyset\right]\sin\bar{\theta}}$.

11. $\bar{x} = x\cos 90^\circ + y\sin 90^\circ = y = -7$,

$\bar{y} = -x\sin 90^\circ + y\cos 90^\circ = -x = -4$,

so $(\bar{x},\bar{y}) = (-7,-4)$.

12. From $x = \bar{x}\cos\emptyset - \bar{y}\sin\emptyset$ and $y = \bar{x}\sin\emptyset + \bar{y}\cos\emptyset$, we have $2 = \cos\emptyset - \sqrt{3}\sin\emptyset$ and $0 = \sin\emptyset + \sqrt{3}\cos\emptyset$. From the latter equation, $\tan\emptyset = \dfrac{\sin\emptyset}{\cos\emptyset} = -\sqrt{3}$, so $\emptyset = -60^\circ$ or $\emptyset = 120^\circ$. Substitution of $\emptyset = -60^\circ$ into the former equation $2 = \cos\emptyset - \sqrt{3}\sin\emptyset$ gives $2 = 2$, while substitution of $\emptyset = 120^\circ$ into this equation gives $2 = -2$; hence, $\emptyset = -60^\circ$.

13. $x = \bar{x}\cos\frac{\pi}{3} - \bar{y}\sin\frac{\pi}{3} = (-3)(\frac{1}{2}) - (-3)\frac{\sqrt{3}}{2}$

$= \dfrac{3\sqrt{3}-3}{2}$,

$y = \bar{x}\sin\frac{\pi}{3} + \bar{y}\cos\frac{\pi}{3} = (-3)\frac{\sqrt{3}}{2} + (-3)(\frac{1}{2})$

$= \dfrac{-(3\sqrt{3}+3)}{2}$, so

$(x,y) = (\frac{3\sqrt{3}-3}{2}, -\frac{3\sqrt{3}+3}{2})$.

14. $\bar{x} = x\cos 45^\circ + y\sin 45^\circ$

$= (5\sqrt{2})(\frac{\sqrt{2}}{2}) + (\sqrt{2})(\frac{\sqrt{2}}{2}) = 6$,

$\bar{y} = -x\sin 45^\circ + y\cos 45^\circ$

$= -(5\sqrt{2})(\frac{\sqrt{2}}{2}) + (\sqrt{2})(\frac{\sqrt{2}}{2}) = -4$, so

$(\bar{x},\bar{y}) = (6,-4)$.

15. $x = \bar{x}\cos 30^\circ - \bar{y}\sin 30^\circ$

$= (-4)(\frac{\sqrt{3}}{2}) - (-2)(\frac{1}{2}) = 1 - 2\sqrt{3}$,

$y = \bar{x}\sin 30^\circ + \bar{y}\cos 30^\circ$

$= (-4)(\frac{1}{2}) + (-2)(\frac{\sqrt{3}}{2}) = -2-\sqrt{3}$, so

$(x,y) = (1-2\sqrt{3}, -2-\sqrt{3})$.

16. $x = \bar{x}\cos\frac{3\pi}{4} - \bar{y}\sin\frac{3\pi}{4}$

$= (-3\sqrt{2})(-\frac{\sqrt{2}}{2}) - \sqrt{2}(\frac{\sqrt{2}}{2}) = 2$,

$y = \bar{x}\sin\frac{3\pi}{4} + \bar{y}\cos\frac{3\pi}{4}$

$= (-3\sqrt{2})(\frac{\sqrt{2}}{2}) + \sqrt{2}(-\frac{\sqrt{2}}{2}) = -4$, so

$(x,y) = (2,-4)$.

17. $\bar{x} = x\cos\pi + y\sin\pi = -x = -(-4) = 4$,

$\bar{y} = -x\sin\pi + y\cos\pi = -y = -0 = 0$,

so $(\bar{x},\bar{y}) = (4,0)$.

18. $\bar{x} = x\cos 240^\circ + y\sin 240^\circ$

$= (1)(-\frac{1}{2}) + (-7)(-\frac{\sqrt{3}}{2}) = \frac{7\sqrt{3}-1}{2}$,

$\bar{y} = -x\sin 240^\circ + y\cos 240^\circ$

$-(1)(-\frac{\sqrt{3}}{2}) + (-7)(-\frac{1}{2}) = \frac{\sqrt{3}+7}{2}$, so

$(\bar{x},\bar{y}) = (\frac{7\sqrt{3}-1}{2}, \frac{\sqrt{3}+7}{2})$.

19. $\bar{x} = x\cos 360^\circ + y\sin 360^\circ = x = 0$ and

$\bar{y} = -x\sin 360^\circ + y\cos 360^\circ = y = 8$,

so $(\bar{x},\bar{y}) = (0,8)$.

20. (a) For $\phi = 90^\circ$, $\bar{x} = x\cos\phi + y\sin\phi = x(0) + y(1) = y$ and $\bar{y} = -x\sin\phi + y\cos\phi = -x(1) + y(0) = -x$.

(b) For $\phi = 180^\circ$, $\bar{x} = x\cos\phi + y\sin\phi = x(-1) + y(0) = -x$ and

$\bar{y} = -x\sin\phi + y\cos\phi = -x(0) + y(-1) = -y$.

(c) $\bar{x} = x \cos\phi + y \sin\phi = y$ must hold for all x and y. Put $x = 0$ and $y = 1$ to conclude that $\sin\phi = 1$. Also, $\bar{y} = -x \sin\phi + y \cos\phi = x$ must hold for all x and y. Put $y = 0$ and $x = 1$ to conclude that $\sin\phi = -1$. Since we cannot have $\sin\phi = 1$ and $\sin\phi = -1$ at the same time, no such angle exists.

(d) Reasoning as in (c) above, no such angle exists.

21. $\left(\frac{\bar{x}+\sqrt{3}\bar{y}}{2}\right)^2 = 3\left(\frac{\sqrt{3}\bar{x}-\bar{y}}{2}\right)$,

$\frac{\bar{x}^2+2\sqrt{3}\bar{x}\bar{y}+3\bar{y}^2}{4} = \frac{3\sqrt{3}\bar{x}}{2} - \frac{3\bar{y}}{2}$,

$\bar{x}^2 + 2\sqrt{3}\bar{x}\cdot\bar{y} + 3\bar{y}^2 = 6\sqrt{3}\bar{x}-6\bar{y}$, or

$\bar{x}^2 + 2\sqrt{3}\bar{x}\cdot\bar{y} + 3\bar{y}^2 - 6\sqrt{3}\bar{x} + 6\bar{y} = 0$.

22. $-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)$, $\frac{\sqrt{3}-3}{2}y = \frac{3\sqrt{3}+1}{2}x$,

or $y = \frac{3\sqrt{3}+1}{\sqrt{3}-3}x$

23. $\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)^2 + \left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2 = 1$,

$\frac{3}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{1}{4}y^2 + \frac{1}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2 = 1$,

or $x^2 + y^2 = 1$.

24. $5\left(\frac{\sqrt{3}}{2}\bar{x} - \frac{1}{2}\bar{y}\right) - \left(\frac{1}{2}\bar{x} + \frac{\sqrt{3}}{2}\bar{y}\right) = 4$,

$\left(\frac{5\sqrt{3}-1}{2}\right)\bar{x} - \left(\frac{5+\sqrt{3}}{2}\right)\bar{y} = 4$, or

$\bar{y} = \frac{(5\sqrt{3}-1)\bar{x}-8}{5+\sqrt{3}}$.

25. $\left(\frac{\sqrt{3}}{2}\bar{x} - \frac{1}{2}\bar{y}\right)^2 + \left(\frac{1}{2}\bar{x} + \frac{\sqrt{3}}{2}\bar{y}\right)^2 = 1$,

$\frac{3}{4}\bar{x}^2 - \frac{\sqrt{3}}{2}\bar{x}\,\bar{y} + \frac{1}{4}\bar{y}^2 + \frac{1}{4}\bar{x}^2 + \frac{\sqrt{3}}{2}\bar{x}\,\bar{y} + \frac{3}{4}\bar{y}^2 = 1$,

or $\bar{x}^2 + \bar{y}^2 = 1$.

26. $\left(\frac{\sqrt{3}}{2}\bar{x} - \frac{1}{2}\bar{y}\right)^2 = 25$, $\frac{3}{4}\bar{x}^2 - \frac{\sqrt{3}}{2}\bar{x}\,\bar{y} + \frac{1}{4}\bar{y}^2 = 25$,

or $3\bar{x}^2 - 2\sqrt{3}\bar{x}\cdot\bar{y} + \bar{y}^2 - 100 = 0$.

27. $x = \frac{1}{\sqrt{2}}(\bar{x} - \bar{y})$ and $y = \frac{1}{\sqrt{2}}(\bar{x}+\bar{y})$; so $xy = \frac{1}{2}(\bar{x}^2-\bar{y}^2) = 1$. This is a hyperbola with asymptotes $\frac{\bar{x}}{\sqrt{2}} - \frac{\bar{y}}{\sqrt{2}} = 0$ and $\frac{\bar{x}}{\sqrt{2}} + \frac{\bar{y}}{\sqrt{2}} = 0$ or $\bar{y} = \pm\bar{x}$ which are the old coordinate axes in the $\bar{x}\,\bar{y}$ system.

28. (a) Let $P = (x,y)$ be a point in the xy coordinate system and θ the angle $\overrightarrow{OP}$ makes with the positive x axis. Then $x = r\cos\theta$ and $y = r\sin\theta$, where $r = \sqrt{x^2+y^2}$. Rotate the coordinate system through an angle ϕ; so $\theta = \bar{\theta} + \phi$ and $r = \bar{r}$. Now the x and y coordinates are $x = \bar{r}\cos(\bar{\theta}+\phi)$ and $y = \bar{r}\sin(\theta+\phi)$. Thus, $x = \bar{r}\cos\bar{\theta}\cos\phi - r\sin\bar{\theta}\sin\phi$

$= \bar{x}\cos\phi - \bar{y}\sin\phi$, and

$y = \bar{r}\sin\bar{\theta}\,\text{sos}\,\phi + \bar{r}\cos\bar{\theta}\sin\phi$

$= \bar{y}\cos\phi + \bar{x}\sin\phi$.

(b) To solve $x = \bar{x}\cos\phi - \bar{y}\sin\phi$ and $y = \bar{x}\sin\phi + \bar{y}\cos\phi$ for $\bar{x}$ and $\bar{y}$, we multiply the first equation by $\cos\phi$ and the second equation by $\sin\phi$. Then adding the resulting equations, we obtain $x\cos\phi + y\sin\phi = \bar{x}(\cos^2 x + \sin^2\phi)$; that is, $\bar{x} = x\cos\phi + y\sin\phi$. Similarly, we multiply the first equation by $-\sin\phi$ and the second equation by $\cos\phi$ and add to obtain $\bar{y} = -x\sin\phi + y\cos\phi$.

29. (a) $A = 1$, $B = 4$, $C = -2$, $\cot 2\phi$

$= \frac{A-C}{B} = \frac{3}{4}$,

$\cos 2\phi = \frac{\cot 2\phi}{\sqrt{\cot^2 2\phi + 1}} = \frac{3}{5}$,

$\cos\phi = \sqrt{\frac{1+\cos 2\phi}{2}} = \frac{2\sqrt{5}}{5}$,

$\sin\phi = \sqrt{\frac{1-\cos 2\phi}{2}} = \frac{\sqrt{5}}{5}$, and so ϕ

$= \sin^{-1}\frac{\sqrt{5}}{5} \approx 26.57°$.

(b) $x = \bar{x}\cos\phi - \bar{y}\sin\phi = \frac{\sqrt{5}}{5}(2\bar{x}-\bar{y})$,

$y = \bar{x}\sin\phi + \bar{y}\cos\phi = \frac{\sqrt{5}}{5}(\bar{x}+2\bar{y})$.

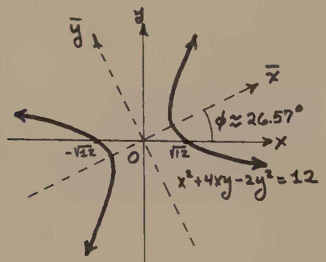
(c) Substituting from (b) into

$x^2 + 4xy - 2y^2 = 12$, we obtain

$\frac{1}{5}(4\bar{x}^2-4\bar{x}\bar{y}+\bar{y}^2) + \frac{4}{5}(2\bar{x}^2+3\bar{x}\bar{y}-2\bar{y}^2)$

$- \frac{2}{5}(\bar{x}^2+4\bar{x}\cdot\bar{y}+4\bar{y}^2) = 12$, or

$2\bar{x}^2 - 3\bar{y}^2 = 12$. The latter is equivalent to $\frac{\bar{x}^2}{6} - \frac{\bar{y}^2}{4} = 1$, a hyperbola whose graph is shown.



30. (a) $A = 1$, $B = 2$, $C = 1$, $\cot 2\phi = \frac{A-C}{B} = 0$,

$\phi = 45°$, $\sin\phi = \cos\phi = \frac{\sqrt{2}}{2}$.

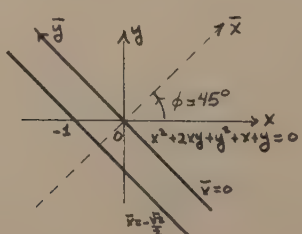
(b) $x = \frac{\sqrt{2}}{2}(\bar{x}-\bar{y})$ and $y = \frac{\sqrt{2}}{2}(\bar{x}+\bar{y})$.

(c) Substituting from (b) into

$x^2 + 2xy + y^2 + x + y = 0$, we obtain

$\frac{1}{2}(\bar{x}^2-2\bar{x}\cdot\bar{y}+\bar{y}^2)+(\bar{x}^2-\bar{y}^2)+\frac{1}{2}(\bar{x}^2+2\bar{x}\cdot\bar{y}+\bar{y}^2)+\sqrt{2}\bar{x}=0$

or $2\bar{x}^2 + \sqrt{2}\bar{x} = 0$. The latter is equivalent to $x(2x+2) = 0$. The graph, which is shown, consists of the line $\bar{x} = 0$ and the line $\bar{x} = -\frac{\sqrt{2}}{2}$.



31. (a) $A = 1$, $B = 2$, $C = 1$, $\cot 2\phi = \frac{A-C}{B} =$

$\phi = 45°$, $\sin\phi = \cos\phi = \frac{\sqrt{2}}{2}$.

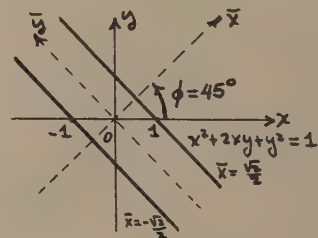
(b) $x = \frac{\sqrt{2}}{2}(\bar{x}-\bar{y})$, $y = \frac{\sqrt{2}}{2}(\bar{x}+\bar{y})$.

(c) Substituting from (b) into

$x^2+2xy+y^2 = 1$, we obtain

$\frac{1}{2}(\bar{x}^2-2\bar{x}\cdot\bar{y}+\bar{y}^2)+(\bar{x}^2-\bar{y}^2)+\frac{1}{2}(\bar{x}^2+2\bar{x}\cdot\bar{y}+\bar{y}^2)=1$, o

$2\bar{x}^2 = 1$. The latter is equivalent to $x = \pm\frac{\sqrt{2}}{2}$. The graph, which is shown, consists of two parallel lines, $\bar{x} = \frac{\sqrt{2}}{2}$ and $\bar{x} = -\frac{\sqrt{2}}{2}$.



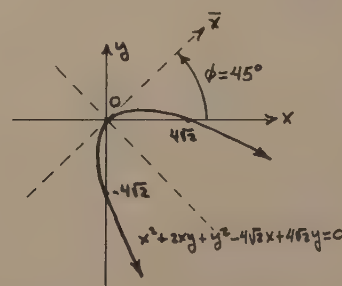
32. (a) $A = 1$, $B = 2$, $C = 1$, $\cot 2\phi = \frac{A-C}{B} =$ (

$\phi = 45°$.

(b) $x = \frac{\sqrt{2}}{2}(\bar{x}-\bar{y})$, $y = \frac{\sqrt{2}}{2}(\bar{x}+\bar{y})$

(c) Substituting from (b) into

$x^2 + 2xy + y^2 - 4\sqrt{2}x + 4\sqrt{2}y = 0$ and simplifying, we obtain $\bar{x}^2 + 8\bar{y} = 0$, or $-\bar{x}^2 = 4\bar{y}$, a parabola whose graph is shown.



33. (a) $A = 9$, $B = -24$, $C = 16$,

$\cot 2\phi = \frac{A-C}{B} = \frac{7}{24}$,

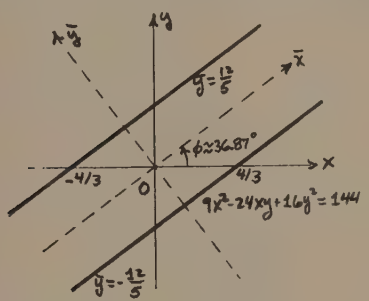
$$\cos 2\phi = \frac{\frac{7}{24}}{\sqrt{(\frac{7}{24})^2 + 1}} = \frac{7}{25},$$

$$\cos\phi = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5},$$

$$\sin\phi = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5},\ \phi = \sin^{-1}\frac{3}{5} \approx 36.87^\circ$$

(b) $x = \frac{4\bar{x}-3\bar{y}}{5}$, $\quad y = \frac{3\bar{x}+4\bar{y}}{5}$

(c) Substituting from (b) into $9x^2-24xy+16y^2 = 144$ and simplifying, we obtain $25\bar{y}^2 = 144$, or $\bar{y} = \pm\frac{12}{5}$, a pair of parallel lines as shown in the graph.



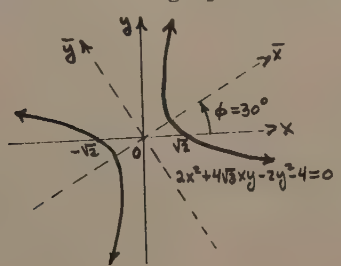
34. (a) $A = 2$, $B = 4\sqrt{3}$, $C = -2$,

$\cot 2\phi = \frac{A-C}{B} = \frac{1}{\sqrt{3}}$, $2\phi = 60^\circ$, $\phi = 30^\circ$,

$\cos\phi = \frac{\sqrt{3}}{2}$, $\sin\phi = \frac{1}{2}$.

(b) $x = \frac{\sqrt{3}\bar{x}-\bar{y}}{2}$, $\quad y = \frac{\bar{x}+\sqrt{3}\bar{y}}{2}$

(c) Substituting from (b) into $2x^2+4\sqrt{3}xy-2y^2-4 = 0$ and simplifying, we obtain $4\bar{x}^2-4\bar{y}^2-4 = 0$, or $\bar{x}^2-\bar{y}^2 = 1$, a hyperbola whose graph is shown.



35. (a) $A = 6$, $B = -6$, $C = 14$,

$\cot 2\phi = \frac{A-C}{B} = \frac{4}{3}$,

$$\cos 2\phi = \frac{\frac{4}{3}}{\sqrt{\frac{16}{9} + 1}} = \frac{4}{5},$$

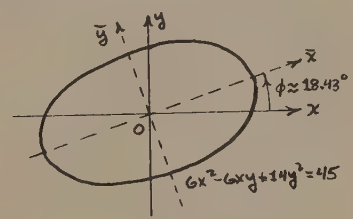
$$\cos\phi = \sqrt{\frac{1 + \frac{4}{5}}{2}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10},$$

$$\sin\phi = \sqrt{\frac{1 - \frac{4}{5}}{2}} = \frac{\sqrt{10}}{10},\ \phi = \sin^{-1}\frac{\sqrt{10}}{10} \approx 18.43^\circ.$$

(b) $x = \frac{\sqrt{10}}{10}(3\bar{x}-\bar{y})$,

$y = \frac{\sqrt{10}}{10}(\bar{x}+3\bar{y})$

(c) Substituting from (b) into $6x^2-6xy+14y^2 = 45$ and simplifying, we obtain $5\bar{x}^2+15\bar{y}^2 = 45$, or $\frac{\bar{x}^2}{9} + \frac{\bar{y}^2}{3} = 1$, an ellipse whose graph is shown.



36. (a) $A = 17$, $B = -12$, $C = 8$, $D = -68$

$E = 24$, $F = -12$.

$$\cot 2\phi = \frac{A-C}{B} = -\frac{3}{4},\ \cos 2\phi = \frac{-3/4}{\sqrt{\frac{9}{16} + 1}} = -\frac{3}{5},$$

$$\cos\phi = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{\sqrt{5}}{5},\ \sin\phi = \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2\sqrt{5}}{5},$$

$$\phi = \sin^{-1}\frac{2\sqrt{5}}{5} \approx 63.43^\circ$$

(b) $x = \frac{\sqrt{5}}{5}(\bar{x}-2\bar{y})$ and $y = \frac{\sqrt{5}}{5}(2\bar{x}+\bar{y})$.

(c) Making direct use of the formulas on page 579, we have,

$\bar{A} = 17(\frac{1}{5}) - 12(\frac{2}{5}) + 8(\frac{4}{5}) = 5$,

$\bar{C} = 17(\frac{4}{5}) + 12(\frac{2}{5}) + 8(\frac{1}{5}) = 20$,

$\bar{D} = -68(\frac{\sqrt{5}}{5}) + 24(\frac{2\sqrt{5}}{5}) = -4\sqrt{5}$,

$\overline{E} = 68(\frac{2\sqrt{5}}{5}) + 24(\frac{\sqrt{5}}{5}) = 32\sqrt{5}$,

and $\overline{F} = F$, so the "new" equation is

$5\overline{x}^2 + 20\overline{y}^2 - 4\sqrt{5}\overline{x} + 32\sqrt{5}\overline{y} - 12 = 0$.

Completing the squares gives

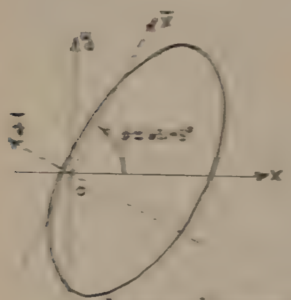
$5(\overline{x}^2 - \frac{4\sqrt{5}\overline{x}}{5} + \frac{4}{5}) + 20(\overline{y}^2 + \frac{32\sqrt{5}\overline{y}}{20} + \frac{16}{5}) =$

$12 + 4 + 64$, or $(\overline{x} - \frac{2\sqrt{5}}{5})^2 + 4(\overline{y} + \frac{4\sqrt{5}}{5})^2 = 16$.

The latter equation is equivalent to

$\frac{(\overline{x} - \frac{2\sqrt{5}}{5})^2}{16} + \frac{(\overline{y} + \frac{4\sqrt{5}}{5})^2}{4} = 1$, an ellipse

as in the figure.



37. (a) $A = 2$, $B = 6$, $C = -6$, $D = 2\sqrt{10}$,

$E = 3\sqrt{10}$, $F = -16$, so

$\cot 2\phi = \frac{A-C}{B} = \frac{8}{6} = \frac{4}{3}$, $\cos 2\phi = \frac{4/3}{\sqrt{\frac{16}{9}+1}} = \frac{4}{5}$,

$\cos\phi = \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3\sqrt{10}}{10}$, $\sin\phi = \sqrt{\frac{1-\frac{4}{5}}{2}} = \frac{\sqrt{10}}{10}$.

$\phi = \sin^{-1}\frac{\sqrt{10}}{10} \approx 18.43°$.

(b) $x = \frac{\sqrt{10}}{10}(3\overline{x}-\overline{y})$ and $y = \frac{\sqrt{10}}{10}(\overline{x}+3\overline{y})$.

(c) The formulas on page 579 give

$\overline{A} = 2(\frac{9}{10}) + 6(\frac{3}{10}) - 6(\frac{1}{10}) = 3$,

$\overline{C} = 2(\frac{1}{10}) - 6(\frac{3}{10}) - 6(\frac{9}{10}) = -7$,

$\overline{D} = 2\sqrt{10}(\frac{3\sqrt{10}}{10}) + 3\sqrt{10}(\frac{\sqrt{10}}{10}) = 9$,

$\overline{E} = -2\sqrt{10}(\frac{\sqrt{10}}{10}) + 3\sqrt{10}(\frac{3\sqrt{10}}{10}) = 7$,

$\overline{F} = F = -16$.

Thus, the "new" equation is

$3\overline{x}^2 - 7\overline{y}^2 + 9\overline{x} + 7\overline{y} - 16 = 0$.

Completing the squares, we obtain

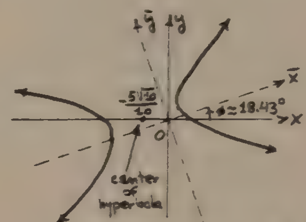
$3(\overline{x}^2 + 3\overline{x} + \frac{9}{4}) - 7(\overline{y}^2 - \overline{y} + \frac{1}{4}) =$

$\frac{27}{4} - \frac{7}{4} + 16$, or $3(\overline{x} + \frac{3}{2})^2 - 7(\overline{y} - \frac{1}{2})^2 = 21$

The latter equation is equivalent to

$\frac{(\overline{x} + \frac{3}{2})^2}{7} - \frac{(\overline{y} - \frac{1}{2})^2}{3} = 1$, a hyperbola

whose center <u>in the "new" coordinate system</u> is $(-\frac{3}{2},\frac{1}{2})$. Thus, the "old" coordinates of the center are $x = \frac{\sqrt{10}}{10}(-\frac{9}{2} - \frac{1}{2}) = -\frac{5\sqrt{10}}{10}$ and $y = \frac{\sqrt{10}}{10}(-\frac{3}{2} + \frac{3}{2}) = 0$. The graph is shown.



38. $A = 4$, $B = -12$, $C = 9$, $D = -52$, $E = 26$, and $F = 27$. $\cot 2\phi = \frac{A-C}{B} = \frac{4-9}{-12} = \frac{5}{12}$, so

$\cos 2\phi = \frac{5/12}{\sqrt{(\frac{5}{12})^2+1}} = \frac{5/12}{\sqrt{\frac{25+144}{144}}} = \frac{5}{12}\cdot\frac{12}{13} = \frac{5}{13}$

Thus, $\sin\phi = \sqrt{\frac{1-\frac{5}{13}}{2}} = \frac{2}{\sqrt{13}}$ and

$\cos\phi = \sqrt{\frac{1+\frac{5}{13}}{2}} = \frac{3}{\sqrt{13}}$ and $\phi \approx 33.69°$.

Using the formulas on page 983:

$\overline{A} = 4(\frac{9}{13}) + (-12)(\frac{6}{13}) + 9(\frac{4}{13}) = 0$

$\overline{C} = 4(\frac{4}{13}) - (-12)(\frac{6}{13}) + 9(\frac{9}{13}) = 13$

$\overline{D} = (-52)(\frac{3}{\sqrt{13}}) + 26(\frac{2}{\sqrt{13}}) = -\frac{104}{\sqrt{13}} \approx -28.8$

$\overline{E} = -(-52)(\frac{2}{\sqrt{13}}) + 26(\frac{3}{\sqrt{13}}) = \frac{182}{\sqrt{13}} \approx 50.5$

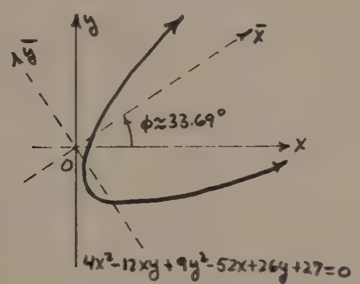
$\overline{F} = 27$. Therefore, the new equation is:

$13\overline{y}^2 - \frac{104}{\sqrt{13}}\overline{x} + \frac{182}{\sqrt{13}}\overline{y} + 27 = 0$;

$13(\overline{y}^2 + \frac{14}{\sqrt{13}}\overline{y} + (\frac{7}{\sqrt{13}})^2) = \frac{104}{\sqrt{13}}\overline{x} - 27 + 13(\frac{49}{13})$;

$13(\overline{y} + \frac{7}{\sqrt{13}})^2 = \frac{104}{\sqrt{13}}(\overline{x} + \frac{22\sqrt{13}}{104})$.

Thus, $(\overline{y} + \frac{7}{\sqrt{13}})^2 = \frac{8}{\sqrt{13}}(\overline{x} + \frac{11\sqrt{13}}{52})$ is the equation and is a parabola with vertex $\overline{V} = (-\frac{11\sqrt{13}}{52}, -\frac{7}{\sqrt{13}}) \approx (-0.76, -1.94)$ in $\overline{x},\overline{y}$ system and $p = \frac{2}{\sqrt{13}}$.



9. Substituting $x = \overline{x}\cos\phi - \overline{y}\sin\phi$ and $y = \overline{x}\sin\phi + \overline{y}\cos\phi$ into equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, we obtain:

$A(\overline{x}^2\cos^2\phi - 2\overline{x}\,\overline{y}\cos\phi\sin\phi + \overline{y}^2\sin^2\phi) +$
$B(\overline{x}^2\cos\phi\sin\phi + \overline{x}\,\overline{y}\cos^2\phi - \overline{x}\,\overline{y}\sin^2\phi -$
$\overline{y}^2\sin\phi\cos\phi) + C(\overline{x}^2\sin^2\phi + 2\overline{x}\,\overline{y}\sin\phi\cos\phi$
$+\overline{y}^2\cos^2\phi) + D(\overline{x}\cos\phi - \overline{y}\sin\phi) +$
$E(\overline{x}\sin\phi + \overline{y}\cos\phi) + F = 0$.

Multiplying out, we find the coefficients of $\overline{x}^2$, $\overline{x}\,\overline{y}$, $\overline{y}^2$, $\overline{x}$, $\overline{y}$ called $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$, and $\overline{E}$, respectively, are:

$\overline{A} = A\cos^2\phi + B\cos\phi\sin\phi + C\sin^2\phi$;

$\overline{B} = -2A\cos\phi\sin\phi + 2C\sin\phi\cos\phi + B\cos^2\phi - B\sin^2\phi = 2(C-A)\cos\phi\sin\phi + B(\cos^2\phi - \sin^2\phi)$;

$\overline{C} = A\sin^2\phi - B\sin\phi\cos\phi + C\cos^2\phi$;

$\overline{D} = E\sin\phi + D\sin\phi$;

$\overline{E} = -D\sin\phi + E\cos\phi$.

Also the constant $\overline{F} = F$. Thus, we have the equations on page 579.

40. Rotation of the coordinate system as in Theorem 1 puts the equation in the form $\overline{A}\,\overline{x}^2 + \overline{C}\,\overline{y}^2 + \overline{D}\,\overline{x} + \overline{E}\,\overline{y} + \overline{F} = 0$, $B = 0$. By Theorem 3, $\overline{B}^2 - 4\overline{A}\,\overline{C} = B^2 - 4AC$, so $\overline{A}\,\overline{C} = -\frac{1}{4}(B^2 - 4AC)$. Consider separately the three cases:

(a) $B^2 - 4AC < 0$,

(b) $B^2 - 4AC = 0$, and

(c) $B^2 - 4AC > 0$.

In case (c), $\overline{A}\,\overline{C} > 0$, so $\overline{A}$ and $\overline{C}$ have the same algebraic sign. In case (b), $\overline{A}\,\overline{C} = 0$, so either $\overline{A} = 0$ or $\overline{B} = 0$ (or both). In case (c), $\overline{A}\,\overline{C} < 0$, so $\overline{A}$ and $\overline{C}$ have opposite algebraic signs. We consider case (b) first. If both $\overline{A} = 0$ and $\overline{C} = 0$, the equation becomes $\overline{D}x + \overline{E}y + \overline{F} = 0$, contrary to the hypothesis that the conic is non-degenerate. Thus one, but not both, of $\overline{A}$ and $\overline{C}$ is zero. Suppose $\overline{A} \neq 0$, $\overline{C} = 0$. (The other situation is handled similarly.) The equation becomes $\overline{A}x^2 + \overline{D}x + \overline{E}y + \overline{F} = 0$, or completing the square, $\overline{A}(\overline{x}^2 + \frac{\overline{D}}{\overline{A}}\overline{x} + \frac{\overline{D}^2}{4\overline{A}^2}) = \frac{\overline{D}^2}{4\overline{A}} - \overline{E}\,\overline{y} - \overline{F}$.

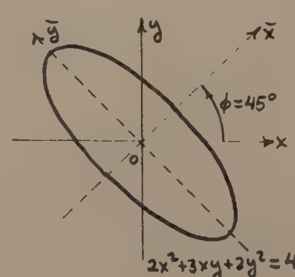
The latter equation is equivalent to $(\overline{x} + \frac{\overline{D}}{2\overline{A}})^2 = (\frac{\overline{D}^2}{4\overline{A}^2} - \frac{\overline{F}}{\overline{A}}) - \overline{E}\,\overline{y}$, which is the equation of a parabola (provided $\overline{E} \neq 0$). (Notice that, if $E = 0$, the graph would be a degenerate conic.) We now consider cases (a) and (b). Since, in these cases, $\overline{A} \neq 0$ and $\overline{C} \neq 0$, we can complete the squares as follows:

$\bar{A}(\bar{x}^2 + \frac{\bar{D}}{\bar{A}}\bar{x} + \frac{\bar{D}^2}{4\bar{A}^2}) + \bar{C}(\bar{y}^2 + \frac{\bar{E}}{\bar{C}} + \frac{\bar{E}^2}{4\bar{C}^2}) =$ $\frac{\bar{D}^2}{4\bar{A}} + \frac{\bar{E}^2}{4\bar{C}} - \bar{F}$. Put $h = -\frac{\bar{D}}{2\bar{A}}$, $k = -\frac{\bar{E}}{2\bar{C}}$, $C = \frac{\bar{D}^2}{4\bar{A}} + \frac{\bar{E}^2}{4\bar{C}} - \bar{F}$ and rewrite the equation as $\bar{A}(\bar{x}-h)^2 + \bar{C}(\bar{y}-k)^2 = C$. If $C = 0$, we again have a degenerate conic. (Why?) Thus, we can suppose $C \neq 0$. The equation can be written as $\frac{(\bar{x}-h)^2}{(\frac{C}{\bar{A}})} + \frac{(\bar{y}-k)^2}{(\frac{C}{\bar{C}})} = 1$. In case (a), both $\frac{C}{\bar{A}}$ and $\frac{C}{\bar{C}}$ have the same algebraic sign, and the graph (since it is not degenerate) is either a circle or an ellipse. In case (c), $\frac{C}{\bar{A}}$ and $\frac{C}{\bar{C}}$ have opposite algebraic signs, so the graph is a hyperbola. Finally, if $A = C$ and $B = 0$, then by completing the squares as usual, the original equation is seen to represent a circle.

41. (a) $B^2 - 4AC = 16 - 4(6)(3)$
$= -56 < 0$, ellipse.
(b) $B^2 - 4AC = 12^2 - 4(18)(2) = 0$, parabola.
(c) $A = C$ and $B = 0$, so the graph is a circle.
(d) $B^2 - 4AC = 9 - 4(1)(-3)$
$= 21 > 0$, hyperbola.

42. (a) Obvious, since all x and y satisfy the equation.
(b) There is no point (x,y) such that $x^2 + y^2 = -1$.
(c) Rotating the coordinate system through $\phi = 45^\circ$, we find that the equation becomes $4\bar{y}^2 = 0$; that is, $\bar{y} = 0$. This is the equation of a line (the $\bar{x}$ axis).
(d) Completing the squares, we have $4(x^2+4x+4) - (y^2-2y+1) = -15+16-1$, or $4(x+2)^2 - (y-1)^2 = 0$. Thus, $4(x+2)^2 = (y-1)^2$, or $2(x+2) = \pm(y-1)$. The graph consists of the two intersecting lines $2(x+2) = y-1$ and $2(x+2) = -(y-1)$.
(e) Rotating the coordinate system through $\phi = 45^\circ$, we find that the equation becomes $2\bar{y}^2-18 = 0$, or $\bar{y} = \pm 9$. The graph consists of the two parallel lines $\bar{y} = 9$ and $\bar{y} = -9$.
(f) Completing the squares, we have $(x^2-6x+9) + (y^2+4y+4) = -13+9+4$, or $(x-3)^2 + (y+2)^2 = 0$.
The only solution is the single point $(x,y) = (3,-2)$.

43. $A = 2$, $B = 3$, $C = 2$, $\cot 2\phi = \frac{A-C}{B} = 0$, $2\phi = 90^\circ$, $\phi = 45^\circ$, $x = \frac{\sqrt{2}}{2}(\bar{x}-\bar{y})$, $y = \frac{\sqrt{2}}{2}(\bar{x}+\bar{y})$, and the given equation becomes $(\bar{x}-\bar{y})^2 + \frac{3}{2}(\bar{x}-\bar{y})(\bar{x}+\bar{y}) + (\bar{x}+\bar{y})^2 = 4$, or $\frac{7}{2}\bar{x}^2 + \frac{1}{2}\bar{y}^2 = 4$. This can be rewritten as $\frac{\bar{x}^2}{(\frac{8}{7})} + \frac{\bar{y}^2}{8} = 1$, an ellipse.



44. By Problem 39, we have $\bar{F} = F$. We also have,

$\bar{A} = A\cos^2\phi + B\cos\phi\sin\phi + C\sin^2\phi$

$\bar{B} = 2(C-A)\cos\phi\sin\phi + B(\cos^2\phi - \sin^2\phi)$

$= (C-A)\sin 2\phi + B\cos 2\phi$, and

$\bar{C} = A\sin^2\phi - B\cos\phi\sin\phi + C\cos^2\phi$. Thus,

$\bar{B}^2 = (C-A)^2\sin^2 2\phi + 2(C-A)B\sin 2\phi\cos 2\phi +$
$B^2\cos^2 2\phi$,

$\bar{A} = A(\frac{1+\cos 2\phi}{2}) + B(\frac{\sin 2\phi}{2}) + C(\frac{1-\cos 2\phi}{2})$

$= \frac{1}{2}\left[(A+C) + (A-C)\cos 2\phi + B\sin 2\phi\right]$, and

$\bar{C} = A(\frac{1-\cos 2\phi}{2}) - B(\frac{\sin 2\phi}{2}) + C(\frac{1+\cos 2\phi}{2})$

$= \frac{1}{2}\left[(A+C) - (A-C)\cos 2\phi - B\sin 2\phi\right]$

Therefore, $4\bar{A}\bar{C} =$

$(A+C)^2 - \left[(A-C)\cos 2\phi + B\sin 2\phi\right]^2 =$

$(A+C)^2 - (A-C)^2\cos^2 2\phi - 2(A-C)B\sin 2\phi\cos 2\phi -$
$B^2\sin^2 2\phi$.

It follows that

$\bar{B}^2 - 4\bar{A}\cdot\bar{C} = (C-A)^2\sin^2 2\phi + 2(C-A)B\sin 2\phi\cos 2\phi$
$+ B^2\cos^2 2\phi - (A+C)^2 + (A-C)^2\cos^2 2\phi$
$+ 2(A-C)B\sin 2\phi\cos 2\phi + B^2\sin^2 2\phi$
$= (A-C)^2(\sin^2 2\phi + \cos^2 2\phi) + B^2(\sin^2 2\phi$
$+ \cos^2 2\phi) - (A+C)^2$
$= (A-C)^2 + B^2 - (A+C)^2 = B^2 - 4AC.$

Also,

$\bar{A} + \bar{C} = A\cos^2\phi + B\cos\phi\sin\phi + C\sin^2\phi$
$+ A\sin^2\phi - B\cos\phi\sin\phi + C\cos^2\phi$
$= A(\cos^2\phi + \sin^2\phi) + C(\cos^2\phi + \sin^2\phi)$
$= A + C.$

45. In Problem 29,

$B^2 - 4AC = 4^2 - 4(1)(-2) = 24$ and

$\bar{B}^2 - 4\bar{A}\,\bar{C} = 0^2 - 4(2)(-3) = 24$. Also,

$A + C = 1 + (-2) = -1$ and

$\bar{A} + \bar{C} = 2 + (-3) = -1$. $\bar{F} = F = -12$.

In Problem 31,

$B^2 - 4AC = 2^2 - 4(1)(1) = 0$ and

$\bar{B}^2 - 4\bar{A}\,\bar{C} = 0^2 - 4(2)(0) = 0$.

Also, $A + C = 1 + 1 = 2$ and

$\bar{A} + \bar{C} = 2 + 0 = 2$.

$\bar{F} = F = -1$.

In Problem 33,

$B^2 - 4AC = (-24)^2 - 4(9)(16) = 0$ and

$\bar{B}^2 - 4\bar{A}\bar{C} = 0^2 - 4(0)(25) = 0$. Also,

$A + C = 9 + 16 = 25$ and

$\bar{A} + \bar{C} = 0 + 25 = 25$. $\bar{F} = F = -144$.

In Problem 35,

$B^2 - 4AC = (-6)^2 - 4(6)(14) = -300$ and

$\bar{B}^2 - 4\bar{A}\bar{C} = 0^2 - 4(5)(15) = -300$. Also

$A + C = 6 + 14 = 20$ and

$\bar{A} + \bar{C} = 5 + 15 = 20$. $\bar{F} = F = -45$.

In Problem 37,

$B^2 - 4AC = 6^2 - 4(2)(-6) = 84$ and

$\bar{B}^2 - 4\bar{A}\bar{C} = 0^2 - 4(3)(-7) = 84$. Also,

$A + C = 2 + (-6) = -4$ and

$\bar{A} + \bar{C} = 3 + (-7) = -4$. $\bar{F} = F = -16$.

46. Since $AC > 0$, then both A and C have the same algebraic sign. Multiply both sides of the equation by (-1) if necessary, so that we can assume that both A and C are positive. Now complete the squares as follows:

$A(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}) + C(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2})$

$= \frac{D^2}{4A} + \frac{E^2}{4C} - F.$

Put $h = -\frac{D}{2A}$, $k = -\frac{E}{2C}$, $q = \frac{D^2}{4A} + \frac{E^2}{4C} - F$,

so that the equation becomes

$A(x-h)^2 + C(y-k)^2 = q$. If $q < 0$, this equation has no solution. If $q = 0$, it has only the single point (h,k) as a solution. Thus, suppose $q > 0$. Then the equation becomes

$\frac{(x-h)^2}{(\frac{q}{a})} + \frac{(y-k)^2}{(\frac{q}{C})} = 1$, where $\frac{q}{A} > 0$ and

$\frac{q}{C} > 0$. Clearly, this is either a circle or an ellipse.

47. (a) The center is at the origin and the distance c from the center to a focus is $\sqrt{3^2 + 1^2} = \sqrt{10}$ units. Since the distance from the center to either focus is less than the distance a from the center to either vertex, the conic is an ellipse. A point $P = (x,y)$ will belong to the ellipse if and only if

$$\sqrt{(x+3)^2 + (y+1)^2} + \sqrt{(x-3)^2 + (y-1)^2} = 2a=8.$$

Removing the square roots by squaring twice as usual and then simplifying, we get $7x^2-6xy+15y^2 = 96$. The line through the foci makes an angle ϕ with the x axis, where $\sin\phi = \frac{\sqrt{10}}{10}$ and $\cos\phi = \frac{3\sqrt{10}}{10}$. Rotation of the coordinate system through the angle ϕ gives $x = \frac{\sqrt{10}}{10}(3\bar{x}-\bar{y})$, $y = \frac{\sqrt{10}}{10}(\bar{x}+3\bar{y})$. Substitution into $7x^2 - 6xy + 15y^2 = 96$ gives $3\bar{x}^2 + 8\bar{y}^2 = 48$ or $\frac{\bar{x}^2}{16} + \frac{\bar{y}^2}{6} = 1$.

(b) Arguing as above, we see that the conic is a hyperbola with center at the origin. A point $P = (x,y)$ will belong to the hyperbola if and only if

$$\sqrt{(x+4)^2 + (y+3)^2} - \sqrt{(x-4)^2 + (y-3)^2} = 2a = 6.$$

Removing the square roots by squaring twice as usual and then simplifying, we get $7x^2 + 24xy = 144$. The line through the foci makes an angle ϕ with the axis, where $\sin\phi = \frac{3}{5}$ and $\cos\phi = \frac{4}{5}$. Thus, rotation through the angle $\phi = \sin^{-1}\frac{3}{5}$ is accomplished by putting $x = \frac{4\bar{x}-3\bar{y}}{5}$ and $y = \frac{3\bar{x}+4\bar{y}}{5}$. Substitution into $7x^2 + 24xy =$ 144 gives $16\bar{x}^2 - 9\bar{y}^2 = 144$, or $\frac{\bar{x}^2}{9} - \frac{\bar{y}^2}{16} = 1$.

48. Since $AC < 0$, then A and C have opposite algebraic signs. Complete the squares to get $A(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2})+C(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}) = \frac{D^2}{4A} + \frac{E^2}{4C} - F$. Put $h = -\frac{D}{2A}$, $k = -\frac{E}{2C}$ and $q = \frac{D^2}{4A} + \frac{E^2}{4C} - F$, so that the equation becomes $A(x-h)^2 + C(y-k)^2 = q$. If $q = 0$, the equation is equivalent to $(x-h)^2 = -\frac{C}{A}(y-k)^2$, or $x-h = \pm\sqrt{-\frac{C}{A}}(y-k)$ (Note that $-\frac{C}{A}$ is positive, since A and C have opposite algebraic signs.) The latter equation has the two intersecting straight lines $x-h = \sqrt{-\frac{C}{A}}(y-k)$ and $x-h = -\sqrt{-\frac{C}{A}}(y-k)$ as its graph. If $q \neq 0$, the equation can be written $\frac{(x-h)^2}{(\frac{q}{A})} + \frac{(y-k)^2}{(\frac{q}{C})} = 1$. Since $\frac{q}{A}$ and $\frac{q}{C}$ must have opposite algebraic signs, this is the equation of a hyperbola.

49. $(y-3x)(y+3x) = 0$, or $y^2-9x^2 = 0$.

50. Complete the square to obtain $A(x^2 + \frac{Dx}{A} + \frac{D^2}{4A^2}) = \frac{D^2}{4A} - Ey - F$. Put $h = -\frac{D}{2A}$, and rewrite the latter equation as $(x-h)^2 = (\frac{D^2}{4A^2} - \frac{F}{A}) - \frac{E}{A}y$. If $E = 0$, this equation becomes $(x-h)^2 = \frac{D^2}{4A^2} - \frac{F}{A}$, so if $\frac{D^2}{4A^2} - \frac{F}{A} < 0$, there is no solution and the graph is empty. If $E = 0$ and $\frac{D^2}{4A^2} - \frac{F}{A} \geq 0$, the equation becomes

$x = h \pm \sqrt{\frac{D^2}{4A^2} - \frac{F}{A}}$, which is either two parallel lines or (when $\frac{D^2}{4A^2} - \frac{F}{A} = 0$) a single line. Suppose $E \neq 0$; then the equation $(x-h)^2 = (\frac{D^2}{4A^2} - \frac{F}{A}) - \frac{E}{A}y$ becomes $(x-h)^2 = -\frac{E}{A}(y-k)$, where $k = \frac{A}{E}(\frac{D^2}{4A^2} - \frac{F}{A})$. The equation $(x-h)^2 = -\frac{E}{A}(y-k)$ clearly represents a parabola.

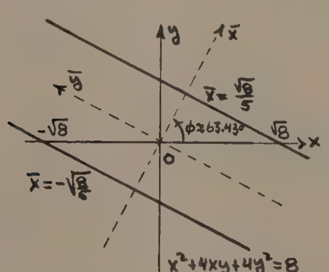
51. $A = 1$, $B = 4$, $C = 4$, $\cot 2\phi = \frac{A-C}{B} = -\frac{3}{4}$,

$\cos 2\phi = \frac{(-3/4)}{\sqrt{(-\frac{3}{4})^2+1}} = -\frac{3}{5}$,

$\cos\phi = \sqrt{\frac{1+(-3/5)}{2}} = \frac{\sqrt{5}}{5}$,

$\sin\phi = \sqrt{\frac{1-(-3/5)}{2}} = \frac{2\sqrt{5}}{5}$,

$\phi = \sin^{-1}\frac{2\sqrt{5}}{5} \approx 63.43°$, $x = \frac{\sqrt{5}}{5}(\bar{x}-2\bar{y})$, $y = \frac{\sqrt{5}}{5}(2\bar{x}+\bar{y})$. Substituting the latter two equations into $x^2 + 4xy + 4y^2 = 8$, we obtain $\frac{1}{5}(\bar{x}-2\bar{y})^2 + \frac{4}{5}(\bar{x}-2\bar{y})(2\bar{x}+\bar{y}) + \frac{4}{5}(2\bar{x}+\bar{y})^2 = 8$, or $5\bar{x}^2 = 8$. Thus, the graph consists of the two parallel lines $\bar{x} = \sqrt{\frac{8}{5}}$ and $\bar{x} = -\sqrt{\frac{8}{5}}$.



52. $2A^2+B^2+2C^2 = 2A^2+B^2-4AC+2C^2+4AC$

$= B^2-4AC+2(A^2+2AC+C^2)$

$= B^2-4AC+2(A+C)^2$

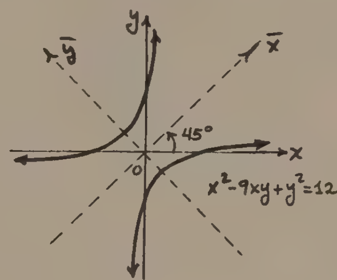
$= \bar{B}^2-4\bar{A}\bar{C}+2(\bar{A}+\bar{C})^2$

$= 2\bar{A}^2+\bar{B}^2+2\bar{C}^2$.

53. $A = 1$, $B = -9$, $C = 1$, $\cot 2\phi = \frac{A-C}{B} = 0$,

$\phi = 45°$, $x = \frac{\sqrt{2}}{2}(\bar{x}-\bar{y})$, $y = \frac{\sqrt{2}}{2}(\bar{x}+\bar{y})$.

Substituting into $x^2-9xy+y^2 = 12$, we obtain $\frac{1}{2}(\bar{x}-\bar{y})^2-\frac{9}{2}(\bar{x}-\bar{y})(\bar{x}+\bar{y})+\frac{1}{2}(\bar{x}+\bar{y})^2 = 12$, or $-\frac{7}{2}\bar{x}^2+\frac{11}{2}\bar{y}^2 = 12$. This can be rewritten as $\frac{\bar{y}^2}{(\frac{24}{11})} - \frac{\bar{x}^2}{(\frac{24}{7})} = 1$, a hyperbola.



54. We have $\cos 2\theta = \frac{(\frac{A-C}{B})}{\sqrt{\frac{(A-C)^2}{B^2}+1}}$

$= \frac{A-C}{B\sqrt{\frac{(A-C)^2+B^2}{B^2}}} = \frac{A-C}{\frac{B}{B^2}\sqrt{(A-C)^2+B^2}}$

$= \frac{A-C}{\frac{B}{|B|}\sqrt{(A-C)^2+B^2}} = \frac{A-C}{S\sqrt{(A-C)^2+B^2}}$.

Since $0 < 2\theta < \pi$, then $0 < \sin 2\theta$, so $\sin 2\theta = \sqrt{1-\cos^2 2\theta} = \sqrt{1-(\frac{A-C}{S\sqrt{(A-C)^2+B^2}})^2}$

$= \sqrt{1-(\frac{(A-C)^2}{S^2[(A-C)^2+B^2]})}$. Notice that $S = \frac{B}{|B|} = \pm 1$, so $S^2 = 1$ and $\sin 2\theta$

$= \sqrt{1-\frac{(A-C)^2}{(A-C)^2+B^2}} = \sqrt{\frac{(A-C)^2+B^2-(A-C)^2}{(A-C)^2+B^2}}$

$= \frac{\sqrt{B^2}}{\sqrt{(A-C)^2+B^2}} = \frac{|B|}{\sqrt{(A-C)^2+B^2}}$.

Using the equation for $\bar{A}$ on page 579,

$\bar{A} = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta =$

$A(\frac{1+\cos 2\theta}{2}) + B(\frac{\sin 2\theta}{2}) + C(\frac{1-\cos 2\theta}{2}) =$

$\frac{1}{2}\cdot\left[A\ 1+\frac{A-C}{s\cdot\sqrt{(A-C)^2+B^2}} + B\left(\frac{|B|}{\sqrt{(A-C)^2+B^2}}\right) + C\left(1-\frac{A-C}{s\cdot\sqrt{(A-C)^2+B^2}}\right)\right].$ Thus, $\bar{A} =$

$$(\tfrac{1}{2})\left[\frac{sA\sqrt{(A-C)^2+B^2}+A^2-AC+B|B|s+sC\sqrt{(A-C)^2+B^2}-AC+C^2}{s\cdot\sqrt{(A-C)^2+B^2}}\right]$$

$$= (\tfrac{1}{2})\left[\frac{s(A+C)\sqrt{(A-C)^2+B^2}+A^2-2AC+C^2+B^2}{s\sqrt{(A-C)^2+B^2}}\right]$$

$$= \frac{1}{2}\left[(A+C) + \frac{(A-C)^2+B^2}{s\sqrt{(A-C)^2+B^2}}\right]$$

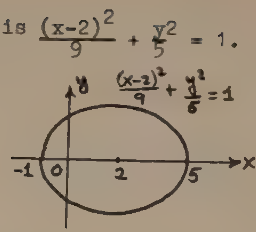
$$= \frac{1}{2}\left[(A+C) + \frac{1}{s}\sqrt{(A-C)^2+B^2}\right].$$

Since $s = \pm 1$, then $s = \frac{1}{s}$, so $\bar{A} =$ $\frac{1}{2}\left[(A+C) + s\sqrt{(A-C)^2+B^2}\right]$. By Theorem 3, page 582, $\bar{A} + \bar{C} = A + C$, so $\bar{C} = A + C - \bar{A} =$ $A+C - \frac{1}{2}\left[(A+C) + s\cdot\sqrt{(A-C)^2+B^2}\right] =$ $\frac{1}{2}\left[(A+C) - s\cdot\sqrt{(A-C)^2+B^2}\right]$

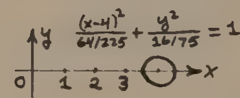
55. Suppose the graph is a circle. The equation will be of the form $(\bar{x}-h)^2 + (\bar{y}-k)^2 = r^2$ after any rotation. Multiplying and arranging the terms, we obtain $\bar{x}^2+\bar{y}^2-2h\bar{x}-2k\bar{y}+h^2+k^2-y^2 = 0$, which is of the form $Ax^2+Bxy+Cy^2+Dx+Ey+F = 0$ and $B = 0$. But $B \neq 0$; hence, the equation cannot be a circle. We have a circle only if $A = C$ and $B = 0$.

Problem Set 9.8, page 591

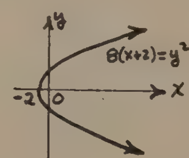
1. By case (i), $e = \frac{2}{3}$, $d = \frac{5}{2}$, $a = \frac{ed}{1-e^2} = 3$, $b = \frac{ed}{\sqrt{1-e^2}} = \sqrt{5}$, $c = ae = 2$. The equation is $\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$.



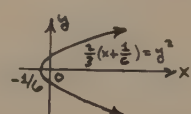
2. By case (i), $e = \frac{1}{2}$, $d = \frac{4}{5}$, $a = \frac{ed}{1-e^2} = \frac{8}{15}$, $b = \frac{ed}{\sqrt{1-e^2}} = \frac{4}{5\sqrt{3}}$, $c = ae = \frac{4}{15}$. The equation is $\frac{(x-4)^2}{\frac{64}{225}} + \frac{y^2}{\frac{16}{75}} = 1$.



3. By case (ii), $e = 1$, $d = 4$, so $\frac{d}{2} = p = 2$. The equation is $8(x+2) = y^2$.

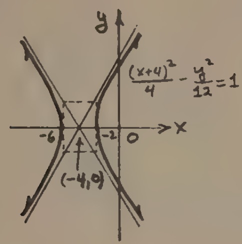


4. By case (ii), $e = 1$, $\frac{d}{2} = p = \frac{1}{6}$ since $d = \frac{1}{3}$. The equation is $\frac{2}{3}(x + \frac{1}{6}) = y^2$.



5. By case (iii), $e = 2$, $d = 3$, $a = \frac{ed}{e^2 - 1} = 2$, $b = \frac{ed}{\sqrt{e^2 - 1}} = 2\sqrt{3}$, $c = ae = 4$. The equation is

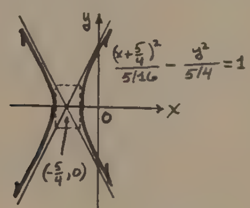
$$\frac{(x+4)^2}{4} - \frac{y^2}{12} = 1.$$



6. By case (iii), $e = \sqrt{5}$, $d = 1$,

$a = \frac{ed}{e^2-1} = \frac{\sqrt{5}}{4}$, $b = \frac{ed}{\sqrt{e^2-1}} = \frac{\sqrt{5}}{2}$, $c = ae = \frac{5}{4}$.

The equation is $\dfrac{(x+\frac{5}{4})^2}{\frac{5}{16}} - \dfrac{y^2}{\frac{5}{4}} = 1$.



7. $r = \dfrac{20}{4-3\cos\theta} = \dfrac{5}{1-\frac{3}{4}\cos\theta}$, so $e = \frac{3}{4}$,

and the polar conic is an ellipse with polar point $F_1 = (0,0)$ as a focus.

$V_1 = (\frac{20}{4+3}, \pi) = (-\frac{20}{7}, 0)$,

$V_2 = (20,0)$ are vertices. Also, since $de = 5$ and $e = \frac{3}{4}$, we have $d = \frac{20}{3}$. Also

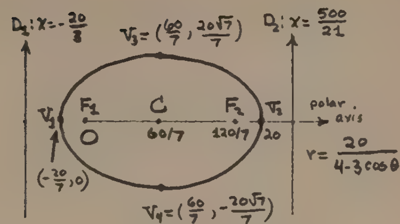
$a = \dfrac{5}{1-\frac{9}{16}} = \frac{80}{7}$ and $c = ae = (\frac{80}{7})(\frac{3}{4}) = \frac{60}{7}$.

The center is $(\frac{60}{7}, 0)$. Since $b = \sqrt{(\frac{80}{7})^2 - (\frac{60}{7})^2} = \frac{20}{7}\sqrt{2}$, then

$V_3 = (\frac{60}{7}, \frac{20\sqrt{7}}{7})$ and $V_4 = (\frac{60}{7}, -\frac{20\sqrt{7}}{7})$. The directrix with $F_1 = (0,0)$ is D_1: $x = -\frac{20}{3}$.

The directrix with $F_2 = (\frac{120}{7}, 0)$ is

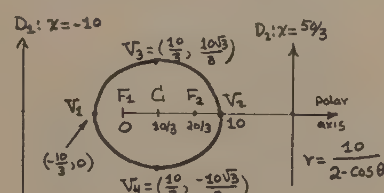
D_2: $x = c+d$ or $x = \frac{120}{7} + \frac{20}{3} = \frac{500}{21}$.



8. $r = \dfrac{10}{2-\cos\theta} = \dfrac{5}{1-\frac{1}{2}\cos\theta}$; $e = \frac{1}{2}$ and $ed = 5$, so $d = 10$. The conic is an ellipse since $e = \frac{1}{2} < 1$. Let $\theta = 0$ and $\theta = \pi$ to find the vertices; $V_2 = (10,0)$, $V_1 = (\frac{10}{3}, \pi) = (-\frac{10}{3}, 0)$. The center is the midpoint of $\overline{V_1V_2}$; hence, $(\frac{10}{3}, 0)$ is the center; and $a = \frac{20}{3}$, $c = (\frac{20}{3})(\frac{1}{2}) = \frac{10}{3}$. Thus, the foci are: $F_1 = (0,0)$, $F_2 = (\frac{20}{3}, 0)$. The directrices are D_1: $x = -10$ and D_2: $x = \frac{20}{3} + d = \frac{50}{3}$.

Now $b^2 = a^2 - c^2$ so $b^2 = \frac{400}{9} - \frac{100}{9} = \frac{300}{9}$ or $b = \frac{10\sqrt{3}}{3}$. Thus, V_3 and V_4 are $\frac{10\sqrt{3}}{3}$ units above and below the center:

$V_3 = (\frac{10}{3}, \frac{10\sqrt{3}}{3})$, $V_4 = (\frac{10}{3}, -\frac{10\sqrt{3}}{3})$.



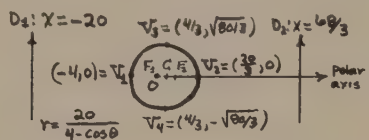
9. $r = \dfrac{20}{4-\cos\theta} = \dfrac{5}{1-\frac{1}{4}\cos\theta}$. Thus $e, = \frac{1}{4}$ and the polar conic is an ellipse. $de = 5$ and $e = \frac{1}{4}$ gives $d = 20$; so the focus $F_1 = (0,0)$ and directrix D_1 is $x = -20$. The polar points $V_1 = (\frac{5}{1+\frac{1}{4}}, 0)$

$= (-4,0)$ and $V_2 = (\frac{5}{1-\frac{1}{4}},0) = (\frac{20}{3},0)$ are the vertices on the polar axis.

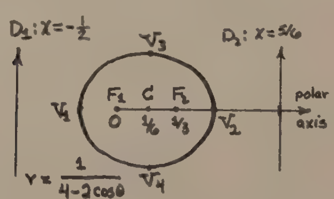
$a = \frac{5}{1-\frac{1}{16}} = \frac{16}{3}$ and $c = (\frac{16}{3})(\frac{1}{4}) = \frac{4}{3}$.

The center of the conic $C = (\frac{4}{3},0)$ and $F_2 = (\frac{8}{3},0)$ in polar form. The directrix D_2 is given by $x = 2c+d = \frac{8}{3} + 20 = \frac{68}{3}$.

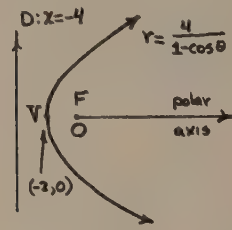
Also, the number $b = \sqrt{a^2-c^2} = \sqrt{\frac{256}{9} - \frac{16}{9}} = \sqrt{\frac{80}{3}}$. $V_3 = (\frac{4}{3},\sqrt{\frac{80}{3}})$ and $V_4 = (\frac{4}{3},-\sqrt{\frac{80}{3}})$.



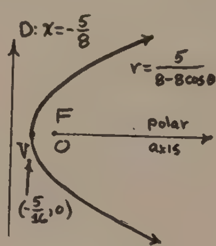
10. $r = \frac{1}{4-2\cos\theta} = \frac{\frac{1}{4}}{1-\frac{1}{2}\cos\theta}$; $e = \frac{1}{2}$, $ed = \frac{1}{4}$, so $d = \frac{1}{2}$. The conic is an ellipse since $e = \frac{1}{2}$. Let $\theta = 0$ and $\theta = \pi$ to find the vertices: $V_1 = (\frac{1}{2},0)$ and $V_2 = (\frac{1}{6},\pi) = (-\frac{1}{6},0)$. Thus, the center is $(\frac{1}{6},0)$. $a = \frac{1}{3}$, $c = (\frac{1}{3})(\frac{1}{2}) = \frac{1}{6}$, and $b = \sqrt{a^2-c^2} = \sqrt{\frac{1}{9} - \frac{1}{36}} = \frac{\sqrt{3}}{6}$. So the foci are: $F_1 = (0,0)$ and $F_2 = (\frac{1}{3},0)$. Also $V_3 = (\frac{1}{6},\frac{\sqrt{3}}{6})$, $V_4 = (\frac{1}{6},-\frac{\sqrt{3}}{6})$. The directrices are $x = -\frac{1}{2}$ and $x = 2c+d = \frac{5}{6}$.



11. $r = \frac{4}{1-\cos\theta}$; $e = 1$, $de = 4$, so $d = 4$. Since $e = 1$, the graph is a parabola. The vertex is $(2,\pi) = (-2,0)$, the directrix is $x = -4$, and the focus is $(0,0)$.

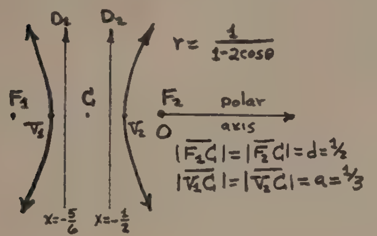


12. $r = \frac{5}{8-8\cos\theta} = \frac{5/8}{1-\cos\theta}$; $e = 1$, $ed = \frac{5}{8}$; so $d = \frac{5}{8}$. The conic is a parabola with focus at $(0,0)$, directrix $x = -\frac{5}{8}$ and vertex $V = (-\frac{5}{16},0)$. The y intercepts are $(\frac{5}{8},\frac{\pi}{2})$ and $(\frac{5}{8},\frac{3\pi}{2})$.

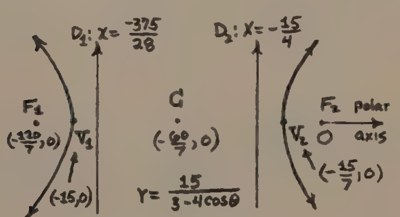


13. $r = \frac{1}{1-2\cos\theta}$; $e = 2$, so the conic is a hyperbola with $F_2 = (0,0)$ as a focus. Directrix D_2 is $x = -\frac{1}{2}$ since $de = 1$ and $e = 2$. Polar points $V_1 = (-1,0)$ and $V_2 = (\frac{1}{3},\pi) = (-\frac{1}{3},0)$ are the vertices on the polar axis. $a = \frac{1}{4-1} = \frac{1}{3}$ and $c = (\frac{1}{3})(2) = \frac{2}{3}$. The polar point $(-\frac{2}{3},0)$ is the center of the conic. The other focus is $F_1 = (-\frac{4}{3},0)$ and directrix D_1 is $x = -\frac{5}{6}$ because F_1 is $2c = \frac{4}{3}$ units

to the left of $F_2 = (0,0)$ and D_1 is $d = \frac{1}{2}$ units to the right of F_1.

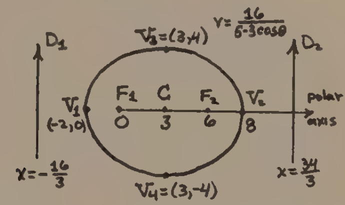


14. $r = \dfrac{15}{3-4\cos\theta} = \dfrac{15/3}{1 - \frac{4}{3}\cos\theta}$; $e = \frac{4}{3}$, $de = \frac{15}{3}$, so $d = \frac{15}{4}$. The conic is a hyperbola since $e = \frac{4}{3}$. The vertices are $V_1 = (-15,0)$ and $V_2 = (-\frac{15}{7},0)$ and the center is $(-\frac{60}{7},0)$. $a = \frac{45}{7}$, $c = ae = (\frac{45}{7})(\frac{4}{3})$, so $c = \frac{60}{7}$. The foci are $F_1 = (-\frac{120}{7},0)$ and $F_2 = (0,0)$. The directrices are $x = -\frac{15}{4}$ and $x = -\frac{120}{7} + \frac{15}{4} = -\frac{375}{28}$; two more points are $(\frac{15}{3}, \frac{\pi}{2})$ and $(\frac{15}{3}, -\frac{3\pi}{2})$.



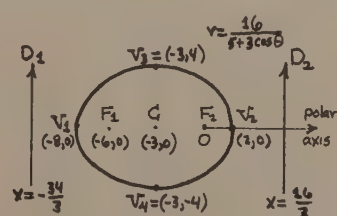
15. $r = \dfrac{16/5}{1 - \frac{3}{5}\cos\theta}$; $e = \frac{3}{5}$. The conic is an ellipse. $de = \frac{16}{5}$ and $d = \frac{16}{3}$. Directrix D_1 is $x = -\frac{16}{3}$. $a = \dfrac{16/5}{1 - \frac{9}{25}} = 5$, $c = (5)(\frac{3}{5}) = 3$, and $b = \sqrt{a^2-c^2} = \sqrt{25-9} = 4$. $F_2 = (2c,0) = (6,0)$ and $F_1 = (0,0)$. Center is $(3,0)$. The directrix $D_2 = 2c+d = \frac{34}{3}$. $V_1 = (2,\pi) = (-2,0)$, $V_2 = (8,0)$, $V_3 = (3,4)$, and $V_4 = (3,-4)$.

Note: V_3 and V_4 are 4 units above and below the center.



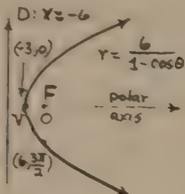
16. $r = \dfrac{16}{5+3\cos\theta} = \dfrac{16/5}{1 + \frac{3}{5}\cos\theta}$;

$e = \frac{3}{5}$, so the conic is an ellipse. $ed = \frac{16}{5}$, so $d = \frac{16}{3}$; $a = \dfrac{ed}{1-e^2} = \dfrac{16/5}{1 - \frac{9}{25}} = 5$ and $c = ae = 5(\frac{3}{5}) = 3$. So $b = \sqrt{a^2-c^2} = 4$. One focus is $(0,0)$, the other is $(-2c,0) = (-6,0)$. Center is $(-3,0)$. One directrix is $x = \frac{16}{3}$ and the other is $x = -(2c+d) = -(6 + \frac{16}{3}) = -\frac{34}{3}$. The vertices are $V_1 = (-8,0)$, $V_2 = (2,0)$, $V_3 = (-3,4)$, and $V_4 = (-3,-4)$.



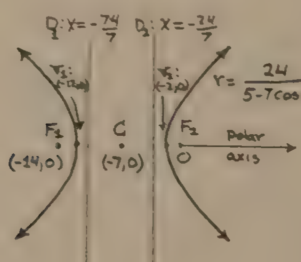
17. $e = 1$. The conic is a parabola. The directrix is perpendicular to the polar axis and $d = 6$ units to the left of the pole: $x = -6$. The focus is at the pole. The vertex is $(-\frac{d}{2},0) =$

(-3,0) in Cartesian coordinates or $(3,\pi)$ in the polar coordinates.

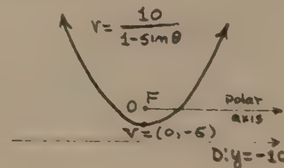


18. $r = \dfrac{24/5}{1 - \frac{7}{5}\cos\theta} = \dfrac{7/5(24/7)}{1 - \frac{7}{5}\cos\theta}$;

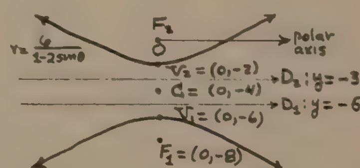
$e = \frac{7}{5}$. The conic is a hyperbola. One directrix is perpendicular to the polar axis and $d = \frac{24}{7}$ units to the left of the pole: $x = -\frac{24}{7}$. Here we use $a = \frac{ed}{e^2-1} = \frac{24/5}{\frac{49}{25}-1} = 5$, and $c = ae = 7$. Hence, the second directrix is $2c-d = 14 - \frac{24}{7} = \frac{74}{7}$ units to the left of the pole: $x = -\frac{74}{7}$. One focus is at the pole. The second focus is $2c = 14$ units to the left of the pole. The vertices have polar coordinates $(c-a,\pi) = (2,\pi)$ or $(-2,0)$ in Cartesian coordinates; $(c+a,\pi) = (12,\pi)$ or $(-12,0)$ in Cartesian coordinates.



19. $e = 1$. The conic is a parabola. The directrix is $d = 10$ units below the polar axis and parallel to the polar axis: $y = -10$. The focus is at the pole. The vertex is $(\frac{d}{2},\frac{3\pi}{2}) = (5,\frac{3\pi}{2})$ in polar coordinates, or $(0,-5)$ in Cartesian coordinates.

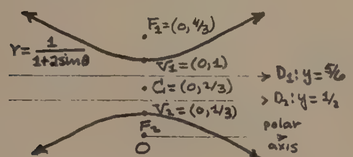


20. $r = \dfrac{2\cdot 3}{1-2\sin\theta}$. $e = 2$. The conic is a hyperbola. One directrix is $d = 3$ units below the polar axis and parallel to the polar axis: $y = -3$. Here $a = \frac{ed}{e^2-1} = \frac{6}{3} = 2$, and $c = 2\cdot 2 = 4$. Hence, the second directrix is $2c-d = 5$ units below the pole: $y = -5$. One focus is at the pole. The second focus is $2c = 8$ units below the pole and on the ray $\theta = \frac{3\pi}{2}$. The vertices have polar coordinates $(c-a,\frac{3\pi}{2}) = (2,\frac{3\pi}{2})$ and $(-(c+a),-\frac{\pi}{2}) = (-6,\frac{\pi}{2})$.



21. $e = 2$. The conic is a hyperbola. Here we have $d = \frac{1}{2}$. $a = \frac{ed}{e^2-1} = \frac{1}{3}$ and $c = ae = \frac{2}{3}$. One directrix is parallel to the polar axis and $d = \frac{1}{2}$ unit above the pole, while the second directrix is parallel to the polar axis and $2c-d = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$ units

above the pole. Their equations are $y = \frac{1}{2}$ and $y = \frac{5}{6}$. Once focus is at the pole and the other focus is $2c = \frac{4}{3}$ units above the pole on the ray $\theta = \frac{\pi}{2}$. Thus, $F_2 = (0,0)$ and $F_1 = (0,\frac{4}{3})$. The vertices have polar coordinates $(c-a,\frac{\pi}{2}) = (\frac{1}{3},\frac{\pi}{2})$ or $(0,\frac{1}{3})$ in Cartesian coordinates; and $(-(c+a), \frac{3\pi}{2}) = (-1,\frac{3\pi}{2})$, or $(0,1)$ in Cartesian coordinates.

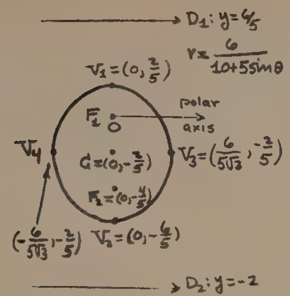


22. $r = \dfrac{3/5}{1+\frac{1}{2}\sin\theta} = \dfrac{1/2(6/5)}{1+\frac{1}{2}\sin\theta}$. $e = \frac{1}{2}$. The conic is an ellipse. One directrix is $d = \frac{6}{5}$ units above the pole and parallel to the polar axis. Here we have

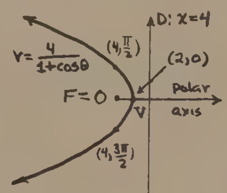
$a = \dfrac{ed}{1-e^2} = \dfrac{3/5}{1-\frac{1}{4}} = \frac{4}{5}$, $b = \dfrac{ed}{\sqrt{1-e^2}} = \dfrac{3/5}{\sqrt{1-\frac{1}{4}}} = \dfrac{6}{5\sqrt{3}}$,

$c = ae = \frac{2}{5}$. The second directrix is $2c+d = 2$ units below the pole and parallel to the polar axis. One focus is at the pole: $F_1 = (0,0)$. The second focus is $2c = \frac{4}{5}$ units below the pole on the ray $\theta = \frac{3\pi}{2}$; that is, $F_2 = (0,-\frac{4}{5})$. The center is $(0,-\frac{2}{5})$. The vertices in Cartesian coordinates are $V_1 = (0,a-c) = (0,\frac{2}{5})$; $V_2 = (0,-c-a) = (0,-\frac{6}{5})$; $V_3 = (b,-c) = (\frac{6}{5\sqrt{3}},-\frac{2}{5})$; $V_4 = (-b,-c) = (-\frac{6}{5\sqrt{3}},-\frac{2}{5})$.

Note: $\frac{6}{5\sqrt{3}} \approx 0.69$.



23. $e = 1$. The conic is a parabola. The directrix is perpendicular to the polar axis and $d = 4$ units to the right of the pole: $x = 4$. The focus is at the pole. The vertex is $(\frac{d}{2},0) = (2,0)$ in polar coordinates and in Cartesian coordinates.

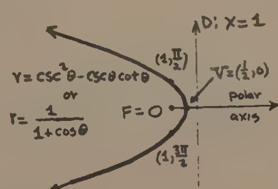


24. $r = \csc^2\theta - \csc\theta\cot\theta =$

$\dfrac{1}{\sin^2\theta} - (\dfrac{1}{\sin\theta})(\dfrac{\cos\theta}{\sin\theta}) = \dfrac{1-\cos\theta}{\sin^2\theta} = \dfrac{1-\cos\theta}{1-\cos^2\theta}$

$= \dfrac{1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)} = \dfrac{1}{1+\cos\theta}$.

$e = 1$. The conic is a parabola. The directrix is perpendicular to the polar axis and $d = 1$ unit to the right of the pole: $x = 1$. The focus is at the pole. The vertex is $(\frac{d}{2},0) = (\frac{1}{2},0)$ in polar and in Cartesian coordinates.



25. $e = \frac{1}{3}$ and $d = 3$. The conic is an ellipse with focus at (0,0) and has an equation of the form $r = \frac{de}{1-e\cos\theta}$. Thus,
$$r = \frac{1}{1 - \frac{1}{3}\cos\theta} = \frac{3}{3-\cos\theta}.$$

26. $e = \frac{1}{4}$ and $d = 3$. The conic is an ellipse with focus at (0,0) and an equation of the form $r = \frac{de}{1+e\cos\theta} = \frac{3/4}{1+\frac{1}{4}\cos\theta}$ or
$$r = \frac{3}{4+\cos\theta}.$$

27. $e = 1$. The conic is a parabola opening to the left with directrix $x = 3$ and focus at (0,0). The equation is of the form $r = \frac{ed}{1+e\cos\theta} = \frac{3}{1+\cos\theta}$.

28. $e = 1$. The conic is a parabola opeming upward with directrix $y = -3$ and focus at (0,0) and with an equation of the form
$$r = \frac{ed}{1-e\sin\theta} = \frac{3}{1-\sin\theta}.$$

29. $e = 3$. The conic is a hyperbola. The vertical directrix to the left of the focus (0,0) requires an equation of the form $r = \frac{ed}{1-e\cos\theta}$ with $d = 2$. Thus, $r = \frac{6}{1-3\cos\theta}$ is its equation.

30. $e = 2.5 = \frac{5}{2}$. The conic is a hyperbola. The horizontal directrix 4 units above the focus (0,0) requires an equation of the form $r = \frac{ed}{1+e\sin\theta}$ with $d = 4$. Thus,
$$r = \frac{(5/2)4}{1+\frac{5}{2}\sin\theta} = \frac{20}{2+5\sin\theta}.$$

31. $e = \frac{1}{3}$. The conic is an ellipse. The horizontal directrix 5 units above $F =$ (0,0) requires an equation of the form $r = \frac{ed}{1+e\sin\theta}$ with $d = 5$. Thus,
$$r = \frac{(1/3)(5)}{1+\frac{1}{3}\sin\theta} = \frac{5}{3+\sin\theta}.$$

32. The two vertical directrices on either side of the focus at (0,0) indicates an ellipse whose equation is of the form $r = \frac{ed}{1+e\cos\theta}$ with $d = 4$. The other directrix is $x = -6 = -(2c+4)$, so $c = 1$. If $\theta = 0$, $r = a-c$, so $a-1 = \frac{4e}{1+e}$. If $\theta = \pi$, $r = a+c$, so $a+1 = \frac{4e}{1-e}$. Subtrac[ting] these equations, we obtain $2 = -\frac{4e}{1+e} + \frac{4e}{1-e}$[.] Solving for e, we obtain $10e^2 = 2$, so $e = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$. Thus, the equation is
$$r = \frac{(\sqrt{5}/5)(4)}{1+\frac{\sqrt{5}}{5}\cos\theta} \text{ or } r = \frac{4\sqrt{5}}{5+\sqrt{5}\cos\theta}.$$

33. $r = ed(1-e\cos\theta)^{-1}$.
$$\frac{dr}{d\theta} = -e^2d(1-e\cos\theta)^{-2}\sin\theta = \frac{-e^2d\sin\theta}{(1-e\cos\theta)^2}$$
$\frac{dr}{d\theta} = 0$ if $\sin\theta = 0$; that is, $\theta = 0$ or $\theta = \pi$. If $-\frac{\pi}{2} < \theta < 0$, then $\frac{dr}{d\theta} > 0$. If $0 < \theta < \pi$, then $\frac{dr}{d\theta} < 0$. Thus, $\theta = 0$ gives a maximum value of $r = \frac{ed}{1-e}$. Similarly if $\pi < \theta < 2\pi$, $\frac{dr}{d\theta} > 0$, so $\theta = \pi$ gives a minimum value for $r = \frac{ed}{1+e}$.

34. From page 587, we have that $a = \frac{1}{2}\left(\frac{ed}{1+e} + \frac{ed}{1-e}\right)$. Simplifying, we have
$$a = \frac{1}{2}\left(\frac{ed(1-e)+ed(1+e)}{1-e^2}\right) = \frac{1}{2}\left(\frac{2ed}{1-e^2}\right) = \frac{ed}{1-e^2}.$$
Also, we have that $c = \frac{ed}{1-e^2} - \frac{ed}{1+e}$. Thus, simplifying we have $c = \frac{ed-ed(1-e)}{1-e^2}$
$$= \frac{ed(e)}{1-e^2} = ae.$$

35. Since $a = \frac{ed}{1-e^2}$, we have $a(1-e^2) = ed$ or $a = ae^2+ed = (ae)e+ed = ce + ed$.

36. $\frac{b^2}{c} = \frac{a^2-c^2}{c} = \frac{a^2(1 - \frac{c^2}{a^2})}{c} = \frac{a^2(1-e^2)}{ae}$

$= \frac{a(1-e^2)}{e} = \frac{ed}{1-e^2}\,\frac{(1-e^2)}{e} = d.$

37. $c = ae$, so that $e = \frac{c}{a} = \frac{2c}{2a} = \frac{|\overline{F_1F_2}|}{|\overline{V_1V_2}|}$.

38. $x = -d$ and $x = 2c + d$ are the equations of the directrices. The distance between the two directrices will be $2c+d-(-d) = 2c + 2d = 2(c+d) = 2(c + \frac{b^2}{c})$ by Problem 36. Thus, $2c + 2d = \frac{2(c^2+b^2)}{c} = \frac{2a^2}{c}$.

39. $\sqrt{1-(\frac{b}{a})^2} = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{c^2}{a^2}} = \sqrt{e^2} = e$ since $e > 1$.

40. By Problem 35, $ec + ed = a$. Since $a = \frac{ed}{1-e^2}$, $e(c+d) = \frac{ed}{1-e^2}$ or $c+d = \frac{d}{1-e^2}$. Solving for e, we have $(c+d)(1-e^2) = d$; $e^2 = \frac{c}{c+d}$, so that $e = \sqrt{\frac{c}{c+d}}$.

41. Let C be the center of the hyperbola. Then $c = |\overline{FC}| = |\overline{CV_2}| + |\overline{V_2F}| = a + \frac{ed}{1-e\cos\pi}$ $= a + \frac{ed}{1+e}$. Thus, $c = \frac{ed}{e^2-1} + \frac{ed}{1+e} =$ $\frac{ed-(1-e)ed}{e^2-1} = \frac{e^2d}{e^2-1} = e(\frac{ed}{e^2-1}) = ae.$

42. $a = \frac{ed}{e^2-1}$. Solving for d, we obtain $\frac{a(e^2-1)}{e} = d$. Thus, $d = a(e - \frac{1}{e}) =$ $ae - \frac{(a)(a)}{(e)(a)} = c - \frac{a^2}{c}$, since $c = ae$.

43. $a = \frac{ed}{e^2-1}$, so $ae^2-a = ed$. Since $c = ae$, we write $ce-a = ed$, so that $a = ce-ed$.

44. The equation of the directrices are $x = -d$ and $x = -2c+d$. The distance between them is given by $-d-(-2c+d) = 2c-2d$. Problem 42 says $d = c - \frac{a^2}{c^2}$. Thus, $2c-2d = 2c-2(c - \frac{a^2}{c}) =$ $\frac{2c^2-2c^2+2a^2}{c} = \frac{2a^2}{c}$.

45. $c = ae$ or $e = \frac{c}{a} = \frac{2c}{2a} = \frac{|\overline{F_1F_2}|}{|\overline{V_1V_2}|}$.

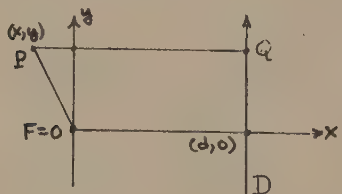
46. $a = ce - ed$ by Problem 43. Solving for e, we have $e = \frac{a}{c-d} = \frac{\frac{ed}{e^2-1}}{c-d}$, so that $1 = \frac{d}{(e^2-1)(c-d)}$. Solving for e, we get $e^2-1 = \frac{d}{c-d}$ or $e^2 = 1 + \frac{d}{c-d} = \frac{c}{c-d}$. Thus, $e = \sqrt{\frac{c}{c-d}}$.

47. Place the sun at the focus F_1 of an ellipse with vertices V_1 and V_2 on the polar axis. The ratio $\frac{29}{30} \approx \frac{|\overline{F_1V_1}|}{|\overline{F_1V_2}|}$ or $|\overline{F_1V_1}| \approx \frac{29}{30}|\overline{F_1V_2}|$. Using the standard notation, we write $|\overline{F_1V_1}| = a-c$, $|\overline{F_1V_2}| = c+a$; since $c = ae$, we have $a-ae \approx \frac{29}{30}(ae+a)$ which we can solve for e to obtain $a \approx 59ae$ or $e \approx \frac{1}{59} \approx 0.017$.

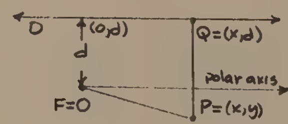
48. $c = ae$. Now $c = \frac{a}{2}$ since F is midway between the center and the vertex. Therefore, $\frac{a}{2} = ae$ or $e = \frac{1}{2}$.

49. Let $P = (r,\theta)$ be an arbitrary point in the plane, and let Q be the point at the foot of the perpendicular from P to the directrix D. Switching for a moment to Cartesian coordinates, so that $P = (x,y)$,

$Q = (d,y)$, $x = r\cos\theta$, and $y = r\sin\theta$, we see that $|\overline{PQ}| = |d-x| = |d-r\cos\theta|$ and $|\overline{PF}| = \sqrt{x^2+y^2} = |r|$. By definition, P belongs to the conic if and only if $\frac{|\overline{PF}|}{|\overline{PQ}|} = e$, that is, $\frac{|r|}{|d-r\cos\theta|} = e$. Thus, $\frac{\pm r}{d-r\cos\theta} = e$. If $P = (r_1,\theta_1)$ satisfies the equation $\frac{-r}{d-r\cos\theta} = e$, then $P = (-r_1, \theta_1+\pi)$ also satisfies the equation $\frac{r}{d-r\cos\theta} = e$. Hence, we lose no points on the conic by writing its equation as $\frac{r}{d-r\cos\theta} = e$. Now solving for r, we obtain $r = \frac{ed}{1+e\cos\theta}$.



50. We will show that if the directrix is d units above the polar axis and parallel to the polar axis, with focus at the pole, then the equation of the conic is $r = \frac{ed}{1+e\sin\theta}$. If the directrix is below the polar axis, a similar argument will yield the equation $r = \frac{ed}{1-e\sin\theta}$. Let $P = (r,\theta)$ be an arbitrary point in the plane, and let Q be the point at the foot of the perpendicular from P to the directrix D. Writing $P = (x,y)$, $Q = (x,d)$, $x = r\cos\theta$, and $y = r\sin\theta$, we see that $|\overline{PQ}| = |d-y| = |d-r\sin\theta|$ and $|\overline{PF}| = \sqrt{x^2+y^2} = |r|$. By definition, P belongs to the conic if and only if $\frac{|\overline{PF}|}{|\overline{PQ}|} = e$, that is, $\frac{|r|}{|d-r\sin\theta|} = e$. Thus, $\frac{\pm r}{d-r\sin\theta} = e$. If (r_1,θ_1) satisfies the equation $\frac{-r}{d-r\sin\theta} = e$, then $P = (-r_1, \theta_1+\pi)$ also satisfies $\frac{r}{d-r\sin\theta} = e$. Hence, we lose no points on the conic by writing its equation as $\frac{r}{d-r\sin\theta} = e$. Now solving for r, we obtain $r = \frac{ed}{1+e\sin\theta}$.



51. For any ellipse, $b^2 = a^2 - c^2 = a^2 - a^2e^2$, so that $\frac{b^2}{a^2} = 1 - e^2$. As $e \to 0$, $\frac{b^2}{a^2} \to 1$ and $b \to a$. Hence, the ellipse becomes more circular. Furthermore, by Problem 38, the distance from the center to the directrix would be $\frac{1}{2}\left(\frac{2a^2}{c}\right) = \frac{a^2}{c}$ and is a constant value K. Therefore, $K = \frac{a^2}{ae}$ or $eK = a$. As $e \to 0$, a gets smaller. So the ellipse gets smaller as $e \to 0$, and eventually shrinks to a point, as it becomes more circular.

52. For any hyperbola $b^2 = c^2-a^2 = a^2e^2-a^2$ or $\frac{b^2}{a^2} = e^2-1$. Therefore, the slope of an asymptote which is $\frac{b}{a}$ gets large as $e \to \infty$. Just as in Problem 51, $a = Ke$ where K is a constant. So as $e \to \infty$ so does a. Thus, as e gets large the asymptotes "open up" and the vertices move away from the origin.

53. (1) Since the hyperbola does not pass through the pole, O is not a point of intersection.

(2) We solve the simultaneous equations

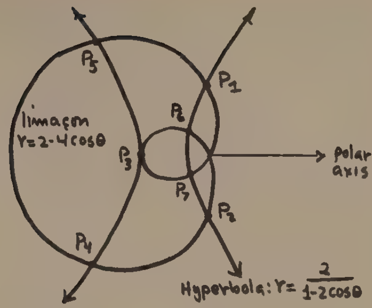
$$\begin{cases} r = \dfrac{2}{1-2\cos\theta} \\ r = 2-\cos(\theta + 2n\pi), \end{cases}$$

where $r \neq 0$ and n is an integer. Noting that $\cos(\theta + 2n\pi) = \cos\theta$, we rewrite the above equations as $\frac{2}{1-2\cos\theta} = 2-4\cos\theta = r$. Thus, $2 = (1-2\cos\theta)(2-4\cos\theta)$, $2 = 2(1-2\cos\theta)^2$, $1-2\cos\theta = \pm 1$, $\cos\theta = \frac{1\pm 1}{2}$, $\cos\theta = 0$ or $\cos\theta = 1$. For $\cos\theta = 0$, we have $r = 2-4\cos\theta = 2$. Hence, we obtain the two points $P_1 = (2, \frac{\pi}{2})$ and $P_2 = (2, -\frac{\pi}{2})$ corresponding to $\cos\theta = 0$. Similarly, for $\cos\theta = 1$, we have $r = 2-4\cos\theta = -2$ and we obtain the point $P_3 = (-2,0) = (2,\pi)$.

(3) We solve the simultaneous equations

$$\begin{cases} r = \dfrac{2}{1-2\cos\theta} \\ r = -\left[2-4\cos(\theta+2n\pi+\pi)\right], \\ r \neq 0,\ n \text{ an integer.} \end{cases}$$

Noting that $\cos(\theta + 2n\pi + \pi) = \cos(\theta+\pi) = -\cos\theta$, we rewrite the above equations as $\frac{2}{1-2\cos\theta} = -(2+4\cos\theta) = r$. Thus, $2 = -(1-2\cos\theta)(2+4\cos\theta) = -2(1-4\cos^2\theta)$, so that $-1 = 1-4\cos^2\theta$, or $\cos^2\theta = \frac{1}{2}$. Therefore, $\cos\theta = \frac{\sqrt{2}}{2}$ or $\cos\theta = -\frac{\sqrt{2}}{2}$. For $\cos\theta = \frac{\sqrt{2}}{2}$, we have $r = -(2+4\cos\theta) = -(2+2\sqrt{2})$; hence, we obtain the two points $P_4 = (-2-2\sqrt{2}, \frac{\pi}{4})$ and $P_5 = (-2-2\sqrt{2}, \frac{5\pi}{4})$ corresponding to $\cos\theta = \frac{\sqrt{2}}{2}$. Similarly, for $\cos\theta = -\frac{\sqrt{2}}{2}$, we have $r = -(2+4\cos\theta) = -2+2\sqrt{2}$ and we obtain the two additional points $P_6 = (-2+2\sqrt{2}, \frac{3\pi}{4})$ and $P_7 = (-2+2\sqrt{2}, \frac{5\pi}{4})$.



54. The left vertex of the ellipse $r = \frac{ed}{1-e\cos\theta}$ has polar coordinates $(\frac{ed}{1+d}, \pi)$ and the intersection of the inner loop of the limaçon $r = a-b\cos\theta$ with the extension of the polar axis to the left of the pole is $(a-b,0)$ or $(b-a,\pi)$. Thus, for $0 < a < b$, the left vertex of the ellipse lies on the inner loop of the limaçon if and only if $\frac{ed}{1+d} = b-a$.

55. The distance from the center (the origin) to either directrix is $c+d$ units. By Problem 36, $d = \frac{b^2}{c}$ and, from the fact that $a^2+b^2 = c^2$, we have $c+d = c + \frac{b^2}{c} = \frac{c^2+b^2}{c} = \frac{a^2}{\sqrt{a^2-b^2}}$. Thus, the equations of the two directrices are $x = \pm \frac{a^2}{\sqrt{a^2-b^2}}$.

56. In the figure below, $|\overline{PF}| = \sqrt{x^2+y^2}$ and $|\overline{PQ}| = |x+d|$; hence, the equation $\frac{|\overline{PF}|}{|\overline{PQ}|} = e$ of the conic can be written as $\frac{\sqrt{x^2+y^2}}{|x+d|} = e$, or $\sqrt{x^2+y^2} = e|x+d|$. Squaring both sides of the latter equation yields $x^2+y^2 = e^2(x^2+2dx+d^2)$, or $(1-e^2)x^2-2e^2dx+y^2 = e^2d^2$. For $e = 1$, this equation clearly reduces to $y^2-2dx = d^2$, or $y^2-4px = 4p^2$, where

we have substituted d = 2p. Thus, for $e = 1$, the equation becomes $y^2 = 4p(x+p)$. This is case (ii). Now suppose $e \neq 1$ and rewrite the equation as

$$(1-e^2)\left[x^2-2\frac{e^2d}{1-e^2}x\right]+y^2 = e^2d^2$$

Completing the square in the latter equation gives,

$$(1-e^2)\left[x^2-2\frac{e^2d}{1-e^2}x+\frac{e^4d^2}{(1-e^2)^2}\right]+y^2 = (1-e^2)\frac{e^4d^2}{(1-e^2)^2}+e^2d^2,$$ or

$$(1-e^2)\left[x-\frac{e^2d}{1-e^2}\right]^2+y^2 = \frac{e^4d^2}{1-e^2}+e^2d^2.$$

Since $\frac{e^4d^2}{1-e^2}+e^2d^2 = \left[\frac{e^2}{1-e^2}+1\right]e^2d^2 = \left[\frac{e^2+(1-e^2)}{1-e^2}\right]e^2d^2 = \frac{e^2d^2}{1-e^2}$, then the equation of the conic becomes

$$(1-e^2)\left[x-\frac{e^2d}{1-e^2}\right]^2+y^2 = \frac{e^2d^2}{1-e^2}.$$

For case (i), suppose that $0 < e < 1$, so that $1-e^2 > 0$ and the latter equation can be rewritten as

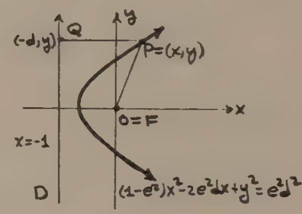
$$\frac{\left[x-\frac{e^2d}{1-e^2}\right]^2}{\frac{e^2d^2}{(1-e^2)^2}}+\frac{y^2}{\frac{e^2d^2}{1-e^2}} = 1, \text{ or } \frac{(x-c)^2}{a^2}+\frac{y^2}{b^2}=1,$$

with $c = \frac{e^2d}{1-e^2}$, $a = \frac{ed}{1-e^2}$, and $b = \frac{ed}{\sqrt{1-e^2}}$.

Note that $a^2-b^2 = \frac{e^2d^2}{(1-e^2)^2}-\frac{e^2d^2}{1-e^2} = e^2d^2\left[\frac{1}{(1-e^2)^2}-\frac{1-e^2}{(1-e^2)^2}\right] = \frac{e^4d^2}{(1-e^2)^2} = c^2$ and $c = \frac{e^2d}{1-e^2} = ea$. Finally, for case (iii), suppose that $1 < e$, so that $e^2-1 > 0$ and the equation can be rewritten as

$$\frac{\left[x+\frac{e^2d}{e^2-1}\right]^2}{\frac{e^2d^2}{(e^2-1)^2}}-\frac{y^2}{\frac{e^2d^2}{e^2-1}} = 1,$$ or

$\frac{(x+c)^2}{a^2}-\frac{y^2}{b^2} = 1$ with $c = \frac{e^2d}{e^2-1}$, $a = \frac{ed}{e^2-1}$, and $b = \frac{ed}{\sqrt{e^2-1}}$. Calculating as above, we find that $a^2+b^2 = c^2$. Again we have $c = \frac{e^2d}{e^2-1} = ea$.

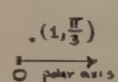


57. The distance from the center (the origin) to either directrix is c-d units. By Problem 42, $c-d = \frac{a^2}{c} = \frac{a^2}{\sqrt{a^2+b^2}}$, so the equations of the two directrices are $x = \pm\frac{a^2}{\sqrt{a^2+b^2}}$.

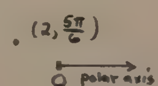
58. By Problem 43, ec-ed = a; hence $e = \frac{a}{c-d} = \frac{a}{\left(\frac{a^2}{c}\right)} = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{1+\left(\frac{b}{a}\right)^2}$.

Review Problem Set, Chapter 9, page 592

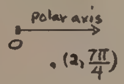
1. (a) $(-1,\frac{4\pi}{3})$. (b) $(1,-\frac{5\pi}{3})$. (c) $(-1,-\frac{2\pi}{3})$.



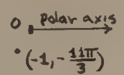
2. (a) $(-2, \frac{11\pi}{6})$. (b) $(2, -\frac{7\pi}{6})$. (c) $(-2, -\frac{\pi}{6})$.



3. (a) $(-2, \frac{3\pi}{4})$.   (b) $(2, -\frac{\pi}{4})$.
   (c) $(-2, -\frac{5\pi}{4})$.

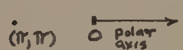


4. (a) $(-1, \frac{\pi}{3})$.   (b) $(1, -\frac{2\pi}{3})$.
   (c) $(+1, -\frac{5\pi}{3})$.



5. (a) $(-3, 0)$.   (b) $(3, -\pi)$.
   (c) $(-3, -2\pi)$.

6. (a) $(-\pi, 0)$.   (b) $(\pi, -\pi)$.
   (c) $(-\pi, -2\pi)$.



7. $(0,0)$.

8. $(17,0)$.

9. $(-17,0)$.

10. $(-\frac{3\sqrt{3}}{2}, -\frac{3}{2})$.

11. $(-\frac{11\sqrt{2}}{2}, \frac{11\sqrt{2}}{2})$.

12. $(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$.

13. $(17, \tan^{-1}0) = (17,0)$.

14. $(\sqrt{4+9}, \tan^{-1}\frac{3}{2}) = (\sqrt{13}, \tan^{-1}\frac{3}{2})$.

15. $(\sqrt{4+12}, \tan^{-1} - \frac{2\sqrt{3}}{2}) = (4, -\frac{\pi}{3})$.

16. $(\sqrt{3+1}, \tan^{-1}\frac{1}{\sqrt{3}} - \pi) = (2, -\frac{5\pi}{6})$.

17. $(\sqrt{289+289}, \tan^{-1}-1+\pi) = (17\sqrt{2}, \frac{3\pi}{4})$.

18. $(1, -\frac{\pi}{2})$.

19. $r^2\cos 2\theta = 1$ can be expressed as $r^2(\cos^2\theta - \sin^2\theta) = 1$ or $r^2\cos^2\theta - r^2\sin^2\theta = 1$, so that $x^2 - y^2 = 1$.

20. $r = \frac{1}{3\cos\theta - 4\sin\theta}$ can be rewritten as $3r\cos\theta - 4r\sin\theta = 1$ or $3x - 4y = 1$.

21. $r^2 = |\sec\theta|$ so that $r^2 = \frac{1}{|\cos\theta|}$ and so $r^2|\cos\theta| = 1$ or $|r(r\cos\theta)| = 1$. Thus, $|\sqrt{x^2+y^2} \cdot x| = 1$ or $x^2(x^2+y^2) = 1$, so that $y^2 = \frac{1-x^4}{x^2}$.

22. $r - er\cos\theta = de$ so that $\sqrt{x^2+y^2} - ex = de$ or $\sqrt{x^2+y^2} = ex+de$. Now, $x^2+y^2 = e^2x^2 + 2e^2dx + d^2e^2$ or $(1-e^2)x^2 - 2e^2dx + y^2 = d^2e^2$.

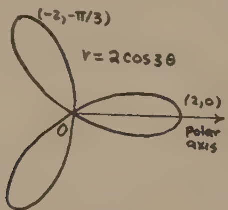
23. $y = 2x-1$ can be rewritten as $r\sin\theta = 2r\cos\theta - 1$ or $r = \frac{1}{2\cos\theta - \sin\theta}$.

24. $(x-1)^2 + (y-3)^2 = 4$ is equivalent to $x^2 - 2x + 1 + y^2 - 6y + 9 = 4$, so that $r^2 - 2r\cos\theta - 6r\sin\theta + 6 = 0$ or $r^2 - 2r(\cos\theta + 3\sin\theta) + 6 = 0$.

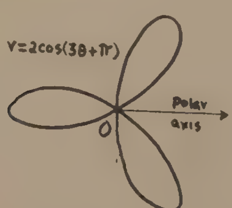
25. $y^2 = 4x$ can be converted to $r^2\sin^2\theta = 4r\cos\theta$ so that $r\sin^2\theta = 4\cos\theta$, and so $r = 4\cot\theta\csc\theta$. (We do not lose $r = 0$, since $\theta = \frac{\pi}{2}$ will yield $r = 0$)

26. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be converted to $\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$ or $r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$, so that $r = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$.

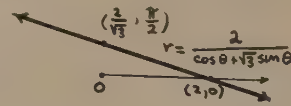
27. We apply the tests given in Section 9.2: $r = 2\cos 3(-\theta) = 2\cos 3\theta$, so there is symmetry about the polar axis. $r = 2\cos 3(\pi-\theta) = 2(\cos 3\pi \cos 3\theta + \sin 3\pi \sin 3\theta) = -2\cos 3\theta$, which is not equivalent to the given equation; $-r = 2\cos 3(-\theta)$ or $-r = 2\cos 3\theta$, which is not equivalent to the given equation. Now $r = 2\cos 3(\theta+\pi) = -2\cos 3\theta$ is not equivalent to the original, and neither is $-r = 2\cos 3\theta$. The graph is a three-leaved rose.



28. $r = 2\cos(3\theta+\pi) = 2\left[\cos 3\theta \cos\pi - \sin 3\theta \sin\pi\right] = -2\cos 3\theta$. Now we apply the symmetry tests of Section 9.2 to $r = -2\cos 3\theta$. First, $r = -2\cos(-3\theta) = -2\cos 3\theta$ is equivalent to $r = -2\cos 3\theta$, so there is symmetry about the polar axis. Now $r = -2\cos 3(\pi-\theta) = 2\cos 3\theta$ is not equivalent to $r = -2\cos 3\theta$; $-r = -2\cos 3(-\theta) = 2\cos 3\theta$ is not equivalent to $r = -2\cos 3\theta$. Now $r = -2\cos(3\theta+3\pi) = 2\cos 3\theta$, which is not equivalent to the given equation ; $-r = -2\cos 3\theta$ is not either.



29. This is a straight line whose Cartesian equation is $x + \sqrt{3}\cdot y - 2 = 0$ and it does not go through O, so there is no symmetry.



30. Replace θ by $-\theta$: $r = \frac{1}{2} + \sin(-\theta)$

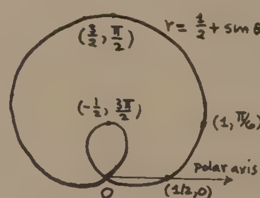
$= \frac{1}{2} - \sin\theta$.

Replace θ by $\pi - \theta$ and r by $-r$: $-r = \frac{1}{2} + \sin(\pi-\theta) = \frac{1}{2} + \sin\theta$.

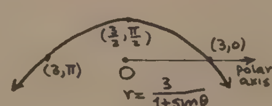
Replace θ by $\pi - \theta$: $r = \frac{1}{2} + \sin(\pi-\theta) = \frac{1}{2} + \sin\theta$, so there is symmetry about the line $\theta = \frac{\pi}{2}$.

Replace θ by $\theta+\pi$: $r = \frac{1}{2} + \sin(\theta+\pi) = \frac{1}{2} - \sin\theta$.

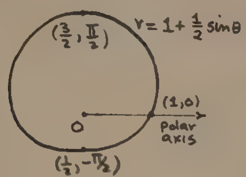
Replace r by $-r$: $-r = \frac{1}{2} + \sin\theta$.



31. This is the equation of a parabola with directrix parallel to the polar axis and above the polar axis, with focus at O. Hence, it is symmetric about the line $\theta = \frac{\pi}{2}$.



32. This is a limaçon which is symmetric about $\theta = \frac{\pi}{2}$.



33. $\tan\alpha = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} =$

$$\frac{\frac{5\sin\theta + 3\cos\theta}{(5\cos\theta - 2\sin\theta)^2}\cdot(\sin\theta) + r\cos\theta}{\frac{5\sin\theta + 3\cos\theta}{(5\cos\theta - 3\sin\theta)^2}\cdot(\cos\theta) - r\sin\theta}.$$

When $r = \frac{1}{5}$ and $\theta = 0$, then $\tan\alpha =$

$$\frac{\frac{3}{25}(0) + \frac{1}{5}(1)}{\frac{3}{25}(1) - 0} = \frac{5}{3}.$$

34. Since $(0, \frac{\pi}{k})$ is just O, then $\tan\alpha = \tan\theta$. Thus, $\tan\theta = \tan\frac{\pi}{k}$.

35. $\tan\alpha = \dfrac{(3\sin\theta)\sin\theta + r\cos\theta}{(3\sin\theta)\cos\theta - r\sin\theta}$.

For $r = 2$ and $\theta = \frac{\pi}{2}$, $\tan\alpha = \frac{3}{-2} = -\frac{3}{2}$.

36. $\tan\alpha = \dfrac{\frac{9\sin 2\theta}{r}(\sin\theta) + r\cos\theta}{\frac{9\sin 2\theta}{r}(\cos\theta) - r\sin\theta}$.

When $\theta = \frac{\pi}{2}$ and $r = 3$, then $\tan\alpha = \frac{0}{-3} = 0$.

37. $\tan\alpha = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} =$

$$\frac{(4\sin\theta)(\sin\theta) + (3-4\cos\theta)\cos\theta}{(4\sin\theta)(\cos\theta) - (3-4\cos\theta)\sin\theta}.$$

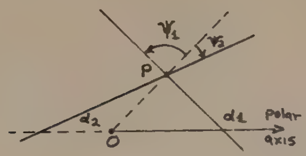
The tangent line is horizontal provided $4\sin^2\theta - 4\cos^2\theta + 3\cos\theta = 0$, that is, $4 - 8\cos^2\theta + 3\cos\theta = 0$ or $8\cos^2\theta - 3\cos\theta - 4 = 0$. So $\cos\theta = \frac{3 \pm \sqrt{137}}{16}$ and $\theta = \cos^{-1}(\frac{3+\sqrt{137}}{16})$ or $\theta = \cos^{-1}(\frac{3-\sqrt{137}}{16})$; or $\theta = 2\pi - \cos^{-1}(\frac{3+\sqrt{137}}{16})$ or $\theta = 2\pi - \cos^{-1}(\frac{3-\sqrt{137}}{16})$. The tangent line is vertical provided $4\sin\theta\cos\theta - 3\sin\theta + 4\sin\theta\cos\theta = 0$, that is, $8\sin\theta\cos\theta - 3\sin\theta = 0$, so that $\sin\theta(8\cos\theta - 3) = 0$. Hence, $\sin\theta = 0$ or $\cos\theta = \frac{3}{8}$. Thus, we have $\theta = 0, \pi$, $\cos^{-1}\frac{3}{8}$ or $2\pi - \cos^{-1}\frac{3}{8}$. Therefore, the points where the tangent to the limaçon are horizontal are $(3 - \frac{3+\sqrt{137}}{4}, \cos^{-1}(\frac{3+\sqrt{137}}{16}))$, $(3 - \frac{3-\sqrt{137}}{4}, \cos^{-1}(\frac{3-\sqrt{137}}{16}))$ and $(3 - \frac{3+\sqrt{137}}{4}, 2\pi - \cos^{-1}(\frac{3+\sqrt{137}}{16}))$ and $(3 - \frac{3-\sqrt{137}}{4}, 2\pi - \cos^{-1}(\frac{3-\sqrt{137}}{16}))$. The points where the tangent is vertical are $(-1,0)$, $(7,\pi)$, $(\frac{3}{2}, \cos^{-1}\frac{3}{8})$, and $(\frac{3}{2}, 2\pi - \cos^{-1}\frac{3}{8})$.

38. $\tan\psi = \dfrac{r}{\frac{dr}{d\theta}}$. Therefore, in Problem 35 we have $\tan\psi = \frac{2-3\cos\theta}{3\sin\theta} = \frac{2}{3}$. In Problem 36, $\tan\psi = \dfrac{r}{\frac{-9\sin 2\theta}{r}} = \dfrac{r^2}{-9\sin 2\theta} = \dfrac{9\cos 2\theta}{-9\sin 2\theta} = -\cot 2\theta$. At $\theta = \frac{\pi}{2}$, $\tan\psi$ is undefined.

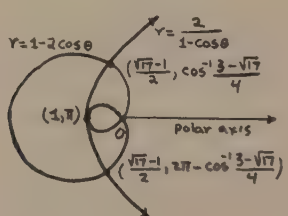
39. $\tan\psi = \dfrac{r}{\frac{dr}{d\theta}} = \dfrac{\sin^3\theta}{3\sin^2\theta\cos\theta} = \frac{1}{3}\tan\theta$.

40. Let ψ_1 be the angle from the radial line to the tangent line to $r = f(\theta)$ at P, and let ψ_2 be the angle from the radial line to the tangent line to $r = g(\theta)$ at P, as shown. Now $\phi = \psi_1 - \psi_2$, and so $\tan\phi = \tan(\psi_1 - \psi_2)$

$$= \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1\tan\psi_2}.$$



41. We solve $r = 1-2\cos(\theta+2n\pi)$ and $r = \frac{2}{1-\cos\theta}$ simultaneously; $1-2\cos(\theta+2n\pi) = \frac{2}{1-\cos\theta}$ becomes $1-2\cos\theta = \frac{2}{1-\cos\theta}$ and yields $2\cos^2\theta-3\cos\theta-1 = 0$. Thus $\cos\theta = \frac{3\pm\sqrt{17}}{4}$. But we cannot use the plus since $\cos\theta$ is not greater than 1. Therefore, $\cos\theta = \frac{3-\sqrt{17}}{4}$. The points of intersection are $(\frac{\sqrt{17}-1}{2}, \cos^{-1}\frac{3-\sqrt{17}}{4})$ and $(\frac{\sqrt{17}-1}{2}, 2\pi-\cos^{-1}\frac{3-\sqrt{17}}{4})$. Now we solve $-r = 1-2\cos(\theta+(2n+1)\pi)$ and $r = \frac{2}{1-\cos\theta}$ simultaneously. Thus, $1+2\cos\theta = \frac{-2}{1-\cos\theta}$, and so $2\cos^2\theta-\cos\theta-3 = 0$ or $(2\cos\theta-3)(\cos\theta+1) = 0$ yields only the solution $\cos\theta = -1$ or $\theta = \pi$. The point of intersection is $(1,\pi)$. This point is represented by $(-1,0)$ on the limaçon.



42. Let $r = 0$. Then $\theta = \frac{\pi}{2}$ for $r_1 = 2\cos\theta$ and $\theta = \frac{\pi}{4}$ for $r_2 = 2\cos2\theta$. So $(0,0)$ is a point of intersection. Now solve $r_1 = 2\cos(\theta+2n\pi)$ and $r_2 = 2\cos2\theta$ simultaneously. Thus,

$$2\cos(\theta+2n\pi) = 2\cos2\theta$$
$$2\cos\theta = 2\cos^2\theta-1$$
$$2\cos^2\theta-\cos\theta-1 = 0$$
$$(2\cos\theta+1)(\cos\theta-1) = 0$$

Now $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$. Hence, $\theta = 0$, 2π or $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. So the points of intersection are $(0,0)$, $(2,0)$, $(-1,\frac{4\pi}{3})$, $(-1,\frac{2\pi}{3})$. Now solve $-r_1 = 2\cos(\theta+(2n+1)\pi)$ and $r_2 = 2\cos2\theta$ simultaneously, so $-2\cos(\theta+\pi) = 2\cos2\theta$ or $2\cos\theta = 2\cos2\theta$ which gives the same solution as above.

43. $A = \frac{1}{2}\int_0^{\pi/4} r^2 d\theta = \frac{1}{2}\int_0^{\pi/4}(1+\cos\theta)^2 d\theta =$

$\frac{1}{2}\int_0^{\pi/4}(1+2\cos\theta+\cos^2\theta)d\theta =$

$\frac{1}{2}\int_0^{\pi/4}(1+2\cos\theta + \frac{1+\cos2\theta}{2})d\theta =$

$\frac{1}{2}\int_0^{\pi/4}(\frac{3}{2} + 2\cos\theta + \frac{\cos2\theta}{2})d\theta =$

$\frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin2\theta}{4}\Big|_0^{\pi/4}\right] =$

$\frac{1}{2}(\frac{3\pi}{8} + \frac{2\sqrt{2}}{2} + \frac{1}{4}) = \frac{3\pi}{16} + \frac{\sqrt{2}}{2} + \frac{1}{8}$ square units.

44. $A = \frac{1}{2}\int_0^{\pi/2}(-\theta)^2 d\theta = \frac{1}{2}(\frac{\theta^3}{3})\Big|_0^{\pi/2} = \frac{\pi^3}{48}$ square units.

45. $A = \frac{1}{2}\int_0^{\pi} 16\sin^2\theta d\theta = 8\int_0^{\pi}\frac{1-\cos2\theta}{2}d\theta$

$= 4\left[\theta - \frac{1}{2}\sin2\theta\right]\Big|_0^{\pi} = 4\pi$ square units.

46. $A = \frac{1}{2}\int_0^{\pi/6}(\frac{1}{\sin\theta + \cos\theta})^2 d\theta =$

$\frac{1}{2}\int_0^{\pi/6}\frac{1}{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta}d\theta$

$$= \frac{1}{2}\int_0^{\pi/6} \frac{1}{1+\sin 2\theta}d\theta = \frac{1}{2}\int_0^{\pi/6} \frac{1-\sin 2\theta}{1-\sin^2 2\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} \frac{1}{\cos^2 2\theta}\,d\theta - \frac{1}{2}\int_0^{\pi/6} \frac{\sin 2\theta}{1-\sin^2 2\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} \sec^2 2\theta d\theta - \frac{1}{2}\int_0^{\pi/6} \frac{\sin 2\theta}{\cos^2 2\theta}\,d\theta$$

$$= \frac{1}{4}\tan 2\theta\Big|_0^{\pi/6} - \frac{1}{2}\int_0^{\pi/6} \tan 2\theta\sec 2\theta d\theta$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{4}\sec 2\theta\Big|_0^{\pi/6} = \frac{\sqrt{3}}{4} - \frac{1}{4}(2-1) =$$

$\frac{\sqrt{3}-1}{4}$ square unit.

47. $A = \frac{1}{2}\int_0^{\pi}(1-\cos\theta)^2 d\theta$

$$= \frac{1}{2}\int_0^{\pi}(1-2\cos\theta + \cos^2\theta)d\theta$$

$$= \frac{1}{2}(\theta - 2\sin\theta)\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}(\pi) + \frac{1}{2}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\pi}$$

$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} \text{ square units.}$$

48. $A = \frac{1}{2}\int_0^{\pi} e^{4\theta}d\theta = \frac{1}{2}\left[\frac{e^{4\theta}}{4}\Big|_0^{\pi}\right] = \frac{e^{4\pi}}{8} - \frac{1}{8} =$

$\frac{1}{8}(e^{4\pi} - 1)$ square units.

49. (a) The intersection points of $r = 4\sin\theta$ and $r = \csc\theta = \frac{1}{\sin\theta}$ (whose Cartesian equation is $y = 1$) are solutions of $\frac{1}{\sin\theta} = 4\sin\theta$. So $\sin\theta = \pm\frac{1}{2}$ or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. The polar points are $(2,\frac{\pi}{6})$ and $(2,\frac{5\pi}{6})$.

(b) $A = 2\left[\frac{1}{2}\int_0^{\pi/6}(4\sin\theta)^2 d\theta\right] +$

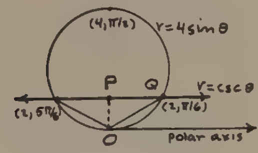
$2\left[\text{area of triangle OPQ}\right]$

$$= 16\int_0^{\pi/6}\sin^2\theta d\theta + 2\left[\frac{(2\cos\pi/6)(1)}{2}\right]$$

$$= 16\int_0^{\pi/6}\left(\frac{1-\cos 2\theta}{2}\right)d\theta + \frac{2\sqrt{3}}{2}$$

$$= 8\left[\theta - \frac{\sin 2\theta}{2}\Big|_0^{\pi/6}\right] + \sqrt{3}$$

$$= 8\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] + \sqrt{3} = \frac{4\pi}{3} - \sqrt{3} \text{ square units.}$$



50. To find points of intersection: First, (0,0) is a point of intersection. Now solve: $6\sin\theta = 2(1-\sin\theta)$, $8\sin\theta = 2$, $\sin\theta = \frac{1}{4}$. Thus, $(\frac{3}{2},\sin^{-1}\frac{1}{4})$ and $(\frac{3}{2},\pi-\sin^{-1}\frac{1}{4})$ are the other points of intersection.

Now $A = 2\left[\frac{1}{2}\int_{\sin^{-1}\frac{1}{4}}^{\pi/2} 36\sin^2\theta\, d\theta -\right.$

$$\left.\left(\frac{1}{2}\int_{\sin^{-1}\frac{1}{4}}^{\pi/2} 4(1-\sin\theta)^2 d\theta\right)\right]$$

$$= \frac{36}{2}\int_{\sin^{-1}\frac{1}{4}}^{\pi/2}(1-\cos 2\theta)d\theta -$$

$$4\int_{\sin^{-1}\frac{1}{4}}^{\pi/2}(1-2\sin\theta + \sin^2\theta)d\theta$$

$$= 18\left(\theta - \frac{\sin 2\theta}{2}\Big|_{\sin^{-1}\frac{1}{4}}^{\pi/2}\right) -$$

$$4\left(\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_{\sin^{-1}\frac{1}{4}}^{\pi/2}\right)$$

$$= 18\left[\frac{\pi}{2} - \sin^{-1}\frac{1}{4} + \sin(\sin^{-1}\frac{1}{4})\cos(\sin^{-1}\frac{1}{4})\right]$$

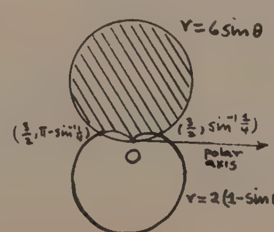
$$- 4\left[\frac{\pi}{2} + \frac{\pi}{4} - (\sin^{-1}\frac{1}{4} + 2\cos(\sin^{-1}\frac{1}{4}) + \right.$$

$$\left.\frac{\sin^{-1}\frac{1}{4}}{2} - \frac{2}{4}\sin(\sin^{-1}\frac{1}{4})\cos^{-1}(\sin^{-1}\frac{1}{4}))\right]$$

$$= 9\pi - 18\sin^{-1}\frac{1}{4} + \frac{18\sqrt{15}}{16} - 2\pi - \pi + 4\sin^{-1}\frac{1}{4}$$

$$+ \frac{8\sqrt{15}}{4} + 2\sin^{-1}\frac{1}{4} - \frac{2\sqrt{15}}{16}$$

$$= 6\pi - 12\sin^{-1}(\tfrac{1}{4}) + 3\sqrt{15} \text{ square units.}$$



51. By Problem 28 in Section 9.3, the area of the inner loop is given by $(2a^2+b^2)(\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{a}{b}) + \frac{9a}{4}\sqrt{b^2-a^2}$. Now

$A = \frac{1}{2}\int_0^{2\pi} (a+b\sin\theta)^2 d\theta - 2$(area of the inner loop)

$= \frac{1}{2}\int_0^{2\pi} (a^2+2ab\sin\theta+b^2\sin^2\theta)d\theta -$

2(area of the inner loop)

$= \frac{1}{2}(a^2\theta - 2ab\cos\theta + \frac{b^2\theta}{2} - \frac{b^2\sin 2\theta}{4})\Big|_0^{2\pi} -$

2(area of the inner loop)

$= \frac{1}{2}(2\pi a^2 - 2ab + b^2\pi + 2ab) - 2$(area of inner loop)

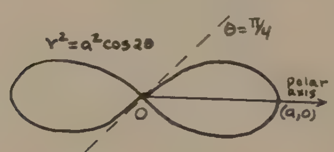
$= \frac{\pi}{2}(2a^2+b^2) - (2a^2+b^2)(\frac{\pi}{2} - \sin^{-1}\frac{a}{b}) -$

$\frac{9}{2}a\sqrt{b^2-a^2}$.

$= (2a^2+b^2)\sin^{-1}\frac{a}{b} - \frac{9}{2}a\sqrt{b^2-a^2}$ square units.

52. $A = \frac{4}{2}\int_0^{\pi/4} \left[a^2\cos 2\theta\right] d\theta$

$= 2a^2(\frac{\sin 2\theta}{2})\Big|_0^{\pi/4} = a^2$ square units.



53. $s = \int_0^{2\pi}\sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$

$= \int_0^{2\pi}\sqrt{9e^{-6\theta} + e^{-6\theta}}\, d\theta$

$= \sqrt{10}\int_0^{2\pi} e^{-3\theta}d\theta = -\frac{\sqrt{10}}{3}e^{-3\theta}\Big|_0^{2\pi}$

$= -\frac{\sqrt{10}}{3}(e^{-6\pi}-1) = \frac{\sqrt{10}}{3}(1-e^{-6\pi})$ units.

54. $s = \int_0^{\pi}\sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$

$= \int_0^{\pi}\sqrt{(5\cos\theta)^2 + (5\sin\theta)^2}\, d\theta$

$= 5\int_0^{\pi} d\theta = 5(\theta)\Big|_0^{\pi} = 5\pi$ units.

55. $s = \int_0^{\pi}\sqrt{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} + \cos^4\frac{\theta}{2}}\, d\theta$

$= \int_0^{\pi}\cos\frac{\theta}{2}\sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}\, d\theta$

$= 2\sin\frac{\theta}{2}\Big|_0^{\pi} = 2$ units.

56. $s = \int_0^{2\pi}\sqrt{(\sin\theta)^2 + (1-\cos\theta)^2}\, d\theta$

$= \int_0^{2\pi}\sqrt{2-2\cos\theta}\, d\theta$

$= \sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2\frac{\theta}{2}}\, d\theta = 2\int_0^{2\pi}\sin\frac{\theta}{2}\, d\theta$

$= -4\cos\frac{\theta}{2}\Big|_0^{2\pi} = 4 + 4 = 8$ units.

57. $e = 1$, so that the conic is a parabola. The directrix is 17 units to the left of the pole and perpendicular to the polar axis; that is, D: $x = -17$.

58. $r = \frac{15}{3+5\sin\theta} = \frac{5}{1+\frac{5}{3}\sin\theta} = \frac{5/3(3)}{1+\frac{5}{3}\sin\theta}$.

$e = \frac{5}{3}$, so that the conic is a hyperbola. The directrix is parallel to the polar axis and 3 units above the pole; that is, D: $y = 3$.

59. $r = \frac{2}{1-\frac{2}{5}\sin\theta} = \frac{2/5(5)}{1-\frac{2}{5}\sin\theta}$.

$e = \frac{2}{5}$, so that the conic is an ellipse. The directrix is parallel to the polar axis and 5 units below the pole; that is, D: $y = -5$.

60. $r = \frac{3}{2}\csc^2\frac{\theta}{2} = \frac{3/2}{\sin^2\frac{\theta}{2}} = \frac{3/2}{\frac{1-\cos\theta}{2}} = \frac{3}{1-\cos\theta}$.

$e = 1$, so that the conic is a parabola. The directrix is 3 units to the left of the pole and perpendicular to the polar axis; that is, D: $x = -3$.

61. $r = \frac{1}{\frac{1}{2}+\sin\theta} = \frac{2}{1+2\sin\theta}$.

$e = 2$ so that the conic is a hyperbola. The directrix is parallel to the polar axis and 1 unit above the pole; that is, D: $y = 1$.

62. $e = 1$, so that the conic is a parabola; its axis is the line $\theta = \frac{\pi}{4}$. The directrix is perpendicular to the line $\theta = \frac{\pi}{4}$ and 1 unit above the focus on the ray $\theta = \frac{\pi}{4}$.

63. $c = 3$, $a = 5$. $c^2 = a^2 - b^2$, $9 = 25 - b^2$, $b^2 = 16$. $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

64. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. So $\frac{16}{a^2} + \frac{9}{b^2} = 1$; $\frac{36}{a^2} + \frac{4}{b^2} = 1$.

Solving $16b^2 + 9a^2 = a^2b^2$ and $36b^2 + 4a^2 = a^2b^2$ simultaneously, we get $a^2 = 4b^2$. So $\frac{16}{4b^2} + \frac{36}{4b^2} = 1$, and so $4b^2 = 52$, $b^2 = 13$. Equation is $\frac{x^2}{52} + \frac{y^2}{13} = 1$.

65. $2a = 16$, $a = 8$. $2b = 8$, $b = 4$. Equation is $\frac{x^2}{16} + \frac{y^2}{64} = 1$.

66. $2a = 20$, $a = 10$. $2b = 12$, $b = 6$. $\frac{x^2}{100} + \frac{y^2}{36} = 1$.

67. Since the center is half-way between the vertices and on the major axis, the center is $(5,0)$. So $a = 5$, $c = 4$ and since $c^2 = a^2-b^2$, $b^2 = 9$. Equation is $\frac{(x-5)^2}{25} + \frac{y^2}{9} = 1$.

68. Center is the midpoint of the line segment joining the foci. So the center is $(6,-2)$. $c = 3$. $2b = 8$ and $b = 4$. Since $c^2 = a^2-b^2$, $9 = a^2-16$, $25 = a^2$.

Equation is $\frac{(x-6)^2}{25} + \frac{(y+2)^2}{16} = 1$.

69. Center: $(0,0)$. Vertices: $(0,2\sqrt{3})$, $(0,-2\sqrt{3})$, $(2\sqrt{2},0)$, $(-2\sqrt{2},0)$

Foci: $(0,2)$, $(0,-2)$. Eccentricity: $e = \frac{c}{a} = \frac{2}{\sqrt{12}} = \frac{\sqrt{12}}{6} = \frac{\sqrt{3}}{3}$.

Directrices: $y = \pm \frac{a^2}{c}$, $y = \frac{12}{2} = 6$ and $y = -6$. (Note that $c^2 = a^2-b^2$, $c^2 = 12-8 = 4$.)

70. $\frac{x^2}{169} + \frac{y^2}{144} = 1$. $c^2 = a^2-b^2$, $c^2 = 169-144$, $c^2 = 25$.

Center: $(0,0)$. Vertices: $(13,0)$, $(-13,0)$ $(0,-12)$, $(0,12)$. Foci: $(5,0)$, $(-5,0)$.

Eccentricity: $e = \frac{5}{13}$.

Directrices: $x = \pm \frac{a^2}{c}$, $x = \frac{169}{5}$ and $x = -\frac{169}{5}$.

71. $9(x^2+2x) + 25(y^2-2y) = 191$,

$9(x+1)^2 + 25(y-1)^2 = 191 + 9 + 25 = 225$.

$\frac{(x+1)^2}{25} + \frac{(y-1)^2}{9} = 1$. Center: $(-1,1)$.

Vertices: $(4,1)$, $(-6,1)$, $(1,4)$, $(-1,-2)$.

$c^2 = a^2-b^2$, $c^2 = 25-9 = 16$, $c = 4$.

Foci: $(3,1)$, $(-5,1)$.

Eccentricity: $e = \frac{c}{a} = \frac{4}{5}$.

Directrices: $x = -1 \pm \frac{a^2}{c} = -1 \pm \frac{25}{4}$, $x = \frac{21}{4}$ and $x = -\frac{29}{4}$.

72. $3(x^2-\frac{28}{3}x) + 4(y^2-4y) + 48 = 0$,

$3(x-\frac{28}{6})^2 + 4(y-2)^2 = -48 + \frac{196}{3} + 16$,

$3(x - \frac{14}{3})^2 + 4(y-2)^2 = \frac{100}{3}$,

$$\frac{(x - \frac{14}{3})^2}{\frac{100}{9}} + \frac{(y-2)^2}{\frac{25}{3}} = 1.$$

Center: $(\frac{14}{3},2)$. Vertices: $(\frac{4}{3},2)$, $(8,2)$,

$(\frac{14}{3}, 2 - \frac{5}{\sqrt{3}}), (\frac{14}{3}, 2 + \frac{5}{\sqrt{3}})$.

Foci: $c^2 = a^2 - b^2$, $c^2 = \frac{100}{9} - \frac{75}{9} = \frac{25}{9}$,

$c = \frac{5}{3}$. So foci are $(\frac{19}{3}, 2)$, $(3,2)$.

Eccentricity: $e = \frac{c}{a} = \frac{5/3}{10/3} = \frac{1}{2}$.

Directrices: $x = \frac{14}{3} + \frac{a^2}{c}$,

$x = \frac{14}{3} + \frac{100/9}{\frac{5}{3}} = \frac{14}{3} + \frac{20}{3} = \frac{34}{3}$; also,

$x = \frac{14}{3} - \frac{20}{3} = -\frac{6}{3} = -2$.

73. $9(x^2+8x) + 4(y^2-12y) = -144$,

$9(x+4)^2 + 4(y-6)^2 = -144 + 144 + 144$,

$\frac{(x+4)^2}{16} + \frac{(y-6)^2}{36}$ 1.

Center: $(-4,6)$. Vertices: $(-4,12)$, $(-4,0)$, $(-8,6)$, $(0,6)$.

Foci: $c^2 = a^2 - b^2$, $c^2 = 36 - 16 = 20$,

$c = \sqrt{20} = 2\sqrt{5}$; $(-4, 6+2\sqrt{5})$, $(-4, 6-2\sqrt{5})$.

Eccentricity: $e = \frac{c}{a} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$.

Directrices: $y = 6 \pm \frac{a^2}{c} = 6 \pm \frac{18}{\sqrt{5}} =$

$6 \pm \frac{18\sqrt{5}}{5} = \frac{30 \pm 18\sqrt{5}}{5}$, so $y = \frac{30+18\sqrt{5}}{5}$,

$y = \frac{30-18\sqrt{5}}{5}$.

74. $2x+6y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{x}{3y}$. $\frac{dy}{dx} = -\frac{3}{6} = \frac{1}{2}$ at $(3,-2)$.

Tangent Line: $y+2 = \frac{1}{2}(x-3)$, or

$2y-x+7 = 0$.

Normal Line: $y+2 = -2(x-3)$, or

$y+2x-4 = 0$.

75. $32x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0$. Given that $\frac{dy}{dt} = -\frac{dx}{dt}$, $32x\frac{dx}{dt} - 18y\frac{dx}{dt} = 0$, $\frac{dx}{dt}(16x-9y) = 0$,

$16x = 9y$, $x = \frac{9}{16}y$. Substituting into the equation of the ellipse: $81y^2+144y^2 = 6400$, $y = \pm\frac{16}{3}$. For $y = \frac{16}{3}$, $x = 3$; for $y = -\frac{16}{3}$, $x = -3$. The points are $(3, \frac{16}{3})$ and $(-3, -\frac{16}{3})$.

76. A(trapezoid) $= \frac{1}{2}(b_1+b_2)h$,

Area $= \frac{1}{2}(2a+2x)y = (a+x)y$.

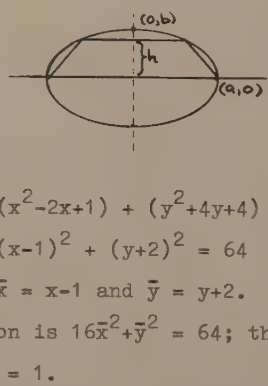
Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the ellipse. Then $A(x) = (a+x)(b\sqrt{1 - \frac{x^2}{a^2}})$.

$A'(x) = b\sqrt{1 - \frac{x^2}{a^2}} + (x+a)\left(\frac{-bx}{a^2\sqrt{1-\frac{x^2}{a^2}}}\right) = 0$.

Simplifying, we get $1 - \frac{x^2}{a^2} - \frac{x^2}{a^2} - \frac{x}{a} = 0$,

$2x^2 + ax - a^2 = 0$, $(2x-a)(x+a) = 0$,

$2x = a$ or $x = -a$. We reject $x = -a$ for obvious reasons. Hence, $x = \frac{a}{2}$ yields trapezoid of maximum area; so its upper base has length $2x = a$, which is half the dimensions of the lower base.



77. (a) $16(x^2-2x+1) + (y^2+4y+4) = 44 + 16 + 4$,

$16(x-1)^2 + (y+2)^2 = 64$

Thus, $\bar{x} = x-1$ and $\bar{y} = y+2$. So the equation is $16\bar{x}^2+\bar{y}^2 = 64$; that is, $\frac{\bar{x}^2}{4} + \frac{\bar{y}^2}{64} = 1$.

(b) $9(x^2+4x+4) + 4(y^2-6y+9) = 252 + 36+36$,

$9(x+2)^2 + 4(y-3)^2 = 324$.

Thus, $\bar{x} = x+2$ and $\bar{y} = y-3$. So the equation is $9\bar{x}^2+4\bar{y}^2 = 324$; that is, $\frac{\bar{x}^2}{36} + \frac{\bar{y}^2}{81} = 1$.

78. $\frac{y+2}{x-3} \cdot \frac{y-1}{x+2} = -6$. $y^2+y-2 = -6x^2+6x+36$,

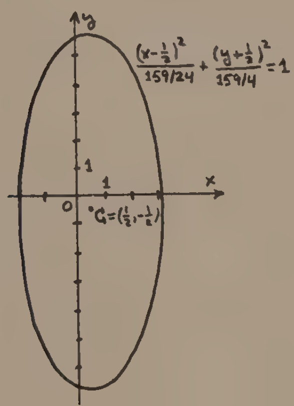
$6x^2-6x+y^2+y = 38.$

$6(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 = \frac{159}{4}.$

$\frac{(x - \frac{1}{2})^2}{\frac{159}{24}} + \frac{(y + \frac{1}{2})^2}{\frac{159}{4}} = 1.$

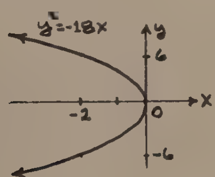
$\frac{159}{24} = 6.24 = b^2$, $b \approx 2.57.$

$a^2 = 39.75$, so $a \approx 6.3.$



79. $x = -\frac{1}{4p}y^2$, so $-2 = -\frac{1}{4p}(36)$, so $p = \frac{9}{2}.$

The equation is $x = -\frac{y^2}{18}$ or $y^2 = -18x.$



80. $x-h = \frac{1}{4p}(y-k)^2.$ Substituting the three points into the equation: $-2-h = \frac{1}{4p}(1-k)^2;$ $1-h = \frac{1}{4p}(2-k)^2;$ $-1-h = \frac{1}{4p}(3-k)^2.$

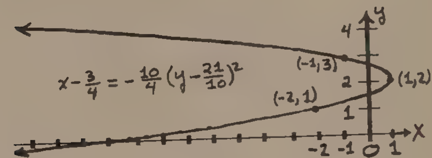
Subtracting the second equation from the first: $-3 = \frac{1}{4p}(-3+2k).$

Subtracting the third equation from the second: $2 = \frac{1}{4p}(-5+2k).$

Subtracting these last two equations, we get $p = -\frac{1}{10}.$

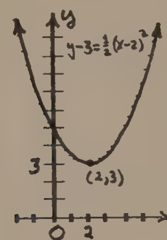
Substituting, we find $k = \frac{21}{10}$ and $h = \frac{3}{4}.$

So $x - \frac{3}{4} = -\frac{10}{4}(y - \frac{21}{10})^2.$

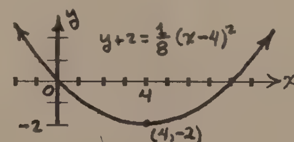


81. $y-3 = \frac{1}{4p}(x-2)^2.$ $2 = \frac{1}{4p} \cdot 4$, so $p = \frac{1}{2}.$

The equation is $y-3 = \frac{1}{2}(x-2)^2.$



82. The distance from $(6,-2)$ to the line $x = 2$ is 4 units, that is, $2p = 4$, $p = 2.$ Hence, the vertex is $(4,-2).$ The equation is $y+2 = \frac{1}{8}(x-4)^2.$

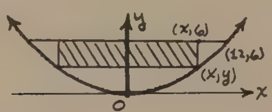


83. $4p = 8.$ $p = 2.$ So the endpoints of the focal chord are $(2,4)$ and $(2,-4).$ The vertex is $(0,0).$ We want the equation of the circle containing $(0,0)$, $(2,4)$ and $(2,-4).$ Use $x^2+y^2+Ax+By+C = 0.$ Thus, $C = 0;$ $4+16+2A+4B = 0;$ $4+16+2A-4B = 0.$ So $4A = -40$, $A = -10.$ Hence, $4+16-20+4B = 0$, $4B = 0$, $B = 0.$ The equation is $x^2 + y^2 - 10x = 0.$

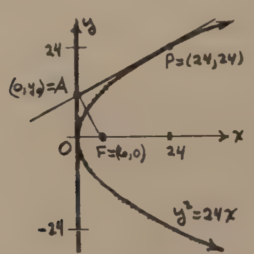
84. (a) $2y\frac{dy}{dx} + 5 = 0$, $10\frac{dy}{dx} + 5 = 0$. So $\frac{dy}{dx} = -\frac{1}{2}$. The tangent line has equation $y-5 = -\frac{1}{2}(x+5)$, or $2y+x-5 = 0$. The normal line has equation $y-5 = 2(x+5)$, or $y-2x-15 = 0$.

(b) $4\frac{dy}{dx} = 16-2x$, $\frac{dy}{dx} = 4 - \frac{1}{2}x$, $\frac{dy}{dx} = 4 - \frac{1}{2} = \frac{7}{2}$. The tangent line has equation $y - \frac{23}{4} = \frac{7}{2}(x-1)$, or $4y-14x-9 = 0$. The normal line has equation $y - \frac{23}{4} = -\frac{2}{7}(x-1)$, or $28y+8x-169 = 0$.

85. Area $= \ell w = (2x)(6-y)$. So $A(x) = (2x)(6 - \frac{x^2}{24}) = 12x - \frac{x^3}{12}$.
$A'(x) = 12 - \frac{x^2}{4} = 0$. $x^2 = 48$, $x = \sqrt{48} = 4\sqrt{3}$. So $y = 2$. The dimensions are $8\sqrt{3}$ by 4.



86. Show $\overline{AF} \perp \overline{AP}$. $F = (6,0)$.
Slope of AP is $\frac{dy}{dx} = \frac{12}{y} = \frac{1}{2}$.
So $\frac{24-y_0}{24} = \frac{1}{2}$ and $y_0 = 12$. Slope of $\overline{AF} = \frac{y_0}{-6} = \frac{12}{-6} = -2$. Hence, the lines are $\perp$.



87. $y = -2x + 6$ is the equation of a line with slope -2. We want, therefore, $\frac{dy}{dx} = 2-2x = -2$, $x = 2$, $y = 4+4-4 = 4$. The point is $(2,4)$.

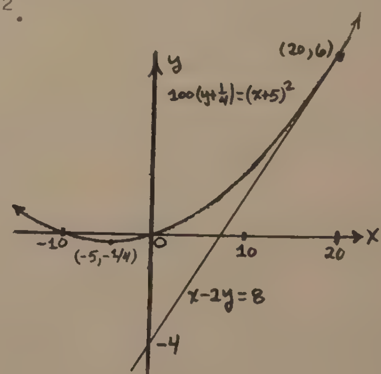
88. The equation must be of the form $4p(y-K) = (x-h)^2$. We obtain the three equations below:

(1) $-4pK = h^2$, since $(0,0)$ is a point on the graph.

(2) $4p(6-K) = (20-h)^2$, since $(20,6)$ is a point on the graph.

(3) $4p(\frac{1}{2}) = 2(20-h)$ or $p = 20-h$, since $\frac{dy}{dx} = \frac{1}{2}$ when $x = 20$.

To solve this system of equations, subtract the first two equations and obtain equation (4): $24p = 400-40h$. Now multiply equation (3) by -40 and add to equation (4). We obtain $16p = 400$ or $p = 25$. This gives $b = -5$ by equation (3). From equation (1); $-4(25)K = 25$, so $K = -\frac{1}{4}$. Thus the equation is $4(25)(y + \frac{1}{4}) = (x+5)^2$ or $100(y + \frac{1}{4}) = (x+5)^2$.



89. The center is $(0,0)$; $c = 5$, $b = 2$, so $a = \sqrt{c^2-b^2} = \sqrt{25-4} = \sqrt{21}$. Thus, the equation is $\frac{x^2}{21} - \frac{y^2}{4} = 1$.

90. $\frac{y^2}{b^2} - \frac{(x-3)^2}{a^2} = 1$. $c^2 = a^2+b^2$, $100 = 36+b^2$, $64 = b^2$. $\frac{y^2}{64} - \frac{(x-3)^2}{36} = 1$.

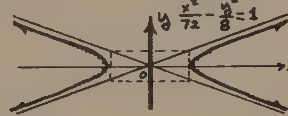
91. $a = \frac{1}{2}(6-(-2)) = \frac{1}{2}(8) = 4$. The focus (7,3) is one unit to the right of (6,3) which is 4 units from the center (since $a = 4$). So $c = 5$. $c^2 = a^2+b^2$, $25 = 16+b^2$, $b^2 = 9$.
$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$.

92. The equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{b}{a} = 2$. So $b = 2a$. Hence, $\frac{x^2}{a^2} - \frac{y^2}{4a^2} = 1$. Since (1,1) is on the hyperbola, $\frac{1}{a^2} - \frac{1}{4a^2} = 1$, and so $4-1 = 4a^2$, $\frac{3}{4} = a^2$. $b^2 = 3$. The equation is $\frac{4x^2}{3} - \frac{y^2}{3} = 1$.

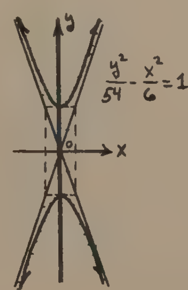
93. $2c = 24$, so $c = 12$. $a^2+b^2 = 144$. Length of focal chord is $\frac{2a^2}{b} = 36$. Solving the two equations simultaneously:
$\frac{2(144-b^2)}{b} = 36$, $2b^2+36b-288 = 0$,
$b^2+18b-144 = 0$, $(b+24)(b-6) = 0$.
So $b = 6$ here. $a^2+36 = 144$; hence, $a^2 = 108$. The equation is
$\frac{y^2}{36} - \frac{x^2}{108} = 1$.

94. $c = 8$, $a = 6$. $a^2+b^2 = c^2$, $36+b^2 = 64$, $b^2 = 28$. The equation is
$\frac{x^2}{36} - \frac{y^2}{28} = 1$.

95. $\frac{x^2}{72} - \frac{y^2}{8} = 1$. Center: (0,0).
Vertices: $(6\sqrt{2},0)$, $(-6\sqrt{2},0) \approx (8.49,0)$, $(-8.49,0)$, respectively.
Foci: $a^2+b^2 = c^2$, $72+8 = c^2$, $80 = c^2$.
The foci are $(4\sqrt{5},0)$, $(-4\sqrt{5},0) \approx (8.94,0)$, $(-8.94,0)$, respectively.
Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{10}}{3}$.
Asymptotes: $y = \pm\sqrt{\frac{8}{72}}\,x$. Thus, $y = \frac{x}{3}$ and $y = -\frac{x}{3}$.



96. $\frac{y^2}{54} - \frac{x^2}{6} = 1$. Center: (0,0).
Vertices: $(0,\sqrt{54})$, $(0,-\sqrt{54}) \approx (0,7.3)$, $(0,-7.3)$, respectively.
Foci: $a^2+b^2 = c^2$, $6+54 = c^2$, $\sqrt{60} = c$.
$(0,\sqrt{60})$, $(0,-\sqrt{60}) \approx (0,7.7)$, $(0,-7.7)$, respectively.
Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{60}}{\sqrt{54}} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}$.
Asymptotes: $y = \sqrt{\frac{54}{6}}\,x$ and $y = -\sqrt{\frac{54}{6}}\,x$, that is, $y = 3x$ and $y = -3x$.



97. $x^2+4x+4-4(y^2-6y+9) = 48 + 4 - 36$.
$(x+2)^2 - 4(y-3)^2 = 16$.
$\frac{(x+2)^2}{16} - \frac{(y-3)^2}{4} = 1$.
Center: (-2,3).
Vertices: (2,3), (-6,3).
Foci: $a^2+b^2 = c^2$, $16+4 = c^2$, $c = \sqrt{20}$.

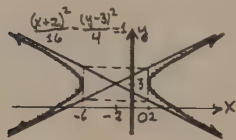
$(-2+\sqrt{20},3)$, $(-2-\sqrt{20},3)$.

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{20}}{4} = \frac{\sqrt{5}}{2}$.

Asymptotes: $y-3 = \frac{1}{2}(x+2)$ and

$y-3 = -\frac{1}{2}(x+2)$ or

$2y-x = 8$ and $2y + x = 4$.



98. $16(x^2-6x+9)-9y^2 = 0+144$, $\frac{(x-3)^2}{9} - \frac{y^2}{16} = 1$.

Center: $(3,0)$.

Vertices: $(6,0)$, $(0,0)$.

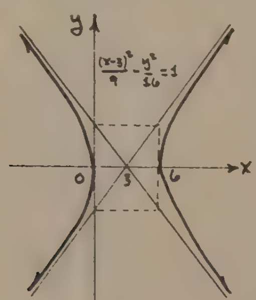
Foci: $a^2+b^2 = c^2$, $25 = c^2$, $c = 5$.

$(8,0)$, $(-2,0)$.

Eccentricity: $e = \frac{c}{a} = \frac{5}{3}$.

Asymptotes: $y = \frac{4}{3}(x-3)$ and $y = -\frac{4}{3}(x-3)$,

or $3y-4x+12 = 0$ and $3y+4x-12 = 0$.



99. $4(y^2-6y+9) - (x^2-2x+1) = 34+36-1 = 1$,

$\frac{(y-3)^2}{\frac{1}{4}} - \frac{(x-1)^2}{1} = 1$.

Center: $(1,3)$.

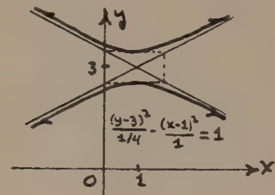
Vertices: $(1,3.5)$, $(1,2.5)$.

Foci: $a^2+b^2 = c^2$, $1 + \frac{1}{4} = c^2$, $\frac{\sqrt{5}}{2} = c$.

$(1,3+\frac{\sqrt{5}}{2})$, $(1,3-\frac{\sqrt{5}}{2})$.

Eccentricity: $e = \frac{c}{a} = \frac{\frac{\sqrt{5}}{2}}{1} = \frac{\sqrt{5}}{2}$.

Asymptotes: $y-3 = \frac{1}{2}(x-1)$ and $y-3 = -\frac{1}{2}(x-1)$, or $2y-x-5 = 0$ and $2y+x-7 = 0$.



100. Since $y = -\frac{5}{3}x + 3$ is the equation of given line, the slope of the tangent is $\frac{3}{5}$. Now, $2x-2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{x}{y}$. Hence, $\frac{x}{y} = \frac{3}{5}$, $x = \frac{3}{5}y$. Substituting into equation of hyperbola, we can find the points of tangency: $\frac{9}{25}y^2-y^2 = -16$, $y^2 = 25$, $y = \pm 5$, and the points are $(3,5)$ and $(-3,-5)$. The equations are $y-5 = \frac{3}{5}(x-3)$ and $y+5 = \frac{3}{5}(x+3)$ or $5y-3x-16 = 0$ and $5y-3x+16 = 0$.

101. $2x=16y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{x}{8y}$. At $(3,1)$ the slope of tangent line is $\frac{3}{8}$ and the slope of normal line is $-\frac{8}{3}$. Equation of tangent line: $y-1 = \frac{3}{8}(x-3)$, or $8y-3x+1 = 0$. Equation of normal line: $y-1 = -\frac{8}{3}(x-3)$, or $3y+8x-27 = 0$.

102. Distance from (x,y) to $(0,6)$ $= \sqrt{(x-0)^2 + (y-6)^2}$.

$D(y) = \sqrt{(y^2+16) + (y-6)^2} = \sqrt{2y^2-12y+52}$.

$D'(y) = \frac{2y-6}{\sqrt{2y^2-12y+52}} = 0$ for $y = 3$. So

$x = 5$ or $x = -5$. The points are $(5,3)$ and $(-5,3)$.

103. $2x\,\frac{dx}{dt} - 8y\,\frac{dy}{dt} = 0$, $12(3)-8(-2)\frac{dy}{dt} = 0$.

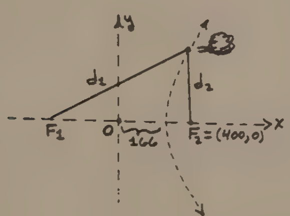
So $\frac{dy}{dt} = -\frac{36}{16} = -\frac{9}{4}$. y is decreasing at the rate of $\frac{9}{4}$ units per second.

104. $\frac{d_1}{332} - \frac{d_2}{332} = 1$, $d_1-d_2 = 332$.

$|\overline{PF}_1| - |\overline{PF}_2| = 332 = 2a$.

According to this last statement, which is just the definition of a hyperbola, P is on a certain hyperbola. We determine which hyperbola as follows: $2a = 332$, $a = 166$ and $c = 400$, so $a^2+b^2 = c^2$, and $b^2 = 132{,}444$. The equation is

$$\frac{x^2}{27{,}556} - \frac{y^2}{132{,}444} = 1.$$



105. $b = 195$, $a = 229$, $c^2 = a^2-b^2 = (229)^2-(195)^2 = 14416$, so $c \approx 120.067$.

$e = \frac{c}{a} \approx 0.52$.

106. (a) With the origin at the lowest point of the cable of sag H and span L, the points $(\frac{L}{2},H)$ and $(-\frac{L}{2},H)$ are points on the parabola representing the cable. The equation of the parabola is of the form $x^2 = 4py$. Thus, $\frac{L^2}{4} = 4pH$ and $p = \frac{L^2}{16H}$; so the equation of the parabolic cable is $x^2 = \frac{L^2}{4H}y$ or $y = \frac{4H}{L^2}x^2$.

(b) $ds = \sqrt{1+[f'(x)]^2}\,dx = \sqrt{1+(\frac{8Hx}{L^2})^2}\,dx$

Thus, $s = 2\int_0^{L/2} \sqrt{1+(\frac{8Hx}{L^2})}\,dx$. Using the trigonometric substitution $x = \frac{L^2}{8H}\tan\theta$, we have $dx = \frac{L^2}{8H}\sec^2\theta\,d\theta$, so that

$$s = 2\int_0^{\tan^{-1}(\frac{4H}{L})} \sqrt{1+\tan^2\theta}\,(\frac{L^2}{8H}\sec^2\theta)\,d\theta =$$

$$\frac{L^2}{4H}\int_0^{\tan^{-1}(\frac{4H}{L})} \sec^3\theta\,d\theta =$$

$$\frac{L^2}{4H}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_0^{\tan^{-1}(\frac{4H}{L})} =$$

(front cover Formulas 33 and 11.)

$$\frac{L^2}{8H}\left[(\frac{\sqrt{L^2+16H^2}}{L})(\frac{4H}{L})+\ln\left|\frac{\sqrt{L^2+16H^2}}{L}+\frac{4H}{L}\right|\right] =$$

$$\frac{1}{2}\sqrt{L^2+16H^2} + \frac{L^2}{8H}\ln\left(\frac{\sqrt{L^2+16H^2}+4H}{L}\right),$$

where we used the fact that, if $\tan\theta = \frac{4H}{L}$, then $\sec\theta = \frac{\sqrt{L^2+16H^2}}{L}$.

(c) Using part (b), we first let $L = 564$ and $H = 63$. Then the length of the cable $s = \frac{1}{2}\sqrt{(564)^2+16(63)^2} + \frac{(564)^2}{8(63)}\ln\left(\frac{\sqrt{(564)^2+16(63)^2}+252}{564}\right)$

$= 308.8689+631.14286\ln(1.5420883)$

≈ 582 meters.

107. $a = 74$ and $b = 42$. By Problem 55, Section 9.4, the area of an ellipse is πab. Thus, the area is 3108π square meters.

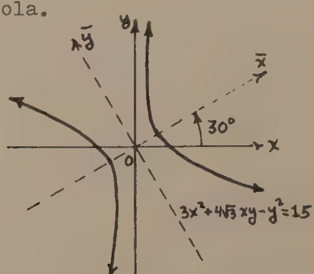
108. $e = \frac{c}{a}$, so if $e \to 0$, then $\frac{c^2}{a^2} = e^2 \to 0$; hence, $1 - \frac{b^2}{a^2} = \frac{c^2}{a^2} \to 0$. This shows that as $e \to 0$, $\frac{b^2}{a^2} \to 1$, so $\frac{b}{a} \to 1$. But, $\frac{b}{a} \to 1$ means that the values of a and b are relatively close to each other. An ellipse in which $a \approx b$ is quite circular in shape. If $a = b$, then

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of a circle of radius a. This would suggest that a conic with eccentricity e = 0 should be regarded as being a circle.

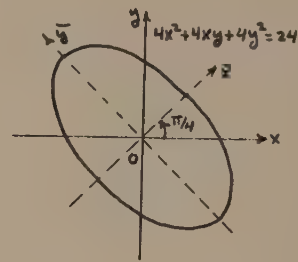
109. $r^2\sin 2\theta = 2$ becomes $\bar{r}^2\sin 2(\bar{\theta} + \frac{\pi}{4}) = 2$ or $\bar{r}^2\sin(2\bar{\theta} + \frac{\pi}{2}) = 2$. Therefore, $\bar{r}^2\cos 2\bar{\theta} = 2$.

110. If $\sqrt{x} + \sqrt{y} = 2$, then $\sqrt{y} = 2-\sqrt{x}$ for $0 \le x \le 4$; or $\sqrt{x} = 2-\sqrt{y}$ for $0 \le y \le 4$. Now, $(\sqrt{y})^2 = (2-\sqrt{x})^2$ or $y = 4-4\sqrt{x}+x$. Then square again: $(y-x-4)^2 = (-4\sqrt{x})^2$ or $x^2-2xy+y^2-8x-8y+16 = 0$. Now substitute: $x = \frac{\sqrt{2}}{2}(\bar{x}-\bar{y})$ and $y = \frac{\sqrt{2}}{2}(\bar{x}+\bar{y})$ and the equation becomes $\bar{y}^2 = 4\sqrt{2}(\bar{x}-\sqrt{2})$ which is a parabola subject to $0 \le x \le 4$ and $0 \le y \le 4$. These conditions are equivalent to $\sqrt{2} \le \bar{x} \le 2\sqrt{2}$ and $-2\sqrt{2} \le \bar{y} \le 2\sqrt{2}$. The graph of the equation is just a portion of a parabola.

111. $\cot 2\phi = \frac{A-C}{B} = \frac{1}{\sqrt{3}}$, $2\phi = 60°$, $\phi = 30°$, $x = \frac{\sqrt{3}}{2}\bar{x} - \frac{1}{2}\bar{y}$, $y = \frac{1}{2}\bar{x} + \frac{\sqrt{3}}{2}\bar{y}$. Substituting into $3x^2+4\sqrt{3}xy-y^2 = 15$ and simplifying, we obtain $5\bar{x}^2-3\bar{y}^2 = 15$, or $\frac{\bar{x}^2}{3} - \frac{\bar{y}^2}{5} = 1$, a hyperbola.



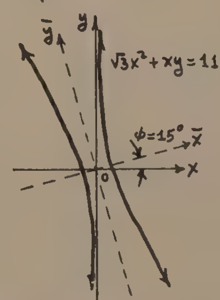
112. $\cot 2\phi = \frac{A-C}{B} = 0$, $\phi = 45°$, $x = \frac{\sqrt{2}}{2}(\bar{x}-\bar{y})$, $y = \frac{\sqrt{2}}{2}(\bar{x}+\bar{y})$. Substituting into $4x^2+4xy+4y^2 = 24$ and simplifying, we obtain $6\bar{x}^2+2\bar{y}^2 = 24$, or $\frac{\bar{x}^2}{4} + \frac{\bar{y}^2}{12} = 1$, an ellipse.



113. $\cot 2\phi = \frac{A-C}{B} = \sqrt{3}$, $\cos 2\phi = \frac{\sqrt{3}}{\sqrt{3+1}} = \frac{\sqrt{3}}{2}$, $\cos\phi = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$, $\sin\phi = \sqrt{\frac{1-\sqrt{3}/2}{2}}$ $= \frac{\sqrt{2-\sqrt{3}}}{2}$.

$\phi = \sin^{-1}\frac{\sqrt{2-\sqrt{3}}}{2} = 15°$, $x = \bar{x}\cos 15° - \bar{y}\sin 15$

$y = \bar{x}\sin 15° + \bar{y}\cos 15°$. Substituting into $\sqrt{3}\,x^2+xy = 11$ and simplifying, we obtain $\frac{(\sqrt{3}+2)}{2}\bar{x}^2 - \frac{(2-\sqrt{3})}{2}\bar{y}^2 = 11$, or $\frac{\bar{x}^2}{a^2} - \frac{\bar{y}^2}{b^2} = 1$, where $a = \sqrt{\frac{22}{\sqrt{3}+2}} \approx 2.43$ and $b = \sqrt{\frac{22}{2-\sqrt{3}}} \approx 9.06$



114. $\cot 2\phi = \frac{A-C}{B} = \frac{0-1}{1} = -1$, $\cos 2\phi = \frac{-1}{\sqrt{1+1}}$, $\cos\phi = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\sin\phi = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} =$ $\frac{\sqrt{2+\sqrt{2}}}{2}$, $x = \frac{\sqrt{2-\sqrt{2}}}{2}\cdot\bar{x} - \frac{\sqrt{2+\sqrt{2}}}{2}\cdot\bar{y}$, and

$y = \frac{(\sqrt{2+\sqrt{2}})}{2}\bar{x} + \frac{(\sqrt{2-\sqrt{2}})}{2}\bar{y}$. Substitution into the original equation gives

$\frac{2+\sqrt{2}}{4}\bar{x}^2 + \frac{2\sqrt{2}}{4}\bar{x}\bar{y} + \frac{2-\sqrt{2}}{4}\bar{y}^2 + \frac{\sqrt{2}}{4}\bar{x}^2 - \frac{2\sqrt{2}}{4}\bar{x}\bar{y} -$

$\frac{\sqrt{2}}{4}\bar{y}^2 - \frac{3\sqrt{2-\sqrt{2}}}{2} \cdot \bar{x} + \frac{3\sqrt{2+\sqrt{2}}}{2} \cdot \bar{y} = 7$, or

$(\frac{1+\sqrt{2}}{2})\bar{x}^2 + \frac{(1-\sqrt{2})}{2}\bar{y}^2 - \frac{(3\sqrt{2-\sqrt{2}})}{2}\bar{x} + \frac{(3\sqrt{2+\sqrt{2}})}{2}\bar{y} = 7.$

Completing the squares gives

$\frac{(1+\sqrt{2})}{2}(\bar{x}^2 - (\frac{3\sqrt{2-\sqrt{2}}}{1+\sqrt{2}})\bar{x} + \frac{9(2-\sqrt{2})}{4(3+2\sqrt{2})}) +$

$(\frac{1-\sqrt{2}}{2})(\bar{y}^2 + (\frac{3\sqrt{2+\sqrt{2}}}{1-\sqrt{2}})\bar{y} + \frac{9(2+\sqrt{2})}{4(3-2\sqrt{2})}) =$

$7 + \frac{9(2-\sqrt{2})}{8(1+\sqrt{2})} + \frac{9(2+\sqrt{2})}{8(1-\sqrt{2})}$, or

$\frac{(1+\sqrt{2})}{2}\left[\bar{x} - \frac{3\sqrt{2-\sqrt{2}}}{2(1+\sqrt{2})}\right]^2 + \frac{(1-\sqrt{2})}{2}\left[\bar{y} + \frac{3\sqrt{2+\sqrt{2}}}{2(1-\sqrt{2})}\right]^2 =$

$\frac{-56+9(2-\sqrt{2})(1-\sqrt{2}) + 9(2+\sqrt{2})(1+\sqrt{2})}{-8}.$

The latter equation simplifies to

$\frac{(\sqrt{2}+1)}{2}\left[\bar{x} - \frac{3\sqrt{2-\sqrt{2}}}{2(1+\sqrt{2})}\right]^2 - \frac{(\sqrt{2}-1)}{2}\left[\bar{y} - \frac{3\sqrt{2+\sqrt{2}}}{2(\sqrt{2}-1)}\right]^2$

$= -2$, or $\frac{(\bar{y}-k)^2}{b^2} - \frac{(\bar{x}-h)^2}{a^2} = 1$, where

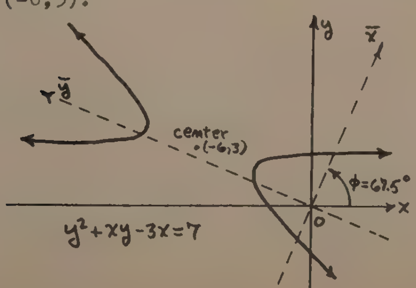
$a = \sqrt{\frac{4}{\sqrt{2}+1}} \approx 1.29$, $b = \sqrt{\frac{4}{\sqrt{2}-1}} \approx 3.11$,

$h = \frac{3\sqrt{2-\sqrt{2}}}{2(\sqrt{2}+1)} \approx 0.48$, and

$k = \frac{3\sqrt{2+\sqrt{2}}}{2(\sqrt{2}-1)} \approx 6.69$. Also, since

$\cot 2\phi = -1$, $2\phi = 135^\circ$ and $\phi = 67.5^\circ$.

The center has "old" xy coordinates $(-6,3)$.



115. There is no mixed term so no rotation is necessary.

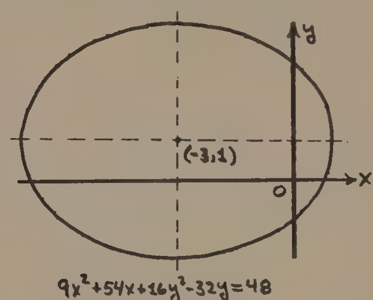
$9(x^2+6x)+16(y^2-2y) = 48$ or

$9(x^2+6x+9)+16(y^2-2y+1) = 48+81+16.$

Thus, $9(x+3)^2+16(y-1)^2 = 145.$

This is an ellipse with center $(-3,1)$, with $a = \frac{\sqrt{145}}{3}$ and $b = \frac{\sqrt{145}}{4}$ or

$a \approx 4.01$ and $b \approx 3.01$.



116. $\cot 2\phi = \frac{1-9}{-6} = \frac{-8}{-6} = \frac{4}{3}$, $\cos 2\phi =$

$\frac{4/3}{\sqrt{(\frac{4}{3})^2+1}} = \frac{4/3}{\sqrt{\frac{25}{9}}} = \frac{4}{3} \cdot \frac{3}{5} = \frac{4}{5}$ so

$\cos \phi = \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$ and

$\sin \phi = \sqrt{\frac{1-\frac{4}{5}}{10}} = \frac{1}{\sqrt{10}}$; so that $\phi \approx 18.43^\circ$.

Now, $x = \frac{1}{\sqrt{10}}(3\bar{x}-\bar{y})$ and $y = \frac{1}{\sqrt{10}}(\bar{x}+3\bar{y})$.

Substituting into

$x^2-6xy+9y^2+x-3y = 4$, we obtain,

$\frac{1}{10}(9\bar{x}^2-6\bar{x}\bar{y}+\bar{y}^2)-6(\frac{1}{10})(3\bar{x}^2+8\bar{x}\bar{y}-3\bar{y}^2) +$

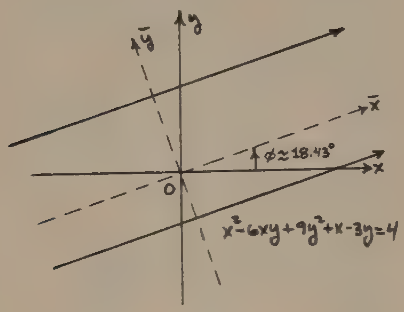
$\frac{9}{10}(\bar{x}^2+6\bar{x}\bar{y}+9\bar{y}^2)+ \frac{1}{\sqrt{10}}(3\bar{x}-\bar{y}) - \frac{3}{\sqrt{10}}(\bar{x}+3\bar{y}) = 4,$

which simplifies to

$\frac{100}{10}\bar{y}^2 + \frac{3}{\sqrt{10}}\bar{x} - \frac{1}{\sqrt{10}}\bar{y} - \frac{3}{\sqrt{10}}\bar{x} - \frac{9}{\sqrt{10}}\bar{y} = 4$

or $10\bar{y}^2 - \frac{10}{\sqrt{10}}\bar{y} = 4$ or $10\bar{y}^2 - \sqrt{10}\bar{y} - 4 = 0.$

Thus the conic is a degenerate one and consists of two lines: $\bar{y} = \frac{\sqrt{10}+\sqrt{170}}{20} \approx 0.81$ and $\bar{y} = \frac{\sqrt{10}-\sqrt{170}}{20} \approx -0.49$.



117. $B^2-4AC = (6\sqrt{3})^2 - 4(13)(7)$
$= -256 < 0$, <u>ellipse</u>.

118. $B^2-4AC = 6^2 - 4(1)(1)$
$= 32 > 0$, <u>hyperbola</u>.

119. $B^2-4AC = (2\sqrt{3})^2 - 4(7)(5)$
$= -128 < 0$, <u>ellipse</u>.

120. $B^2-4AC = (-3)^2 - 4(1)(5)$
$= -11 < 0$, <u>ellipse</u>.

121. $B^2-4AC = (-18)^2 - 4(81)(1)$
$= 0$, <u>parabola</u>.

10

INDETERMINATE FORMS, IMPROPER INTEGRALS, AND TAYLOR'S FORMULA

Problem Set 10.1, page 602

1. $\lim\limits_{x\to 0}\dfrac{x+\sin 2x}{x-\sin 2x}=\lim\limits_{x\to 0}\dfrac{1+2\cos 2x}{1-2\cos 2x}=\dfrac{3}{-1}=-3.$

2. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin x-1}{\frac{\pi}{2}-x}=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{-1}=\dfrac{0}{-1}=0.$

3. $\lim\limits_{x\to -2}\dfrac{2x^2+3x-2}{3x^2-x-14}=\lim\limits_{x\to -2}\dfrac{4x+3}{6x-1}=\dfrac{-5}{-13}=\dfrac{5}{13}.$

4. $\lim\limits_{x\to 1}\dfrac{x^3-3x^2+5x-3}{x^2+x-2}=\lim\limits_{x\to 1}\dfrac{3x^2-6x+5}{2x+1}=\dfrac{2}{3}.$

5. $\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{\sin\sqrt{x}}=\lim\limits_{x\to 0^+}\dfrac{1/(2\sqrt{x})}{\left(\frac{1}{2\sqrt{x}}\right)\cos\sqrt{x}}=\lim\limits_{x\to 0^+}\dfrac{1}{\cos\sqrt{x}}=1.$

6. $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{\sin x^2}=\lim\limits_{x\to 0}\dfrac{-\sin x+3\sin 3x}{2x\cos x^2}=$

   $\lim\limits_{x\to 0}\dfrac{-\cos x+9\cos 3x}{-4x\sin x^2+2\cos x^2}=\dfrac{8}{2}=4.$

7. $\lim\limits_{t\to\frac{\pi}{2}}\dfrac{\sin t-1}{\cos t}=\lim\limits_{t\to\frac{\pi}{2}}\dfrac{\cos t}{-\sin t}=\dfrac{0}{-1}=0.$

8. $\lim\limits_{x\to 0}\dfrac{xe^{3x}-x}{1-\cos 2x}=\lim\limits_{x\to 0}\dfrac{e^{3x}+3xe^{3x}-1}{2\sin 2x}=$

   $\lim\limits_{x\to 0}\dfrac{3e^{3x}+3e^{3x}+9xe^{3x}}{4\cos 2x}=\dfrac{6}{4}=\dfrac{3}{2}.$

9. $\lim\limits_{y\to 0}\dfrac{e^y-1}{y^3}=\lim\limits_{y\to 0}\dfrac{e^y}{3y^2}=+\infty.$

10. $\lim\limits_{t\to 0}\dfrac{t-\sin t}{t^3}=\lim\limits_{t\to 0}\dfrac{1-\cos t}{3t^2}=\lim\limits_{t\to 0}\dfrac{\sin t}{6t}=$

    $\lim\limits_{t\to 0}\dfrac{\cos t}{6}=\dfrac{1}{6}.$

11. $\lim\limits_{x\to 7}\dfrac{\ln\frac{x}{7}}{7-x}=\lim\limits_{x\to 7}\dfrac{\frac{1}{x}}{-1}=-\dfrac{1}{7}.$

12. $\lim\limits_{x\to 0}\dfrac{x-\tan^{-1}x}{x-\sin^{-1}x}=\lim\limits_{x\to 0}\dfrac{1-\dfrac{1}{1+x^2}}{1-\dfrac{1}{\sqrt{1-x^2}}}=\lim\limits_{x\to 0}\dfrac{\dfrac{x^2}{1+x^2}}{1-\dfrac{1}{\sqrt{1-x^2}}}=$

    $\lim\limits_{x\to 0}\dfrac{\dfrac{2x(1+x^2)-x^2\cdot 2x}{(1+x^2)^2}}{\dfrac{-2x}{2(1-x^2)^{3/2}}}=-2.$

13. $\lim\limits_{x\to 1}\dfrac{\ln x-\sin(x-1)}{(x-1)^2}=\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-\cos(x-1)}{2(x-1)}=$

    $\lim\limits_{x\to 1}\dfrac{-\frac{1}{x^2}+\sin(x-1)}{2}=-\dfrac{1}{2}.$

14. $\lim\limits_{t\to 0^+}\dfrac{\ln(e^t+1)-\ln 2}{t^2}=\lim\limits_{t\to 0^+}\dfrac{\dfrac{e^t}{e^t+1}}{2t}=+\infty.$

15. $\lim\limits_{t\to 0}\dfrac{e^t-e^{-t}-2\sin t}{4t^3}=\lim\limits_{t\to 0}\dfrac{e^t+e^{-t}-2\cos t}{12t^2}=$

    $\lim\limits_{t\to 0}\dfrac{e^t-e^{-t}+2\sin t}{24t}=\lim\limits_{t\to 0}\dfrac{e^t+e^{-t}+2\cos t}{24}=\dfrac{1}{6}.$

16. $\lim\limits_{y\to 0}\dfrac{y^2-y\sin y}{e^y+e^{-y}-y^2-2}=\lim\limits_{y\to 0}\dfrac{2y-\sin y-y\cos y}{e^y-e^{-y}-2y}=$

    $\lim\limits_{y\to 0}\dfrac{2-\cos y-\cos y+y\sin y}{e^y+e^{-y}-2}=$

    $\lim\limits_{y\to 0}\dfrac{2\sin y+\sin y+y\cos y}{e^y-e^{-y}}=$

    $\lim\limits_{y\to 0}\dfrac{3\cos y+\cos y-y\sin y}{e^y+e^{-y}}=\dfrac{4}{2}=2.$

17. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\ln(\sin x)}{(\pi-2x)^2}=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x}{2(\pi-2x)(-2)}=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{-\csc^2 x}{8}=-\dfrac{1}{8}.$

18. $\lim\limits_{x\to+\infty} \dfrac{\ln x}{x - \sqrt{x}} = \lim\limits_{x\to+\infty} \dfrac{\frac{1}{x}}{1 - \frac{1}{2\sqrt{x}}} = 0.$

19. $\lim\limits_{x\to+\infty} \dfrac{\sin \frac{7}{x}}{\frac{5}{x}} = \lim\limits_{x\to+\infty} \dfrac{-\frac{7}{x^2} \cos \frac{7}{x}}{-\frac{5}{x^2}} = \dfrac{7}{5}.$

20. $\lim\limits_{x\to+\infty} \dfrac{\sin \frac{3}{x}}{\tan^{-1} \frac{2}{x}} = \lim\limits_{x\to+\infty} \dfrac{-\frac{3}{x^2} \cos \frac{3}{x}}{-\frac{2}{x^2} \frac{1}{1 + \frac{4}{x^2}}} = \dfrac{3}{2}.$

21. $\lim\limits_{x\to+\infty} \dfrac{1 - \cos \frac{2}{x}}{\tan \frac{3}{x}} = \lim\limits_{x\to+\infty} \dfrac{-2x^{-2}\sin \frac{2}{x}}{-3x^{-2}\sec^2 \frac{3}{x}} = \lim\limits_{x\to+\infty} \dfrac{2 \sin \frac{2}{x}}{3 \sec^2 \frac{3}{x}} =$

$\frac{0}{3} = 0.$

22. $\lim\limits_{x\to+\infty} \dfrac{\sin \frac{1}{x}}{\sin \frac{2}{x}} = \lim\limits_{x\to+\infty} \dfrac{-x^{-2}\cos \frac{1}{x}}{-2x^{-2}\cos \frac{2}{x}} = \lim\limits_{x\to+\infty} \dfrac{\cos \frac{1}{x}}{2 \cos \frac{2}{x}} = \dfrac{1}{2}.$

23. $\lim\limits_{x\to 0} \dfrac{\int_0^x 3 \cos^4 7t \, dt}{\int_0^x e^{5t^2} \, dt} = \lim\limits_{x\to 0} \dfrac{D_x \int_0^x 3 \cos^4 7t \, dt}{D_x \int_0^x e^{5t^2} \, dt} =$

$\lim\limits_{x\to 0} \dfrac{3 \cos^4 7x}{e^{5x^2}} = 3.$

24. $\lim\limits_{x\to 0} \dfrac{\int_0^x e^{7t}(4t^3 + t^2 + 11) \, dt}{\int_0^x e^{7t}(-7t^3 + 6t + 8) \, dt} =$

$\lim\limits_{x\to 0} \dfrac{D_x \int_0^x e^{7t}(4t^3 + t^2 + 11) \, dt}{D_x \int_0^x e^{7t}(-7t^3 + 6t + 8) \, dt} =$

$\lim\limits_{x\to 0} \dfrac{e^{7x}(4x^3 + x^2 + 11)}{e^{7x}(-7x^2 + 6x + 8)} = \dfrac{11}{8}.$

25. $f'(c) = 6c^2$ and $g'(c) = 6c$. We want c in $(0,2)$ such that $\dfrac{f(2) - f(0)}{g(2) - g(0)} = \dfrac{f'(c)}{g'(c)}$, or $\dfrac{16 - 0}{11 - (-1)} = \dfrac{6c^2}{6c}$. Thus, $c = \dfrac{4}{3}$.

26. $f'(c) = \cos c$ and $g'(c) = -\sin c$. We want c in $(0, \pi/4)$ such that $\dfrac{f(\pi/4) - f(0)}{g(\pi/4) - g(0)} = \dfrac{f'(c)}{g'(c)}$, or

$\dfrac{\frac{1}{2}\sqrt{2} - 0}{\frac{1}{2}\sqrt{2} - 1} = \dfrac{\cos c}{-\sin c}$. Thus, $c = \cot^{-1}\left(\dfrac{1}{\sqrt{2} - 1}\right)$.

27. $f'(c) = \frac{1}{c}$ and $g'(c) = -\frac{1}{c^2}$. $\dfrac{f(e) - f(1)}{g(e) - g(1)} = \dfrac{f'(c)}{g'(c)}$,

or $\dfrac{1 - 0}{\frac{1}{e} - 1} = \dfrac{(\frac{1}{c})}{(-\frac{1}{c^2})}$, so that $c = \dfrac{e}{e - 1}$.

28. $f'(c) = \dfrac{1}{\sqrt{1 - c^2}}$ and $g'(c) = 1$. $\dfrac{f(1) - f(0)}{g(1) - g(0)} =$

$\dfrac{f'(c)}{g'(c)}$, or $\dfrac{\frac{\pi}{2} - 0}{1 - 0} = \dfrac{(\frac{1}{\sqrt{1 - c^2}})}{1}$. Thus, $\sqrt{1 - c^2} = \dfrac{2}{\pi}$,

so that $c = \dfrac{\sqrt{\pi^2 - 4}}{\pi}$.

29. $f'(c) = \sec^2 c$ and $g'(c) = \frac{4}{\pi}$. $\dfrac{f(\frac{\pi}{4}) - f(-\frac{\pi}{4})}{g(\frac{\pi}{4}) - g(-\frac{\pi}{4})} =$

$\dfrac{\sec^2 c}{(\frac{4}{\pi})}$, or $\dfrac{1 - (-1)}{1 - (-1)} = \dfrac{\pi \sec^2 c}{4}$. Thus, $\sec^2 c = \dfrac{4}{\pi}$

and $c = \sec^{-1} \dfrac{2}{\sqrt{\pi}}$.

30. $f'(c) = 4c^3 - 4c$ and $g'(c) = 1$. $\dfrac{f(1) - f(-1)}{g(1) - g(-1)} =$

$\dfrac{4c^3 - 4c}{1}$, or $\dfrac{-1 - (-1)}{1 - (-1)} = 4c(c^2 - 1)$. The equation $0 = 4c(c^2 - 1)$ has solutions -1, 0, and 1; however, only $c = 0$ satisfies $-1 < c < 1$.

31. $\lim\limits_{x\to 0} \dfrac{\int_0^x \frac{t^2 dt}{\sqrt{a + t}}}{bx - \sin x} = \lim\limits_{x\to 0} \dfrac{D_x \int_0^x \frac{t^2 dt}{\sqrt{a + t}}}{b - \cos x} = \lim\limits_{x\to 0} \dfrac{(\frac{x^2}{\sqrt{a + x}})}{b - \cos x}$.

Since the numerator of the latter fraction approaches zero, the denominator must also approach zero if the fraction is to approach 1. Therefore, $b = 1$.

Now we require $1 = \lim\limits_{x\to 0} \dfrac{\frac{x^2}{\sqrt{a + x}}}{1 - \cos x} =$

$\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{a + x}\,(1 - \cos x)} =$

$\lim\limits_{x\to 0} \dfrac{2x}{\frac{1 - \cos x}{2\sqrt{a + x}} + (\sin x)\sqrt{a + x}} =$

$\lim\limits_{x\to 0} \dfrac{2x(2\sqrt{a + x})}{1 - \cos x + 2(\sin x)(a + x)} =$

$\lim\limits_{x\to 0} \dfrac{2(2\sqrt{a + x}) + 2x(\frac{2}{2\sqrt{a + x}})}{\sin x + 2(\cos x)(a + x) + 2 \sin x} = \dfrac{4\sqrt{a}}{2a} = 2\,\dfrac{\sqrt{a}}{a}$.

Thus, $a = 4$.

32. $\lim\limits_{x\to 1} \dfrac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} =$

$\lim\limits_{x\to 1} \dfrac{n(n+1)x^n - n(n+1)x^{n-1}}{2(x-1)} =$

$\dfrac{n(n+1)}{2} \lim\limits_{x\to 1} \dfrac{x^n - x^{n-1}}{x-1} =$

$\dfrac{n(n+1)}{2} \lim\limits_{x\to 1} \dfrac{nx^{n-1} - (n-1)x^{n-2}}{1} =$

$\dfrac{n(n+1)}{2} [n - (n-1)] = \dfrac{n(n+1)}{2}$.

33. Let the functions f and g be defined and differentiable on an open interval (b,a) and suppose that $g(x) \neq 0$ for $b < x < a$. Assume that $\lim\limits_{x\to a^-} f(x) = 0$, $\lim\limits_{x\to a^-} g(x) = 0$, and $g'(x) \neq 0$ for $b < x < a$. Then, if $\lim\limits_{x\to a^-} \dfrac{f'(x)}{g'(x)}$ exists, so does $\lim\limits_{x\to a^-} \dfrac{f(x)}{g(x)}$ and $\lim\limits_{x\to a^-} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a^-} \dfrac{f'(x)}{g'(x)}$. Proof. Define functions F and G as follows:

$$F(x) = \begin{cases} f(x) & \text{if } b < x < a \\ 0 & \text{if } x = a \end{cases}, \qquad G(x) = \begin{cases} g(x) & \text{if } b < x < a \\ 0 & \text{if } x = a \end{cases}$$

for all values of x in (b,a]. Just as in the proof of Theorem 2, we choose any number x in (b,a) and apply Theorem 1 to F and G on the interval [x,a]. Thus, there exists a number c with $x < c < a$ such that $\dfrac{F(a) - F(x)}{G(a) - G(x)} = \dfrac{F'(c)}{G'(c)}$; that is, $\dfrac{-f(x)}{-g(x)} = \dfrac{f'(c)}{g'(c)}$. Now, let $x \to a^-$, so that $c \to a^-$ and we have $\lim\limits_{x\to a^-} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a^-} \dfrac{f'(c)}{g'(c)} = \lim\limits_{c\to a^-} \dfrac{f'(c)}{g'(c)} = \lim\limits_{x\to a^-} \dfrac{f'(x)}{g'(x)}$.

34. The definition of the derivative is $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. Now as $h \to 0$, then the denominator approaches 0. Since f is differentiable, then f is continuous, so that $f(x+h) - f(x)$ approaches 0 as $h \to 0$. Thus, the evaluation of the derivative from the definition involves an indeterminate of the form $\frac{0}{0}$.

35. In order to use L'Hôpital's rule to calculate $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$, we would have to find the derivative of sin x; but we need the $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ to find the derivative of the sine function. This would be circular reasoning.

36. By repeated applications of L'Hôpital's rule the numerator remains the same, but the denominator gets "worse"; that is, the exponent on t in the denominator gets higher. In fact, after n applications of the rule, $\lim\limits_{t\to 0^+} \dfrac{e^{-\frac{1}{t}}}{t}$ looks like $\lim\limits_{t\to 0^+} \dfrac{e^{-\frac{1}{t}}}{n!t^{n+1}}$.

37. $\lim\limits_{t\to 0^+} \dfrac{e^{-\frac{1}{t}}}{t} = \lim\limits_{x\to +\infty} \dfrac{e^{-x}}{\frac{1}{x}} = \lim\limits_{x\to +\infty} xe^{-x} = 0$.

38. Let the functions f and g be defined and differentiable on the open interval $(-\infty,k)$, where $k < 0$ with $g(x) \neq 0$ for $x < k$, and suppose that $\lim\limits_{x\to -\infty} f(x) = 0$ and $\lim\limits_{x\to -\infty} g(x) = 0$ and that $g'(x) \neq 0$ for $x < k$. Then, if $\lim\limits_{x\to -\infty} \dfrac{f'(x)}{g'(x)}$ exists, so does $\lim\limits_{x\to -\infty} \dfrac{f(x)}{g(x)}$ and $\lim\limits_{x\to -\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to -\infty} \dfrac{f'(x)}{g'(x)}$.

Proof. We put $t = \frac{1}{x}$ for $x < k$, so that $\frac{1}{k} < t < 0$ and that $t \to 0^-$ as $x \to -\infty$. We define F and G on $(\frac{1}{k},0]$ by the equations

$$F(t) = \begin{cases} f(\frac{1}{t}) & \text{for } \frac{1}{k} < t < 0 \\ 0 & \text{for } t = 0 \end{cases} \quad \text{and } G(t) = \begin{cases} g(\frac{1}{t}) & \text{for } \frac{1}{k} < t < 0 \\ 0 & \text{for } t = 0 \end{cases}$$

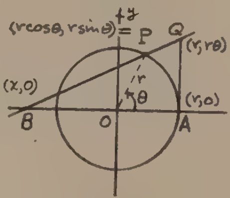
Now $\lim\limits_{t\to 0^-} F(t) = \lim\limits_{t\to 0^-} f(\frac{1}{t}) = \lim\limits_{x\to -\infty} f(x) = 0$ and similarly $\lim\limits_{t\to 0^-} G(t) = 0$. Since f and g are differentiable on $(-\infty,k)$, then by the chain rule, F and G are differentiable on $(\frac{1}{k},0)$ and $F'(t) = \dfrac{-f'(\frac{1}{t})}{t^2}$ and $G'(t) = \dfrac{-g'(\frac{1}{t})}{t^2}$ for $\frac{1}{k} < t < 0$. Hence, by L'Hôpital's rule of Theorem 2, $\lim\limits_{t\to 0^-} \dfrac{F(t)}{G(t)} =$

$\lim\limits_{t\to 0^-} \frac{F'(t)}{G'(t)}$. Therefore, $\lim\limits_{x\to -\infty} \frac{f(x)}{g(x)} = \lim\limits_{t\to 0^-} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} =$

$$\lim_{t\to 0^-} \frac{F(t)}{G(t)} = \lim_{t\to 0^-} \frac{F'(t)}{G'(t)} = \lim_{t\to 0^-} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{x\to -\infty} \frac{f'(x)}{g'(x)}.$$

39. $\lim\limits_{R\to 0} \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) = \lim\limits_{R\to 0} \frac{E\left(\frac{t}{L}\, e^{-\frac{Rt}{L}}\right)}{1} = \frac{Et}{L}$.

40. Call angle AOP = θ.



Then P has coordinates $(r \cos \theta, r \sin \theta)$, and Q has coordinates $(r, r\theta)$. Call B = $(x,0)$. The slope of the line through P and Q can be written in two equivalent ways:

$\frac{r\theta - r \sin \theta}{r - r \cos \theta} = \frac{r\theta - 0}{r - x}$. Solving for x, we have

$x = r - \frac{r\theta(1 - \cos \theta)}{\theta - \sin \theta}$. We want to find $\lim\limits_{\theta\to 0} x$.

Now $\lim\limits_{\theta\to 0} x = \lim\limits_{\theta\to 0} r - \lim\limits_{\theta\to 0} \frac{r\theta(1 - \cos \theta)}{\theta - \sin \theta} =$

$r - \lim\limits_{\theta\to 0} \frac{r(1 - \cos \theta) + r\theta(\sin \theta)}{1 - \cos \theta} =$

$r - r \lim\limits_{\theta\to 0} \frac{\sin \theta + \sin \theta + \theta \cos \theta}{\sin \theta} =$

$r - r \lim\limits_{\theta\to 0} \frac{2 \cos \theta + \cos \theta - \theta \sin \theta}{\cos \theta} = r - r(3) = -2r.$

The limiting position of B as P approaches A is $(-2r,0)$.

41. $\lim\limits_{p\to w} \frac{A}{p^2 - w^2} (\sin wt - \sin pt) = \lim\limits_{p\to w} \frac{A(-t \cos pt)}{2p} =$

$\frac{-At \cos wt}{2w}$.

42. $V_n = \int_0^1 2\pi x(x^n)dx = \int_0^1 2\pi x^{n+1}dx$ and $H_n =$

$\pi\int_0^1 (x^n)^2 dx = \pi\int_0^1 x^{2n}dx.$

(a) $\lim\limits_{n\to +\infty} V_n = \lim\limits_{n\to +\infty} \left[\frac{2\pi\ x^{n+2}}{n + 2}\Big|_0^1\right] = \lim\limits_{n\to +\infty} \frac{2\pi}{n + 2} = 0.$

(b) $\lim\limits_{n\to +\infty} H_n = \lim\limits_{n\to +\infty} \left[\frac{\pi\ x^{2n+1}}{2n + 1}\Big|_0^1\right] = \lim\limits_{n\to +\infty} \frac{\pi}{2n + 1} = 0.$

(c) $\lim\limits_{n\to +\infty} \frac{V_n}{H_n} = \lim\limits_{n\to +\infty} \frac{\frac{2\pi}{n + 2}}{\frac{\pi}{2n + 1}} = \lim\limits_{n\to +\infty} \frac{2(2n + 1)}{n + 2} =$

$\lim\limits_{n\to +\infty} \frac{4}{1} = 4.$

Problem Set 10.2, page 607

1. $\lim\limits_{x\to \frac{\pi}{2}} \frac{1 + \sec x}{\tan x} = \lim\limits_{x\to \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim\limits_{x\to \frac{\pi}{2}} \sin x = 1.$

2. $\lim\limits_{x\to \frac{1}{2}} \frac{\sec 3\pi x}{\tan 3\pi x} = \lim\limits_{x\to \frac{1}{2}} \frac{1}{\sin 3\pi x} = \frac{1}{-1} = -1.$

3. $\lim\limits_{x\to +\infty} \frac{\ln(17 + x)}{x} = \lim\limits_{x\to +\infty} \frac{\frac{1}{17 + x}}{1} = 0.$

4. $\lim\limits_{x\to 0^+} \frac{1 - \ln x}{e^{1/x}} = \lim\limits_{x\to 0^+} \frac{-1/x}{(-1/x^2)e^{1/x}} = \lim\limits_{x\to 0^+} \frac{x}{e^{1/x}} = 0.$

5. $\lim\limits_{x\to +\infty} \frac{e^x + 1}{x^4 + x^3} = \lim\limits_{x\to +\infty} \frac{e^x}{4x^3 + 3x^2} = \lim\limits_{x\to +\infty} \frac{e^x}{12x^2 + 6x} =$

$\lim\limits_{x\to +\infty} \frac{e^x}{24x + 6} = \lim\limits_{x\to +\infty} \frac{e^x}{24} = +\infty.$

6. $\lim\limits_{x\to +\infty} \frac{2^x}{x^3} = \lim\limits_{x\to +\infty} \frac{(\ln 2)2^x}{3x^2} = \lim\limits_{x\to +\infty} \frac{(\ln 2)^2 2^x}{6x} =$

$\lim\limits_{x\to +\infty} \frac{(\ln 2)^3 2^x}{6} = +\infty.$

7. $\lim\limits_{x\to +\infty} \frac{2x^4}{e^{3x}} = \lim\limits_{x\to +\infty} \frac{8x^3}{3e^{3x}} = \lim\limits_{x\to +\infty} \frac{24x^2}{9e^{3x}} = \lim\limits_{x\to +\infty} \frac{48x}{27e^{3x}} =$

$\lim\limits_{x\to +\infty} \frac{48}{81e^x} = 0.$

8. $\lim\limits_{x\to +\infty} \frac{\ln(x + e^x)}{x} = \lim\limits_{x\to +\infty} \frac{\frac{1 + e^x}{x + e^x}}{1} = \lim\limits_{x\to +\infty} \frac{e^x}{1 + e^x} =$

$\lim\limits_{x\to +\infty} \frac{1}{\frac{1}{e^x} + 1} = 1.$

9. $\lim\limits_{t\to +\infty} \frac{t \ln t}{(t + 2)^2} = \lim\limits_{t\to +\infty} \frac{\ln t + 1}{2(t + 2)} = \lim\limits_{t\to +\infty} \frac{\frac{1}{t}}{2} = 0.$

10. $\lim\limits_{x\to +\infty} \frac{x + e^{2x}}{\ln x + e^{2x}} = \lim\limits_{x\to +\infty} \frac{1 + 2e^{2x}}{\frac{1}{x} + 2e^{2x}} = \lim\limits_{x\to +\infty} \frac{4e^{2x}}{\frac{-1}{x^2} + 4e^{2x}} =$

$\lim\limits_{x\to +\infty} \frac{4}{-\frac{1}{x^2 e^{2x}} + 4} = 1.$

11. $\lim_{x\to+\infty} x(e^{-x} - 1) = -\infty.$

12. $\lim_{t\to 0} \sin 3t \cos 2t = \lim_{t\to 0} \frac{\sin 3t}{\tan 2t} = \lim_{t\to 0} \frac{3\cos 3t}{2\sec^2 2t} = \frac{3}{2}.$

13. $\lim_{x\to 0^+} xe^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-\frac{1}{x^2}e^{\frac{1}{x}}}{-\frac{1}{x^2}} = +\infty.$

14. $\lim_{x\to+\infty} x \sin\frac{\pi}{x} = \lim_{x\to+\infty} \frac{\sin\frac{\pi}{x}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{-\frac{\pi}{x^2}\cos\frac{\pi}{x}}{-\frac{1}{x^2}} = \pi.$

15. $\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{2\ln x}{x}}{-\frac{1}{x^2}} =$

$\lim_{x\to 0^+} \frac{2\ln x}{-\frac{1}{x}} = \lim_{x\to 0^+} \frac{2/x}{-\frac{1}{x^2}} = 0.$

16. $\lim_{x\to\frac{\pi}{2}} \cos 3x \sec 5x = \lim_{x\to\frac{\pi}{2}} \frac{\cos 3x}{\cos 5x} = \lim_{x\to\frac{\pi}{2}} \frac{-3\sin 3x}{-5\sin 5x} =$

$\frac{-3(-1)}{-5(+1)} = -\frac{3}{5}.$

17. $\lim_{x\to\frac{\pi}{2}} \tan x \tan 2x = \lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{\cot x} = \lim_{x\to\frac{\pi}{2}} \frac{2\sec^2 2x}{-\csc^2 x} =$

$\frac{2}{-1} = -2.$

18. $\lim_{x\to 0} \csc x \sin^{-1} x = \lim_{x\to 0} \frac{\sin^{-1} x}{\sin x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x} = 1.$

19. $\lim_{x\to 1} \left[\frac{1}{x-1} - \frac{1}{\ln x}\right] = \lim_{x\to 1} \frac{x\ln x - x + 1}{x\ln x - \ln x} =$

$\lim_{x\to 1} \frac{(\ln x) + 1 - 1}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}.$

20. $\lim_{x\to 1} \left[\frac{x}{\ln x} - \frac{1}{x\ln x}\right] = \lim_{x\to 1} \frac{x^2 - 1}{x\ln x} = \lim_{x\to 1} \frac{2x}{1 + \ln x} = 2.$

21. $\lim_{x\to 0^+} (\csc x - \csc 2x) = \lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{\sin 2x}\right) =$

$\lim_{x\to 0^+} \frac{\sin 2x - \sin x}{\sin x \sin 2x} =$

$\lim_{x\to 0^+} \frac{2\cos 2x - \cos x}{\cos x \sin 2x + 2\sin x \cos 2x} = +\infty.$

22. $\lim_{t\to 0} \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) = \lim_{t\to 0} \frac{t - e^t + 1}{t(e^t - 1)} =$

$\lim_{t\to 0} \frac{1 - e^t}{e^t + te^t - 1} = \lim_{t\to 0} \frac{-e^t}{e^t + te^t + e^t} = -\frac{1}{2}.$

23. Let $x = \frac{1}{t}$, so that $t \to 0^+$ as $x \to +\infty$. Thus,

$\lim_{x\to+\infty} (x^2 - \sqrt{x^4 + x^2 + 7}) = \lim_{t\to 0^+} \left(\frac{1}{t^2} - \sqrt{\frac{1}{t^4} + \frac{1}{t^2} + 7}\right) =$

$\lim_{t\to 0^+} \left(\frac{1 - \sqrt{7t^4 + t^2 + 1}}{t^2}\right) = \lim_{t\to 0^+} \frac{-14t^3 - t}{2t\sqrt{7t^4 + t^2 + 1}} =$

$\lim_{t\to 0^+} \frac{-14t^2 - 1}{2\sqrt{7t^4 + t^2 + 1}} = \frac{-1}{2}.$

24. $\lim_{x\to\frac{\pi}{2}} (x\tan x - \frac{\pi}{2}\sec x) = \lim_{x\to\frac{\pi}{2}} \left(\frac{x\sin x - \frac{\pi}{2}}{\cos x}\right) =$

$\lim_{x\to\frac{\pi}{2}} \left(\frac{x\cos x + \sin x}{-\sin x}\right) = \frac{0 + 1}{-1} = -1.$

25. $\lim_{x\to 4} \left(\frac{7}{x^2 - x - 12} - \frac{1}{x - 4}\right) = \lim_{x\to 4} \frac{7 - (x + 3)}{x^2 - x - 12} =$

$\lim_{x\to 4} \frac{-1}{2x - 1} = -\frac{1}{7}.$

26. $\lim_{x\to 1} \left[\frac{n}{x^n - 1} - \frac{m}{x^m - 1}\right] = \lim_{x\to 1} \left(\frac{nx^m - n - mx^n + m}{x^{n+m} - x^m - x^n + 1}\right) =$

$\lim_{x\to 1} \frac{nmx^{m-1} - mnx^{n-1}}{(n + m)x^{n+m-1} - mx^{m-1} - nx^{n-1}} =$

$\lim_{x\to 1} \frac{nmx^{m-n} - mn}{(n + m)x^m - mx^{m-n} - n}$, dividing by x^{n-1}, $=$

$\lim_{x\to 1} \frac{(nm)(m - n)x^{m-n-1}}{m(n + m)x^{m-1} - m(m - n)x^{m-n-1}} =$

$\frac{(nm)(m - n)}{m(n + m) - m(m - n)} = \frac{n(m - n)}{n + m - m + n} = \frac{m - n}{2}$. The same result is obtained if one divides by x^{m-1} in the third step.

27. $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}$. Now, $\lim_{x\to 0^+} x\ln x =$

$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0.$ Hence, $\lim_{x\to 0^+} x^x = e^0 = 1.$

28. $\lim_{x\to 0^+} (\sinh x)^x = \lim_{x\to 0^+} e^{x\ln(\sinh x)}$. Now

$\lim_{x\to 0^+} x\ln(\sinh x) = \lim_{x\to 0^+} \frac{\ln(\sinh x)}{1/x} =$

$\lim_{x\to 0^+} \frac{\frac{\cosh x}{\sinh x}}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2\cosh x}{\sinh x} =$

$\lim_{x\to 0^+} \frac{-2x\cosh x - x^2\sinh x}{\cosh x} = 0.$ Hence,

$\lim_{x\to 0^+} (\sinh x)^x = e^0 = 1.$

29. $\lim_{y\to 0^+} (e^y - 1)^y = \lim_{y\to 0^+} e^{y \ln(e^y - 1)}$. Now

$\lim_{y\to 0^+} y \ln (e^y - 1) = \lim_{y\to 0^+} \frac{\ln(e^y - 1)}{1/y} =$

$\lim_{y\to 0^+} \frac{\frac{e^y}{e^y - 1}}{-1/y^2} = \lim_{y\to 0^+} \frac{-y^2 e^y}{e^y - 1} = \lim_{y\to 0^+} \frac{-y^2 e^y - 2ye^y}{e^y} = 0.$

Hence, $\lim_{y\to 0^+} (e^y - 1)^y = e^0 = 1$.

30. $\lim_{x\to\frac{\pi}{2}^-} (\frac{5\pi}{2} - 5x)^{\cos x} = \lim_{x\to\frac{\pi}{2}^-} e^{\cos x \ln(\frac{5\pi}{2} - 5x)}$.

Now $\lim_{x\to\frac{\pi}{2}^-} \cos x \ln (\frac{5\pi}{2} - 5x) = \lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\frac{5\pi}{2} - 5x)}{\sec x} =$

$\lim_{x\to\frac{\pi}{2}^-} \frac{(\frac{-5}{5\pi/2 - 5x})}{\sec x \tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{-5 \cos^2 x}{\sin x(\frac{5\pi}{2} - 5x)} =$

$\lim_{x\to\frac{\pi}{2}^-} \frac{10 \cos x \sin x}{-5 \sin x + (\cos x)(\frac{5\pi}{2} - 5x)} = \frac{0}{-5} = 0.$ Thus,

$\lim_{x\to\frac{\pi}{2}^-} (\frac{5\pi}{2} - 5x)^{\cos x} = e^0 = 1.$

31. $\lim_{x\to\frac{\pi}{2}^-} (\cos x)^{(x-\frac{\pi}{2})} = \lim_{x\to\frac{\pi}{2}^-} e^{(x-\frac{\pi}{2})\ln \cos x}$. Now,

$\lim_{x\to\frac{\pi}{2}^-} (x - \frac{\pi}{2})\ln \cos x = \lim_{x\to\frac{\pi}{2}^-} \frac{\ln \cos x}{\frac{1}{x - \frac{\pi}{2}}} =$

$\lim_{x\to\frac{\pi}{2}^-} \frac{-\tan x}{\frac{-1}{(x - \frac{\pi}{2})^2}} = \lim_{x\to\frac{\pi}{2}^-} \frac{(\sin x)(x - \pi/2)^2}{\cos x} =$

$\lim_{x\to\frac{\pi}{2}^-} \frac{2(\sin x)(x - \frac{\pi}{2}) + (\cos x)(x - \frac{\pi}{2})^2}{-\sin x} = \frac{0}{-1} = 0.$

Hence, $\lim_{x\to\frac{\pi}{2}^-} (\cos x)^{(x-\frac{\pi}{2})} = e^0 = 1.$

32. $\lim_{x\to\frac{\pi}{4}^-} (\frac{\pi}{4} - x)^{\cos 2x} = \lim_{x\to\frac{\pi}{4}^-} e^{\cos 2x \ln(\frac{\pi}{4} - x)}$. Now

$\lim_{x\to\frac{\pi}{4}^-} (\cos 2x) \ln(\frac{\pi}{4} - x) = \lim_{x\to\frac{\pi}{4}^-} \frac{\ln(\frac{\pi}{4} - x)}{\sec 2x} =$

$\lim_{x\to\frac{\pi}{4}^-} \frac{\frac{-1}{\frac{\pi}{4} - x}}{2 \sec 2x \tan 2x} = \lim_{x\to\frac{\pi}{4}^-} \frac{-\cos^2 2x}{2(\frac{\pi}{4} - x)\sin 2x} =$

$\lim_{x\to\frac{\pi}{4}^-} \frac{4 \cos 2x \sin 2x}{-2 \sin 2x + 4 (\frac{\pi}{4} - x) \cos 2x} = \frac{0}{-2} = 0.$ Thus,

$\lim_{x\to\frac{\pi}{4}^-} (\frac{\pi}{4} - x)^{\cos 2x} = e^0 = 1.$

33. $\lim_{x\to 0^+} (\cot x)^{x^2} = \lim_{x\to 0^+} e^{x^2 \ln \cot x}$. Now

$\lim_{x\to 0^+} x^2 \ln \cot x = \lim_{x\to 0^+} \frac{\ln \cot x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\frac{-\csc^2 x}{\cot x}}{-\frac{2}{x^3}} =$

$\lim_{x\to 0^+} \frac{x^3}{2 \sin x \cos x} = \lim_{x\to 0^+} \frac{3x^2}{2 \cos^2 x - 2 \sin^2 x} = \frac{0}{2} = 0.$

Hence, $\lim_{x\to 0^+} (\cot x)^{x^2} = e^0 = 1.$

34. $\lim_{x\to 0^+} (\cot x)^{\sin x} = \lim e^{\sin x \ln \cot x}$. Now

$\lim_{x\to 0^+} \sin x \ln \cot x = \lim_{x\to 0^+} \frac{\ln \cot x}{\csc x} =$

$\lim_{x\to 0^+} \frac{-\frac{\csc^2 x}{\cot x}}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{\sin x}{\cos^2 x} = 0.$ Hence,

$\lim_{x\to 0^+} (\cot x)^{\sin x} = e^0 = 1.$

35. $\lim_{x\to+\infty} (\frac{x}{x - 2})^x = \lim_{x\to+\infty} e^{x \ln \frac{x}{x-2}}$. Now

$\lim_{x\to+\infty} x \ln \frac{x}{x - 2} = \lim_{x\to+\infty} \frac{\ln \frac{x}{x - 2}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{\frac{-2}{x(x - 2)}}{-\frac{1}{x^2}} =$

$\lim_{x\to+\infty} \frac{2x}{x - 2} = \lim_{x\to+\infty} \frac{2}{1} = 2.$ Hence,

$\lim_{x\to+\infty} (\frac{x}{x - 2})^x = e^2.$

36. $\lim_{x\to+\infty} (e^x + x)^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\frac{1}{x} \ln(e^x+x)}$. Now,

$\lim_{x\to+\infty} \frac{1}{x} \ln(e^x + x) = \lim_{x\to+\infty} \frac{\ln(e^x + x)}{x} = \lim_{x\to+\infty} \frac{\frac{e^x + 1}{e^x + x}}{1} =$

$\lim_{x\to+\infty} \frac{1 + \frac{1}{e^x}}{1 + \frac{x}{e^x}} = 1$ since $\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0.$

Hence, $\lim_{x\to+\infty} (e^x + x)^{\frac{1}{x}} = e^1 = e.$

37. $\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} e^{(1/n)\ln n}$. Now $\lim_{n\to+\infty} \frac{1}{n} \ln n =$

$\lim_{n\to+\infty} \frac{\ln n}{n} = \lim_{n\to+\infty} \frac{1/n}{1} = 0.$ Hence, $\lim_{n\to+\infty} \sqrt[n]{n} = e^0 = 1.$

38. $\lim_{x\to0^+} (-\ln x)^x = \lim_{x\to0^+} e^{x\ln(-\ln x)}$. Now

$\lim_{x\to0^+} x\ln(-\ln x) = \lim_{x\to0^+} \frac{\ln(-\ln x)}{1/x} = \lim_{x\to0^+} \frac{\frac{-1}{\ln x}\cdot\frac{-1}{x}}{-\frac{1}{x^2}} =$

$\lim_{x\to0^+} \frac{-x}{\ln x} = 0$. Hence $\lim_{x\to0^+} (-\ln x)^x = e^0 = 1$.

39. $\lim_{x\to0} (1+\tan x)^{\frac{1}{x}} = \lim_{x\to0} e^{\frac{1}{x}\ln(1+\tan x)}$. Now,

$\lim_{x\to0} \frac{1}{x}\ln(1+\tan x) = \lim_{x\to0} \frac{\frac{\sec^2 x}{1+\tan x}}{1} = 1$. Hence,

$\lim_{x\to0} (1+\tan x)^{\frac{1}{x}} = e^1 = e$.

40. $\lim_{x\to+\infty} (1+\frac{3}{x})^x = \lim_{x\to+\infty} e^{x\ln(1+\frac{3}{x})}$. Now

$\lim_{x\to+\infty} x\ln(1+\frac{3}{x}) = \lim_{x\to+\infty} \frac{\ln(1+\frac{3}{x})}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{\frac{-\frac{3}{x^2}}{1+\frac{3}{x}}}{-\frac{1}{x^2}} =$

$\lim_{x\to+\infty} \frac{3}{1+\frac{3}{x}} = 3$. Thus, $\lim_{x\to+\infty} (1+\frac{3}{x})^x = e^3$.

41. $\lim_{x\to0} (1+2x)^{\frac{3}{x}} = \lim_{x\to0} e^{\frac{3}{x}\ln(1+2x)}$. Now

$\lim_{x\to0} \frac{3\ln(1+2x)}{x} = \lim_{x\to0} \frac{\frac{6}{1+2x}}{1} = 6$. Hence,

$\lim_{x\to0} (1+2x)^{\frac{3}{x}} = e^6$.

42. $\lim_{x\to0^+} (1+x)^{\ln x} = \lim_{x\to0^+} e^{(\ln x)\ln(1+x)}$. Now

$\lim_{x\to0^+} (\ln x)\ln(1+x) = \lim_{x\to0^+} \frac{\ln(1+x)}{\frac{1}{\ln x}} =$

$\lim_{x\to0^+} \frac{\frac{1}{1+x}}{\frac{1}{x(\ln x)^2}} = \lim_{x\to0^+} \frac{(\ln x)^2}{\frac{1+x}{x}} = \lim_{x\to0^+} \frac{\frac{2\ln x}{x}}{-\frac{1}{x^2}} =$

$\lim_{x\to0^+} \frac{2\ln x}{-\frac{1}{x}} = \lim_{x\to0^+} \frac{\frac{2}{x}}{\frac{1}{x^2}} = \lim_{x\to0^+} 2x = 0$. Thus,

$\lim_{x\to0^+} (1+x)^{\ln x} = e^0 = 1$.

43. $\lim_{x\to+\infty} (\cos\frac{2}{x})^{x^2} = \lim_{x\to+\infty} e^{x^2\ln(\cos\frac{2}{x})}$. Now

$\lim_{x\to+\infty} x^2\ln(\cos\frac{2}{x}) = \lim_{x\to+\infty} \frac{\ln(\cos\frac{2}{x})}{\frac{1}{x^2}} =$

$\lim_{x\to+\infty} \frac{\left(\frac{(\frac{-2}{x^2})(-\sin\frac{2}{x})}{\cos\frac{2}{x}}\right)}{-2/x^3} = \lim_{x\to+\infty} \frac{-x\sin\frac{2}{x}}{\cos\frac{2}{x}} = \lim_{x\to+\infty} -x\tan\frac{2}{x} =$

$\lim_{x\to+\infty} \frac{\tan\frac{2}{x}}{-1/x} = \lim_{x\to+\infty} \frac{(\sec^2\frac{2}{x})(\frac{-2}{x^2})}{\frac{1}{x^2}} = \lim_{x\to+\infty} -2\sec^2\frac{2}{x} = -2$.

Hence, $\lim_{x\to+\infty} (\cos\frac{2}{x})^{x^2} = e^{-2}$.

44. $\lim_{y\to0} (\frac{\sin y}{y})^{1/y} = \lim_{y\to0} e^{\frac{1}{y}\ln(\frac{\sin y}{y})}$. Now

$\lim_{y\to0} \frac{1}{y}\ln(\frac{\sin y}{y}) = \lim_{y\to0} \frac{\ln(\sin y) - \ln y}{y} =$

$\lim_{y\to0} \frac{\frac{\cos y}{\sin y} - \frac{1}{y}}{1} = \lim_{y\to0} \frac{y\cos y - \sin y}{y\sin y} =$

$\lim_{y\to0} \frac{\cos y - y\sin y - \cos y}{\sin y + y\cos y} = 0$. Hence,

$\lim_{y\to0} (\frac{\sin y}{y})^{1/y} = e^0 = 1$.

45. $\lim_{x\to0} (1+x)^{\cot x} = \lim_{x\to0} e^{(\cot x)\ln(1+x)}$. Now,

$\lim_{x\to0} (\cot x)\ln(1+x) = \lim_{x\to0} \frac{\cos x\ln(1+x)}{\sin x} =$

$\lim_{x\to0} \frac{\frac{\cos x}{1+x} - \sin x\ln(1+x)}{\cos x} = \frac{1}{1} = 1$. Thus,

$\lim_{x\to0} (1+x)^{\cot x} = e^1 = e$.

46. $\lim_{x\to2} (1-\frac{x}{2})^{\tan\pi x} = \lim_{x\to2} e^{\tan\pi x\ln(1-\frac{x}{2})}$. Now,

$\lim_{x\to2} \tan\pi x\ln(1-\frac{x}{2}) = \lim_{x\to2} \frac{\ln(1-\frac{x}{2})}{\cot\pi x} =$

$\lim_{x\to2} \frac{-\frac{1}{2}(\frac{1}{1-\frac{x}{2}})}{-\pi\csc^2\pi x} = \lim_{x\to2} \frac{\sin^2\pi x}{2\pi(1-\frac{x}{2})} =$

$\lim_{x\to2} \frac{2\pi\sin\pi x\cos\pi x}{-\pi} = 0$. Hence,

$\lim_{x\to2} (1-\frac{x}{2})^{\tan\pi x} = e^0 = 1$.

47. $\lim_{x\to0^-} (1+x)^{\ln|x|} = \lim_{x\to0^-} e^{(\ln|x|)[\ln(1+x)]}$. Now

$\lim_{x\to0^-} \ln|x|\ln(1+x) = \lim_{x\to0^-} \frac{\ln|x|}{\frac{1}{\ln(1+x)}} =$

$$\lim_{x\to 0^-} \frac{\frac{1}{x}}{\frac{-1}{(1+x)[\ln(1+x)]^2}} = \lim_{x\to 0^-} \frac{-(1+x)[\ln(1+x)]^2}{x} =$$

$$\lim_{x\to 0^-} \frac{-2\ln(1+x) - [\ln(1+x)]^2}{1} = 0. \text{ Thus,}$$

$$\lim_{x\to 0^-} (1+x)^{\ln|x|} = e^0 = 1.$$

48. $\lim_{x\to 0} (e^{2x} + 2x)^{\frac{1}{4x}} = \lim_{x\to 0} e^{\frac{1}{4x}\ln(e^{2x}+2x)}$. Now

$$\lim_{x\to 0} \frac{1}{4x}\ln(e^{2x} + 2x) = \lim_{x\to 0} \frac{\frac{2e^{2x}+2}{e^{2x}+2x}}{4} = 1. \text{ Hence,}$$

$$\lim_{x\to 0} (e^{2x} + 2x)^{\frac{1}{4x}} = e^1 = e.$$

49. $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\sec x} = \lim_{x\to\frac{\pi}{2}} e^{\sec x \ln(\sin x)}$. Now,

$$\lim_{x\to\frac{\pi}{2}} \sec x \ln(\sin x) = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\cot x}{-\sin x} =$$

$\frac{0}{-1} = 0$. Thus, $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\sec x} = e^0 = 1$.

50. $\lim_{x\to 1} x^{\frac{1}{1-x}} = \lim_{x\to 1} e^{\frac{1}{1-x}\ln x}$. Now $\lim_{x\to 1} \frac{1}{1-x}\ln x =$

$\lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$. Therefore, $\lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1} = \frac{1}{e}$.

51. (a) Try $x = 10^{10}$. Then $\frac{\ln(17+x)}{x} \approx 2.3\times 10^{-9} \approx 0$.

(b) Try $x = 0.01$. Then $xe^{1/x} \approx 2.7\times 10^{41}$; if $x = 0.001$, $xe^{1/x} \approx 10\times 10^{96}$.

(c) Try $x = 10^{-10}$. Then $x^x = 1$.

52. $g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x} = \lim_{x\to 0} \frac{x^2 \sin 1/x}{x} =$

$\lim_{x\to 0} x \sin(\frac{1}{x}) = 0$ (since $|\sin \frac{1}{x}| \leq 1$).

53. $\lim_{x\to+\infty} \frac{xf'(x)}{f(x)} = \lim_{x\to+\infty} \frac{xf''(x) + f'(x)}{f'(x)} =$

$$\lim_{x\to+\infty} \frac{xf'''(x) + f''(x) + f''(x)}{f''(x)} = \lim_{x\to+\infty} \frac{xf'''(x)}{f''(x)} +$$

$$\lim_{x\to+\infty} 2 = 1 + 2 = 3.$$

54. $\lim_{x\to+\infty} (\frac{x+c}{x-c}) = \lim_{x\to+\infty} e^{x\ln(\frac{x+c}{x-c})}$. Now $\lim_{x\to+\infty} \frac{\ln(\frac{x+c}{x-c})}{\frac{1}{x}} =$

$$\lim_{x\to+\infty} \frac{\frac{-2c}{(x+c)(x-c)}}{-1/x^2} = \lim_{x\to+\infty} \frac{2cx^2}{x^2 - c^2} = \lim_{x\to+\infty} \frac{4cx}{2x} = 2c.$$

Thus, $\lim_{x\to+\infty} (\frac{x+c}{x-c})^x = e^{2c} = 4$ provided $c = \ln 2$.

55. $\lim_{x\to+\infty} \frac{1}{x}\int_0^x e^t dt = \lim_{x\to+\infty} \frac{\int_0^x e^t dt}{x} = \lim_{x\to+\infty} \frac{e^x - 1}{1} = +\infty.$

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1. $\int_1^\infty \frac{dx}{x\sqrt{x}} = \lim_{b\to+\infty} \int_1^b x^{-3/2}dx = \lim_{b\to+\infty} (-2x^{-\frac{1}{2}})\Big|_1^b =$

$$\lim_{b\to+\infty} (-\frac{2}{\sqrt{b}} + \frac{2}{1}) = 2.$$

2. $\int_1^\infty \frac{dx}{(4x+3)^2} = \lim_{b\to+\infty} \int_1^b (4x+3)^{-2}dx =$

$$\lim_{b\to+\infty} \tfrac{1}{4}(\frac{-1}{4x+3}\Big|_1^b) = \lim_{b\to+\infty} \tfrac{1}{4}(\frac{-1}{4b+3} + \frac{1}{7}) = \frac{1}{28}.$$

3. $\int_3^\infty \frac{dx}{x^2+9} = \lim_{b\to+\infty} \int_3^b \frac{dx}{x^2+9} = \lim_{b\to+\infty} (\frac{1}{3}\tan^{-1}\frac{x}{3})\Big|_3^b =$

$$\lim_{b\to+\infty} (\frac{1}{3}\tan^{-1}\frac{b}{3} - \frac{\pi}{12}) = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}.$$

4. $\int_{-\infty}^0 \frac{dx}{x^2+16} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{x^2+16} = \lim_{a\to-\infty} (\tfrac{1}{4}\tan^{-1}\frac{x}{4})\Big|_a^0 =$

$$\lim_{a\to-\infty} (0 - \tfrac{1}{4}\tan^{-1}\frac{a}{4}) = -\tfrac{1}{4}(-\frac{\pi}{2}) = \frac{\pi}{8}.$$

5. $\int_1^\infty \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty} \ln x\Big|_1^b =$

$\lim_{b\to+\infty} (\ln b - 0) = +\infty$. The integral is divergent.

6. $\int_{-\infty}^2 \frac{dx}{(4-x)^2} = \lim_{a\to-\infty} \int_a^2 \frac{dx}{(4-x)^2} = \lim_{a\to-\infty} \frac{1}{4-x}\Big|_a^2 =$

$$\lim_{a\to-\infty} \left[\frac{1}{2} - \frac{1}{4-a}\right] = \frac{1}{2}.$$

7. $\int_0^\infty \frac{dx}{(x+1)(x+2)} = \lim_{b\to+\infty} \int_0^b \frac{dx}{(x+1)(x+2)} =$

$$\lim_{b\to+\infty} \left[\int_0^b \frac{1}{x+1}dx + \int_0^b \frac{-1}{x+2}dx\right] =$$

$$\lim_{b\to+\infty} (\ln|x+1| - \ln|x+2|)\Big|_0^b =$$

$$\lim_{b\to+\infty} [\ln(\frac{b+1}{b+2}) + \ln 2] = 0 + \ln 2 = \ln 2.$$

8. $\int_2^\infty \frac{x\,dx}{(x+1)(x+2)} = \lim_{b\to+\infty} \int_2^b \frac{x\,dx}{(x+1)(x+2)} =$

$\lim_{b\to+\infty} \left[\int_2^b \frac{-1}{x+1}\,dx + \int_2^b \frac{2}{x+2}\,dx\right] =$

$\lim_{b\to+\infty} (-\ln(x+1) + 2\ln(x+2))\Big|_2^b =$

$\lim_{b\to+\infty} \ln\left[\frac{(b+2)^2}{b+1}\right] - \ln\frac{16}{3} = +\infty$. The integral is divergent.

9. $\int_0^\infty 4e^{8x}dx = \lim_{b\to+\infty} \int_0^b 4e^{8x}dx = \lim_{b\to+\infty} (\frac{e^{8x}}{2})\Big|_0^b =$

$\lim_{b\to+\infty} (\frac{e^{8b}}{2} - \frac{1}{2}) = +\infty$. The interval is divergent.

10. $\int_0^\infty \frac{e^{-\sqrt{u}}}{\sqrt{u}}\,du = \lim_{b\to+\infty} \int_0^b \frac{e^{-\sqrt{u}}}{\sqrt{u}}\,du = \lim_{b\to+\infty} -2e^{-\sqrt{u}}\Big|_0^b =$

$\lim_{b\to+\infty} (-2e^{-\sqrt{b}} + 2) = 2.$

11. $\int_1^\infty \frac{x\,dx}{1+x^4} = \lim_{b\to+\infty} \int_1^b \frac{x}{1+x^4}\,dx$. Now $\int \frac{x}{1+x^4}\,dx =$ $\frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}x^2 + C$, where $u = x^2$. Thus, $\int_1^\infty \frac{x\,dx}{1+x^4} = \lim_{b\to+\infty} (\frac{1}{2}\tan^{-1}x^2)\Big|_1^b =$

$\lim_{b\to+\infty} (\frac{1}{2}\tan^{-1}b^2 - \frac{1}{2}\tan^{-1}1) = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}.$

12. $\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{b\to+\infty} \int_e^b \frac{dx}{x(\ln x)^2} = \lim_{b\to+\infty} (-\frac{1}{\ln x})\Big|_e^b =$

$\lim_{b\to+\infty} (\frac{1}{1} - \frac{1}{\ln b}) = 1.$

13. $\int_e^\infty \frac{dx}{x\ln x} = \lim_{b\to+\infty} \int_e^b \frac{dx}{x\ln x} = \lim_{b\to+\infty} \ln(\ln x)\Big|_e^b = +\infty.$

The integral is divergent.

14. $\int_0^\infty e^{-x}\sin x\,dx = \lim_{b\to+\infty} \int_0^b e^{-x}\sin x\,dx$. Integrating by parts, we get $\lim_{b\to+\infty} [-\frac{1}{2}(e^{-x}\sin x + e^{-x}\cos x)\Big|_0^b]$.

The limit does not exist; hence, the integral is divergent.

15. $\int_{-\infty}^{-2} \frac{dx}{(x-1)^4} = \lim_{a\to-\infty} \int_a^{-2} \frac{dx}{(x-1)^4} = \lim_{a\to-\infty} \frac{-1}{3(x-1)^3}\Big|_a^{-2} =$

$\lim_{a\to-\infty} [\frac{1}{81} + \frac{1}{3(a-1)^3}] = \frac{1}{81}.$

16. $\int_{-\infty}^1 \frac{3t\,dt}{(3t^2+1)^3} = \lim_{a\to-\infty} \int_a^1 \frac{3t\,dt}{(3t^2+1)^3} =$

$\lim_{a\to-\infty} \frac{-1}{4(3t^2+1)^2}\Big|_a^1 = \lim_{a\to-\infty} [\frac{-1}{64} + \frac{1}{4(3a^2+1)^2}] = \frac{-1}{64}.$

17. $\int_{-\infty}^0 xe^x dx = \lim_{a\to-\infty} \int_a^0 xe^x dx = \lim_{a\to-\infty} ((xe^x - e^x)\Big|_a^0) =$

$\lim_{a\to-\infty} (-1 - ae^a + e^a) = -1 - \lim_{-t\to+\infty} \frac{-t}{e^t} + 0 =$

$-1 - \lim_{-t\to+\infty} \frac{-1}{e^t} = -1.$

18. $\int_{-\infty}^\infty (x^2 + 2x + 2)^{-1}dx = \lim_{a\to-\infty} \int_a^0 (x^2 + 2x + 2)^{-1}dx +$

$\lim_{b\to+\infty} \int_0^b (x^2 + 2x + 2)^{-1}dx = \lim_{a\to-\infty} \int_a^0 \frac{1}{(x+1)^2 + 1}\,dx +$

$\lim_{b\to+\infty} \int_0^b \frac{1}{(x+1)^2 + 1}\,dx = \lim_{a\to-\infty} \tan^{-1}(x+1)\Big|_a^0 +$

$\lim_{b\to+\infty} \tan^{-1}(x+1)\Big|_0^b = \lim_{a\to-\infty} [\tan^{-1}1 - \tan^{-1}(a+1)] +$

$\lim_{b\to+\infty} [\tan^{-1}(b+1) - \tan^{-1}1] = \frac{\pi}{4} - (-\frac{\pi}{2}) + \frac{\pi}{2} - \frac{\pi}{4} = \pi.$

19. $\int_{-\infty}^\infty \frac{x\,dx}{1+x^4} = \lim_{a\to-\infty} \int_a^0 \frac{-x}{1+x^4}\,dx + \lim_{b\to+\infty} \int_0^b \frac{x}{1+x^4}\,dx =$

$\lim_{a\to-\infty} (-\frac{1}{2}\tan^{-1}x^2\Big|_a^0) + \lim_{b\to+\infty} (\frac{1}{2}\tan^{-1}x^2\Big|_0^b) =$

$\lim_{a\to-\infty} (\frac{\tan^{-1}a^2}{2}) + \lim_{b\to+\infty} \frac{\tan^{-1}b^2}{2} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$

20. $\int_{-\infty}^\infty x^2e^{-x^3}dx = \lim_{a\to-\infty} \int_a^0 x^2e^{-x^3}dx + \lim_{b\to+\infty} \int_0^b x^2e^{-x^3}dx =$

$\lim_{a\to-\infty} \frac{-e^{-x^3}}{3}\Big|_a^0 + \lim_{b\to+\infty} \frac{-e^{-x^3}}{3}\Big|_0^b = \lim_{a\to-\infty} (-\frac{1}{3} + \frac{e^{-a^3}}{3}) +$

$\lim_{b\to+\infty} (\frac{-e^{-b^3}}{3} + \frac{1}{3}) = +\infty$. Hence, the integral is divergent.

21. $\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)^2} = \lim_{a\to-\infty} \int_a^0 \frac{x\,dx}{(x^2+1)^2} + \lim_{b\to+\infty} \int_0^b \frac{x\,dx}{(x^2+1)^2} =$

$\lim_{a\to-\infty} \frac{-1}{2(x^2+1)}\Big|_a^0 + \lim_{b\to+\infty} \frac{-1}{2(x^2+1)}\Big|_0^b =$

$\lim_{a\to-\infty} (-\frac{1}{2} + \frac{1}{2(a^2+1)}) + \lim_{b\to+\infty} (\frac{-1}{2(b^2+1)} + \frac{1}{2}) =$

$-\frac{1}{2} + \frac{1}{2} = 0.$

22. $\int_{-\infty}^{\infty} \frac{e^x}{\cosh x}\,dx = \lim_{a\to-\infty} \int_a^0 \frac{e^x}{\frac{e^x + e^{-x}}{2}} +$

$\lim_{b\to+\infty} \int_0^b \frac{e^x}{\frac{e^x + e^{-x}}{2}}\,dx$. Now $\int \frac{2e^x}{e^x + e^{-x}}\,dx =$

$\int \frac{2e^x(e^x dx)}{e^{2x} + 1} = \int \frac{2u\,du}{u^2 + 1} = \ln(u^2 + 1) + C$ (where

$u = e^x) = \ln(e^{2x} + 1) + C$. Thus, $\int_{-\infty}^{\infty} \frac{e^x}{\cosh x}\,dx =$

$\lim_{a\to-\infty} \ln(e^{2x} + 1)\Big|_a^0 + \lim_{b\to+\infty} \ln(e^{2x} + 1)\Big|_0^b =$

$\lim_{a\to-\infty} [\ln(2) - \ln(e^{2a} + 1)] + \lim_{b\to+\infty} [\ln(e^{2b} + 1) - \ln(2)] =$

$\ln 2 + \infty = +\infty$. Hence, the integral is divergent.

23. $\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{a^2 + x^2} + \lim_{b\to+\infty} \int_0^b \frac{dx}{a^2 + x^2} =$

$\lim_{t\to-\infty} (\frac{1}{a} \tan^{-1} \frac{x}{a})\Big|_t^0) + \lim_{b\to+\infty} ((\frac{1}{a} \tan^{-1} \frac{x}{a})\Big|_0^b) =$

$\lim_{t\to-\infty} (\frac{1}{a} \tan^{-1} 0 - \frac{1}{a} \tan^{-1} \frac{t}{a}) +$

$\lim_{b\to+\infty} (\frac{1}{a} \tan^{-1} \frac{b}{a} - \frac{1}{a} \tan^{-1} \frac{0}{a}) = 0 + \frac{1}{a} \cdot \frac{\pi}{2} +$

$\frac{1}{a} \cdot \frac{\pi}{2} - 0 = \frac{\pi}{a}$.

24. $\int_{-\infty}^{\infty} \operatorname{sech} x\,dx = \int_{-\infty}^{\infty} \frac{1}{\cosh x}\,dx = \int_{-\infty}^{\infty} \frac{2}{e^x + e^{-x}}\,dx =$

$\int_{-\infty}^{\infty} \frac{2e^x}{e^{2x} + 1}\,dx = \int_{-\infty}^{0} \frac{2e^x}{e^{2x} + 1}\,dx + \int_0^{\infty} \frac{2e^x}{e^{2x}+1}dx =$

$\lim_{a\to-\infty} \int_a^0 \frac{2e^x}{e^{2x}+1}dx + \lim_{b\to\infty} \int_0^b \frac{2e^x dx}{e^{2x}+1} = \lim_{a\to-\infty} [2 \tan^{-1}(e^x)]\Big|_a^0 +$

$\lim_{b\to\infty} [2 \tan^{-1}(e^x)]\Big|_0^b = 2[\frac{\pi}{4} - 0] + 2[\frac{\pi}{2} - \frac{\pi}{4}] = \pi$.

25. $\int_0^{\infty} 2^{-x}\,dx = \lim_{b\to+\infty} \int_0^b 2^{-x}\,dx = \lim_{b\to+\infty} \frac{-2^{-x}}{\ln 2}\Big|_0^b =$

$\lim_{b\to+\infty} (\frac{-2^{-b}}{\ln 2} + \frac{1}{\ln 2}) = \frac{1}{\ln 2}$ square units.

26. $\int_0^{\infty} xe^{-x}\,dx = \lim_{b\to+\infty} \int_0^b xe^{-x}\,dx = \lim_{b\to+\infty} (-xe^{-x} - e^{-x})\Big|_0^b =$

$\lim_{b\to+\infty} (-be^{-b} - e^{-b} + 1) = 1$.

27. $\int_0^{\infty} (\frac{2}{x + 1} - \frac{n}{x + 3})dx = \lim_{b\to+\infty} \int_0^b (\frac{2}{x + 1} - \frac{n}{x + 3})\,dx =$

$\lim_{b\to+\infty} (2 \ln|x + 1| - n \ln|x + 3|)\Big|_0^b =$

$\lim_{b\to+\infty} [\ln(b + 1)^2 - n \ln(b + 3) + n \ln 3] =$

$\lim_{b\to+\infty} (\ln\left[\frac{(b + 1)^2}{(b + 3)^n}\right] + n \ln 3)$. The integral converges provided $n = 2$. Thus for $n = 2$,

$\lim_{b\to+\infty} [\ln (\frac{b + 1}{b + 3})^2 + 2 \ln 3] = 2 \ln 3$ and so the integral has the value $2 \ln 3$.

28. $\int_{-\infty}^{\infty} f(x)dx$ is convergent means that $\int_{-\infty}^{0} f(x)dx$ is convergent and so is $\int_0^{\infty} f(x)dx$; and that $\int_{-\infty}^{\infty} f(x)dx =$

$\int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$. Now $\int_{-\infty}^{0} f(x)dx =$

$\lim_{a\to-\infty} \int_a^0 f(x)dx = \lim_{-c\to-\infty} \int_{-c}^0 f(x)dx = \lim_{c\to+\infty} \int_{-c}^0 f(x)dx$.

$\int_0^{\infty} f(x)dx = \lim_{c\to+\infty} \int_0^c f(x)dx$. Thus, $\int_{-\infty}^{\infty} f(x)dx =$

$\lim_{c\to+\infty} \int_{-c}^0 f(x)dx + \lim_{c\to+\infty} \int_0^c f(x)dx =$

$\lim_{c\to+\infty} \left[\int_{-c}^0 f(x)dx + \int_0^c f(x)dx\right] = \lim_{c\to+\infty} \int_{-c}^c f(x)dx$.

29. $\lim_{c\to+\infty} \int_{-c}^{c} \sin x\,dx = \lim_{c\to+\infty} (-\cos x)\Big|_{-c}^{c} =$

$\lim_{c\to+\infty} [-\cos c + \cos(-c)] = \lim_{c\to+\infty} (-\cos c + \cos c) =$

$\lim_{c\to+\infty} 0 = 0$.

30. No, since $\int_{-\infty}^{\infty} \sin x\,dx$ is not convergent. The reason for the divergence is that $\int_{-\infty}^{0} \sin x\,dx =$

$\lim_{a\to-\infty} (-\cos x)\Big|_a^0 = \lim_{a\to-\infty} (-1 + \cos a)$ and this limit does not exist; hence, $\int_{-\infty}^{0} \sin x\,dx$ is divergent.

31. $A = 2\int_0^{+\infty} \frac{1}{e^x + e^{-x}}\,dx = 2 \lim_{b\to+\infty} \int_0^b \frac{e^x}{e^{2x} + 1}\,dx$. Put

$u = e^x$. Then $A = 2 \lim_{e^b\to+\infty} \int_1^{e^b} \frac{du}{u^2 + 1} =$

$2 \lim_{e^b\to+\infty} (\tan^{-1} u)\Big|_1^{e^b} = 2 \lim_{e^b\to+\infty} (\tan^{-1} e^b - \tan^{-1} 1) =$

$2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{2}$ square units.

32. $V = \pi\int_0^{+\infty}\left[\sqrt{x}e^{-x^2}\right]^2 dx = \pi \lim_{b\to+\infty} \int_0^b xe^{-2x^2}dx$. Put $u = -2x^2$, so that $du = -4x\,dx$. Then $\int xe^{-2x^2}dx = \int -\tfrac{1}{4}e^u du = -\tfrac{1}{4}e^u + C = -\tfrac{1}{4}e^{-2x^2} + C$. Then $V = \pi \lim_{b\to+\infty} (-\tfrac{1}{4}e^{-2x^2})\Big|_0^b = \pi \lim_{b\to+\infty} (-\tfrac{1}{4}e^{-2b^2} + \tfrac{1}{4}) = \frac{\pi}{4}$ cubic unit.

33. $V = \pi\int_1^{+\infty} (\frac{1}{x})^2 dx = \pi \lim_{b\to+\infty} \int_1^b \frac{1}{x^2} dx = \pi \lim_{b\to+\infty} (-\frac{1}{x})\Big|_1^b = \pi \lim_{b\to+\infty} (-\frac{1}{b} + 1) = \pi$ cubic units.

34. $S = \int_1^\infty 2\pi(\frac{1}{x}) \sqrt{1 + (-\frac{1}{x^2})^2}\, dx > \int_1^\infty 2\pi(\frac{1}{x})\, dx$. Now $\int_1^\infty 2\pi(\frac{1}{x})dx = \lim_{b\to+\infty} 2\pi \ln x\Big|_1^b = 2\pi \lim_{b\to+\infty} \ln b = +\infty$. Hence, S is infinite.

35. $A = \int_1^\infty (\frac{1}{x^2} - e^{-2x})dx = \lim_{b\to+\infty} (-\frac{1}{x} + \frac{e^{-2x}}{2})\Big|_1^b = \lim_{b\to+\infty} (-\frac{1}{b} + \frac{1}{2e^{2b}} + 1 - \frac{1}{2e^2}) = 1 - \frac{1}{2e^2}$ square unit.

36. (a) The region under the curve $y = \frac{1}{x}$ on $[1,\infty)$ has infinite area, since $A = \lim_{b\to+\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to+\infty} (\ln x)\Big|_1^b = \lim_{b\to+\infty} (\ln b - 0) = +\infty$. But this region rotated about the x axis yields $V = \lim_{b\to+\infty} \int_1^b \frac{\pi}{x^2} dx = \lim_{b\to+\infty} (-\frac{\pi}{x})\Big|_1^b = \pi$ cubic units.

(b) The solid obtained by rotating the region under the curve $y = \frac{1}{x}$ over $[1,\infty)$ about the x axis has finite volume, by Part (a), but infinite surface area by Problem 34.

37. $P(r) = \int_0^\infty e^{-rt}(A + Bt)dt = \lim_{b\to+\infty} \int_0^b e^{-rt}(A + Bt)dt = \lim_{b\to+\infty} \int_0^b (Ae^{-rt} + Bte^{-rt})dt = \lim_{b\to+\infty} [-\frac{A}{r} e^{-rt} - B(\frac{te^{-rt}}{r} + \frac{e^{-rt}}{r^2})]\Big|_0^b = \lim_{b\to+\infty} [-\frac{A}{r} e^{-rt} - B(\frac{be^{-rb}}{r} + \frac{e^{-rb}}{r^2}) + \frac{A}{r} + \frac{B}{r^2}] = \frac{A}{r} + \frac{B}{r^2}$.

38. $P(r) = \int_0^\infty Ae^{-rt}dt = A \lim_{b\to+\infty} \int_0^b e^{-rt}dt = -\frac{A}{r} \lim_{b\to+\infty} e^{-rt}\Big|_0^b = -\frac{A}{r} \lim_{b\to+\infty} [e^{-rb} - 1] = \frac{A}{r}$. Now, for $r = 0.10$ and $A = \$10{,}000$, $P(0.10) = \frac{10{,}000}{0.10} = \$100{,}000$.

39. The Laplace transform of $af + bg$ is $\int_0^\infty e^{-rt}[af(t) + bg(t)]dt = a\int_0^\infty e^{-rt}f(t)dt + b\int_0^\infty e^{-rt}g(t)dt = aP + bQ$.

40. $Q(r) = \int_0^\infty e^{-rt}f'(t)dt = \lim_{b\to+\infty} \int_0^b e^{-rt}f'(t)dt = \lim_{b\to+\infty} [e^{-rt}f(t)\Big|_0^b - \int_0^b [-re^{-rt}f(t)]dt = \lim_{b\to+\infty} [e^{-rb}f(b) - f(0)] + r \lim_{b\to+\infty} \int_0^b e^{-rt}f(t)dt = -f(0) + rP(r) = rP(r) - f(0)$.

41. $\Gamma(n + 1) = \int_0^\infty e^{-x}x^n dx = \lim_{b\to+\infty} \int_0^b e^{-x}x^n dx = \lim_{b\to+\infty} [-e^{-x}x^n\Big|_0^b - \int_0^b [-e^{-x}nx^{n-1}]dx = \lim_{b\to+\infty} [-e^{-b}b^n + n\int_0^b e^{-x}x^{n-1}dx = n \lim_{b\to+\infty} \int_0^b e^{-x}x^{n-1}dx = n\Gamma(n)$.

42. By Problem 41, $\Gamma(k + 1) = k\Gamma(k)$ for all positive real numbers k. Thus,

$$\begin{aligned}\Gamma(n + 1) &= n\Gamma(n)\\ &= n(n - 1)\Gamma(n - 1)\\ &= n(n - 1)(n - 2)\Gamma(n - 2)\\ &\ \ \vdots\\ &= n(n - 1)(n - 2)\ldots3\cdot2\cdot\Gamma(2).\end{aligned}$$

Now $\Gamma(2) = \int_0^\infty e^{-x}x\, dx = 1$ by Problem 26. Thus $\Gamma(n + 1) = n!$.

43. $\Gamma(n + 1) = \int_0^\infty e^{-x}x^n dx$. Let $x = rt$. Then $\Gamma(n + 1) = \int_0^\infty e^{-rt}(rt)^n r\, dt = r^{n+1} \int_0^\infty e^{-rt}t^n dt$.

44. $P(r) = \int_0^\infty e^{-rt}t^n dt = \frac{\Gamma(n + 1)}{r^{n+1}}$ by Problem 43.

45. $P(r) = \int_0^\infty e^{-rt} \sin t\, dt = \lim_{b\to+\infty} \int_0^b e^{-rt} \sin t\, dt$.

Integrating by parts, we get $P(r) =$

$$\lim_{b\to+\infty} \frac{-e^{-rt}}{r^2+1}(\cot t + r\sin t)\Big|_0^b =$$

$$\lim_{b\to+\infty}\left[\frac{-e^{-rb}}{r^2+1}(-\cos b - r\sin b) + \frac{1}{r^2+1}\right] = \frac{1}{r^2+1}.$$

46. $P(r) = \int_0^\infty e^{-rt}t^n dt = \frac{\Gamma(n+1)}{r^{n+1}}$ (by Problem 44) $=$

$\frac{n!}{r^{n+1}}$ (by Problem 42).

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1. $\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to0^+}\int_{0+\varepsilon}^4 x^{-\frac{1}{2}}dx = \lim_{\varepsilon\to0^+}(2\sqrt{x})\Big|_\varepsilon^4 =$

$\lim_{\varepsilon\to0^+}(4 - 2\sqrt{\varepsilon}) = 4.$

2. $\int_0^9 \frac{dx}{x\sqrt{x}} = \lim_{\varepsilon\to0^+}\int_\varepsilon^9 x^{-3/2}dx = \lim_{\varepsilon\to0^+}(-2x^{-\frac{1}{2}})\Big|_\varepsilon^9 =$

$\lim_{\varepsilon\to0^+}\frac{-2}{\sqrt{9}} + \frac{2}{\sqrt{\varepsilon}} = +\infty.$ The integral is divergent.

3. $\int_1^{28}\frac{dx}{\sqrt[3]{x-1}} = \lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^{28}(x-1)^{-1/3}dx =$

$\lim_{\varepsilon\to0^+}\frac{3}{2}(x-1)^{2/3}\Big|_{1+\varepsilon}^{28} = \lim_{\varepsilon\to0^+}[\frac{3}{2}(27)^{2/3} - \frac{3}{2}(\varepsilon)^{2/3}] =$

$\frac{27}{2}.$

4. Let $u = \sin x$. Then, $\int\frac{\cos x}{\sqrt{\sin x}}dx = \int u^{-1/2}du =$

$2u^{1/2} + C = 2\sqrt{\sin x} + C.$ Therefore,

$\int_0^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}dx =$

$\lim_{\varepsilon\to0^+}(2\sqrt{\sin x}\Big|_\varepsilon^{\pi/2}) = \lim_{\varepsilon\to0^+}(2\sqrt{\sin(\pi/2)} - 2\sqrt{\sin\varepsilon}) =$

$2\sqrt{\sin(\pi/2)} = 2.$

5. $\int_0^1\frac{\cos\sqrt[3]{x}}{\sqrt[3]{x^2}}dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^1\frac{\cos\sqrt[3]{x}}{\sqrt[3]{x^2}}dx =$

$\lim_{\varepsilon\to0^+}(3\sin\sqrt[3]{x})\Big|_\varepsilon^1 = \lim_{\varepsilon\to0^+}(3\sin 1 - 3\sin\sqrt[3]{\varepsilon}) =$

$3\sin 1.$

6. $\int_0^1\frac{dx}{(1+x)\sqrt{x}} = \lim_{\varepsilon\to0^+}\int_\varepsilon^1\frac{dx}{(1+x)\sqrt{x}} =$

$\lim_{\varepsilon\to0^+}\int_{\sqrt{\varepsilon}}^1\frac{2}{1+u^2}du = \lim_{\varepsilon\to0^+}(2\tan^{-1}u)\Big|_{\sqrt{\varepsilon}}^1 =$

$\lim_{\varepsilon\to0^+}(2\tan^{-1}1 - 2\tan^{-1}\sqrt{\varepsilon}) = 2(\frac{\pi}{4}) = \frac{\pi}{2}.$

7. $\int_0^{\pi/2}\csc^2 x\,dx = \lim_{\varepsilon\to0^+}\int_{0+\varepsilon}^{\pi/2}\csc^2 x\,dx =$

$\lim_{\varepsilon\to0^+}(-\cot x)\Big|_{0+\varepsilon}^{\pi/2} = \lim_{\varepsilon\to0^+}\cot\varepsilon = +\infty.$ The integral is divergent.

8. $\int_0^1\frac{(\ln x)^2}{x}dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^1\frac{(\ln x)^2}{x}dx = \lim_{\varepsilon\to0^+}\frac{(\ln x)^3}{3}\Big|_\varepsilon^1 =$

$\lim_{\varepsilon\to0^+}\left[0 - \frac{(\ln\varepsilon)^3}{3}\right] = +\infty.$ The integral is divergent.

9. $\int_0^4\frac{dx}{\sqrt{16-x^2}} = \lim_{\varepsilon\to0^+}\int_0^{4-\varepsilon}\frac{dx}{\sqrt{16-x^2}} = \lim_{\varepsilon\to0^+}(\sin^{-1}\frac{x}{4})\Big|_0^{4-\varepsilon}$

$\lim_{\varepsilon\to0^+}(\sin^{-1}\frac{(4-\varepsilon)}{4} - \sin^{-1}0) = \frac{\pi}{2}.$

10. $\int_0^5\frac{x\,dx}{\sqrt{25-x^2}} = \lim_{\varepsilon\to0^+}\int_0^{5-\varepsilon}\frac{x\,dx}{\sqrt{25-x^2}} = \lim_{\varepsilon\to0^+}(-\sqrt{25-x^2})\Big|_0^5$

$\lim_{\varepsilon\to0^+}[5 - \sqrt{25-(5-\varepsilon)^2}] = 5.$ The integration is performed by putting $u = 25 - x^2$.

11. $\int_0^4\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^4\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{\varepsilon\to0^+}(-2e^{-\sqrt{x}})\Big|_\varepsilon^4 =$

$\lim_{\varepsilon\to0^+}(-\frac{2}{e^2} + 2e^{-\sqrt{\varepsilon}}) = -\frac{2}{e^2} + 2.$

12. $\int_2^5\frac{2x-6}{x^2-6x+5}dx = \lim_{\varepsilon\to0^+}\int_2^{5-\varepsilon}\frac{2x-6}{x^2-6x+5}dx =$

$\lim_{\varepsilon\to0^+}\ln|x^2-6x+5|\Big|_2^{5-\varepsilon} =$

$\lim_{\varepsilon\to0^+}(\ln|(5-\varepsilon)^2 - 6(5-\varepsilon) + 5| - \ln|-3|) = -\infty.$

The integral is divergent.

13. $\int_{\frac{1}{2}}^1\frac{dt}{t(\ln t)^{2/7}} = \lim_{\varepsilon\to0^+}\int_{\frac{1}{2}}^{1-\varepsilon}\frac{dt}{t(\ln t)^{2/7}} =$

$\lim_{\varepsilon\to0^+}\frac{7}{5}(\ln t)^{5/7}\Big|_{\frac{1}{2}}^{(1-\varepsilon)} = \lim_{\varepsilon\to0^+}\frac{7}{5}[\ln(1-\varepsilon)]^{5/7} -$

$\frac{7}{5}(\ln\frac{1}{2})^{5/7} = \frac{7}{5}(\ln 2)^{5/7}.$

14. $\int_0^1\frac{1}{x^2}\sin\frac{1}{x}dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^1\frac{1}{x^2}\sin\frac{1}{x}dx = \lim_{\varepsilon\to0^+}\cos\frac{1}{x}\Big|_\varepsilon^1$

$$\lim_{\varepsilon\to0^+}\left(\cos 1 - \cos\frac{1}{\varepsilon}\right).$$ This limit does not exist.

The integral is divergent.

15. $$\int_{-1}^{1}\frac{dx}{x^3} = \lim_{\varepsilon\to0^+}\int_{-1}^{-\varepsilon}\frac{dx}{x^3} + \lim_{\varepsilon\to0^+}\int_{\varepsilon}^{1}\frac{dx}{x^3} = \lim_{\varepsilon\to0^+}\left(-\frac{1}{2x^2}\right)\Big|_{-1}^{-\varepsilon} +$$
$$\lim_{\varepsilon\to0^+}\left(-\frac{1}{2x^2}\right)\Big|_{\varepsilon}^{1}.$$ Now $\lim_{\varepsilon\to0^+}\left(-\frac{1}{2x^2}\right)\Big|_{-1}^{-\varepsilon} = -\infty$. Hence, the integral diverges.

16. $$\int_{1}^{3}\frac{x\,dx}{2-x} = \lim_{\varepsilon\to0^+}\int_{1}^{2-\varepsilon}\frac{x\,dx}{2-x} + \lim_{\varepsilon\to0^+}\int_{2+\varepsilon}^{3}\frac{x\,dx}{2-x} =$$
$$\lim_{\varepsilon\to0^+}(-x - 2\ln|2-x|)\Big|_{1}^{2-\varepsilon} +$$
$$\lim_{\varepsilon\to0^+}(-x - 2\ln|2-x|)\Big|_{2+\varepsilon}^{3} =$$
$$\lim_{\varepsilon\to0^+}[(\varepsilon - 2 - 2\ln\varepsilon) - (-1 - 2\ln 1)] +$$
$$\lim_{\varepsilon\to0^+}[(-3 - 2\ln|-1|) - (-2 - \varepsilon - 2\ln|\varepsilon|)] = +\infty.$$
The integral is divergent.

17. $$\int_{0}^{\pi}\frac{\sin x}{\sqrt[5]{\cos x}}\,dx = \int_{0}^{\pi/2}\frac{\sin x}{\sqrt[5]{\cos x}}\,dx + \int_{\pi/2}^{\pi}\frac{\sin x}{\sqrt[5]{\cos x}}\,dx =$$
$$\lim_{\varepsilon\to0^+}\int_{0}^{\frac{\pi}{2}-\varepsilon}\frac{\sin x}{\sqrt[5]{\cos x}}\,dx + \lim_{\varepsilon\to0^+}\int_{\frac{\pi}{2}+\varepsilon}^{\pi}\frac{\sin x}{\sqrt[5]{\cos x}} =$$
$$\lim_{\varepsilon\to0^+}\left[-\frac{5}{4}(\cos x)^{4/5}\right]\Big|_{0}^{\frac{\pi}{2}-\varepsilon} + \lim_{\varepsilon\to0^+}\left(-\frac{5}{4}(\cos x)^{4/5}\Big|_{\frac{\pi}{2}+\varepsilon}^{\pi}\right) =$$
$$\lim_{\varepsilon\to0^+}\left[-\frac{5}{4}\cos^{4/5}\left(\frac{\pi}{2}-\varepsilon\right) + \frac{5}{4}\right] +$$
$$\lim_{\varepsilon\to0^+}\left[-\frac{5}{4}\cos^{4/5}\pi + \frac{5}{4}\cos^{4/5}\left(\frac{\pi}{2}+\varepsilon\right)\right] = 0 + \frac{5}{4} +$$
$$\left(-\frac{5}{4} + 0\right) = 0.$$

18. $$\int_{0}^{2}\frac{x\,dx}{(x-1)^{2/3}} = \lim_{\varepsilon\to0^+}\int_{0}^{1-\varepsilon}\frac{x\,dx}{(x-1)^{2/3}} +$$
$$\lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^{2}\frac{x\,dx}{(x-1)^{2/3}} = \lim_{\varepsilon\to0^+}\int_{-1}^{-\varepsilon}u^{-2/3}(u+1)du +$$
$$\lim_{\varepsilon\to0^+}\int_{\varepsilon}^{1}u^{-2/3}(u+1)du = \lim_{\varepsilon\to0^+}\left(\frac{3}{4}u^{4/3} + 3u^{1/3}\right)\Big|_{-1}^{-\varepsilon} +$$
$$\lim_{\varepsilon\to0^+}\left(\frac{3}{4}u^{4/3} + 3u^{1/3}\right)\Big|_{\varepsilon}^{1} =$$
$$\lim_{\varepsilon\to0^+}\left[\frac{3}{4}(-\varepsilon)^{4/3} - 3\varepsilon^{1/3} - \frac{3}{4} + 3\right] +$$
$$\lim_{\varepsilon\to0^+}\left(\frac{3}{4} + 3 - \frac{3}{4}\varepsilon^{4/3} - 3\varepsilon^{1/3}\right) = \frac{9}{4} + \frac{15}{4} = 6.$$

19. $$\int_{0}^{\pi/2}\sec 2x\,dx = \lim_{\varepsilon\to0^+}\int_{0}^{\pi/4-\varepsilon}\sec 2x\,dx +$$
$$\lim_{\varepsilon\to0^+}\int_{\frac{\pi}{4}+\varepsilon}^{\pi/2}\sec 2x\,dx =$$
$$\lim_{\varepsilon\to0^+}\frac{1}{2}\ln|\sec 2x + \tan 2x|\Big|_{0}^{\frac{\pi}{4}-\varepsilon} +$$
$$\lim_{\varepsilon\to0^+}\frac{1}{2}\ln|\sec 2x + \tan 2x|\Big|_{\pi/4+\varepsilon}^{\pi/2} =$$
$$\lim_{\varepsilon\to0^+}\frac{1}{2}[\ln|\sec(\pi/2 - 2\varepsilon) + \tan(\pi/2 - 2\varepsilon)| - \ln 1] +$$
$$\lim_{\varepsilon\to0^+}\frac{1}{2}[\ln 1 - \ln|\sec(\pi/2 + 2\varepsilon) + \tan(\pi/2 + 2\varepsilon)|].$$
The integral diverges.

20. $$\int_{-\pi}^{\pi}\frac{dt}{1-\cos t} = \lim_{\varepsilon\to0^+}\int_{-\pi}^{-\varepsilon}\tfrac{1}{2}\csc^2\frac{t}{2}\,dt +$$
$$\lim_{\varepsilon\to0^+}\int_{\varepsilon}^{\pi}\tfrac{1}{2}\csc^2\frac{t}{2}\,dt = \lim_{\varepsilon\to0^+}\left(-\cot\frac{t}{2}\right)\Big|_{-\pi}^{-\varepsilon} +$$
$$\lim_{\varepsilon\to0^+}\left(-\cot\frac{t}{2}\right)\Big|_{\varepsilon}^{\pi} = \lim_{\varepsilon\to0^+}\left[-\cot\left(-\frac{\varepsilon}{2}\right) + \cot\left(-\frac{\pi}{2}\right)\right] +$$
$$\lim_{\varepsilon\to0^+}\left(-\cot\frac{\pi}{2} + \cot\frac{\varepsilon}{2}\right).$$ The limit in either case is infinite. The integral is divergent.

21. $$\int_{0}^{\pi}\frac{\sec^2 t}{\sqrt{1-\tan t}}\,dt = \lim_{\varepsilon\to0^+}\int_{0}^{\frac{\pi}{4}-\varepsilon}\frac{\sec^2 t}{\sqrt{1-\tan t}}\,dt =$$
$$-\lim_{\varepsilon\to0^+}\left(2\sqrt{1-\tan t}\right)\Big|_{0}^{\frac{\pi}{4}-\varepsilon} =$$
$$-\lim_{\varepsilon\to0^+}\left[2\sqrt{1 - \tan\left(\frac{\pi}{4}-\varepsilon\right)} - 2\right] = 2.$$

22. $$\int_{e/3}^{2}\frac{dx}{x(\ln x)^3} = \lim_{\varepsilon\to0^+}\int_{e/3}^{1-\varepsilon}\frac{dx}{x(\ln x)^3} +$$
$$\lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^{2}\frac{dx}{x(\ln x)^3} = \lim_{\varepsilon\to0^+} -\frac{1}{2(\ln x)^2}\Big|_{e/3}^{1-\varepsilon} +$$
$$\lim_{\varepsilon\to0^+} -\frac{1}{2(\ln x)^2}\Big|_{1+\varepsilon}^{2} = \lim_{\varepsilon\to0^+}\left(-\frac{1}{2\ln(1-\varepsilon)^2} + \frac{1}{2(\ln\frac{e}{3})^2}\right) +$$
$$\lim_{\varepsilon\to0^+}\left(\frac{-1}{2(\ln 2)^2} + \frac{1}{2(\ln 1+\varepsilon)^2}\right).$$ The first limit is $-\infty$. Hence, the integral is divergent.

23. $$\int_{-1}^{1}\frac{e^x}{\sqrt[5]{e^x - 1}}\,dx = \lim_{\varepsilon\to0}\int_{-1}^{\varepsilon}\frac{e^x}{\sqrt[5]{e^x-1}}\,dx +$$
$$\lim_{\varepsilon\to0}\int_{\varepsilon}^{1}\frac{e^x}{\sqrt[5]{e^x-1}}\,dx = \frac{5}{4}\lim_{\varepsilon\to0}(e^x - 1)^{4/5}\Big|_{-1}^{\varepsilon} +$$
$$\frac{5}{4}\lim_{\varepsilon\to0}(e^x - 1)^{4/5}\Big|_{\varepsilon}^{1} =$$
$$\frac{5}{4}\left[\lim_{\varepsilon\to0}(e^{\varepsilon} - 1)^{4/5} - (e^{-1} - 1)^{4/5}\right] +$$

$\frac{5}{4}[\lim_{\varepsilon\to 0}((e-1)^{4/5}-(e^{\varepsilon}-1)^{4/5})] = -\frac{5}{4}(e^{-1}-1)^{4/5} + \frac{5}{4}(e-1)^{4/5} = \frac{5}{4}[\sqrt[5]{(e-1)^4} - \sqrt[5]{(e^{-1}-1)^4}]$.

24. $\int_{-1}^{1} \frac{e^{-1/x}}{x^2}\,dx = \lim_{\varepsilon\to 0^+}\int_{-1}^{-\varepsilon}\frac{e^{-1/x}}{x^2}\,dx + \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{e^{-1/x}}{x^2}\,dx = \lim_{\varepsilon\to 0^+}(e^{-1/x}\Big|_{-1}^{-\varepsilon}) + \lim_{\varepsilon\to 0^+}(e^{-1/x}\Big|_{\varepsilon}^{1}) = \lim_{\varepsilon\to 0^+}(e^{1/\varepsilon}-e) + \lim_{\varepsilon\to 0^+}(e^{-1}-e^{-1/\varepsilon})$. Since $\lim_{\varepsilon\to 0^+}(e^{1/\varepsilon}-e) = +\infty$, it follows that the integral is divergent.

25. First suppose that $n \neq -1$. Put $u = \ln x$, $dv = x^n dx$, so that $du = \frac{dx}{x}$, $v = \frac{x^{n+1}}{n+1}$ and $\int_0^1 x^n \ln x\,dx = \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1} x^n \ln x\,dx = \lim_{\varepsilon\to 0^+}\left[\frac{x^{n+1}}{n+1}\ln x\Big|_{\varepsilon}^{1} - \int_{\varepsilon}^{1}\frac{x^n}{n+1}dx\right] = \lim_{\varepsilon\to 0^+}\left[-\frac{\varepsilon^{n+1}}{n+1}\ln\varepsilon - \left(\frac{x^{n+1}}{[n+1]^2}\right)\Big|_{\varepsilon}^{1}\right] = \lim_{\varepsilon\to 0^+}\left[-\frac{\varepsilon^{n+1}\ln\varepsilon}{n+1} - \frac{1}{(n+1)^2} + \frac{\varepsilon^{n+1}}{(n+1)^2}\right]$. If $n < -1$, then $\lim_{\varepsilon\to 0^+}\varepsilon^{n+1}\ln\varepsilon = -\infty$, and the integral diverges. If $n > -1$, then, by L'Hôpital's rule, $\lim_{\varepsilon\to 0^+}\varepsilon^{n+1}\ln\varepsilon = \lim_{\varepsilon\to 0^+}\frac{\ln\varepsilon}{\varepsilon^{-(n+1)}} = \lim_{\varepsilon\to 0^+}\frac{1/\varepsilon}{-(n+1)\varepsilon^{-n-2}} = \lim_{\varepsilon\to 0^+}\frac{\varepsilon^{n+1}}{-(n+1)} = 0$, and the integral converges to $\frac{-1}{(n+1)^2}$. Now, suppose that $n = -1$. Then $\int_0^1 x^n \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{\ln x}{x}\,dx = \lim_{\varepsilon\to 0^+}\frac{(\ln x)^2}{2}\Big|_{\varepsilon}^{1} = \lim_{\varepsilon\to 0^+}\frac{-(\ln\varepsilon)^2}{2} = -\infty$, and the integral is divergent.

26. (a) $\lim_{\varepsilon\to 0^+}\left(\int_{-1}^{-\varepsilon}\frac{dx}{x} + \int_{\varepsilon}^{1}\frac{dx}{x}\right) = \lim_{\varepsilon\to 0^+}\left(\ln|x|\Big|_{-1}^{-\varepsilon} + \ln|x|\Big|_{\varepsilon}^{1}\right) = \lim_{\varepsilon\to 0^+}[(\ln\varepsilon - \ln 1) + (\ln 1 - \ln\varepsilon)] = \lim_{\varepsilon\to 0^+} 0 = 0$.

(b) No, since $\int_{-1}^{1}\frac{dx}{x} = \lim_{\varepsilon\to 0^+}\int_{-1}^{-\varepsilon}\frac{dx}{x} + \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{dx}{x}$ and either limit diverges. Hence, $\int_{-1}^{1}\frac{dx}{x}$ is divergent.

27. The fundamental theorem of calculus is not applicable to improper integrals.

28. $\int_0^1 \frac{dx}{1-x^4} = \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\frac{dx}{(1+x^2)(1-x)(1+x)} = \lim_{\varepsilon\to 0^+}\left[\int_0^{1-\varepsilon}\frac{\frac{1}{2}}{1+x^2}dx + \int_0^{1-\varepsilon}\frac{\frac{1}{4}}{1+x}dx + \int_0^{1-\varepsilon}\frac{\frac{1}{4}}{1-x}dx\right] = \lim_{\varepsilon\to 0^+}\left(\tfrac{1}{2}\tan^{-1}x + \tfrac{1}{4}\ln|1+x| - \tfrac{1}{4}\ln|1-x|\right)\Big|_0^{1-\varepsilon} = \lim_{\varepsilon\to 0^+}\left(\tfrac{1}{2}\tan^{-1}(1-\varepsilon) + \tfrac{1}{4}\ln(2-\varepsilon) - \tfrac{1}{4}\ln\varepsilon\right) = +\infty$, because of $\ln\varepsilon$. Hence, the integral is divergent.

29. $A = \int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{\varepsilon\to 0^+}\left[\int_{\varepsilon}^{1}\frac{1}{\sqrt{x(2-x)}}\,dx\right] + \lim_{\varepsilon\to 0^+}\left[\int_1^{2-\varepsilon}\frac{1}{\sqrt{x(2-x)}}\,dx\right] = \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{1}{\sqrt{1-(x-1)^2}}\,dx + \lim_{\varepsilon\to 0^+}\int_1^{2-\varepsilon}\frac{1}{\sqrt{1-(x-1)^2}}\,dx = \lim_{\varepsilon\to 0^+}\int_{\varepsilon-1}^{0}\frac{du}{\sqrt{1-u^2}} + \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\frac{du}{\sqrt{1-u^2}} = \lim_{\varepsilon\to 0^+}(\sin^{-1}u)\Big|_{\varepsilon-1}^{0} + \lim_{\varepsilon\to 0^+}(\sin^{-1}u)\Big|_0^{1-\varepsilon} = \lim_{\varepsilon\to 0^+}[\sin^{-1}0 - \sin^{-1}(\varepsilon-1)] + \lim_{\varepsilon\to 0^+}[\sin^{-1}(1-\varepsilon) - \sin^{-1}0] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ square units.

30. $V = \int_0^2 \pi\left(\frac{1}{\sqrt{x(2-x)}}\right)^2 dx = \int_0^2 \frac{\pi}{x(2-x)}\,dx = \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{\pi}{x(2-x)}\,dx + \lim_{\varepsilon\to 0^+}\int_1^{2-\varepsilon}\frac{\pi}{x(2-x)}\,dx = \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\left[\frac{\frac{\pi}{2}}{x} + \frac{\frac{\pi}{2}}{(2-x)}\right]dx + \lim_{\varepsilon\to 0^+}\int_1^{2-\varepsilon}\left(\frac{\frac{\pi}{2}}{x} + \frac{\frac{\pi}{2}}{2-x}\right)dx = \lim_{\varepsilon\to 0^+}\left(\frac{\pi}{2}\ln|x| - \frac{\pi}{2}\ln|2-x|\right)\Big|_{\varepsilon}^{1} + \lim_{\varepsilon\to 0^+}\left(\frac{\pi}{2}\ln|x| - \frac{\pi}{2}\ln|2-x|\right)\Big|_1^{2-\varepsilon}$. But $\lim_{\varepsilon\to 0^+}\left[-\frac{\pi}{2}\ln\varepsilon + \frac{\pi}{2}\ln(2-\varepsilon)\right] = +\infty$. The volume is infinite.

31. $V = \pi\int_2^4 \frac{1}{(x-2)^2}\,dx = \lim_{\varepsilon\to 0^+}\pi\int_{2+\varepsilon}^{4}\frac{1}{(x-2)^2}\,dx = \lim_{\varepsilon\to 0^+}\pi\left(-\frac{1}{x-2}\right)\Big|_{2+\varepsilon}^{4} = \lim_{\varepsilon\to 0^+}\pi\left(-\tfrac{1}{2} + \frac{1}{2+\varepsilon-2}\right) = +\infty$. The volume is infinite.

32. $A = \int_0^1 \ln\frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\ln\frac{1}{x}\,dx =$

$\lim_{\varepsilon\to0^+} [x \ln \frac{1}{x}\Big|_\varepsilon^1 - \int_\varepsilon^1 (-dx)] =$

$\lim_{\varepsilon\to0^+} [(0 - \varepsilon \ln \frac{1}{\varepsilon}) + (1 - \varepsilon)]$. By L'Hôpital's rule, $\lim_{\varepsilon\to0^+} \varepsilon \ln \frac{1}{\varepsilon} = \lim_{\varepsilon\to0^+} \frac{\ln \frac{1}{\varepsilon}}{\frac{1}{\varepsilon}} = \lim_{\varepsilon\to0^+} \frac{-\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} =$

$\lim_{\varepsilon\to0^+} \varepsilon = 0$. Hence, $A = 1$ square unit.

33. If $p > 0$ and $p \neq \frac{1}{2}$, then $V = \pi\int_0^1 (\frac{1}{x^p})^2 dx =$

$\pi \lim_{\varepsilon\to0^+} \int_\varepsilon^1 x^{-2p}dx = \pi \lim_{\varepsilon\to0^+} \frac{x^{1-2p}}{1 - 2p}\Big|_\varepsilon^1 =$

$\pi \lim_{\varepsilon\to0^+} (\frac{1}{1 - 2p} - \frac{\varepsilon^{1-2p}}{1 - 2p})$. If $0 < p < \frac{1}{2}$, then $V = \frac{\pi}{1 - 2p}$ cubic units. If $p > \frac{1}{2}$, then V is infinite. If $p = \frac{1}{2}$, then $V = \pi\int_0^1 \frac{1}{x}\, dx =$

$\lim_{\varepsilon\to0^+} \int_\varepsilon^1 \frac{\pi}{x}\, dx = \lim_{\varepsilon\to0^+} \pi \ln x\Big|_\varepsilon^1 = \lim_{\varepsilon\to0^+} \pi(\ln 1 - \ln \varepsilon) =$

$+\infty$.

34. $\int_a^b f(x)dx = \int_a^{a+\varepsilon} f(x)dx + \int_{a+\varepsilon}^b f(x)dx$. Now

$\lim_{\varepsilon\to0^+} \int_a^b f(x)dx = \lim_{\varepsilon\to0^+} \int_a^{a+\varepsilon} f(x)dx + \lim_{\varepsilon\to0^+} \int_{a+\varepsilon}^b f(x)dx$.

Since $\int_a^b f(x)dx$ is a constant, $\lim_{\varepsilon\to0^+} \int_a^b f(x)dx =$

$\int_a^b f(x)dx$. Also, by the mean value theorem for integrals, Section 5.3, there exists c in $[a, a + \varepsilon]$ such that $\int_a^{a+\varepsilon} f(x)dx = f(c)[a + \varepsilon - a] = f(c)(\varepsilon)$.

Now, $\lim_{\varepsilon\to0^+} \int_a^{a+\varepsilon} f(x)dx = \lim_{\varepsilon\to0^+} [f(c)(\varepsilon)] = f(a)\cdot 0 = 0$.

Hence, $\int_a^b f(x)dx = \lim_{\varepsilon\to0^+} \int_{a+\varepsilon}^b f(x)dx$.

35. If $p \neq 2$, then $V = \int_0^1 2\pi x(\frac{1}{x^p})dx = \lim_{\varepsilon\to0^+} \int_\varepsilon^1 \frac{2\pi}{x^{p-1}}\, dx =$

$\lim_{\varepsilon\to0^+} \frac{2\pi x^{2-p}}{2 - p}\Big|_\varepsilon^1 = \lim_{\varepsilon\to0^+} 2\pi(\frac{1}{2 - p} - \frac{\varepsilon^{2-p}}{2 - p})$. If $p > 2$, then V is infinite. If $0 < p < 2$, then $V = \frac{2\pi}{2 - p}$ cubic units. If $p = 2$, then $V = \int_0^1 \frac{2\pi}{x}\, dx =$

$\lim_{\varepsilon\to0^+} 2\pi \ln x\Big|_\varepsilon^1 = \lim_{\varepsilon\to0^+} 2\pi(\ln 1 - \ln \varepsilon) = +\infty$.

36. No. Consider the region R under the curve $y = \frac{1}{x}$ on $[1,\infty)$. Revolving R about the x axis, we have

$V = \pi\int_1^\infty (\frac{1}{x})^2\, dx = \pi \lim_{b\to+\infty} \int_1^b x^{-2}\, dx = \pi \lim_{b\to+\infty} (-\frac{1}{x})\Big|_1^b =$

$\pi \lim_{b\to+\infty} (1 - \frac{1}{b}) = \pi$ cubic units. However, revolving R about the y axis, we have $V = 2\pi\int_1^\infty x(\frac{1}{x})dx =$

$2\pi \lim_{b\to+\infty} \int_1^b dx = 2\pi \lim_{b\to+\infty} x\Big|_1^b = 2\pi \lim_{b\to+\infty} (b - 1) = +\infty$.

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1. $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$, $f^{(4)}(x) = 24x^{-5}$, $f^{(5)}(x) = -120x^{-6}$, $f^{(6)}(x) = 720x^{-7}$, $f^{(7)}(x) = -5040x^{-8}$. $f(2) = \frac{1}{2}$, $f'(2) = -\frac{1}{4}$, $f''(2) = \frac{1}{4}$, $f'''(2) = -\frac{6}{16}$, $f^{(4)}(2) = \frac{24}{32}$, $f^{(5)}(2) = -\frac{120}{64}$, $f^{(6)}(2) = \frac{720}{128}$. Hence, $P_6(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3 + \frac{f^{(4)}(2)}{4!}(x - 2)^4 + \frac{f^{(5)}(2)}{5!}(x - 2)^5 + \frac{f^{(6)}(2)}{6!}(x - 2)^6$,

and so $P_6(x) = \frac{1}{2} - \frac{(x - 2)}{4} + \frac{(x - 2)^2}{8} - \frac{(x - 2)^3}{16} + \frac{(x - 2)^4}{32} - \frac{(x - 2)^5}{64} + \frac{(x - 2)^6}{128}$. $R_6(x) =$

$\frac{f^{(7)}(c)(x - 2)^7}{7!} = \frac{-7!c^{-8}(x - 2)^7}{7!} = -c^{-8}(x - 2)^7$,

c strictly between 2 and x.

2. $g(x) = x^{1/2}$, $g'(x) = \frac{1}{2}x^{-1/2}$, $g''(x) = -\frac{1}{4}x^{-3/2}$, $g'''(x) = \frac{3}{8}x^{-5/2}$, $g^{(4)}(x) = -\frac{15}{16}x^{-7/2}$, $g^{(5)}(x) = \frac{105}{32}x^{-9/2}$, $g^{(6)}(x) = -\frac{945}{64}x^{-11/2}$. $g(4) = 2$, $g'(4) = \frac{1}{4}$, $g''(4) = (-\frac{1}{4})(\frac{1}{8})$, $g'''(4) = \frac{3}{8}(\frac{1}{32})$, $g^{(4)}(4) = -\frac{15}{16}(\frac{1}{128})$, $g^{(5)}(4) = \frac{105}{32}(\frac{1}{512})$. Hence, $P_5(x) = g(4) + g'(4)(x - 4) + \frac{g''(4)}{2!}(x - 4)^2 + \frac{g'''(4)}{3!}(x - 4)^3 + \frac{g^{(4)}(4)}{4!}(x - 4)^4 + \frac{g^{(5)}(4)}{5!}(x - 4)^5$, and so $P_5(x) =$

$2 + \frac{(x - 4)}{4} - \frac{(x - 4)^2}{64} + \frac{(x - 4)^3}{512} - \frac{5(x - 4)^4}{16,384} + \frac{7(x - 4)^5}{131,072}$. $R(x) = \frac{g^{(6)}(c)(x - 4)^6}{6!} =$

$\frac{105}{32} (-\frac{9}{2})c^{-11/2} \frac{(x - 4)^6}{6!} = -\frac{21}{1024}(x - 4)^6 \cdot c^{-11/2}$,

f(

c strictly between 4 and x.

3. $f(x) = x^{-\frac{1}{2}}$, $f'(x) = -\frac{1}{2}x^{-3/2}$, $f''(x) = \frac{3}{4}x^{-5/2}$, $f'''(x) = \frac{-15}{8}x^{-7/2}$, $f^{(4)}(x) = \frac{105}{16}x^{-9/2}$, $f^{(5)}(x) = -\frac{945}{32}x^{-11/2}$. $f(100) = \frac{1}{10}$, $f'(100) = -\frac{1}{2(10^3)}$, $f''(100) = \frac{3}{4(10^5)}$, $f'''(100) = \frac{-15}{8(10^7)}$, $f^{(4)}(100) = \frac{105}{16(10^9)}$. Hence, $P_4(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2 + \frac{f'''(100)}{3!}(x - 100)^3 + \frac{f^{(4)}(100)}{4!}(x - 100)^4$, and so $P_4(x) = \frac{1}{10} - \frac{(x - 100)}{2(10^3)} + \frac{3(x - 100)^2}{8(10)^5} - \frac{5(x - 100)^3}{16(10^7)} + \frac{35(x - 100)^4}{128(10^9)}$. $R_4(x) = \frac{f^5(c)(x - 100)^5}{5!} = -\frac{63}{256}c^{-11/2}(x - 100)^5$, c strictly between 100 and x.

4. $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, $f'''(x) = \frac{10}{27}x^{-8/3}$, $f^{(4)}(x) = -\frac{80}{81}x^{-11/3}$, $f^{(5)}(x) = \frac{880}{243}x^{-14/3}$. $f(1000) = 10$, $f'(1000) = \frac{1}{3(10^2)}$, $f''(1000) = -\frac{2}{9(10^5)}$, $f'''(1000) = \frac{10}{27(10^8)}$, $f^{(4)}(1000) = \frac{-80}{81(10^{11})}$. Hence, $P_4(x) = f(1000) + f'(1000)(x - 1000) + \frac{f''(1000)}{2!}(x - 1000)^2 + \frac{f'''(1000)}{3!}(x - 1000) + \frac{f^{(4)}(1000)}{4!}(x - 1000)^4$, and so $P_4(x) = 10 + \frac{(x - 1000)}{3(10^2)} - \frac{(x - 1000)^2}{9(10^5)} + \frac{5(x - 1000)^3}{81(10^8)} - \frac{10(x - 1000)^4}{243(10^{11})}$. $R_4(x) = \frac{f^{(5)}(c)(x - 1000)^5}{5!} = \frac{22}{729}c^{-14/3}(x - 1000)^5$, c strictly between 1000 and x.

5. $g(x) = (x - 2)^{-2}$, $g'(x) = -2(x - 2)^{-3}$, $g''(x) = 6(x - 2)^{-4}$, $g'''(x) = -24(x - 2)^{-5}$, $g^{(4)}(x) = 120(x - 2)^{-6}$, $g^{(5)}(x) = -720(x - 2)^{-7}$, $g^{(6)}(x) = 5040(x - 2)^{-8}$. $g(3) = 1$, $g'(3) = -2$, $g''(3) = 6$, $g'''(3) = -24$, $g^{(4)}(3) = 120$, $g^{(5)}(3) = -720$, $g^{(6)}(3) = 5040$. Hence, $P_5(x) = g(3) + g'(3)(x - 3) + \frac{g''(3)}{2!}(x - 3)^2 + \frac{g'''(3)}{3!}(x - 3)^3 + \frac{g^{(4)}(3)}{4!}(x - 3)^4 + \frac{g^{(5)}(3)}{5!}(x - 3)^5$, and so $P_5(x) = 1 - 2(x - 3) + 3(x - 3)^2 - 4(x - 3)^3 + 5(x - 3)^4 - 6(x - 3)^5$. $R_5(x) = \frac{g^{(6)}(c)(x - 3)^6}{6!} = \frac{7!(c - 2)^{-8}(x - 3)^6}{6!} = \frac{7(x - 3)^6}{(c - 2)^8}$, c strictly between 3 and x.

6. $f(x) = (1 - x)^{-\frac{1}{2}}$, $f'(x) = \frac{1}{2}(1 - x)^{-3/2}$, $f''(x) = \frac{3}{4}(1 - x)^{-5/2}$, $f'''(x) = \frac{15}{8}(1 - x)^{-7/2}$, $f^{(4)}(x) = \frac{105}{16}(1 - x)^{-9/2}$. $f(0) = 1$, $f'(0) = \frac{1}{2}$, $f''(0) = \frac{3}{4}$, $f'''(0) = \frac{15}{8}$. Hence, $P_3(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3$, and so $P_3(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3$. $R_3(x) = \frac{f^{(4)}(c)(x)^4}{4!} = 35\frac{(1 - c)^{-9/2}(x^4)}{128}$, c strictly between 0 and x.

7. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$, $f^{(6)}(x) = -\sin x$, $f^{(7)}(x) = -\cos x$. $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^4(0) = 0$, $f^5(0) = 1$, $f^6(0) = 0$. Hence, $P_6(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3 + \frac{f^4(0)(x - 0)^4}{4!} + \frac{f^5(0)(x - 0)^5}{5!} + \frac{f^6(0)}{6!}(x - 0)^6$, and so $P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$. $R_6(x) = \frac{f^{(7)}(c)(x^7)}{7!} = \frac{(-\cos c)x^7}{7!}$, c strictly between 0 and x.

8. $g(x) = \cos x$, $g'(x) = -\sin x$, $g''(x) = -\cos x$, $g'''(x) = \sin x$, $g^{(4)}(x) = \cos x$. $g(-\frac{\pi}{3}) = \frac{1}{2}$, $g'(-\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $g''(-\frac{\pi}{3}) = -\frac{1}{2}$, $g'''(-\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. Hence, $P_3(x) = g(-\frac{\pi}{3}) + g'(-\frac{\pi}{3})(x + \frac{\pi}{3}) + \frac{g''(-\pi/3)}{2!}(x + \frac{\pi}{3})^2 + \frac{g'''(-\pi/3)}{3!}(x + \frac{\pi}{3})^3$, and so $P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x + \frac{\pi}{3}) - \frac{1}{4}(x + \frac{\pi}{3})^2 - \frac{\sqrt{3}}{12}(x + \frac{\pi}{3})^3$. $R_3(x) = \frac{g^{(4)}(c)(x + \pi/3)^4}{4!} = \frac{\cos c(x + \pi/3)^4}{4!}$, c strictly between $-\frac{\pi}{3}$ and x.

9. $g(x) = \tan x$, $g'(x) = \sec^2 x$, $g''(x) = 2\sec^2 x \tan x$, $g'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$, $g^{(4)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$, $g^{(5)}(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x$. $g(\frac{\pi}{4}) = 1$, $g'(\frac{\pi}{4}) = 2$, $g''(\frac{\pi}{4}) = 4$, $g'''(\frac{\pi}{4}) = 16$, $g^4(\frac{\pi}{4}) = 80$. Thus, $P_4(x) = g(\frac{\pi}{4}) + g'(\frac{\pi}{4})(x - \frac{\pi}{4}) + \frac{g''(\pi/4)}{2!}(x - \frac{\pi}{4})^2 + \frac{g'''(\pi/4)}{3!}(x - \frac{\pi}{4})^3 + \frac{g^{(4)}(\pi/4)}{4!}(x - \frac{\pi}{4})^4$, and so $P_4(x) = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3 + \frac{10}{3}(x - \frac{\pi}{4})^4$. $R_4(x) = \frac{g^5(c)}{5!}(x - \frac{\pi}{4})^5 = \frac{16\sec^2 c \tan^4 c + 88\sec^4 c \tan^2 c + 16\sec^6 c}{5!}(x - \frac{\pi}{4})^5$, c strictly between $\frac{\pi}{4}$ and x.

10. $f(x) = e^{2x}$, $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, $f'''(x) = 8e^{2x}$, $f^{(4)}(x) = 16e^{2x}$, $f^{(5)}(x) = 32e^{2x}$, $f^{(6)}(x) = 64e^{2x}$. $f(0) = 1$, $f'(0) = 2$, $f''(0) = 4$, $f'''(0) = 8$, $f^{(4)}(0) = 16$, $f^5(0) = 32$. Thus, $P_5(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3 + \frac{f^4(0)}{4!}(x - 0)^4 + \frac{f^5(0)}{5!}(x - 0)^5$, and so $P_5(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5$. $R_5(x) = \frac{f^6(c)(x)^6}{6!} = \frac{64e^{2c}}{6!}x^6 = \frac{4e^{2c}x^6}{45}$, c strictly between 0 and x.

11. $f(x) = xe^x$, $f'(x) = xe^x + e^x$, $f''(x) = xe^x + 2e^x$, $f'''(x) = xe^x + 3e^x$, $f^{(4)}(x) = xe^x + 4e^x$. $f(1) = e$, $f'(1) = 2e$, $f''(1) = 3e$, $f'''(1) = 4e$. Thus, $P_3(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3$, and so $P_3(x) = e + 2e(x - 1) + \frac{3e}{2}(x - 1)^2 + \frac{2}{3}e(x - 1)^3$, $R_3(x) = \frac{f^4(c)(x - 1)^4}{4!} = \frac{(ce^c + 4e^c)(x - 1)^4}{4!}$, c strictly between 1 and x.

12. $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = 4x^2e^{-x^2} - 2e^{-x^2}$, $f'''(x) = -8x^3e^{-x^2} + 12xe^{-x^2}$, $f^{(4)}(x) = 16x^4e^{-x^2} - 48x^2e^{-x^2} + 12e^{-x^2}$. $f(0) = 1$, $f'(0) = 0$, $f''(0) = -2$, $f'''(0) = 0$. Thus, $P_3(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)(x-0)^3}{3!}$, and so $P_3(x) = 1 - x^2$, $R_3(x) = \frac{16c^4e^{-c^2} - 48c^2e^{-c^2} + 12e^{-c^2}}{4!}x^4$, c strictly between 0 and x.

13. $g(x) = 2^x$, $g'(x) = (\ln 2)2^x$, $g''(x) = (\ln 2)^2 2^x$, $g'''(x) = (\ln 2)^3 2^x$, $g^{(4)}(x) = (\ln 2)^4 2^x$. $g(1) = 2$, $g'(1) = 2\ln 2$, $g''(1) = 2(\ln 2)^2$, $g'''(1) = 2(\ln 2)^3$. Hence, $P_3(x) = g(1) + g'(1)(x - 1) + \frac{g''(1)}{2!}(x-1)^2 + \frac{g'''(1)}{3!}(x - 1)^3$ and so $P_3(x) = 2 + 2(\ln 2)(x - 1) + (\ln 2)^2(x - 1)^2 + \frac{(\ln 2)^3}{3}(x - 1)^3$. $R_3(x) = \frac{f^4(c)(x - 1)^4}{4!} = \frac{(\ln 2)^4 2^c(x - 1)^4}{4!}$, c strictly between 1 and x.

14. $f(x) = \ln x$, $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, $f^{(5)}(x) = 24x^{-5}$. $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, $f^{(4)}(x) = -6$. Hence, $P_4(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4$, and so $P_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$. $R_4(x) = \frac{f^{(5)}(c)(x - 1)^5}{5!} = \frac{24c^{-5}(x - 1)^5}{5!} = \frac{(x - 1)^5}{5c^5}$, c strictly between 1 and x.

15. $g(x) = \sinh x$, $g'(x) = \cosh x$, $g''(x) = \sinh x$, $g'''(x) = \cosh x$, $g^{(4)}(x) = \sinh x$, $g^{(5)}(x) = \cosh x$. $g(0) = 0$, $g'(0) = 1$, $g''(0) = 0$, $g'''(0) = 1$, $g^4(0) = 0$. Hence, $P_4(x) = g(0) + g'(0)(x - 0) + \frac{g''(0)}{2!}(x - 0)^2 + \frac{g'''(0)}{3!}(x - 0)^3 + \frac{g^4(0)(x - 0)^4}{4!}$, and so $P_4(x) = x + \frac{x^3}{3!}$. $R_4(x) = \frac{g^{(5)}(c)x^5}{5!} = \frac{(\cosh c)x^5}{5!}$, c between 0 and x.

16. $f(x) = \ln(\cos x)$, $f'(x) = -\tan x$, $f''(x) = -\sec^2 x$, $f'''(x) = -2\sec^2 x \tan x$, $f^{(4)}(x) = -2\sec^4 x - 4\sec^2 x \tan^2 x$. $f(\frac{\pi}{3}) = \ln(\frac{1}{2}) = -\ln 2$, $f'(\frac{\pi}{3}) = -\sqrt{3}$, $f''(\frac{\pi}{3}) = -4$, $f'''(\frac{\pi}{3}) = -8\sqrt{3}$. Hence, $P_3(x) = f(\frac{\pi}{3}) +$

$f'(\frac{\pi}{3})(x - \frac{\pi}{3}) + \frac{f''(\pi/3)}{2!}(x - \frac{\pi}{3})^2 + \frac{f'''(\pi/3)}{3!}(x - \frac{\pi}{3})^3$, and so $P_3(x) = -\ln 2 - \sqrt{3}(x - \frac{\pi}{3}) - 2(x - \frac{\pi}{3})^2 - \frac{4\sqrt{3}}{3}(x - \frac{\pi}{3})^3$. $R_3(x) = \frac{f^{(4)}(c)(x - \pi/3)^4}{4!} = \frac{(-4\sec^2 c\tan^2 c - 2\sec^4 c)(x - \pi/3)^4}{24}$, c strictly between $\frac{\pi}{3}$ and x.

17. Take $f(x) = \sin x$, $a = 0$, and $b = 1$. $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, so that $|f^{(n+1)}(c)|$ is either $\pm\sin c$ or $\pm\cos c$. Hence, $|f^{(n+1)}(c)| \leq 1$; so we can take $M_n = 1$ in Theorem 3. The error in absolute value cannot exceed $\frac{M_n|b - a|^{n+1}}{(n + 1)!} = \frac{1 \cdot |1 - 0|^{n+1}}{(n + 1)!} = \frac{1}{(n+1)!} \leq \frac{1}{10^5}$ provided n is at least 8. $P_8(x) = \sin 0 + (\cos 0)(x - 0) - \frac{\sin 0}{2!}(x - 0)^2 \frac{(\cos 0)(x - 0)^3}{3!} + \frac{\sin 0}{4!}(x - 0)^4 + \frac{\cos 0}{5!}(x - 0)^5 - \frac{\sin 0}{6!}(x - 0)^6 - \frac{\cos 0}{7!}(x - 0)^7 + \frac{\sin 0}{8!}(x - 0)^8$ so $P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$. Hence, $\sin 1 \approx P_8(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} \approx 0.84146 \approx 0.8415$. (The correct value of sin 1 rounded off to six places is 0.841471.)

18. Take $f(x) = \cos x$, $a = \frac{\pi}{6} = 30^\circ$, and $b = \frac{29\pi}{180} = 29^\circ$. $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, so that $|f^{(n+1)}(c)| \leq 1$ since $f^{(n+1)}(c)$ is either $\pm\sin c$ or $\pm\cos x$. Hence, we can take $M_n = 1$, so the error in absolute value cannot exceed $\frac{M_n|b - a|^{n+1}}{(n + 1)!} = \frac{1 \cdot |\frac{29\pi}{180} - \frac{\pi}{6}|^{n+1}}{(n + 1)!} = \frac{(\frac{\pi}{180})^{n+1}}{(n + 1)!} \leq \frac{1}{10^5}$ provided n is at least 2. $P_2(x) = \cos\frac{\pi}{6} - \sin\frac{\pi}{6}(x - \frac{\pi}{6}) - \frac{\cos\pi/6(x - \pi/6)^2}{2!} = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{6}) - \frac{\sqrt{3}}{4}(x - \frac{\pi}{6})^2$. Hence, $\cos 29^\circ = \cos\frac{29\pi}{180} \approx P_2(\frac{29\pi}{180}) = \frac{\sqrt{3}}{2} - \frac{1}{2}(-\frac{\pi}{180}) - \frac{\sqrt{3}}{4}(-\frac{\pi}{180})^2 \approx 0.87462 \approx 0.8746$. (The correct value of $\cos 29^\circ$ rounded off to six places is 0.874620.)

19. Take $f(x) = e^x$, $a = 0$, and $b = 1$. $f^{(n+1)}(c) = e^c$, so that $|f^{(n+1)}(c)| = e^c$. Now $0 < c < 1$, so that $e^0 < e^c < e^1 < 3$. We can take $M_n = 3$, so the error in absolute value cannot exceed $\frac{M_n|b - a|^{n+1}}{(n + 1)!} = \frac{3|1 - 0|^{n+1}}{(n + 1)!} = \frac{3}{(n + 1)!} < \frac{1}{10^5}$ provided n is at least 8. $P_8(x) = e^0 + e^0x + \frac{e^0x^2}{2!} + \frac{e^0x^3}{3!} + \frac{e^0x^4}{4!} + \frac{e^0x^5}{5!} + \frac{e^0x^6}{6!} + \frac{e^0x^7}{7!} + \frac{e^0x^8}{8!}$. Hence, $e^1 \approx P_8(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \approx 2.71827 \approx 2.7183$. (The correct value of e rounded off to six places is 2.718282.)

20. Take $f(x) = e^x$, $a = -1$, and $b = -1.1$. $|f^{(n+1)}(c)| = e^c$. Now $-1.1 < c < -1$, so that $e^c < e^{-1}$ and $\frac{1}{e} < \frac{1}{2}$. Hence, take $M_n = \frac{1}{2}$, so that the error in absolute value cannot exceed $\frac{\frac{1}{2}|b - a|^{n+1}}{(n + 1)!} = \frac{1}{2}\frac{|-0.1|^{n+1}}{(n + 1)!} = \frac{1}{2(10^{n+1})(n + 1)!} < \frac{1}{10^5}$ provided n is at least 3. $P_3(x) = e^{-1} + e^{-1}(x + 1) + \frac{e^{-1}(x + 1)^2}{2!} + \frac{e^{-1}(x + 1)^3}{3!}$. Thus, $e^{-1.1} \approx P_3(-1.1) = e^{-1} + e^{-1}(-0.1) + \frac{e^{-1}(-0.1)^2}{2} + \frac{e^{-1}(-0.1)^3}{3!} \approx 0.33287 \approx 0.3329$. (The correct value of $e^{-1.1}$ rounded off to six places is 0.332871.)

21. Take $f(x) = \ln x$, $a = 1$, and $b = 0.98$. $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -3!x^{-4}$ and so forth, so that $|f^{(n+1)}(c)| = n!c^{-(n+1)}$ where $0.98 < c < 1$. Thus, $c > \frac{1}{2}$, so that $|f^{(n+1)}(c)| = n!c^{-(n+1)} < n!(\frac{1}{2})^{-(n+1)} = n!2^{n+1}$. Take $M_n = n!2^{n+1}$. Hence, we have $\frac{M_n|b - a|^{n+1}}{(n + 1)!} \leq \frac{n!2^{n+1}(0.02)^{n+1}}{(n + 1)!} = \frac{1}{25^{n+1}(n + 1)} \leq \frac{1}{10^5}$ for n at least 3. Hence, $P_n(x) = \ln 1 + 1(x - 1) - \frac{1}{1^2}\frac{(x - 1)^2}{2!} + \frac{2(x - 1)^3}{(1)^3 \cdot 3!}$, and so $\ln(0.98) \approx P_n(0.98) = -(0.02) - \frac{(0.02)^2}{2!} - \frac{1}{3}(0.02)^3 \approx$

$0.02020 \approx -0.0202$. (The correct value of ln (0.98) rounded off to six decimal places is -0.020203.)

22. $\ln 17 = \ln[16(1 + \frac{1}{16})] = \ln 16 + \ln(1 + \frac{1}{16}) = 4 \ln 2 + \ln(1 + \frac{1}{16})$. Now take $f(x) = \ln(1 + x)$, so that $|f^{n+1}(c)| = n!(1 + c)^{-(n+1)} \le \frac{n!}{1}$. If we take $M_n = n!$, then $\frac{M_n|b - a|^{n+1}}{(n + 1)!} = \frac{(\frac{1}{16} - 0)^{n+1}}{n + 1} = \frac{1}{16^{n+1}(n + 1)} \le \frac{1}{10^5}$ for n at least 3. Then, since $P_3(x) = x - \frac{x^2}{2!} + \frac{x^3}{3}$, and so $\ln(1 + \frac{1}{16}) \approx P_3(\frac{1}{16}) = \frac{1}{16} - \frac{1}{2(16)^2} + \frac{1}{16^3(3)}$. Thus, $\ln 17 = \ln(1 + \frac{1}{16}) + 4 \ln 2 \approx \frac{1}{16} - \frac{1}{2(16)^2} + \frac{1}{16^3(3)} + 2.77259 \approx 2.83322 \approx 2.8332$. (The correct value of ln 17 rounded off to six decimal places is 2.833213.)

23. Take $f(x) = \sqrt{x}$, $a = 9$ and $b = 9.04$. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, $f^4(x) = \frac{-3\cdot5}{16}x^{-7/2}$, $f^5(x) = \frac{3\cdot5\cdot7x^{-9/2}}{32}$, and so $f^n(x) = (-1)^{n-1}\frac{1\cdot3\cdot5\ldots(2n - 3)x^{-(2n-1)/2}}{2^n}$, $n \ge 2$. Thus, $|f^{n+1}(c)| = \frac{[1\cdot3\cdot5\ldots(2n - 1)]c^{-(2n+1)/2}}{2^{n+1}}$ (where $9 < c < 9.04$) $\le \frac{[1\cdot3\cdot5\ldots(2n - 1)]}{9^{\frac{2n+1}{2}}(2^{n+1})} = \frac{1\cdot3\cdot5\ldots(2n - 1)}{3^{(2n+1)}\cdot2^{n+1}}$. If $M_n = \frac{1\cdot3\cdot5\ldots(2n - 1)}{3^{2n+1}(2^{n+1})}$, then $\frac{M_n|b - a|^{n+1}}{(n + 1)!} \le \frac{[1\cdot3\cdot5\ldots(2n - 1)](0.04)^{n+1}}{3^{2n+1}\cdot(2^{n+1})(n + 1)!} = \frac{[1\cdot3\cdot5\ldots(2n - 1)]2^{n+1}}{3^{2n+1}(10^{2n+2})(n + 1)!} \le \frac{1}{10^5}$ for n at least 2. $P_n(x) = f(9) + f'(9)(x - 9) + \frac{f''(9)(x - 9)^2}{2!}$ and so $P_n(9.04) = 3 + \frac{1}{2\sqrt{9}}(9.04 - 9) - \frac{1}{4(27)}\frac{(9.04 - 9)^2}{2!} = 3 + \frac{0.04}{6} - \frac{(0.04)^2}{236} \approx 3.00666 \approx 3.0067$. (The correct value of $\sqrt{9.04}$ rounded off to six decimal places is 3.006659.)

24. $P_1(x) = f(a) + f'(a)(x - a)$, so $P_1'(x) = f'(a)$ and $P_1'(a) = f'(a)$. Thus, the result holds for $n = 1$. Assume that the result holds for n. We prove that it holds for $n + 1$. Evidently, $P_{n+1}(x) = P_n(x) + \frac{f^{(n+1)}(a)}{(n + 1)!}(x - a)^{n+1}$. Now, $D_x(x - a)^{n+1} = (n + 1)(x - a)^n$, $D_x^2(x - a)^{n+1} = (n + 1)n(x - a)^{n-1}$, $D_x^3(x - a)^{n+1} = (n + 1)(n)(n - 1)(x - a)^{n-2}$, and so forth; hence, $D_x^k(x - a)^{n+1} = (n + 1)(n)\ldots(n - k + 2)(x - a)^{n-k+1}$. Thus, $P_{n+1}^{(k)}(x) = P_n^{(k)}(x) + \frac{f^{(n+1)}(a)}{(n + 1)!}(n + 1)(n)\ldots(n - k + 2)(x - a)^{n-k+1}$. By the induction hypothesis, if $1 \le k \le n$, then $P_{n+1}^{(k)}(a) = P_n^{(k)}(a) + 0 = P_n^{(k)}(a) = f^{(k)}(a)$. Since $P_n(x)$ has degree n, then $P_n^{(n+1)}(x) = 0$ and $P_{n+1}^{(n+1)}(x) = P^{(n+1)}(x) + \frac{f^{(n+1)}(a)}{(n + 1)!}(n + 1)(n)\ldots1(x - a)^0 = 0 + f^{(n+1)}(a)$; hence, $P_{n+1}^{(n+1)}(a) = f^{(n+1)}(a)$. Thus, $P_{n+1}^{(k)}(a) = f^{(k)}(a)$ holds for $k = 1,2,\ldots,n + 1$.

25. Let $f(x) = \sin x$, $a = 0$, $b = 5^0 = \frac{\pi}{36}$. By the argument in Problem 17, $|f^{(n+1)}(c)| \le 1 = M_n$, so that a bound on the error is given by $\frac{M_n|b - a|^{n+1}}{(n + 1)!} = \frac{(\frac{\pi}{36})^{n+1}}{(n + 1)!} \le \frac{1}{10^{n+1}(n + 1)!}$.

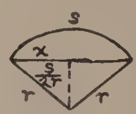
26. Let $f(x) = \sqrt{1 + x}$, $a = 0$, $b = x$, $n = 1$ in Theorem 2. Here $f'(x) = \frac{1}{2\sqrt{1 + x}}$ and $f''(x) = -\frac{1}{4}(1 + x)^{-3/2}$; hence, $P_1(x) = f(0) + f'(0)x = 1 + \frac{1}{2}x$ and $R_1(x) = \frac{f''(c)}{2!}x^2 = -\frac{1}{8}(1 + c)^{-3/2}x^2$, where c is strictly between 0 and x. Thus, since $|x| \le 0.1$ and $(1 + c)^{3/2} \ge (1 - 0.1)^{3/2} = (\frac{9}{10})^{3/2} = \frac{27\sqrt{10}}{100}$, it follows that $|R_1(x)| \le \frac{1}{8}(\frac{100}{27\sqrt{10}})(0.1)^2 = \frac{\sqrt{10}}{2160} \approx 0.0015$.

27. For $f(x) = \cos x$, $a = 0$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$. $P_3(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)(x - 0)^2}{2!} + \frac{f'''(0)(x - 0)^3}{3!} = 1 - \frac{x^2}{2} \approx \cos x$.

$|f^4(c)| = |\cos c| \le 1$. $|R_3| = \left|\frac{f^4(c)(x-0)^4}{4!}\right| \le \frac{x^4}{24}$.

For $n = 3$, the error of the estimate $\cos x = 1 - \frac{x^2}{2}$ does not exceed $\frac{x^4}{24}$ in absolute value.

28. (a) $\sin \frac{s}{2r} = \frac{x}{r}$, so that $x = r \sin \frac{s}{2r}$. Thus, the chord has length $2x = 2r \sin \frac{s}{2r}$. Hence, the difference between the arc length s and the corresponding chord is given by $s - 2r \sin \frac{s}{2r}$.



(b) Take $f(s) = s - 2r \sin \frac{s}{2r}$ and $a = 0$. $f(s) \approx P_3(s) = f(0) + f'(0)(s - 0) + \frac{f''(0)(s^2)}{2!} + \frac{f'''(0)s^3}{3!} = \frac{1}{3!}\left(\frac{1}{4r^2}\right)s^3 = \frac{s^3}{24r^2}$. Since $P_3(s) = P_4(s)$, then a bound for the error is $\frac{M|s-0|^5}{5!}$ where $M \ge |f^{(5)}(c)| = \left|\frac{1}{(2r)^4} \cos \frac{c}{2r}\right|$ so that $M = \frac{1}{16r^4}$. Hence, a bound for the error is $\frac{s^5}{16(120)r^4} = \frac{s^5}{(1920)r^4}$.

29. $A = A$ (sector) $- A$ (triangle) $= \frac{1}{2} rs - \frac{1}{2}(2r \sin \frac{s}{2r})(r \cos \frac{s}{2r})$, so that $A = \frac{1}{2} rs - \frac{1}{2} r^2 \sin \frac{s}{r} = \frac{1}{2} r(s - r \sin \frac{s}{r})$. Let $f(s) = \frac{1}{2} r(s - r \sin \frac{s}{r})$, $f'(s) = \frac{1}{2} r - \frac{1}{2} r \cos \frac{s}{r}$, $f''(s) = \frac{1}{2} \sin \frac{s}{r}$, $f'''(s) = \frac{1}{2r} \cos \frac{s}{r}$. Let $a = 0$. $P_3(s) = f(0) + f'(0)s + \frac{f''(0)s^2}{2!} + \frac{f'''(0)s^3}{3!} = \frac{s^3}{12r}$. Since $P_3(s) = P_4(s)$, then a bound for the error is $\frac{Ms^5}{5!}$, where $|f^{(5)}(c)| = \left|-\frac{1}{2r^3} \cos \frac{s}{r}\right| \le \frac{1}{2r^3} = M$. Hence, the bound for the error is $\frac{s^5}{240r^3}$.

30. By Problem 7, $P_3(x) = x - \frac{x^3}{3!} \approx \sin x$. Now $5(x - \frac{x^3}{3!}) - 4x = 0$ provided $5x - \frac{5x^3}{6} - 4x = 0$ or $6x - 5x^3 = 0$. If $x > 0$, then $x(6 - 5x^2) = 0$ when $x = \sqrt{\frac{6}{5}}$. Hence, $5 \sin x - 4x = 0$ for $x \approx \sqrt{\frac{6}{5}}$.

31. $f(x) = (1 + x)^p$, $f'(x) = p(1 + x)^{p-1}$, $f''(x) = p(p - 1)(1 + x)^{p-2}$, $f'''(x) = p(p - 1)(p - 2)(1 + x)^{p-3}$, $f^{(4)}(x) = p(p - 1)(p - 2)(p - 3)(1 + x)^{p-4}$. $f(0) = 1$, $f'(0) = p$, $f''(0) = p(p - 1)$, $f'''(0) = p(p - 1)(p - 2)$, $f^{(4)}(0) = p(p - 1)(p - 2)(p - 3)$.

(a) $P_1(x) = 1 + px$

(b) $P_2(x) = 1 + px + \frac{p(p-1)}{2} x^2$

(c) $P_3(x) = 1 + px + \frac{p(p-1)}{2} x^2 + \frac{p(p-1)(p-2)}{6} x^3$

(d) $P_4(x) = 1 + px + \frac{p(p-1)}{2} x^2 + \frac{p(p-1)(p-2)}{6} x^3 + \frac{p(p-1)(p-2)(p-3)}{24} x^4$.

32. Let $f(x) = \cosh x$. Then $f'(x) = \sinh x$, $f''(x) = \cosh x$. Since $\frac{wL}{2T}$ is small, we take the Taylor polynomial about $a = 0$; $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$. Then $P_2(x) = 1 + \frac{x^2}{2} \approx \cosh(x)$ for x small; and

$$H \approx \frac{T}{W}\left[1 + \frac{(\frac{wL}{2T})^2}{2} - 1\right] = \frac{T}{W}\left[\frac{w^2L^2}{8T^2}\right] = \frac{L^2w}{8T}.$$

33. From Problem 32, if $f(x) = \cosh x$, then $P_2(x) = 1 + \frac{x^2}{2}$. Thus, $Ts \approx T\left(1 + \frac{(\frac{wL}{2T})^2}{2}\right) = T\left(1 + \frac{w^2L^2}{8T^2}\right) = T + \frac{w^2L^2}{8T}$.

34. Let $Q(x) = f(x) - P_n(x)$. By Problem 24, $Q^{(k)}(a) = 0$ for $k = 1,2,\ldots,n$. Let P_n be the nth degree Taylor polynomial for the function Q at a and let R_n be the corresponding Taylor remainder, so that $Q(x) = P_n(x) + R_n(x)$. Since $Q^{(k)}(a) = 0$ for $k = 1,2,\ldots,n$ and $Q(a) = f(a) - P_n(a) = 0$, it follows that $P_n^{(k)}(a) = 0$ for $k = 1,2,\ldots,n$ and $P_n(x) = 0$ for all x. Therefore, $Q(x) = R_n(x) = \frac{Q^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$, where c is strictly between a and x. Since both $f(x)$ and $P_n(x)$ are polynomials of degree n or less, then Q is a polynomial of degree n or less; hence, $Q^{(n+1)}(c) = 0$. It follows

that $Q(x) = 0$ for all x; hence, $f(x) = P_n(x)$ for all x.

35. (a) Let $f(x) = e^x$ and $a = 0$. $f^{(n)}(x) = e^x$ for all n, so that $f^{(n)}(a) = f^{(n)}(0) = 1$ for all n. Thus, $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + R_4(x)$. But $R_4(x) = \frac{f^{(4+1)}(c)x^{4+1}}{5!} = \frac{e^c x^5}{5!}$ where $x < c < 0$. Since $x < c < 0$, $e^c < e^0$. But $x^5 \le 0$ for $x \le 0$. Hence, $0 \ge e^c x^5 \ge x^5$ and so $0 \ge \frac{e^c x^5}{5!} \ge \frac{x^5}{5!}$. Therefore, $\frac{x^5}{120} \le R_4(x) \le 0$.

(b) $e^{-t^2} = 1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \frac{t^8}{24} - r(t)$. Now $R_4(x) = \frac{e^c(x)^5}{5!} = \frac{e^c t^{10}}{5!} = -r(t)$ where $r(t) = \frac{e^c t^{10}}{120}$. But $0 \le \frac{e^c t^{10}}{120} \le \frac{t^{10}}{120}$, and so $0 \le r(t) \le \frac{t^{10}}{120}$.

(c) By Part (b), $\int_0^b e^{-t^2}\,dt =$ $\int_0^b [1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \frac{t^8}{24} - r(t)]dt =$ $(t - \frac{t^3}{3} + \frac{t^5}{10} - \frac{t^7}{42} + \frac{t^9}{216})\Big|_0^b - \int_0^b r(t)dt = b - \frac{b^3}{3} + \frac{b^5}{10} - \frac{b^7}{42} + \frac{b^9}{216} - \varepsilon$ where $0 \le \int_0^b r(t)dt = \varepsilon \le \int_0^b \frac{t^{10}}{120}\,dt = \frac{t^{11}}{1320}\Big|_0^b = \frac{b^{11}}{1320}$.

(d) $\int_0^{3/4} e^{-t^2}dt \approx \frac{3}{4} - \frac{(\frac{3}{4})^3}{3} + \frac{(\frac{3}{4})^5}{10} - \frac{(\frac{3}{4})^7}{42} + \frac{(\frac{3}{4})^9}{216} \approx 0.63027 \approx 0.630$. Now $0 \le \varepsilon \le \frac{b^{11}}{1320} = \frac{(\frac{3}{4})^{11}}{1320} < 0.00004$, and so the error term does not affect the accuracy of our answer rounded off to three decimal places, since $0.63027 - 0.00004 = 0.63023 \approx 0.630$.

36. First we must show that $R_1(b) = \int_a^b (b - x)f''(x)dx$. Now $R_1(b) = f(b) - P_1(b) = f(b) - [f(a) + \frac{f'(a)(b - a)}{1!}] = f(b) - f(a) - f'(a)(b - a)$. Also, $\int_a^b (b - x)f''(x)dx = f'(x)(b - x)\Big|_a^b + \int_a^b f'(x)dx = f'(b)\cdot 0 - f'(a)(b - a) + f(b) - f(a) = f(b) - f(a) - f'(a)(b - a)$. (The integration was by parts with $u = b - x$ and $dv = f''(x)dx$.) Hence, $R_1(b) = \int_a^b (b - x)f''(x)dx$. Now we assume that $R_k(b) = \frac{1}{k!}\int_a^b (b - x)^k f^{(k+1)}(x)dx$ and show that $R_{k+1}(b) = \frac{1}{(k + 1)!}\int_a^b (b - x)^{(k+1)} f^{(k+2)}(x)dx$. We look at $\frac{1}{(k + 1)!}\int_a^b (b - x)^{(k+1)} f^{(k+2)}(x)dx$. We let $u = (b - x)^{(k+1)}$ and $dv = f^{(k+2)}(x)dx$, so that $du = -(k + 1)(b - x)^k dx$ and $v = f^{(k+1)}(x)$. Thus,

$\frac{1}{(k + 1)!}\int_a^b (b - x)^{(k+1)} f^{(k+2)}(x)dx =$

$\frac{1}{(k + 1)!}[f^{(k+1)}(x)(b - x)^{(k+1)}\Big|_a^b +$

$\int_a^b (k + 1)(b - x)^{k(k+1)} f(x)dx] =$

$\frac{1}{(k + 1)!}[f^{(k+1)}(b)(b - b)^{(k+1)} -$
$f^{(k+1)}(a)(b - a)^{(k+1)}] +$
$\frac{1}{(k + 1)!}\int_a^b (k + 1)(b - x)^k f^{(k+1)}(x)dx =$

$-\frac{1}{(k + 1)!} f^{(k+1)}(a)(b - a)^{(k+1)} +$

$\frac{1}{k!}\int_a^b (b - x)^k f^{(k+1)}(x)dx =$

$-\frac{1}{(k + 1)!} f^{(k+1)}(a)(b - a)^{(k+1)} + R_k(b)$ (by induction hypothesis) $= -\frac{1}{(k + 1)!} f^{(k+1)}(a)(b - a)^{(k+1)} + f(b) - P_k(b) = f(b) - P_{k+1}(b) = R_{k+1}(b)$. Hence, the statement is true for all n.

37. (a) $f(x) = (1 - x)^{-1}$, $f'(x) = (1 - x)^{-2}$, $f''(x) = 2(1 - x)^{-3}$, $f'''(x) = 3\cdot 2(1 - x)^{-4}$, $f^4(x) = 4!(1 - x)^{-5}, \ldots, f^n(x) = n!(1 - x)^{-(n+1)}$. Thus, $P_n(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)(x - 0)^2}{2!} + \frac{f'''(0)(x - 0)^3}{3!} + \ldots + \frac{f^n(0)(x - 0)^n}{n!} = 1 + x + \frac{2!x^2}{2!} + \frac{3!x^3}{3!} + \ldots + \frac{n!x^n}{n!} = 1 + x + x^2 + x^3 + \ldots + x^n$.

(b) $R_n(x) = f(x) - P_n(x) =$
$f(x) - [1 + x + x^2 + x^3 + \ldots + x^n] =$
$\frac{1}{1-x} - 1 - x - x^2 - x^3 - \ldots - x^{n-1} - x^n =$

$$\frac{1-1+x-x+x^2-x^2+x^3-x^3+x^4-\ldots-x^{n-1}+x^n-x^n+x^{n+1}}{1-x} = \frac{x^{n+1}}{1-x}$$

since adjacent pairs cancel out until the last term.

38. Suppose P_n is the nth degree Taylor polynomial for f at a. We consider $P(x) - P_n(x) = Q(x)$. Now by an argument similar to that in Problem 34, $Q(x) = 0$. Hence, P is the nth degree Taylor polynomial for f at a.

39. $g(a) = \int_a^a f(t)dt = 0$; $Q(a) = \int_a^a P_n(x)dx = 0$. Hence, $g(a) = Q(a)$. Now $g'(a) = f(a)$, $g''(a) = f'(a)$ and so forth, so that $g^{k+1}(a) = f^k(a)$ for $k = 1,2,3,4,\ldots,n$. Similarly, $Q^{k+1}(a) = P_n^k(a)$ for $k = 1,2,3,\ldots,n$. But $f^k(a) = P_n^k(a)$ for all k by Problem 34. Hence, $Q^{k+1}(a) = g^{k+1}(a)$ for all $k = 1,2,\ldots,n$. Therefore, by Problem 38, Q is the Taylor polynomial of degree n + 1 for g at a.

40. (a) $g(0) = 0^{n+1}h(0) = 0$. $g'(x) = (n+1)x^n h(x) + x^{n+1}h'(x)$, so that $g'(0) = 0$. $g''(x) = (n+1)(n)x^{n-1}h(x) + (n+1)x^n h'(x) + (n+1)x^n h'(x) + x^{n+1}h''(x)$, so that $g''(0) = 0$. It is clear that successive differentiations will produce more terms, and although the exponents on x will be systematically reduced, an x will still remain in each of the terms of the first n derivatives of g. Hence, $g^k(0) = 0$ for $k = 1,2,3,\ldots,n$.
(b) The nth degree Taylor polynomial for g at a = 0 is the zero polynomial since $P_n(x) = g(0) + g'(0)x + \frac{g''(0)x^2}{2!} + \ldots + \frac{g^n(0)x^n}{n!}$ and each coefficient $g^k(0) = 0$, for $k = 1,2,3,\ldots,n$.

41. $f(0) = P(0) + 0 = P(0)$. By part (a) of Problem 40, the first n derivatives of the function $g(x) = x^{n+1}h(x)$ evaluated at 0 give 0. Hence, $f^k(0) = P^k(0) + 0 = P^k(0)$, $k = 1,2,3,\ldots,n$. Therefore, by Problem 38, P is the nth degree Taylor polynomial for f at 0.

42. (a) By Problem 37, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \frac{x^{n+1}}{1-x}$. Replace x by $-x^2$, so that $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}$.
(b) Let $P(x) = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n}$. By part (a), $f(x) = P(x) + x^{2n+2}\left[\frac{(-1)^{n+1}}{1+x^2}\right]$ or $f(x) = P(x) + x^{2n+1}\left[\frac{(-1)^{n+1}x}{1+x^2}\right] = P(x) + x^{2n+1}h(x)$. By Problem 41, P(x) is the 2nth degree Taylor polynomial for f at 0. Thus $P(x) = P_{2n}(x) = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n}$. Hence, by the definition of the Taylor remainder, $R_{2n}(x) = \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}$.

43. Let $Q(x) = \int_0^x P_{2n}(t)dt = \int_0^x (1 - t^2 + t^4 - t^6 + \ldots + (-1)^n t^{2n})dt = (x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1}) = P_{2n+1}(x)$. Now by Problem 42, $P_{2n}(x)$ is the Taylor polynomial of degree 2n for $f(x) = \frac{1}{1+x^2}$ at a = 0. Hence, by Problem 39, $\int_0^x P_{2n}(t)dt$ is the Taylor polynomial of degree n + 1 for the function $\int_0^x \frac{1}{1+t^2}dt = \tan^{-1}x$. Consequently, $P_{2n+1}(x)$ is the Taylor polynomial of degree 2n + 1 for the inverse tangent of x at a = 0.

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1. $\lim\limits_{x\to 0} \frac{xe^x}{1-e^x} = \lim\limits_{x\to 0} \frac{xe^x + e^x}{-e^x} = -1$.

2. $\lim\limits_{x\to 0} \frac{8^x - 2^x}{4x} = \lim\limits_{x\to 0} \frac{8^x \ln 8 - 2^x \ln 2}{4} = \frac{\ln 8 - \ln 2}{4} = \frac{\ln 2}{2}$.

3. $\lim\limits_{x\to 0} \frac{\ln(\sec 2x)}{\ln(\sec x)} = \lim\limits_{x\to 0} \frac{\frac{2 \sec 2x \tan 2x}{\sec 2x}}{\frac{\sec x \tan x}{\sec x}} = \lim\limits_{x\to 0} \frac{2 \tan 2x}{\tan x} = \lim\limits_{x\to 0} \frac{4 \sec^2 2x}{\sec^2 x} = 4$.

4. $\lim_{x\to 0} \frac{\cos 2x - \cos x}{\sin^2 x} = \lim_{x\to 0} \frac{-2 \sin 2x + \sin x}{2 \sin x \cos x} =$

$\lim_{x\to 0} \frac{-4 \cos 2x + \cos x}{-2 \sin^2 x + 2 \cos^2 x} = -\frac{3}{2}.$

5. $\lim_{x\to 1^+} \left(\frac{x}{\ln x} - \frac{1}{1 - x}\right) = \lim_{x\to 1^+} \frac{x - x^2 - \ln x}{(1 - x) \ln x} =$

$\lim_{x\to 1^+} \frac{1 - 2x - \frac{1}{x}}{(1 - x)\frac{1}{x} - \ln x} = +\infty.$

6. $\lim_{x\to 0} \frac{e^x - 1}{x^2 - x} = \lim_{x\to 0} \frac{e^x}{2x - 1} = -1.$

7. $\lim_{x\to 0^-} \frac{2 - 3e^{-x} + e^{-2x}}{2x^2} = \lim_{x\to 0^-} \frac{3e^{-x} - 2e^{-2x}}{4x} = -\infty.$

8. $\lim_{x\to 1} \frac{2x^3 + 5x^2 - 4x - 3}{x^3 + x^2 - 10x + 8} = \lim_{x\to 1} \frac{6x^2 + 10x - 4}{3x^2 + 2x - 10} = -\frac{12}{5}.$

9. $\lim_{x\to 0} \frac{\sqrt{1 - x} - \sqrt{1 + x}}{x} = \lim_{x\to 0} \frac{-\frac{1}{2\sqrt{1 - x}} - \frac{1}{2\sqrt{1 + x}}}{1} = -1.$

10. $\lim_{x\to 1^+} \left(\frac{1}{x - 1} - \frac{1}{\sqrt{x - 1}}\right) = \lim_{x\to 1^+} \frac{1 - \sqrt{x - 1}}{x - 1} = +\infty.$

Notice that the form was indeterminate, but we did not need L'Hôpital's rule by virtue of the fact that we used the least common denominator $x - 1$ in the second step.

11. $\lim_{x\to 0^+} x^3(\ln x)^3 = \lim_{x\to 0^+} \frac{(\ln x)^3}{\frac{1}{x^3}} = \lim_{x\to 0^+} \frac{\frac{3(\ln x)^2}{x}}{\frac{-3}{x^4}} =$

$\lim_{x\to 0^+} \frac{(\ln x)^2}{-\frac{1}{x^3}} = \lim_{x\to 0^+} \frac{\frac{2 \ln x}{x}}{\frac{3}{x^4}} = \lim_{x\to 0^+} \frac{2 \ln x}{\frac{3}{x^3}} =$

$\lim_{x\to 0^+} \frac{\frac{2}{x}}{-\frac{9}{x^4}} = \lim_{x\to 0^+} -\frac{2}{9}x^3 = 0.$

12. $\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc^2 x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x} =$

$\lim_{x\to 0^+} \frac{-2 \sin x \cos x}{1} = 0.$

13. $\lim_{x\to 1^+} \frac{(\ln x)^2}{\sin(x - 1)} = \lim_{x\to 1^+} \frac{\frac{2 \ln x}{x}}{\cos(x - 1)} =$

$\lim_{x\to 1^+} \frac{2 \ln x}{x \cos(x - 1)} = 0.$

14. $\lim_{x\to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{\tan x} =$

$\lim_{x\to 0} \frac{\frac{\cos x}{2\sqrt{1 + \sin x}} - \frac{-\cos x}{2\sqrt{1 - \sin x}}}{\sec^2 x} = 1.$

15. $\lim_{x\to +\infty} x \sin \frac{a}{x} = \lim_{x\to +\infty} \frac{\sin \frac{a}{x}}{\frac{1}{x}} = \lim_{x\to +\infty} \frac{-\frac{a}{x^2} \cos \frac{a}{x}}{-\frac{1}{x^2}} =$

$\lim_{x\to +\infty} a \cos \frac{a}{x} = a.$

16. $\lim_{x\to 0} \csc x \sin(\tan x) = \lim_{x\to 0} \frac{\sin(\tan x)}{\sin x} =$

$\lim_{x\to 0} \frac{\cos(\tan x)(\sec^2 x)}{\cos x} = 1.$

17. $\lim_{x\to 1} \left[\frac{2}{x^2 - 1} - \frac{1}{x - 1}\right] = \lim_{x\to 1} \frac{2 - (x + 1)}{x^2 - 1} =$

$\lim_{x\to 1} \frac{1 - x}{x^2 - 1} = \lim_{x\to 1} \frac{-1}{2x} = -\frac{1}{2}.$

18. Substitute $\frac{1}{t}$ for x, and since $x \to +\infty$, then $t \to 0^+$.

Thus, $\lim_{x\to +\infty} \frac{\sqrt[3]{1 + x^6}}{1 - x + 2\sqrt{1 + x^2 + x^4}} =$

$\lim_{t\to 0^+} \frac{\sqrt[3]{1 + \frac{1}{t^6}}}{1 - \frac{1}{t} + 2\sqrt{1 + \frac{1}{t^2} + \frac{1}{t^4}}} =$

$\lim_{t\to 0^+} \frac{\frac{1}{t^2}\sqrt[3]{t^6 + 1}}{\frac{t^2 - t + 2\sqrt{t^4 + t^2 + 1}}{t^2}} =$

$\lim_{t\to 0^+} \frac{\sqrt[3]{t^6 + 1}}{t^2 - t + 2\sqrt{t^4 + t^2 + 1}} = \frac{1}{2}.$ Notice that we did not use l'Hôpital's rule here, although we could have. However, the calculation of the limit would have been more complicated.

19. $\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x - \sin x}\right) = \lim_{x\to 0^+} \frac{\sin^2 x}{x^2 - x \sin x} =$

$\lim_{x\to 0^+} \frac{2 \sin x \cos x}{2x - x \cos x - \sin x} =$

$\lim_{x\to 0^+} \frac{-2 \sin^2 x + 2 \cos^2 x}{2 - \cos x + x \sin x - \cos x} = +\infty.$

20. $\lim_{x\to +\infty} (\cosh x - \sinh x) = \lim_{x\to +\infty} \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) =$

$\lim_{x\to +\infty} e^{-x} = 0.$ Notice that we did not use l'Hôpital's rule here since we found a simpler method.

21. $\lim_{x\to 1^-} x^{\frac{1}{1-x^2}} = \lim_{x\to 1^-} e^{\frac{1}{1-x^2}\ln x}$. Now $\lim_{x\to 1^-} \frac{\ln x}{1-x^2} =$ $\lim_{x\to 1^-} \frac{\frac{1}{x}}{-2x} = -\frac{1}{2}$. Thus, $\lim_{x\to 1^-} x^{\frac{1}{1-x^2}} = e^{-\frac{1}{2}}$.

22. $\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^3}} = \lim_{x\to 0^+} e^{\frac{1}{x^3}\ln\frac{\sin x}{x}}$. Now

$$\lim_{x\to 0^+} \frac{\ln\frac{\sin x}{x}}{x^3} = \lim_{x\to 0^+} \frac{\frac{x}{\sin x}\left[\frac{x\cos x - \sin x}{x^2}\right]}{3x^2} =$$

$$\lim_{x\to 0^+} \frac{x\cos x - \sin x}{(\sin x)3x^3} = \lim_{x\to 0^+} \frac{-x\sin x + \cos x - \cos x}{(\cos x)3x^3 + 9x^2\sin x} =$$

$$\lim_{x\to 0^+} \frac{-\sin x}{3x^2\cos x + 9x\sin x} =$$

$$\lim_{x\to 0^+} \frac{-\cos x}{6x\cos x - 3x^2\sin x + 9\sin x + 9x\cos x} = -\infty.$$

Hence, $\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^3}} = 0$.

23. $\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^{x^2} = \lim_{x\to+\infty} e^{x^2\ln(1+\frac{1}{x})}$. Now

$$\lim_{x\to+\infty} x^2\ln\left(1+\frac{1}{x}\right) = \lim_{x\to+\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x^2}} =$$

$$\lim_{x\to+\infty} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{2}{x^3}} = \lim_{x\to+\infty} \frac{x}{2(1+\frac{1}{x})} = +\infty.$$ Hence,

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^{x^2} = +\infty.$$

24. $\lim_{x\to 0} (1+ax^2)^{\frac{a}{x}} = \lim_{x\to 0} e^{\frac{a}{x}\ln(1+ax^2)}$. Now,

$$\lim_{x\to 0} \frac{a\ln(1+ax^2)}{x} = \lim_{x\to 0} \frac{\frac{a(2ax)}{1+ax^2}}{1} = 0.$$ Hence,

$$\lim_{x\to 0} (1+ax^2)^{\frac{a}{x}} = e^0 = 1.$$

25. $\lim_{x\to+\infty} (x^2+4)^{\frac{1}{x}} = \lim_{x\to+\infty} e^{[\frac{1}{x}\ln(x^2+4)]}$. Now,

$$\lim_{x\to+\infty} \frac{\ln(x^2+4)}{x} = \lim_{x\to+\infty} \frac{\frac{2x}{x^2+4}}{1} = \lim_{x\to+\infty} \frac{2}{2x} = 0.$$ Thus,

$$\lim_{x\to+\infty} (x^2+4)^{\frac{1}{x}} = e^0 = 1.$$

26. $\lim_{x\to 4^+} (x-4)^{(x^2-16)} = \lim_{x\to 4^+} e^{(x^2-16)\ln(x-4)}$.

Now $\lim_{x\to 4^+} (x^2-16)\ln(x-4) = \lim_{x\to 4^+} \frac{\ln(x-4)}{\frac{1}{x^2-16}} =$

$$\lim_{x\to 4^+} \frac{\frac{1}{x-4}}{\frac{-2x}{(x^2-16)^2}} = \lim_{x\to 4^+} \frac{(x^2-16)(x^2-16)}{(x-4)(-2x)} =$$

$$\lim_{x\to 4^+} \frac{(x+4)(x^2-16)}{-2x} = 0.$$ Thus,

$$\lim_{x\to 4^+} (x-4)^{(x^2-16)} = e^0 = 1.$$

27. $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln(\cos x)}$. Now,

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x\to 0} \frac{-\tan x}{2x} = \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}.$$

Hence, $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$.

28. $\lim_{x\to 0} (1+\sin x)^{\cot x} = \lim_{x\to 0} e^{\cot x\ln(1+\sin x)}$.

Now, $\lim_{x\to 0} \cot x\ln(1+\sin x) = \lim_{x\to 0} \frac{\ln(1+\sin x)}{\tan x} =$

$\lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x}}{\sec^2 x} = 1$. Thus, $\lim_{x\to 0} (1+\sin x)^{\cot x} =$ $e^1 = e$.

29. $\lim_{x\to 0} [\ln(x+1)]^x = \lim_{x\to 0} e^{x\ln[\ln(x+1)]}$. Now,

$$\lim_{x\to 0} x\ln[\ln(x+1)] = \lim_{x\to 0} \frac{\ln[\ln(x+1)]}{\frac{1}{x}} =$$

$$\lim_{x\to 0} \frac{\frac{1}{(x+1)\ln(x+1)}}{-\frac{1}{x^2}} = \lim_{x\to 0} -\frac{x^2}{(x+1)\ln(x+1)} =$$

$\lim_{x\to 0} \frac{-2x}{1+\ln(x+1)} = 0$. Therefore, $\lim_{x\to 0} [\ln(x+1)]^x =$ $e^0 = 1$.

30. $\lim_{x\to 0^+} (\tan^{-1}x)^{\frac{1}{\ln x}} = \lim_{x\to 0^+} e^{\frac{1}{\ln x}\ln(\tan^{-1}x)}$. Now,

$$\lim_{x\to 0^+} \frac{\ln(\tan^{-1}x)}{\ln x} = \lim_{x\to 0^+} \frac{\frac{1}{\tan^{-1}(x)[1+x^2]}}{\frac{1}{x}} =$$

$$\lim_{x\to0^+} \frac{x}{(\tan^{-1}x)(1+x^2)} = \lim_{x\to0^+}\frac{1}{[(\tan^{-1}x)(2x)]+1} = 1.$$

Thus, $\lim_{x\to0^+}(\tan^{-1}x)^{\frac{1}{\ln x}} = e^1 = e.$

31. $\lim_{x\to0^+}(\ln\frac{1}{x})^x = \lim_{x\to0^+} e^{x\ln(\ln\frac{1}{x})}$. Now,

$\lim_{x\to0^+} x\ln(\ln\frac{1}{x}) = \lim_{x\to0^+}\frac{\ln(\ln\frac{1}{x})}{\frac{1}{x}}$. Put $t=\frac{1}{x}$, so that, as $x\to0^+$, $t\to+\infty$. Then $\lim_{x\to0^+} x\ln(\ln\frac{1}{x}) =$

$\lim_{t\to+\infty}\frac{\ln(\ln t)}{t} = \lim_{t\to+\infty}\frac{\frac{1}{\ln t}\cdot\frac{1}{t}}{1} = 0.$ Hence,

$\lim_{x\to0^+}(\ln\frac{1}{x})^x = e^0 = 1.$

32. $\lim_{x\to0}(\sin^{-1}x)^x = \lim_{x\to0} e^{x\ln\sin^{-1}x}$. Now

$\lim_{x\to0} x\ln\sin^{-1}x = \lim_{x\to0}\frac{\ln\sin^{-1}x}{\frac{1}{x}} =$

$\lim_{x\to0}\frac{(\frac{1}{\sin^{-1}x})(\frac{1}{\sqrt{1-x^2}})}{(-\frac{1}{x^2})} = \lim_{x\to0}\frac{-x^2}{(\sin^{-1}x)\sqrt{1-x^2}} =$

$\lim_{x\to0}(\frac{x}{\sin^{-1}x})(\frac{-x}{\sqrt{1-x^2}}) = (1)(0) = 0.$ Hence,

$\lim_{x\to0}(\sin^{-1}x) = e^0 = 1.$ (To see that $\lim_{x\to0}\frac{x}{\sin^{-1}x} = 1$, let $x = \sin t$, so that, as $x\to0$, $t\to0$. Thus,

$\lim_{x\to0}\frac{x}{\sin^{-1}x} = \lim_{t\to0}\frac{\sin t}{t} = 1.$)

33. $f'(x) = \frac{1}{2\sqrt{x+9}}$, $g'(x) = \frac{1}{2\sqrt{x}}$; $\frac{f(16)-f(0)}{g(16)-g(0)} = \frac{f'(c)}{g'(c)}$,

$\frac{5-3}{4-0} = \frac{(\frac{1}{2\sqrt{c+9}})}{(\frac{1}{2\sqrt{c}})}$, $\frac{1}{2} = \frac{\sqrt{c}}{\sqrt{c+9}}$, $\frac{1}{4} = \frac{c}{c+9}$, $c + 9 = 4c$, $3c = 9$, $c = 3$.

34. $f'(x) = \cos x$, $g'(x) = -\sin x$; $\frac{f(\frac{\pi}{3})-f(\frac{\pi}{6})}{g(\frac{\pi}{3})-g(\frac{\pi}{6})} = \frac{\cos c}{-\sin c}$,

$\frac{\frac{\sqrt{3}}{2}-\frac{1}{2}}{\frac{1}{2}-\frac{\sqrt{3}}{2}} = -\cot c$, $\cot c = 1$, $c = \frac{\pi}{4}$.

35. The expression is indeterminate so that we use l'Hôpital's rule. $\lim_{x\to+\infty}\frac{\int_0^x e^t(t^2-t+5)dt}{\int_0^x e^t(3t^2+7t+1)dt} =$

$\lim_{x\to+\infty}\frac{e^x(x^2-x+5)}{e^x(3x^2+7x+1)} = \lim_{x\to+\infty}\frac{2x-1}{6x+7} = \lim_{x\to+\infty}\frac{2}{6} = \frac{1}{3}.$

36. $\lim_{x\to0}\frac{\int_0^x(\cos^2 t+5\cos t^2)dt}{\int_0^x e^{-t^2}dt} = \lim_{x\to0}\frac{\cos^2 x+5\cos x^2}{e^{-x^2}} = 6.$

37. $\lim_{x\to0^+} x^\alpha\ln x = \lim_{x\to0^+}\frac{\ln x}{\frac{1}{x^\alpha}} = \lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{\alpha}{x^{\alpha+1}}} = \lim_{x\to0^+}\frac{x^\alpha}{-\alpha} = 0$

(since α is positive).

38. $\lim_{t\to0}(\frac{\sin 3t}{t^3}+\frac{a}{t^2}+b) = \lim_{t\to0}(\frac{\sin 3t+at+bt^3}{t^3}) =$

$\lim_{t\to0}\frac{3\cos 3t+a+3bt^2}{3t^2}$. Now we must have $a = -3$ or else the expression is not indeterminate. Now

$\lim_{t\to0}\frac{3\cos 3t-3+3bt^2}{3t^2} = \lim_{t\to0}\frac{-9\sin 3t+6bt}{6t} =$

$\lim_{t\to0}\frac{-27\cos 3t+6b}{6} = 0$ provided $-27 = -6b$, that is, $b = \frac{9}{2}$.

39. $\int_1^\infty\frac{dx}{x\sqrt{2x^2-1}} = \lim_{b\to+\infty}\int_1^b\frac{dx}{x\sqrt{2x^2-1}} =$

$\lim_{b\to+\infty}\int_1^b\frac{\sqrt{2}\,dx}{\sqrt{2}\,x\sqrt{2x^2-1}} = \lim_{b\to+\infty}(\sec^{-1}\sqrt{2}\,x)\Big|_1^b =$

$\lim_{b\to+\infty}(\sec^{-1}\sqrt{2}\,b - \sec^{-1}\sqrt{2}) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}.$

40. $\int_1^\infty\frac{t\,dt}{(1+t^2)^2} = \lim_{b\to+\infty}\int_1^b\frac{t\,dt}{(1+t^2)^2} =$

$\lim_{b\to+\infty}\left[-\frac{1}{2(1+t^2)}\right]\Big|_1^b = \lim_{b\to+\infty}\left[-\frac{1}{2(1+b^2)}+\frac{1}{4}\right] = \frac{1}{4}.$

(We evaluated the integral by putting $u = 1 + t^2$.)

41. $\int_1^\infty\frac{e^{\frac{2}{t^2}}}{t^3}dt = \lim_{b\to+\infty}\int_1^b\frac{e^{\frac{2}{t^2}}dt}{t^3} = \lim_{b\to+\infty}(-\frac{1}{4}e^{\frac{2}{t^2}})\Big|_1^b =$

$\lim_{b\to+\infty}(-\frac{e^{\frac{2}{b^2}}}{4}+\frac{1}{4}e^2) = \frac{e^2}{4}-\frac{1}{4}.$

42. $\int_{-\infty}^0(e^x-e^{2x})dx = \lim_{a\to-\infty}\int_a^0(e^x-e^{2x})dx =$

$\lim_{a\to-\infty}(e^x-\frac{1}{2}e^{2x})\Big|_a^0 = \lim_{a\to-\infty}(e^0-\frac{e^0}{2}-e^a+\frac{e^{2a}}{2}) = \frac{1}{2}.$

43. $\int_1^\infty\frac{x^2-1}{x^4}dx = \lim_{b\to+\infty}\int_1^b(x^{-2}-x^{-4})dx =$

$\lim_{b\to+\infty} \left(-\frac{1}{x} + \frac{1}{3x^3}\right)\Big|_1^b = \lim_{b\to+\infty} \left(-\frac{1}{b} + \frac{1}{3b^3} + 1 - \frac{1}{3}\right) = \frac{2}{3}.$

44. $\int_2^\infty \frac{x\,dx}{(x^2-1)^{3/2}} = \lim_{b\to+\infty} \int_2^b \frac{x\,dx}{(x^2-1)^{3/2}} =$
$\lim_{b\to+\infty} \frac{-1}{\sqrt{x^2-1}}\Big|_2^b = \lim_{b\to+\infty} \left(\frac{-1}{\sqrt{b^2-1}} + \frac{1}{\sqrt{3}}\right) = +\frac{1}{\sqrt{3}}$. (We evaluated the integral by putting $u = x^2 - 1$.)

45. $\int_e^\infty \frac{dx}{x(\ln x)^{7/2}} = \lim_{b\to+\infty}\int_e^b \frac{dx}{x(\ln x)^{7/2}} =$
$\lim_{b\to+\infty} \left(-\frac{2}{5}\frac{1}{(\ln x)^{5/2}}\right)\Big|_e^b = \lim_{b\to+\infty}\left[\frac{2}{5} - \frac{2}{5(\ln b)^{5/2}}\right] = \frac{2}{5}.$

46. $\int_{-\infty}^0 \frac{e^x + 2x}{e^x + x^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{e^x+2x}{e^x+x^2}\,dx =$
$\lim_{a\to-\infty} \ln(e^x + x^2)\Big|_a^0 = \lim_{a\to-\infty} [\ln(1) - \ln(e^a + a^2)] = -\infty.$
The integral diverges.

47. $\int_{-\infty}^1 xe^{3x}dx = \lim_{a\to-\infty}\int_a^1 xe^{3x}dx =$
$\lim_{a\to-\infty}\left[\frac{x}{3}e^{3x}\Big|_a^1 - \int_a^1 \frac{1}{3}e^{3x}\,dx\right] =$
$\lim_{a\to-\infty}\left[\frac{e^3}{3} - \frac{ae^{3a}}{3} - \frac{e^3}{9} + \frac{e^{3a}}{9}\right] = \frac{e^3}{3} - 0 - \frac{e^3}{9} + 0 = \frac{2e^3}{9}.$

48. $\int_{-\infty}^\infty x^3e^{-x}dx = \int_{-\infty}^0 x^3e^{-x}dx + \int_0^\infty x^3e^{-x}dx =$
$\lim_{a\to-\infty}\int_a^0 x^3e^{-x}dx + \lim_{b\to+\infty}\int_0^b x^3e^{-x}dx$. Integrating by the tabular method, we find the limits are as follows: $\lim_{a\to-\infty} (-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x})\Big|_a^0 +$
$\lim_{b\to+\infty} (-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x})\Big|_0^b$. But
$\lim_{a\to+\infty} (-6 + a^3e^{-a} + 3a^2e^{-a} + 6ae^{-a} + 6e^{-a}) =$
$\lim_{a\to-\infty} [-6 + (a^3 + 3a^2 + 6a + 6)e^{-a}] = -\infty$, since $(a^3 + 3a^2 + 6a + 6)$ approaches $-\infty$ as a approaches $-\infty$ and e^{-a} approaches $+\infty$. The integral is divergent.

49. $\int_{-3}^1 \frac{dx}{x+3} = \lim_{\varepsilon\to0^+}\int_{-3+\varepsilon}^1 \frac{dx}{x+3} = \lim_{\varepsilon\to0^+} \ln|x+3|\Big|_{-3+\varepsilon}^1 =$
$\lim_{\varepsilon\to0^+} (\ln 4 - \ln\varepsilon) = +\infty$. The integral diverges.

50. $\int_{-2}^6 \frac{dx}{\sqrt[3]{x+2}} = \lim_{\varepsilon\to0^+}\int_{-2+\varepsilon}^6 \frac{dx}{\sqrt[3]{x+2}} =$
$\lim_{\varepsilon\to0^+} \frac{3}{2}(x+2)^{2/3}\Big|_{-2+\varepsilon}^6 = \lim_{\varepsilon\to0^+}\left[\frac{3}{2}(4) - \frac{3}{2}\varepsilon^{2/3}\right] = 6.$

51. $\int_0^1 \frac{e^t\,dt}{\sqrt[3]{e^t-1}} = \lim_{\varepsilon\to0^+}\int_\varepsilon^1 \frac{e^t\,dt}{\sqrt[3]{e^t-1}} =$
$\lim_{\varepsilon\to0^+} \frac{3}{2}(e^t-1)^{2/3}\Big|_\varepsilon^1 = \lim_{\varepsilon\to0^+}[\frac{3}{2}(e-1)^{2/3} - (e^\varepsilon-1)^{2/3}] =$
$\frac{3}{2}(e-1)^{2/3}.$

52. $\int_{-2}^2 \frac{dx}{\sqrt[5]{x+1}} = \lim_{\varepsilon\to0^+}\int_{-2}^{-1-\varepsilon}\frac{dx}{\sqrt[5]{x+1}} +$
$\lim_{\varepsilon\to0^+}\int_{-1+\varepsilon}^2 \frac{dx}{\sqrt[5]{x+1}} = \lim_{\varepsilon\to0^+} \frac{5}{4}(x+1)^{4/5}\Big|_{-2}^{-1-\varepsilon} +$
$\lim_{\varepsilon\to0^+} \frac{5}{4}(x+1)^{4/5}\Big|_{-1+\varepsilon}^2 = \lim_{\varepsilon\to0^+} [\frac{5}{4}(-\varepsilon)^{4/5} - \frac{5}{4}] +$
$\lim_{\varepsilon\to0^+} [\frac{5}{4}(3)^{4/5} - \frac{5}{4}\varepsilon^{4/5}] = \frac{5}{4} + \frac{5}{4}(3)^{4/5} = \frac{5}{4}(3^{4/5} - 1).$

53. $\int_0^{3a} \frac{2x\,dx}{(x^2-a^2)^{2/3}} = \lim_{\varepsilon\to0^+}\int_0^{a-\varepsilon}\frac{2x\,dx}{(x^2-a^2)^{2/3}} +$
$\lim_{\varepsilon\to0^+}\int_{a+\varepsilon}^{3a}\frac{2x\,dx}{(x^2-a^2)^{2/3}} = \lim_{\varepsilon\to0^+} (3\sqrt[3]{x^2-a^2})\Big|_0^{a-\varepsilon} +$
$\lim_{\varepsilon\to0^+} (3\sqrt[3]{x^2-a^2})\Big|_{a+\varepsilon}^{3a} =$
$\lim_{\varepsilon\to0^+} (3\sqrt[3]{(a-\varepsilon)^2 - a^2} + 3\sqrt[3]{-a^2}) +$
$\lim_{\varepsilon\to0^+} (3\sqrt[3]{8a^2} - 3\sqrt[3]{(a+\varepsilon)^2 - a^2}) =$
$-3\sqrt[3]{a^2} + 6\sqrt[3]{a^2} = 3\sqrt[3]{a^2}.$

54. $\int_a^{2a} \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \lim_{\varepsilon\to0^+}\int_{a+\varepsilon}^{2a}\frac{x^2\,dx}{\sqrt{x^2-a^2}} =$
$\lim_{\varepsilon\to0^+} a^2\left[\frac{1}{2}\frac{x}{a}\frac{\sqrt{x^2-a^2}}{a} + \tfrac{1}{2}\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right|\right]_{a+\varepsilon}^{2a} =$
$\lim_{\varepsilon\to0^+} a^2\left[\sqrt{3} + \tfrac{1}{2}\ln(2+\sqrt{3}) - \left(\frac{a+\varepsilon}{2a}\right)\frac{\sqrt{(a+\varepsilon)^2-a^2}}{a} - \tfrac{1}{2}\ln\left(\frac{a+\varepsilon}{a} + \frac{\sqrt{(a+\varepsilon)^2-a^2}}{a}\right)\right] =$
$a^2[\sqrt{3} + \tfrac{1}{2}\ln(2+\sqrt{3})]$. (We evaluated the integral by the substitution $x = a\sec\theta$.)

55. $\int_0^\infty \frac{1}{x^2+9}\,dx = \lim_{b\to+\infty}\int_0^b \frac{1}{x^2+9}\,dx =$
$\lim_{b\to+\infty} \left(\frac{1}{3}\tan^{-1}\frac{x}{3}\right)\Big|_0^b = \lim_{b\to+\infty}\left(\frac{1}{3}\tan^{-1}\frac{b}{3} - 0\right) =$
$\frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}.$

56. $\int_0^{\infty} \sqrt{x}\, e^{-\sqrt{x}}\, dx = \lim_{b\to+\infty} \int_0^b \sqrt{x}\, e^{-\sqrt{x}}\, dx =$
$\lim_{b\to+\infty} \int_0^{\sqrt{b}} 2u^2 e^{-u} du = \lim_{b\to+\infty} (-2u^2e^{-u} - 4ue^{-u} - 4e^{-u})\Big|_0^{\sqrt{b}} =$
$\lim_{b\to+\infty} (-2be^{-\sqrt{b}} - 4\sqrt{b}\, e^{-\sqrt{b}} - 4e^{-\sqrt{b}} + 4) =$
$\lim_{b\to+\infty} (\frac{-2b}{e^{\sqrt{b}}} - \frac{4\sqrt{b}}{e^{\sqrt{b}}} - \frac{4}{e^{\sqrt{b}}} + 4) = 0 + 4 = 4.$ The limit 0 is obtained by repeated application of l'Hôpital's rule. The integration was by the tabular method of integration by parts.

57. $A = \int_e^{\infty} \frac{1}{x \ln x} = \lim_{b\to+\infty} \int_e^b \frac{1}{x \ln x}\, dx = \lim_{b\to+\infty} \ln(\ln x)\Big|_e^b =$
$\lim_{b\to+\infty} [\ln(\ln\ b) - 0] = +\infty.$ The area is infinite.

58. $A = \int_1^{\infty} \frac{1}{x(x+2)^2}\, dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x(x+2)^2}\, dx =$
$\lim_{b\to+\infty} \left[\int_1^b \frac{\frac{1}{4}}{x}\, dx + \int_1^b \frac{-\frac{1}{4}}{x+2}\, dx + \int_1^b \frac{-\frac{1}{2}}{(x+2)^2}\, dx\right] =$
$\lim_{b\to+\infty} \left[(\tfrac{1}{4} \ln x)\Big|_1^b - \tfrac{1}{4}\ln|x+2|\Big|_1^b + (\tfrac{1}{2}\, \frac{1}{x+2})\Big|_1^b\right] =$
$\lim_{b\to+\infty} [\tfrac{1}{4} \ln b - \tfrac{1}{4} \ln|b+2| + \tfrac{1}{4}\ln 3 + \frac{1}{2(b+2)} - \frac{1}{6}] =$
$\lim_{b\to+\infty} [\tfrac{1}{4} \ln \frac{b}{|b+2|} + \frac{\ln 3}{4} + \frac{1}{2(b+2)} - \frac{1}{6}] =$
$\frac{\ln 3}{4} - \frac{1}{6}.$

59. (a) $V = \pi\int_0^{\infty} (x^2e^{-ax})^2 dx = \pi \lim_{b\to+\infty} \int_0^b x^4e^{-2ax}dx =$
$\pi \lim_{b\to+\infty} \left[-\frac{x^4}{2a}e^{-2ax} - \frac{x^3}{a^2}e^{-2ax} - \frac{3x^2}{2a^3}e^{-2ax} - \frac{3x}{2a^4}e^{-2ax} - \frac{3}{4a^5}e^{-2ax}\right]\Big|_0^b = \pi \lim_{b\to+\infty}\left[-\frac{b^4}{2a}e^{-2ab} - \frac{b^3}{a^2}e^{-2ab} - \frac{3b^2}{2a^3}e^{-2ab} - \frac{3b}{2a^4}e^{-2ab} - \frac{3}{4a^5}e^{-2ab} + \frac{3}{4a^5}\right] = \frac{3\pi}{4a^5}$ cubic units. The limit 0 is obtained by repeated use of l'Hôpital's rule. The integration was by the tabular method of integration by parts.

60. $V = \int_0^{\infty} 2\pi x(x^2e^{-ax})dx = \lim_{b\to+\infty} \int_0^b 2\pi x^3e^{-ax}dx =$
$2\pi \lim_{b\to+\infty} (-\frac{x^3}{a}e^{-ax} - \frac{3x^2}{a^2}e^{-ax} - \frac{6x}{a^3}e^{-ax} - \frac{6}{a^4}e^{-ax})\Big|_0^b =$
$2\pi \lim_{b\to+\infty} \left[-\frac{b^3}{a}e^{-ab} - \frac{3b^2}{a^2}e^{-ab} - \frac{6b}{a^3}e^{-ab} - \frac{6}{a^4}e^{-ab} + \frac{6}{a^4}\right] =$
$2\pi(0 + \frac{6}{a^4}) = \frac{12\pi}{a^4}$ cubic units. The integration was by the tabular method of integration by parts. The limit 0 is obtained by repeated use of l'Hôpital's rule.

60. $W = \int_1^{\infty} F\, ds = \int_1^{\infty} G\frac{m_1m_2}{s^2}\, ds = Gm_1m_2 \lim_{b\to+\infty} \int_1^b \frac{1}{s^2}\, ds =$
$Gm_1m_2 \lim_{b\to+\infty} (-\frac{1}{s})\Big|_1^b = Gm_1m_2 \lim_{b\to+\infty} (-\frac{1}{b} + 1) = Gm_1m_2$ units of work.

61. $f(x) = \sin 2x$, $f'(x) = 2 \cos 2x$, $f''(x) = -4 \sin 2x$, $f'''(x) = -8 \cos 2x$, $f^{(4)}(x) = 16 \sin 2x$. $P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} = 2x - \frac{8x^3}{3!}$.
The corresponding Taylor remainder is $R_3(x) = \frac{f^{(4)}(c)x^4}{4!} = \frac{16(\sin 2c)(x^4)}{4!} = \frac{2(\sin 2c)(x^4)}{3}$, where c is strictly between 0 and x.

62. $f(x) = (1+x)^{-2}$, $f'(x) = -2(1+x)^{-3}$, $f''(x) = 3\cdot 2(1+x)^{-4}$, $f'''(x) = -4!(1+x)^{-5}$, $f^{(4)}(x) = 5!(1+x)^{-6}$. $P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} = \frac{1}{4} + \frac{-2}{8}(x-1) + \frac{3!(x-1)^2}{32} + \frac{4!(x-1)^3}{(32)(3!)} = \frac{1}{4} - \frac{x-1}{4} + \frac{3(x-1)^2}{16} - \frac{(x-1)^3}{8}$. $R_3(x) = \frac{f^{(4)}(c)}{4!}(x-1)^4 = \frac{5!}{4!}(1+c)^{-6}(x-1)^4 = \frac{5(x-1)^4}{(1+c)^6}$, where c is strictly between 1 and x.

63. $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x}$, $f^{(4)}(x) = e^{-x}$, $f^{(5)}(x) = -e^{-x}$, $f^{(6)}(x) = e^{-x}$, $f^{(7)}(x) = -e^{-x}$, $f^{(8)}(x) = e^{-x}$. $P_7(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!}$. $R_7(x) = \frac{f^8(c)x^8}{8!} = \frac{e^{-c}x^8}{8!}$, where c is strictly between 0 and x.

64. $f(x) = \cos 3x$, $f'(x) = -3 \sin 3x$, $f''(x) = -9 \cos 3x$, $f'''(x) = 27 \sin 3x$, $f^{(4)}(x) = 81 \cos 3x$, $f^{(5)}(x) = -243 \sin 3x$, $f^{(6)}(x) = -3^6 \cos 3x$, $f^{(7)}(x) = 3^7 \sin 3x$. $P_6(x) = f(\frac{\pi}{6}) + f'(\frac{\pi}{6})(x - \frac{\pi}{6}) +$

$\frac{f''(\frac{\pi}{6})(x - \frac{\pi}{6})^2}{2!} + \frac{f'''(\frac{\pi}{6})(x - \frac{\pi}{6})^3}{3!} + \frac{f^{(4)}(\frac{\pi}{6})(x - \frac{\pi}{6})^4}{4!} + \frac{f^{(5)}(\frac{\pi}{6})(x - \frac{\pi}{6})^5}{5!} + \frac{f^{(6)}(\frac{\pi}{6})(x - \frac{\pi}{6})^6}{6!} = -3(x - \frac{\pi}{6}) + \frac{9(x - \frac{\pi}{6})^3}{2} - \frac{81}{40}(x - \frac{\pi}{6})^5$. $R_6(x) = \frac{f^7(c)(x - \frac{\pi}{6})^7}{7!} = \frac{(3)^7\sin(3c)(x - \frac{\pi}{6})^7}{7!}$, where c is strictly between x and $\frac{\pi}{6}$.

65. Let $f(x) = \sin(\frac{\pi}{2} - x)$, $a = 90^\circ = \frac{\pi}{2}$, $b = 2^\circ = \frac{\pi}{90}$. $f'(x) = -\cos(\frac{\pi}{2} - x)$, $f''(x) = -\sin(\frac{\pi}{2} - x)$, $f'''(x) = \cos(\frac{\pi}{2} - x)$, etc. $|f^{n+1}(c)| = \pm\sin c$ or $\pm\cos c$. Hence, $|f^{n+1}(c)| \le 1 = M$. Thus in order for the error in absolute value not to exceed $\frac{5}{10^6}$ we must choose n so that $\frac{M|b - a|^{n+1}}{(n + 1)!} = \frac{(\frac{44\pi}{90})^{n+1}}{(n + 1)!} \le \frac{5}{10^6}$. n must be at least 10. $P_{10}(x) = 0 - \cos 0(x - \frac{\pi}{2}) + 0 + \frac{\cos 0(x - \frac{\pi}{2})^3}{3!} + 0 - \frac{\cos 0(x - \frac{\pi}{2})^5}{5!} + \frac{\cos(x - \frac{\pi}{2})^7}{7!} - \frac{\cos 0(x - \frac{\pi}{2})^9}{9!} + 0$. $\sin\frac{44\pi}{90} \approx P_{10}(\frac{\pi}{90}) = \frac{44\pi}{90} - \frac{(44\pi)^3}{6(90)^3} + \frac{(44\pi)^5}{120(90)^5} - \frac{(44\pi)^7}{7!(90)^7} + \frac{(44\pi)^9}{9!(90)^9} \approx 0.99939$. (The correct value rounded off to seven places is 0.9993908.)

66. Define $f(x) = \cos(\frac{\pi}{3} - x)$, let $a = \frac{\pi}{3}$ and $b = \frac{\pi}{180}$. $f'(x) = \sin(\frac{\pi}{3} - x)$, $f''(x) = -\cos(\frac{\pi}{3} - x)$, $f'''(x) = -\sin(\frac{\pi}{3} - x)$, and so forth. $|f^{n+1}(c)| = \pm\sin c$ or $\pm\cos c$, so that the error in absolute value cannot exceed $\frac{|f^{n+1}(c)||b - a|^{n+1}}{(n + 1)!} < \frac{1 \cdot |b - a|^{n+1}}{(n + 1)!} = \frac{(\frac{59\pi}{180})^{n+1}}{(n + 1)!} \le \frac{5}{10^6}$ for n at least 8. $P_8(x) = 1 + 0 - \frac{(\cos 0)(x - \frac{\pi}{3})^2}{2!} + 0 + \frac{(\cos 0)(x - \frac{\pi}{3})^4}{4!} + 0 - \frac{(\cos 0)(x - \frac{\pi}{3})^6}{6!} + \frac{(\cos 0)(x - \frac{\pi}{3})^8}{8!}$. $\cos\frac{59\pi}{180} \approx P_8(\frac{\pi}{180}) = 1 - \frac{(59\pi)^2}{2!(180)^2} + \frac{(59\pi)^4}{4!(180)^4} - \frac{(59\pi)^6}{6!(180)^6} + \frac{(59\pi)^8}{8!(180)^8} \approx 0.51504$. (The correct value rounded off to seven places is 0.5150381.)

67. Let $f(x) = \ln(x + 1)$, $a = 0$, $b = \frac{1}{2}$. $f'(x) = (x + 1)^{-1}$, $f''(x) = -(x + 1)^{-2}$, $f'''(x) = 2(x + 1)^{-3}$, $f^{(4)}(x) = -3!(x + 1)^{-4}$, $f^5(x) = 4!(x + 1)^{-5}$ and so forth. $f^{n+1}(c) = (-1)^n n!(c + 1)^{-(n+1)}$ where $0 < c < \frac{1}{2}$. $|f^{n+1}(c)| \le n!$. The error in absolute value cannot exceed $\frac{M|b - a|^{n+1}}{(n + 1)!} \le \frac{n!(\frac{1}{2})^{n+1}}{(n + 1)!} = \frac{1}{(n + 1)2^{n+1}} \le \frac{5}{10^6}$ for n at least 13. $P_{13}(x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{3!x^4}{4!} + \frac{4!x^5}{5!} - \frac{5!x^6}{6!} + \frac{6!x^7}{7!} - \frac{7!x^8}{8!} + \frac{8!x^9}{9!} - \frac{9!x^{10}}{10!} + \frac{10!x^{11}}{11!} - \frac{11!x^{12}}{12!} + \frac{12!x^{13}}{13!}$. $\ln(1 + \frac{1}{2}) \approx P_{13}(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2(2^2)} + \frac{1}{3(2^3)} - \frac{1}{4(2^4)} + \frac{1}{5(2^5)} - \frac{1}{6(2^6)} + \frac{1}{7(2^7)} - \frac{1}{8(2^8)} + \frac{1}{9(2^9)} - \frac{1}{10(2^{10})} + \frac{1}{11(2^{11})} - \frac{1}{12(2^{12})} + \frac{1}{13(2^{13})} \approx 0.40547$. (The correct value rounded off to seven places is 0.4054651.)

68. Let $f(x) = e^x$, $a = 0$ and $b = \frac{1}{10}$. $f^{n+1}(c) = e^c < e^{\frac{1}{10}} < 4^{\frac{1}{10}} < 4^{\frac{1}{2}} = 2$, since $0 < c < \frac{1}{10}$. Hence, the error in absolute value cannot exceed $\frac{|f^{n+1}(c)||b - a|^{n+1}}{(n + 1)!} \le \frac{2(\frac{1}{10})^{n+1}}{(n + 1)!} \le \frac{5}{10^6}$ for n at least 4. $P_4(x) = e^0 + e^0 x + \frac{e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + \frac{e^0 x^4}{4!}$. Hence, $e^{\frac{1}{10}} \approx P_4(\frac{1}{10}) = 1 + \frac{1}{10} + \frac{1}{2(10^2)} + \frac{1}{6(10^3)} + \frac{1}{24(10^4)} \approx 1.10517$. (The correct value rounded off to seven places is 1.1051709.)

69. Let $f(x) = \sqrt{x + 1}$, $a = 0$ and $b = 0.03$. $f'(x) = \frac{1}{2}(x + 1)^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{2^2}(x + 1)^{-3/2}$, $f'''(x) = \frac{3}{2^3}(x + 1)^{-5/2}$, $f^4(x) = \frac{-3(5)(x + 1)^{-7/2}}{2^4} \ldots f^n(x) = \frac{(-1)^{n+1}[1 \cdot 3 \cdot 5 \ldots (2n - 3)]}{2^n}(x + 1)^{\frac{-(2n - 1)}{2}}$.

$|f^{n+1}(c)| = \frac{1\cdot3\cdot5\ldots(2n-3)(2n-1)}{2^{n+1}}(c+1)^{\frac{-(2n+1)}{2}} \le$

$\frac{1\cdot3\cdot5\ldots(2n-3)(2n-1)}{2^{n+1}}$. Now the error in absolute value cannot exceed

$\frac{1\cdot3\cdot5\ldots(2n-3)(2n-1)(0.03)^{n+1}}{2^{n+1}(n+1)!} =$

$\frac{1\cdot3\cdot5\ldots(2n-1)(3)}{2^{n+1}(10^{2n+2})} \le \frac{5}{10^6}$ for n at least 2. $P_2(x) =$ $f(0) + f'(0)x + \frac{f''(0)x^2}{2!} = 1 + \frac{1}{2}x + (-\frac{x^2}{4(2!)})$. Now $\sqrt{1.03} \approx P_2(0.03) = 1 + \frac{0.03}{2} - \frac{(0.03)^2}{8} \approx 1.01489$.

(The correct value rounded off to seven places is 1.0148892.)

70. First consider $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \frac{f^{2n+1}(c)x^{2n+1}}{(2n+1)!}$, $0 < c < x$. Now replace x by t^2 where $0 \le t \le \frac{1}{2}$. Thus, $\sin t^2 =$ $t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \ldots + \frac{(-1)^{n+1}t^{4n-2}}{(2n-1)!} + \frac{f^{2n-1}(c)t^{4n+2}}{(2n+1)!}$, $0 < c < t^2$. Now $\int_0^{\frac{1}{2}} \sin t^2\, dt =$ $\left[\frac{t^3}{3} - \frac{t^7}{7(3!)} + \frac{t^{11}}{11(5!)} - \frac{t^{15}}{15(7!)} + \ldots + \frac{(-1)^{n+1}t^{4n-1}}{(4n-1)(2n-1)!}\right]\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{f^{2n+1}(c)t^{4n+2}}{(2n+1)!}\, dt$. But $|f^{2n+1}(c)| \le 1$, since $|f^{2n+1}(c)| = \pm\sin c$ or $\pm\cos c$. Hence, $\int_0^{\frac{1}{2}} \frac{t^{4n+2}}{(2n+1)!}\, dt = \frac{t^{4n+3}}{(4n+3)(2n+1)!}\Big|_0^{\frac{1}{2}} = \frac{1}{2^{4n+3}(4n+3)(2n+1)!} \le \frac{5}{10^6}$ for n at least 2.

Hence, $\int_0^{\frac{1}{2}} \sin t^2\, dt \approx \frac{1}{3(2^3)} - \frac{1}{7(3!)2^7} \approx 0.04148$.

11 INFINITE SERIES

Problem Set 11.1, page 640

1. 2, 5, 10, 17, 26, 37. $a_{100} = 100^2 + 1 = 10{,}001$.

2. $\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}$. $a_{100} = -\frac{1}{101}$.

3. $\frac{1}{6}, \frac{2}{9}, \frac{3}{14}, \frac{4}{21}, \frac{5}{30}, \frac{6}{41}$. $a_{100} = \frac{100}{10{,}005} = \frac{20}{2{,}001}$.

4. $3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{11}{5}, \frac{13}{6}$. $a_{100} = \frac{201}{100}$.

5. $a_n = \frac{n+1}{2}$.

6. $a_n = \frac{1}{2} + \frac{(-1)^{n+1}}{2}$.

7. $a_n = \frac{1}{n+1}$.

8. $a_n = (2n-1)^2$.

9. $\lim\limits_{n\to+\infty} \frac{100}{n} = 100 \lim\limits_{n\to+\infty} \frac{1}{n} = 100 \cdot 0 = 0$.

10. $\lim\limits_{n\to+\infty} \frac{n^2}{5n^2+1} = \lim\limits_{n\to+\infty} \frac{1}{5+\frac{1}{n^2}} = \frac{\lim\limits_{n\to+\infty} 1}{\lim\limits_{n\to+\infty} (5+\frac{1}{n^2})} = \frac{1}{\lim\limits_{n\to+\infty} 5 + \lim\limits_{n\to+\infty} \frac{1}{n^2}} = \frac{1}{5}$.

11. $\lim\limits_{n\to+\infty} \frac{n^3-5n}{7n^3+2n} = \lim\limits_{n\to+\infty} \frac{1-\frac{5}{n^2}}{7+\frac{2}{n^2}} = \frac{1 - 5\lim\limits_{n\to+\infty}\frac{1}{n^2}}{7 + 2\lim\limits_{n\to+\infty}\frac{1}{n^2}} = \frac{1-5\cdot 0}{7+2\cdot 0} = \frac{1}{7}$.

12. $\lim\limits_{n\to+\infty} \frac{2n^2+1}{9n^2+5} = \lim\limits_{n\to+\infty} \frac{2+\frac{1}{n^2}}{9+\frac{5}{n^2}} = \frac{\lim\limits_{n\to+\infty} 2 + \lim\limits_{n\to+\infty}\frac{1}{n^2}}{\lim\limits_{n\to+\infty} 9 + 5\lim\limits_{n\to+\infty}\frac{1}{n^2}} = \frac{2+0}{9+5\cdot 0} = \frac{2}{9}$.

13. $\lim\limits_{n\to+\infty} \frac{5n^2}{3n+1} = \lim\limits_{n\to+\infty} \frac{5n}{3+\frac{1}{n^2}}$. Now as $n \to +\infty$, the denominator approaches 3 while the numerator gets larger without bound, so that $\frac{5n}{3+\frac{1}{n^2}}$ gets large without bound. Hence, $\left\{\frac{5n^2}{3n+1}\right\}$ diverges.

14. This sequence $-\frac{1}{10} + \frac{1}{10^2} - \frac{1}{10^3} + \frac{1}{10^4} - \frac{1}{10^5} + \ldots$ alternates signs, but $\lim\limits_{n\to+\infty} \frac{(-1)^n}{10^n} = \pm 1 \lim\limits_{n\to+\infty} (\frac{1}{10})^n = 0$. The sequence converges with limit 0.

15. $\frac{2n^2+n}{n+1} \sin\frac{\pi}{2n} = (\frac{\pi}{2n})(\frac{2n^2+n}{n+1})\frac{\sin \pi/2n}{(\pi/2n)} = \pi(\frac{2n+1}{2n+2})\frac{\sin(\pi/2n)}{(\pi/2n)} = \pi(\frac{2+\frac{1}{n}}{2+\frac{2}{n}})\frac{\sin(\pi/2n)}{(\pi/2n)}$. Hence, $\lim\limits_{n\to+\infty} \frac{2n^2+n}{n+1}\sin\frac{\pi}{2n} = \lim\limits_{n\to+\infty} \pi(\frac{2+\frac{1}{n}}{2+\frac{2}{n}})\frac{\sin(\pi/2n)}{(\pi/2n)} = \pi(\frac{2}{2})(1) = \pi$.

16. $\lim\limits_{n\to+\infty} \frac{e^n+e^{-n}}{e^n-e^{-n}} = \lim\limits_{n\to+\infty} \frac{1+\frac{e^{-n}}{e^n}}{1-\frac{e^{-n}}{e^n}} = \frac{\lim\limits_{n\to+\infty} 1 + \lim\limits_{n\to+\infty} (\frac{1}{e^2})^n}{\lim\limits_{n\to+\infty} 1 - \lim\limits_{n\to+\infty} (\frac{1}{e^2})^n} = \frac{1+0}{1-0} = 1$.

17. $\lim\limits_{x\to+\infty} \frac{\ln(x+1)}{x+1} = \lim\limits_{x\to+\infty} \frac{\frac{1}{x+1}}{1}$ (by l'Hôpital's rule) $= 0$. Hence, by the theorem on convergence of sequences and functions, $\lim\limits_{n\to+\infty} \frac{\ln(n+1)}{n+1} = 0$.

18. $\lim\limits_{n\to+\infty} [1 + (\frac{1}{3})^n - (\frac{3}{4})^n] = \lim\limits_{n\to+\infty} 1 + \lim\limits_{n\to+\infty} (\frac{1}{3})^n - \lim\limits_{n\to+\infty} (\frac{3}{4})^n = 1 + 0 - 0 = 1$.

9. $\lim\limits_{x\to+\infty} \dfrac{\ln\frac{1}{x}}{\ln(x + 4)} = \lim\limits_{x\to+\infty} \dfrac{-\ln x}{\ln(x + 4)} = \lim\limits_{x\to+\infty} \dfrac{\frac{1}{x}}{\frac{1}{x + 4}}$ by l'Hôpital's rule. Now $\lim\limits_{x\to+\infty} -(\frac{x + 4}{x}) = \lim\limits_{x\to+\infty} \frac{-1}{1} = -1$. Hence, by the theorem on convergence of sequences and functions, $\left\{\dfrac{\ln\frac{1}{n}}{\ln(n + 4)}\right\}$ converges to -1.

0. Consider the function $\ln(e^x + 2) - \ln(e^x + 1) = \ln(\frac{e^x + 2}{e^x + 1})$. Now $\lim\limits_{x\to+\infty} \ln(\frac{e^x + 2}{e^x + 1}) = \ln[\lim\limits_{x\to+\infty} \frac{e^x+2}{e^x+1}] =$ $\ln[\lim\limits_{x\to+\infty} \dfrac{1 + \frac{2}{e^x}}{1 + \frac{1}{e^x}}] = \ln[1] = 0$. Thus, by the theorem on convergence of sequences and functions, $\{\ln(e^n + 2) - \ln(e^n + 1)\}$ converges to 0.

1. $\lim\limits_{n\to+\infty} \dfrac{1}{\sqrt{n^2 + 1} - n} = \lim\limits_{n\to+\infty} \dfrac{\sqrt{n^2 + 1} + n}{n^2 + 1 - n^2} =$ $\lim\limits_{n\to+\infty} (\sqrt{n^2 + 1} + n) = +\infty$. The sequence $\left\{\dfrac{1}{\sqrt{n^2 + 1} - n}\right\}$ diverges.

2. $\ln(e^n + 2) - n = \ln(e^n + 2) - \ln e^n = \ln(\frac{e^n + 2}{e^n})$. Now $\lim\limits_{n\to+\infty} [\ln(e^n + 2) - n] =$ $\lim\limits_{n\to+\infty} \ln(\frac{e^n + 2}{e^n}) = \ln[\lim\limits_{n\to+\infty} \frac{e^n + 2}{e^n}] = \ln[\lim\limits_{n\to+\infty} \dfrac{1 + \frac{2}{e^n}}{1}] =$ $\ln 1 = 0$. The sequence converges to 0.

3. Consider $\lim\limits_{x\to+\infty} x^{\frac{1}{\sqrt{x}}} = \lim\limits_{x\to+\infty} e^{\frac{1}{\sqrt{x}}\ln x}$. Now $\lim\limits_{x\to+\infty} \frac{\ln x}{\sqrt{x}} =$ $\lim\limits_{x\to+\infty} \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to+\infty} \frac{2}{\sqrt{x}} = 0$. Thus, $\lim\limits_{x\to+\infty} x^{\frac{1}{\sqrt{x}}} = e^0 = 1$. Hence, the sequence $\left\{n^{\frac{1}{\sqrt{n}}}\right\}$ converges to 1.

4. Consider $\lim\limits_{x\to+\infty} x^{\frac{1}{x^2}} = \lim\limits_{x\to+\infty} e^{\frac{1}{x^2}\ln x}$. Now $\lim\limits_{x\to+\infty} \frac{\ln x}{x^2} =$ $\lim\limits_{x\to+\infty} \dfrac{\frac{1}{x}}{2x} = \lim\limits_{x\to+\infty} \frac{1}{2x^2} = 0$. Thus, $\lim\limits_{x\to+\infty} x^{\frac{1}{x^2}} = e^0 = 1$. Hence, $\left\{n^{\frac{1}{n^2}}\right\}$ converges to 0.

5. $\lim\limits_{x\to+\infty} (1 + \frac{1}{x})^x = \lim\limits_{x\to+\infty} e^{x\ln(1 + \frac{1}{x})}$. Now $\lim\limits_{x\to+\infty} x\ln(1 + \frac{1}{x}) = \lim\limits_{x\to+\infty} \dfrac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} =$ $\lim\limits_{x\to+\infty} \dfrac{\dfrac{-\frac{1}{x^2}}{1 + \frac{1}{x}}}{-\frac{1}{x^2}} = \lim\limits_{x\to+\infty} \dfrac{1}{1 + \frac{1}{x}} = 1$. Hence, $\lim\limits_{x\to+\infty} (1+\frac{1}{x})^x =$ $e^1 = e$. Therefore, $\{(1 + \frac{1}{n})^n\}$ converges to e.

26. $\lim\limits_{x\to+\infty} (1 + \frac{5}{x})^x = \lim\limits_{x\to+\infty} e^{x\ln(1 + \frac{5}{x})}$. Now $\lim\limits_{x\to+\infty} x\ln(1 + \frac{5}{x}) = \lim\limits_{x\to+\infty} \dfrac{\ln(1 + \frac{5}{x})}{\frac{1}{x}} = \lim\limits_{x\to+\infty} \dfrac{\dfrac{-\frac{5}{x^2}}{1 + \frac{5}{x}}}{-\frac{1}{x^2}} =$ $\lim\limits_{x\to+\infty} \dfrac{5}{1 + \frac{5}{x}} = 5$. Hence, $\lim\limits_{x\to+\infty} (1 + \frac{5}{x})^x = e^5$. Therefore, the sequence $\{(1 + \frac{5}{n})^n\}$ converges to e^5.

27. Consider $f(x) = \frac{2x + 1}{3x + 2}$. $f'(x) = \dfrac{1}{(3x + 2)^2} > 0$ for all x, so that f is an increasing function. In particular, $f(n + 1) \geq f(n)$; that is, $\frac{2(n + 1) + 1}{3(n + 1) + 2} \geq \frac{2n + 1}{3n + 2}$ for all n. Hence, $\left\{\frac{2n + 1}{3n + 2}\right\}$ is increasing. Since all the terms are positive, $\left\{\frac{2n + 1}{3n + 2}\right\}$ is bounded below by 0; also, $\frac{2n + 1}{3n + 2} = \dfrac{2 + \frac{1}{n}}{3 + \frac{2}{n}} < \frac{3}{4}$, so that the sequence is bounded above. The sequence converges since it is increasing and bounded above.

28. $\{\sin n\pi\}$ is the sequence 0, 0, 0, 0, ... It is decreasing, increasing, bounded from above and below by 0, and is convergent with limit 0.

29. Consider $f(x) = 3^x - x$. $f'(x) = (\ln 3)3^x - 1 > 0$ for all x, so f is increasing. In particular $3^{n+1} - (n + 1) > 3^n - n$ for all integers n, so that $\{3^n - n\}$ is an increasing sequence. The sequence is bounded below by 0 since all the terms are positive. The sequence is unbounded from above, as can be seen by the following argument: Given any positive number k, no matter how large, there will be a term $3^n - n$ out in the sequence so that

$3^n - n > k$ since $3^n > k + n > k$, and so for $n > \frac{\ln k}{\ln 3}$, $3^n - n > k$. The sequence diverges since

$\lim_{x\to+\infty} (3^x - x) = \lim_{x\to+\infty} (1 - \frac{x}{3^x})3^x =$

$[1 - \lim_{x\to+\infty} \frac{1}{(\ln 3)3^x}] \lim_{x\to+\infty} 3^x = +\infty$.

30. $\frac{3^n}{1 + 3^n} = \frac{1}{\frac{1}{3^n} + 1}$. As n increases, $\frac{1}{3^n}$ decreases, so that $\frac{1}{3^n} + 1$ decreases and $\frac{1}{\frac{1}{3^n} + 1}$ increases. Hence, the sequence increases. The sequence is bounded below by 0 since all terms are positive; the sequence is bounded above by 1 since $\frac{3^n}{1 + 3^n} < \frac{3^n}{3^n} = 1$.

31. The sequence looks like -1, 1, -1, 1, -1, 1, ... $(-1)^{n^2}$ ... . It is nonmonotonic. It is bounded below by -1 and bounded above by 1. The sequence is divergent since by jumping back and forth it does not approach a limit.

32. The sequence looks like $\frac{3}{4}, \frac{48}{11}, \frac{243}{30}, \frac{768}{85}, \frac{1875}{248}, \frac{3888}{735}, \frac{7203}{2194}, \ldots$. The first four terms are increasing but after that the terms decrease. The sequence is nonmonotonic. The sequence is bounded below by 0; it is bounded above, as will be established once we determine its limit: $\lim_{x\to+\infty} \frac{3x^4}{x + 3^x} =$

$\lim_{x\to+\infty} \frac{12x^3}{1 + (\ln 3)3^x} = \lim_{x\to+\infty} \frac{36x^2}{(\ln 3)^2 3^x} =$

$\lim_{x\to+\infty} \frac{72x}{(\ln 3)^3 3^x} = \lim_{x\to+\infty} \frac{72}{(\ln 3)^4 3^x} = 0$. The sequence $\left\{\frac{3n^4}{n + 3^n}\right\}$ is convergent. Since the sequence has a limit L, there exists an N such that for all n > N, $|a_n| < L$. But a finite number of terms are left, $a_1, a_2, \ldots, a_n$, and these numbers have a largest value, say M. Hence the sequence is bounded above by M.

33. As n increases, 5n - 2 increases, so $\frac{3}{5n - 2}$ decreases, so the sequence is decreasing. It is bounded below by 1 since $\frac{3}{5n - 2}$ is positive and bounded above by 2 since $\frac{3}{5n - 2} < \frac{3}{5\cdot 1 - 2} = 1$. Thus, the sequence converges.

34. $a_{n+1}/a_n = 2^{\frac{1}{n+1}}/2^{\frac{1}{n}} = 2^{\frac{1}{n+1} - \frac{1}{n}} = 2^{\frac{n - (n+1)}{n(n+1)}} = 2^{\frac{-1}{n(n+1)}} < 1$, so the sequence is decreasing. It is bounded above by 2 and below by zero; therefore, it converges.

35. Consider $f(x) = \frac{\ln(x + 1)}{x + 2}$. $f'(x) = \frac{\frac{x + 2}{x + 1} - \ln(x+1)}{(x + 2)^2}$ 0 for large enough x, e.g., for $x \geq 4$, so $\left\{\frac{\ln(n + 1)}{n + 2}\right\}$ is decreasing for $n \geq 4$. The sequence is bounded below by 0 and above by 1 since $\frac{\ln(n + 1)}{n + 2} < \frac{\ln(n + 2)}{(n + 2)} < \frac{n + 2}{n + 2} = 1$. Thus, the sequence converges.

36. $(\frac{-n}{\ln n})^n = (-1)^n(\frac{n}{\ln n})^n$ is nonmonotonic since its terms are alternately negative and positive. It is unbounded in both its positive and negative values since we can find $\left|(\frac{-n}{\ln n})^n\right| = (\frac{n}{\ln n})^n > K$ for any $K > 0$ as follows: For large enough n, $\ln n < \sqrt{n}$, so $\frac{n}{\ln n} > n/\sqrt{n} = \sqrt{n}$. Choose $N = K^2$, then for $n > N$ $\frac{n}{\ln n} > \frac{N}{\ln N} > \sqrt{N} = K$. Now since $\ln n < n$ for $n > 0$ $\frac{n}{\ln n} > 1$, so $(\frac{n}{\ln n})^n > \frac{n}{\ln n}$, so $|a_n| > K$ for $n > K^2$. Thus, the sequence is unbounded, and hence divergent.

37. Consider $f(x) = \frac{x + 5}{x^2 + 6x + 4}$. $f'(x) = \frac{-x^2 - 10x - 26}{(x^2 + 6x + 4)^2} = \frac{-(x + 5)^2 - 1}{(x^2 + 6x + 4)^2} < 0$, so f(x) is decreasing for all x, and in particular the sequence $\left\{\frac{n + 5}{n^2 + 6x + 4}\right\}$ is decreasing. The sequence is bounded below by 0 and above by f(1) = 6/11. Hence, the sequence converges.

38. The sequence is decreasing since $a_{n+1}/a_n = \frac{e^{n+2}}{(n + 2)!} \cdot \frac{(n + 1)!}{e^{n+1}} = \frac{e}{n + 2} < 1$ for $n \geq 1$. It is bounded above by $e^2/2$ and below by zero, so the sequence converges.

9. The sequence is nonmonotonic since the signs on the terms alternate. Since $\left|\frac{(-1)^n n}{n + 1}\right| < \left|\frac{n}{n}\right| = 1$, the sequence is bounded above by 1 and bounded below by -1. The odd-numbered terms approach -1 while the even-numbered terms approach 1. Hence, the sequence is divergent.

0. Consider $f(x) = \sqrt{x + 4} - \sqrt{x + 3}$. $f'(x) = \frac{1}{2\sqrt{x + 4}} - \frac{1}{2\sqrt{x + 3}} = \frac{\sqrt{x + 3} - \sqrt{x + 4}}{2\sqrt{(x + 4)(x + 3)}}$. The denominator of $f'(x)$ is positive while the numerator is negative-- $\sqrt{x + 4} > \sqrt{x + 3}$ for all x -- so that f is decreasing for all x and in particular the sequence $\{\sqrt{n + 4} - \sqrt{n + 3}\}$ is decreasing. Since all terms are positive, the sequence is bounded below by 0. Since the sequence is decreasing, $a_1 \geq a_n$ for all n. Hence, the sequence is bounded above by $a_1 = \sqrt{5} - 2$.

Now $\lim_{x\to+\infty} [(\sqrt{x + 4} - \sqrt{x + 3})\frac{(\sqrt{x + 4} + \sqrt{x + 3})}{\sqrt{x + 4} + \sqrt{x + 3}}] = \lim_{x\to+\infty} \frac{1}{\sqrt{x + 4} + \sqrt{x + 3}} = 0$. Thus, the sequence $\{\sqrt{n + 4} - \sqrt{n + 3}\}$ converges with limit 0.

. Let $f(x) = 1 - \frac{2^x}{x}$. $f'(x) = \frac{-x \ln 2(2^x) + 2^x}{x^2} < 0$ for all x since the numerator is always negative while the denominator is always positive. Thus, f is decreasing, and in particular $\left\{1 - \frac{2^n}{n}\right\}$ is decreasing. All the terms are negative, so that the sequence is bounded above by 0. Now $\lim_{x\to+\infty} (1 - \frac{2^x}{x}) = \lim_{x\to+\infty} \frac{(-\ln 2)2^x}{1} = -\infty$; hence, the sequence is not bounded below. The sequence diverges.

. We first show that $\frac{(n + 1)^{n+1}}{(n + 1)!} > \frac{n^n}{n!}$. Indeed, $0 < \frac{n}{n + 1} < 1$; hence, $0 < (\frac{n}{n + 1})^n < 1$, so that $(\frac{1}{n + 1})(\frac{n^n}{(n + 1)^n}) < \frac{1}{n + 1}$, that is, $\frac{n^n}{(n + 1)^{n+1}} < \frac{n!}{(n + 1)!}$. It follows that $\frac{(n + 1)^{n+1}}{(n + 1)!} > \frac{n^n}{n!}$; hence, the sequence $\left\{\frac{n^n}{n!}\right\}$ is increasing. Obviously, the sequence is bounded below by 1. Now $\frac{n^n}{n!} = (\frac{n}{n})(\frac{n}{n - 1})(\frac{n}{n - 2}) \cdots (\frac{n}{1})$. Each factor on the right is greater than or equal to 1 and the last factor is equal to n; hence, $\frac{n^n}{n!} \geq n$. It follows that $\left\{\frac{n^n}{n!}\right\}$ is unbounded above and that it diverges.

43. $\left\{\frac{\sin \frac{n\pi}{4}}{n}\right\}$ looks like $\frac{(\frac{\sqrt{2}}{2})}{1}, \frac{1}{2}, \frac{(\frac{\sqrt{2}}{2})}{3}, \frac{0}{4}, \frac{(-\frac{\sqrt{2}}{2})}{5}, \frac{-1}{6}, \frac{(-\frac{\sqrt{2}}{2})}{7}, \ldots$ and continues in a similar way. Obviously, it is nonmonotonic. Since $\sin\left|\frac{n\pi}{4}\right| \leq 1$, it follows that $\left|\frac{\sin \frac{n\pi}{4}}{n}\right| \leq \frac{1}{n} \leq 1$; hence, the sequence is bounded. Also, since $\left|\frac{\sin \frac{n\pi}{4}}{n}\right| \leq \frac{1}{n}$ and $\lim_{n\to+\infty} \frac{1}{n} = 0$, it follows that $\lim_{n\to+\infty} \frac{\sin \frac{n\pi}{4}}{n} = 0$, so the sequence is convergent.

44. The sequence $\left\{\frac{1\cdot3\cdot5 \ldots (2n - 1)}{n!}\right\}$ is increasing -- $\frac{1\cdot 3\cdot 5 \ldots (2n - 1)(2n + 1)}{(n + 1)!} > \frac{1\cdot 3\cdot 5 \ldots (2n - 1)}{n!}$ or $\frac{n!}{(n + 1)!} > \frac{1\cdot 3\cdot 5 \ldots (2n - 1)}{1\cdot 3\cdot 5 \ldots (2n - 1)(2n + 1)}$ or $\frac{1}{n + 1} > \frac{1}{2n + 1}$ since $n + 1 < 2n + 1$ for all n.

Since all the terms are positive, the sequence is bounded below by 0. Now, $\frac{1\cdot 3\cdot 5 \ldots (2n - 1)}{1\cdot 2\cdot 3 \ldots (n)} > k$, no matter how large k is provided $n > \frac{\ln k}{\ln 3/2}$ as shown by the following: $\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{5}{3}\cdot\frac{7}{5}\cdot\frac{9}{6} \cdots \frac{2n - 1}{n} \geq (\frac{3}{2})^n$, $n > 1$, and $(\frac{3}{2})^n > k$ provided $n \ln \frac{3}{2} > \ln k$ or $n > \frac{\ln k}{\ln 3/2}$. Hence the sequence is not bounded above. The sequence diverges.

45. Consider the sequence $\{n\}$ and the sequence $\{-n\}$. The sum of these unbounded sequences -- the former unbounded above and the latter unbounded below -- is the sequence $\{0\}$ which is a bounded sequence.

46. (a) $a_n = (\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2})\cdot(\frac{4}{2}\cdot\frac{5}{2}\cdot\frac{6}{2} \cdots \frac{n}{2}) = \frac{6}{8}(\frac{4}{2}\cdot\frac{5}{2}\cdot\frac{6}{2} \cdots \frac{n}{2}) \geq \frac{3}{4}(2\cdot 2\cdot 2 \cdots 2) = (3/4)(2^{n-3})$ for $n \geq 3$.

(b) If $n > 3 + \frac{\ln(4k/3)}{\ln 2}$, then $n - 3 > \frac{\ln(4k/3)}{\ln 2}$ and $2^{n-3} > 2^{\ln(4k/3)/\ln 2} = e^{(\ln 2)\cdot\ln(4k/3)/\ln 2} = e^{\ln(4k/3)} = \frac{4k}{3}$, so $a_n \geq 3/4 \cdot 2^{n-3} > 3/4 \cdot \frac{4k}{3} = k$.

(c) Suppose $\{a_n\}$ is bounded above by k. By part (b), if we choose $n > 3 + \frac{\ln(4k/3)}{\ln 2}$, then $a_n > k$. Thus, k cannot be an upper bound for any value of k; so $\{a_n\}$ has no upper bound.

47. (a) 1, 2, 3, 4, 5, 6.

(b) $a_7 = 7 + 6(5)(4)(3)(2)(1) = 727$.

(c) It is difficult to guarantee that the general term of the intended sequence has been determined by an examination of its first few terms.

48. Suppose $a_n = f(n)$ for each positive integer n and $\lim_{x\to+\infty} f(x) = L$. By definition, given $\varepsilon > 0$, there exists a positive number k such that $|f(x) - L| < \varepsilon$ for all $x \geq k$. Now suppose N is an integer such that $N > k$. Then in particular $|f(n) - L| < \varepsilon$ for all $n > N$; that is, $|a_n - L| < \varepsilon$ for all $n > N$. This means that $\lim_{n\to+\infty} a_n = L$ and the sequence $\{a_n\}$ converges to L.

49. Given $\varepsilon > 0$, we must find N so that $|\frac{c}{n^k} - 0| < \varepsilon$ for $n > N$. Choose $N = \left[\!\left[(\frac{|c|}{\varepsilon})^{1/k}\right]\!\right] + 1$, i.e., the next integer after $(\frac{|c|}{\varepsilon})^{1/k}$. Then for $n > N$, $n^k > |c|/\varepsilon$, so $|\frac{c}{n^k} - 0| = \frac{|c|}{n^k} < \varepsilon$.

50. Property 7: We must show that for $\varepsilon > 0$, there exists an integer $N > 0$ such that $n > N$ implies $|c^n - 0| < \varepsilon$. Choose $N > \ln \varepsilon/\ln |c|$. Then $N \ln |c| < \ln \varepsilon$, since $|c| < 1$ so $\ln |c| < 0$. Thus for $n \geq N$, $|c^n - 0| = |c|^n < |c|^N = e^{N \ln |c|} < e^{\ln \varepsilon} = \varepsilon$.

Property 8: First consider $|c| > 0$. We must show that for $k > 0$, there exists an integer N such that $n > N$ implies $c^n > k$. Choose $N = \ln k/\ln |c|$. Then for $n > N$, $c^n > c^N = e^{N \ln c} > e^{\ln k} = k$. Thus, $\{c^n\}$ diverges. If $|c| < 0$, then $\{c^n\}$ alternates in sign while $|c^n|$ is unbounded, by the argument above. So the sequence diverges.

51. (a) Diverges by Property 8.

(b) Diverges, since the sequence is -1,1,-1,1,... .

(c) Converges by Property 7.

(d) Converges, since it is a constant sequence.

(e) Diverges by Property 8.

52. Since $\lim_{n\to+\infty} a_n = L$, then given any $\varepsilon > 0$, there exists N such that $|a_n - L| < \varepsilon$ for all $n \geq N$. If $k \geq N$, then we have $|b_n - L| < \varepsilon$ for all $n \geq k$. If $k \geq N$, then $|b_n - L| < \varepsilon$ for all $n \geq N$. Hence, $\lim_{n\to+\infty} b_n = L$.

53. (a) $1, 3, 2, \frac{5}{2}, \frac{9}{4}, \frac{19}{8}, \frac{37}{16}, \frac{75}{32}$.

(b) First, when $n = 1$, $a_1 = \frac{7}{3} + \frac{(-1)^1}{(3)(2^{-2})} = \frac{7}{3} - \frac{4}{3} = 1$

Now suppose $a_k = \frac{7}{3} + \frac{(-1)^k}{(3)2^{k-3}}$ for all $n \leq k$. We want to show that $a_{k+1} = \frac{7}{3} + \frac{(-1)^{k+1}}{3(2^{k-2})}$. Now by definition, $a_{k+1} = \frac{1}{2}(a_k + a_{k-1}) =$

$$\tfrac{1}{2}[\tfrac{7}{3} + \frac{(-1)^k}{3(2^{k-3})} + \frac{7}{3} + \frac{(-1)^{k-1}}{3(2^{k-4})}] = \frac{7}{3} + \frac{1}{3}[\frac{(-1)^k}{2^{k-2}} + \frac{(-1)^{k-1}}{2^{k-3}}] = \frac{7}{3} + \frac{(-1)^{k+1}}{3}[\frac{(-1)^{-1} + (-1)^{-2}(2)}{2^{k-2}}]$$

$$\frac{7}{3} + \frac{(-1)^{k+1}}{3}[\frac{1+2}{2^{k-2}}] = \frac{7}{3} + \frac{(-1)^{k+1}}{3(2^{k-2})}.$$

Hence the statement $a_n = \frac{7}{3} + \frac{(-1)^n}{3(2^{n-3})}$ is true for all n.

(c) $\lim_{n\to+\infty} [\frac{7}{3} + \frac{(-1)^n}{3(2^{n-3})}] = \frac{7}{3} + 0 = \frac{7}{3}$.

54. Since $\{a_n\}$ is convergent, it has a limit L. This means that for any $\varepsilon > 0$, no matter how small, there exists a positive integer N such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Now the finitely many maining terms $a_1, a_2, a_3, \ldots, a_N$ have a smallest value and a largest value, say a_s and a_ℓ, respectively. Then $a_s \leq a_k \leq a_\ell$ for $k = 1,2,3,\ldots,N$. But

$L - \varepsilon < a_n < L + \varepsilon$ for all $n \geq N$. Choose the smaller of a_s and $L - \varepsilon$, call it m, and the larger of a_ℓ and $L + \varepsilon$, call it M. Then $m < a_n < M$ for all $n \geq 1$. Thus, $\{a_n\}$ is bounded.

5. The sequence 1, 0, 1, 0, 1, 0, ... where $a_n = \frac{1}{2} + \frac{(-1)^{n+1}}{2}$ is bounded but is not convergent.

6. If $|a_n| \leq M$ for all positive integers n, then $\{a_n\}$ is bounded by definition. Suppose now that $\{a_n\}$ is bounded, so that there exist numbers C and D such that $C \leq a_n \leq D$. Choose M to be the larger of $|C|$ and $|D|$. Then we have $-M \leq C \leq \{a_n\} \leq D \leq M$, and so $|a_n| \leq M$ for all n.

7. Suppose $\lim\limits_{n\to+\infty} a_n = L$. Then $L = \lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} (A + Ba_n) = A + BL$. Hence, $L - BL = A$, and so $L = \frac{A}{1 - B}$.

8. Since $\sqrt{2n\pi}(\frac{n}{e})^n < n!$, it follows that $\frac{n^n}{e^n n!} < \frac{1}{\sqrt{2n\pi}} < \frac{1}{\sqrt{2\pi}}$.

9. (a) Since $\lim\limits_{x\to+\infty} a^x = 0$ for $|a| < 1$, then xa^x is indeterminate of the form $\infty\cdot 0$ at $+\infty$. By l'Hôpital's rule, $\lim\limits_{x\to+\infty} xa^x = \lim\limits_{x\to+\infty} \frac{x}{a^{-x}} = \lim\limits_{x\to+\infty} \frac{1}{(-\ln a)a^{-x}} = \lim\limits_{x\to+\infty} \frac{a^x}{(-\ln a)} = 0$. Hence, $\lim\limits_{n\to+\infty} na^n = \lim\limits_{x\to+\infty} xa^x = 0$ for $|a| < 1$.

(b) $\lim\limits_{n\to+\infty} n^2a^n = \lim\limits_{x\to+\infty} x^2a^x = \lim\limits_{x\to+\infty} \frac{x^2}{a^{-x}} = \lim\limits_{x\to+\infty} \frac{2x}{(-\ln a)a^{-x}} = \frac{2}{-\ln a} \lim\limits_{x\to+\infty} xa^x = (\frac{2}{-\ln a})(0) = 0$, by part (a).

0. Assume that $\{a_n\}$ is a decreasing sequence and that $\lim\limits_{n\to+\infty} a_n = L$. We must prove that all terms of the sequence are greater than or equal to L. Suppose not. Then there would be at least one term, say a_q, with $a_q < L$. Thus, let $\varepsilon = L - a_q$, so that $\varepsilon > 0$. By Definition 1, there exists a positive integer N such that $|a_n - L| < \varepsilon$ holds whenever $n \geq N$. Now, choose the integer n to be larger than both N and q. Since $q < n$, it follows that $a_n \leq a_q$, so that $a_n \leq a_q \leq L$ and $L - a_n > 0$. Consequently, $L - a_n = |a_n - L| < \varepsilon = L - a_q$, from which it follows that $a_q < a_n$, contrary to the fact that $a_n \leq a_q$.

61. Consider the function $f(x) = \frac{x^b}{a^x}$. By l'Hôpital's rule, $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{x^b}{a^x} = \ldots = \frac{b(b-1)(b-2)\ldots(b-t+1)x^{b-t}}{(\ln a)^t\, a^x}$, where $t = 1,2,3,\ldots,n$. Now when $b - t \leq 0$, $\lim\limits_{x\to+\infty} \frac{x^{b-t}}{(\ln a)^t a^x} = \lim\limits_{x\to+\infty} \frac{1}{(\ln a)^t a^x x^{t-b}} = 0$. Hence, $\left\{\frac{n^b}{a^n}\right\}$ converges to the limit 0.

Problem Set 11.2, page 649

1. $\sum\limits_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7} + \ldots = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \ldots$. $s_1 = \frac{1}{6}$. $s_2 = \frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$. $s_3 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{6}{20} = \frac{3}{10}$. $s_4 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{10}{30} = \frac{1}{3}$. $s_5 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{15}{42} = \frac{5}{14}$. $s_n = \sum\limits_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \sum\limits_{k=1}^{n} [\frac{1}{k+1} - \frac{1}{k+2}] = \frac{1}{2} - \frac{1}{n+2}$ (since the series is telescoping). $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} (\frac{1}{2} - \frac{1}{n+2}) = \frac{1}{2}$. Thus, the series $\sum\limits_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$ converges and has sum 1/2.

2. $\sum\limits_{k=1}^{\infty} \ln[1 - \frac{2}{2k+3}] = \sum\limits_{k=1}^{\infty} \ln(\frac{2k+1}{2k+3}) = \ln\frac{3}{5} + \ln\frac{5}{7} + \ln\frac{7}{9} + \ln\frac{9}{11} + \ln\frac{11}{13} + \ldots$. $s_n = \ln\frac{3}{5} + \ln\frac{5}{7} + \ldots + \ln\frac{2n+1}{2n+3}$. $s_1 = \ln\frac{3}{5}$. $s_2 = \ln\frac{3}{5} + \ln\frac{5}{7} = \ln(\frac{3}{5})(\frac{5}{7}) = \ln\frac{3}{7}$. $s_3 = \ln\frac{3}{5} + \ln\frac{5}{7} + \ln\frac{7}{9} = \ln(\frac{3}{7})(\frac{7}{9}) = \ln\frac{1}{3}$. $s_4 = \ln\frac{3}{5} + \ln\frac{5}{7} + \ln\frac{7}{9} + \ln\frac{9}{11} = \ln(\frac{1}{3})(\frac{9}{11}) = \ln\frac{3}{11}$. $s_5 = \ln\frac{3}{5} + \ln\frac{5}{7} + \ln\frac{7}{9} +$

$\ln \frac{9}{11} + \ln \frac{11}{13} = \ln (\frac{3}{11})(\frac{11}{13}) = \ln \frac{3}{13}$.

$s_n = \sum_{k=1}^{n} \ln(\frac{2k+1}{2k+3}) = \sum_{k=1}^{n} [\ln(2k+1) - \ln(2(k+1)-1)] = \ln 3 - \ln(2n+3) = \ln \frac{3}{2n+3}$. Now $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \left[\ln (\frac{3}{2n+3})\right] = \ln [\lim_{n\to+\infty} \frac{3}{2n+3}] = -\infty$. The series diverges.

3. $\sum_{k=1}^{\infty} k(k+1) = 2 + 6 + 12 + 20 + 30 + \ldots$.

$s_n = 2 + 6 + 12 + \ldots + n(n+1)$. $s_1 = 2$. $s_2 = 2 + 6 = 8$. $s_3 = 2 + 6 + 12 = 20$. $s_4 = 2 + 6 + 12 + 20 = 40$. $s_5 = 2 + 6 + 12 + 20 + 30 = 70$. $s_n = \sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} (k^2 + k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)}{6}[2n+1+3] = \frac{n(n+1)(n+2)}{3}$. Since $\{s_n\} = \{\frac{n(n+1)(n+2)}{3}\}$ diverges, then the given series diverges.

4. $\sum_{k=1}^{\infty} \frac{1}{k^2+2k} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \ldots$.

$s_n = \sum_{k=1}^{n} \frac{1}{k^2+2k} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \ldots + \frac{1}{n^2+2n}$. $s_1 = \frac{1}{3}$. $s_2 = \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$. $s_3 = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} = \frac{21}{40}$. $s_4 = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} = \frac{17}{30}$. $s_5 = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} = \frac{25}{42}$. $s_n = \sum_{k=1}^{n} \frac{1}{k^2+2k} = \sum_{k=1}^{n} [\frac{-½}{k} - \frac{½}{k+2}]$. Now $\sum_{k=1}^{n} (\frac{½}{k} - \frac{½}{k+2})$ looks like $½[(1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6}) + (\frac{1}{5} - \frac{1}{7}) + (\frac{1}{6} - \frac{1}{8}) + (\frac{1}{9} - \frac{1}{11}) + \ldots + (\frac{1}{n} - \frac{1}{n+2})]$. Consider the fact that $s_2 = ½[1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}]$, $s_3 = ½[1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}]$, $s_4 = ½[1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}]$, $s_5 = ½[1 + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}]$, $s_6 = ½[1 + \frac{1}{2} - \frac{1}{7} - \frac{1}{8}]$. But even $s_1 = ½[1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{3}]$. Hence, it looks as though $s_n = ½[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}]$ for $n \geq 1$. However, mathematical induction would be required to prove that this formula does indeed give s_n for all $n \geq 1$; the interested reader will supply the details. Now $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} ½[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}] = \frac{3}{4}$, and the series converges and has sum $\frac{3}{4}$.

5. $\sum_{k=0}^{\infty} \frac{1}{(2k-1)(2k+1)} = -1 + \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \ldots$.

$s_1 = -1$. $s_2 = -1 + \frac{1}{3} = -\frac{2}{3}$. $s_3 = -1 + \frac{1}{3} + \frac{1}{15} = -\frac{9}{15} = -\frac{3}{5}$. $s_4 = -1 + \frac{1}{3} + \frac{1}{15} + \frac{1}{35} = -\frac{20}{35} = -\frac{4}{7}$. $s_5 = -1 + \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} = -\frac{35}{63} = -\frac{5}{9}$.

$s_n = \sum_{k=0}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=0}^{n} [\frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}] = \sum_{k=0}^{n} [\frac{1}{4k-2} - \frac{1}{4(k+1)-2}] = -\frac{1}{2} - \frac{1}{4n+2}$ (since the series is telescoping) $= -\frac{n-1}{2n+1}$. Thus, the series converges to $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} -\frac{1}{2} - \frac{1}{4n+2} = -\frac{1}{2}$.

6. $\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2} = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} + \frac{11}{900} + \ldots$.

$s_n = \sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \ldots + \frac{2n+1}{n^2(n+1)^2}$. $s_1 = \frac{3}{4}$. $s_2 = \frac{3}{4} + \frac{5}{36} = \frac{8}{9}$. $s_3 = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} = \frac{15}{16}$. $s_4 = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} = \frac{24}{25}$. $s_5 = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} + \frac{11}{900} = \frac{35}{36}$. Now, by partial fractions, $\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{n} (\frac{1}{k^2} - \frac{1}{(k+1)^2}) = 1 - \frac{1}{(n+1)^2}$. Thus, $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} [1 - \frac{1}{(n+1)^2}] = 1$ and the series converges to 1.

7. $a_1 = s_1 = \frac{1}{2}$ and $a_{n+1} = s_{n+1} - s_n = \frac{n+1}{(n+1)+1} - \frac{n}{n+1}$. Thus, $a_{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$. It follows that $a_k = \frac{1}{k(k+1)}$. Since this formula works for $n = 1$, we have $a_k = \frac{1}{k(k+1)}$ for all $k \geq 1$. Since $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{n}{n+1} = 1$, we have $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

8. $a_1 = s_1 = \frac{2}{6} = \frac{1}{3}$ and $a_{n+1} = s_{n+1} - s_n = \frac{2(n+1)}{(n+1)+5} - \frac{2n}{n+5}$. Thus, $a_{n+1} = \frac{(2n^2+12n+10)-(2n^2+12n)}{(n+5)(n+6)} = \frac{10}{(n+5)(n+6)}$, so that $a_k = \frac{10}{(k+4)(k+5)}$. Since this formula also works for $k = 1$, we have $a_k = \frac{10}{(k+4)(k+5)}$ for all $k \geq 1$. Since $\lim_{n\to+\infty} \frac{2n}{n+5} = \lim_{n\to+\infty} \frac{2}{1+\frac{5}{n}} = 2$, we have $\sum_{k=1}^{\infty} \frac{10}{(k+4)(k+5)} = 2$.

9. $a_1 = s_1 - \frac{2}{8} = \frac{1}{4}$ and $a_{n+1} = s_{n+1} - s_n = \frac{2(n+1)^2}{3(n+1)+5} - \frac{2n^2}{3n+5}$. Thus, $a_{n+1} = \frac{(3n+5)(2)(n+1)^2 - 2n^2[3(n+1)+5]}{(3n+5)[3(n+1)+5]} = \frac{6n^2+26n+10}{(3n+5)(3n+8)}$, so that $a_k = \frac{6(k-1)^2+26(k-1)+10}{[3(k-1)+5][3(k-1)+8]} = \frac{2(3k^2+7k-5)}{(3k+2)(3k+5)}$. This formula also works for $k = 1$; hence we have $\sum_{k=1}^{\infty} \frac{2(3k^2+7k-5)}{(3k+2)(3k+5)} = \lim_{n\to+\infty} \frac{2n^2}{3n+5} = +\infty$, so the series diverges.

10. $a_1 = s_1 = 1$ and $a_{n+1} = s_{n+1} - s_n = (n+1) - n = 1$; hence, $a_k = 1$ for $k \geq 1$. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} 1$ diverges since $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} n = +\infty$.

11. $a_1 = s_1 = 2$ and $a_{n+1} = s_{n+1} - s_n = [1 - (-1)^{n+1}] - [1 - (-1)^n]$; hence, $a_{n+1} = (-1)^n - (-1)^{n+1} = (-1)^n[1 - (-1)] = 2(-1)^n$. Thus, $a_k = 2(-1)^{k-1} = 2(-1)^{k-1}(-1)^2 = 2(-1)^{k+1}$. This formula works even for $k = 1$; hence, the desired series, $\sum_{k=1}^{\infty} 2(-1)^{k+1}$, diverges, since $\lim_{n\to+\infty} [1 - (-1)^n]$ does not exist.

12. $a_1 = 1$ and $a_{n+1} = s_{n+1} - s_n = [2 - \frac{1}{2^n}] - [2 - \frac{1}{2^{n-1}}]$; hence, $a_{n+1} = \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{1}{2^{n-1}}[1 - \frac{1}{2}] = \frac{1}{2^{n-1}}(\frac{1}{2}) = \frac{1}{2^n}$. It follows that $a_k = \frac{1}{2^{k-1}}$, a formula that works also for $k = 1$. The series is $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ which converges since $\lim_{n\to+\infty} [2 - \frac{1}{2^{n-1}}] = 2$.

13. $a = \frac{5}{16}$, $r = \frac{1}{4}$. The series converges since $|r| = \frac{1}{4} < 1$. The sum is $\frac{a}{1-r} = \frac{5/16}{1-\frac{1}{4}} = \frac{5}{16} \cdot \frac{4}{3} = \frac{5}{12}$.

14. $a = 1$, $r = \frac{3}{5}$. Since $|r| = \frac{3}{5} < 1$, the series converges, and the sum is $\frac{a}{1-r} = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$.

15. $a = 1$, $r = -\frac{2}{3}$. The series converges since $|r| = \frac{2}{3} < 1$. The sum is $\frac{a}{1-r} = \frac{1}{1+\frac{2}{3}} = 1 \cdot \frac{3}{5} = \frac{3}{5}$.

16. $a = \frac{3}{10}$, $r = \frac{1}{10}$. Since $|r| = \frac{1}{10} < 1$, the series converges, and the sum is $\frac{a}{1-r} = \frac{3/10}{1-\frac{1}{10}} = \frac{1}{3}$.

17. $a = 1$, $r = \frac{2}{7}$. The series converges since $|r| = \frac{2}{7} < 1$. The sum is $\frac{a}{1-r} = \frac{1}{1-\frac{2}{7}} = \frac{7}{5}$.

18. $a = -\frac{5}{8}$, $r = -\frac{5}{8}$. The series converges since $|r| = \frac{5}{8} < 1$. The series has sum $\frac{a}{1-r} = \frac{-\frac{5}{8}}{1-(-\frac{5}{8})} = -\frac{5}{13}$.

19. $a = \frac{7}{6}$, $r = \frac{7}{6}$. The series diverges since $|r| > 1$.

20. $a = 1$, $r = -\frac{5}{3}$. Since $|r| = \frac{5}{3} > 1$, the series diverges.

21. $a = (\frac{9}{10})^2$, $r = \frac{9}{10}$. The series converges since $|r| = \frac{9}{10} < 1$. The sum is $\frac{a}{1-r} = \frac{(9/10)^2}{1-\frac{9}{10}} = \frac{81}{10}$.

22. $a = \frac{1}{4^2} = \frac{1}{16}$, $r = \frac{3}{4}$. The series converges since $|r| = \frac{3}{4} < 1$. The sum is $\frac{a}{1-r} = \frac{\frac{1}{16}}{1-\frac{3}{4}} = \frac{1}{4}$.

23. $a = 1$, $r = -1$. The series diverges since $|r| = 1$.

24. $a = 0.9$, $r = 0.1$. The series converges since $|r| = \frac{1}{10} < 1$. The sum is $\frac{a}{1-r} = \frac{9/10}{1-\frac{1}{10}} = 1$.

25. $a = \frac{1}{5}$, $r = \frac{1}{5}$. The series converges since $|r| = \frac{1}{5} < 1$. The sum is $\frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}$.

26. $a = 1$, $r = \frac{1}{e}$. The series converges since $|r| = \frac{1}{e} < 1$. The sum is $\frac{a}{1-r} = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$.

27. $a = \frac{1}{5}$, $r = \frac{3}{5}$. Since $|r| = \frac{3}{5} < 1$, the series converges. The sum is $\frac{a}{1 - r} = \frac{1/5}{1 - \frac{3}{5}} = \frac{1}{5} \cdot \frac{5}{2} = \frac{1}{2}$.

28. $a = 5$, $r = \frac{5}{6}$. Since $|r| = \frac{5}{6} < 1$, the series converges, and the sum is $\frac{a}{1 - r} = \frac{5}{1 - \frac{5}{6}} = 30$.

29. $a = 1$, $r = -\frac{1}{10}$. The series converges since $|r| = \frac{1}{10} < 1$. The sum is $\frac{a}{1 - r} = \frac{1}{1 + \frac{1}{10}} = \frac{10}{11}$.

30. There is no justification for inserting parentheses in an infinite series.

31. $0.33333\ldots = 0.3 + 0.03 + 0.003 + \ldots$. This series is geometric with $a = 0.3$, $r = 0.1$. The sum is $\frac{a}{1 - r} = \frac{3/10}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}$. Thus $0.3333\ldots = 0.3 + 0.03 + 0.003 + \ldots = \frac{1}{3}$.

32. $1.1111\ldots = 1 + 0.1111\ldots = 1 + [0.1 + 0.01 + 0.001 + \ldots] = 1 + (\frac{0.1}{1 - 0.1}) = 1 + \frac{1}{9} = \frac{10}{9}$.

33. $4.717171\ldots = 4 + [0.71 + 0.0071 + 0.000071 + \ldots] = 4 + [\frac{0.71}{1 - 0.01}] = 4 + \frac{71}{99} = \frac{467}{99}$.

34. $15.712712712\ldots = 15 + [0.712 + 0.000712 + 0.000000712 + \ldots] = 15 + \frac{0.712}{1 - 0.001} = 15 + \frac{712}{999} = \frac{15{,}697}{999}$.

35. Yes, <u>if it converges</u>, because $\sum_{k=1}^{n} a_k$ is the nth partial sum of $\sum_{k=1}^{\infty} a_k$.

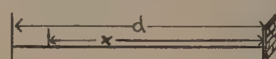
36. $\lim_{n\to+\infty} (1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots + \frac{1}{3^{2n}}) = \sum_{k=1}^{\infty} \frac{1}{3^{2k-2}}$, which is a geometric series with $a = 1$ and $r = \frac{1}{3^2}$ and has sum $\frac{1}{1 - \frac{1}{9}} = \frac{9}{8}$. Thus, $\lim_{n\to+\infty} (1 + \frac{1}{3^2} + \ldots + \frac{1}{3^{2n}}) = \frac{9}{8}$.

37. $0.4929292929\ldots = 0.49 + [0.0029 + 0.000029 + \ldots] = 0.49 + \frac{0.0029}{1 - 0.01} = \frac{49}{100} + \frac{29}{9900} = \frac{4880}{9900} = \frac{244}{495}$.

38. (a) At the start of an execution of the procedure, let the solution in the beaker contain g grams of salt. The concentration is therefore $\frac{g}{1000}$ grams/cm^3. When 250 cm^3 of solution are poured out, 750 cm^3 will remain containing $(\frac{g}{1000})(750) = \frac{3}{4}g$ grams of salt. When 250 cm^3 of pure water are added, the beaker will still contain $\frac{3}{4}g$ grams of salt; that is, $\frac{1}{4}g$ grams of salt have been removed. Thus, on the first execution, $\frac{10}{4}$ grams are removed and $\frac{3}{4}(10)$ grams remain; on the second execution, $\frac{1}{4}(\frac{3}{4})10$ grams are removed and $(\frac{3}{4})^2(10)$ grams remain; on the third execution, $\frac{1}{4}(\frac{3}{4})^2(10)$ grams are removed, while $(\frac{3}{4})^3$(1 grams remain and so forth. After the procedure is re peated n times, the number of grams of salt removed is given by $\frac{10}{4} + \frac{10}{4}(\frac{3}{4}) + \frac{10}{4}(\frac{3}{4})^2 + \ldots \frac{10}{4}(\frac{3}{4})^{n-1}$.
(b) If the procedure is repeated "infinitely often," the amount of salt removed is given by $\frac{a}{1 - r} = \frac{10/4}{1 - \frac{3}{4}} = 10$ grams. Hence, no salt will remain in the beaker.

39. The ball travels a distance of $2 + 2[2(\frac{3}{5}) + 2(\frac{3}{5})^2 + 2(\frac{3}{5})^3 + \ldots]$ meters. Now $[2(\frac{3}{5}) + 2(\frac{3}{5})^2 + 2(\frac{3}{5})^3 + \ldots] = \frac{2(\frac{3}{5})}{1 - \frac{3}{5}} = \frac{\frac{6}{5}}{\frac{2}{5}} = 3$.
Hence, the ball travels $2 + 2(3) = 8$ meters.

40. (a) Let x be Abner's distance from the wall on any one departure of the fly. It takes $\frac{x}{V}$ seconds for the fly to go from man to wall.



Let t be the time it takes the fly to go from wall to man. Abner has walked $(\frac{x}{V} + t)v$ meters. Thus $(\frac{x}{V} + t)v + tV = x$ and we obtain $t = \frac{x(V - v)}{V(V + v)}$. Thus, on any round trip, the fly flies a total distance of $x + tV = x + x(\frac{V - v}{V + v}) = \frac{2xV}{V + v}$ meters and the man ends at a distance $\frac{x}{V} + t = \frac{x(V - v)}{V + v}$ meters from the wall. Thus, on the first trip the fly goes $d\cdot(\frac{2V}{V + v})$ meters and Abner ends $\underbrace{d\cdot(\frac{V - v}{V + v})}_{\text{"new x"}}$ meters from the wall.
On 2nd trip: fly goes $d(\frac{V - v}{V + v})(\frac{2V}{V + v})$ meters --

man ends $\underbrace{d\cdot(\frac{V - v}{V + v})(\frac{V - v}{V + v})}_{\text{"new x"}}$ meters from the wall.

On 3rd trip: fly goes $d\cdot(\frac{V - v}{V + v})(\frac{V - v}{V + v})(\frac{2V}{V + v})$ meters, etc. Thus on the nth trip the fly covers a distance of $d\cdot(\frac{V - v}{V + v})^{n-1}(\frac{2V}{V + v})$ meters.

(b) The fly travels V meters per second; thus, the fly requires $\frac{[\frac{2\cdot d\cdot V}{V + v}(\frac{V - v}{V + v})^{n-1}]}{V}$ or $\frac{2d}{V + v}(\frac{V - v}{V + v})^{n-1}$ seconds.

(c) $\sum_{n=1}^{\infty} \frac{2Vd}{V + v}(\frac{V - v}{V + v})^{n-1}$ is a geometric series with $a = \frac{2Vd}{V + v}$ and $r = \frac{V - v}{V + v}$. Thus, the total distance flown by the fly is given by $\frac{a}{1 - r} = \frac{\frac{2Vd}{V + v}}{1 - \frac{V - v}{V + v}} = \frac{2Vd}{V + v - V + v} = \frac{2Vd}{2v} = \frac{Vd}{v}$ meters.

(d) The total time required for Abner to reach the wall is given by $\sum_{n=1}^{\infty} \frac{2d}{V + v}(\frac{V - v}{V + v})^{n-1}$. It is a geometric series with $a = \frac{2d}{V + v}$ and $r = \frac{V - v}{V + v}$. Thus, the total time for Abner to reach the wall is $\frac{\frac{2d}{V + v}}{1 - \frac{V - v}{V + v}} = \frac{2d}{2v} = \frac{d}{v}$ seconds.

(e) It takes Abner $\frac{d}{v}$ seconds to reach the wall. The fly travels V meters per second. Hence, the fly travels $\frac{dV}{v}$ meters.

1. $(b_1 - b_2) = 0 - (-s_1) = s_1 = a_1$. For $n > 1$, $(b_n - b_{n+1}) = -s_{n-1} - (-s_n) = s_n - s_{n-1} = a_n$. Thus $\sum_{k=1}^{\infty} (b_k - b_{k+1}) = \sum_{k=1}^{\infty} a_k$.

2. $\sum_{k=1}^{n} (b_k - b_{k+2}) = \sum_{k=1}^{n} [(b_k - b_{k+1}) + (b_{k+1} - b_{k+2})] = \sum_{k=1}^{n} (b_k - b_{k+1}) + \sum_{k=1}^{n} (b_{k+1} - b_{k+2}) = b_1 - b_{n+1} + b_{1+1} - b_{n+2} = (b_1 + b_2) - (b_{n+1} + b_{n+2})$.

3. Call s_n the nth partial sum of $\sum_{k=1}^{\infty} (b_k - b_{k+2})$. Now $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} [(b_1 + b_2) - (b_{n+1} + b_{n+2})] = (b_1 + b_2) - (L + L) = b_1 + b_2 - 2L$. Hence, the series $\sum_{k=1}^{\infty} (b_k - b_{k+2})$ converges with sum $b_1 + b_2 - 2L$.

Problem Set 11.3, page 656

1. $\lim_{n\to+\infty} \frac{n}{5n + 7} = \lim_{n\to+\infty} \frac{1}{5 + \frac{7}{n}} = \frac{1}{5} \neq 0$. So $\sum_{k=1}^{\infty} \frac{k}{5k + 7}$ diverges.

2. $\lim_{n\to+\infty} \ln(\frac{5n}{12n + 5}) = \ln[\lim_{n\to+\infty} \frac{5n}{12n + 5}] = \ln[\lim_{n\to+\infty} \frac{5}{12 + \frac{5}{n}}] = \ln \frac{5}{12} \neq 0$. So $\sum_{k=1}^{\infty} \ln(\frac{5k}{12k + 5})$ diverges.

3. $\lim_{n\to+\infty} \frac{3n^2 + 5n}{7n^2 + 13n + 2} = \lim_{n\to+\infty} \frac{3 + \frac{5}{n}}{7 + \frac{13}{n} + \frac{2}{n^2}} = \frac{3}{7} \neq 0$. Thus $\sum_{k=1}^{\infty} \frac{3k^2 + 5k}{7k^2 + 13k + 2}$ diverges.

4. $\lim_{n\to+\infty} \frac{e^n}{3e^n + 7} = \lim_{n\to+\infty} \frac{1}{3 + \frac{7}{e^n}} = \frac{1}{3} \neq 0$. Thus, $\sum_{k=1}^{\infty} \frac{e^k}{3e^k + 7}$ diverges.

5. $\lim_{n\to+\infty} \sin \frac{\pi n}{4}$ does not exist. Thus, $\sum_{k=1}^{\infty} \sin \frac{\pi k}{4}$ diverges.

6. $\lim_{n\to+\infty} \frac{n}{\cos n} = +\infty$ since $|\cos n| \leq 1$. Thus, $\sum_{k=1}^{\infty} \frac{k}{\cos k}$ diverges.

7. $\lim_{n\to+\infty} n \sin \frac{1}{n} = \lim_{n\to+\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$. Hence, $\sum_{k=1}^{\infty} k \sin \frac{1}{k}$ diverges.

8. The sequence $\left\{\frac{n!}{2^n}\right\}$ is unbounded from above. Hence, $\lim_{n\to+\infty} \frac{n!}{2^n} \neq 0$. Thus, $\sum_{k=1}^{\infty} \frac{k!}{2^k}$ diverges.

9. $\sum_{k=1}^{\infty} [(\frac{1}{3})^k + (\frac{1}{4})^k] = \sum_{k=1}^{\infty} (\frac{1}{3})^k + \sum_{k=1}^{\infty} (\frac{1}{4})^k =$

$$\frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

10. $\sum_{k=1}^{\infty} [(\frac{1}{2})^{k-1} - (-\frac{1}{3})^{k+1}] = \sum_{k=1}^{\infty} (\frac{1}{2})^{k-1} - \sum_{k=1}^{\infty} (-\frac{1}{3})^{k+1} =$

$\frac{1}{1-\frac{1}{2}} - \frac{\frac{1}{9}}{1-(-\frac{1}{3})} = 2 - \frac{1}{12} = \frac{23}{12}.$

11. $\sum_{k=1}^{\infty} [\frac{1}{k(k+1)} - (\frac{3}{4})^{k-1}] = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} -$

$\sum_{k=1}^{\infty} (\frac{3}{4})^{k-1} = 1 - \frac{1}{1-\frac{3}{4}} = 1 - 4 = -3.$ We used the result of Example 1, page 643, to write $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$

12. $\sum_{k=0}^{\infty} [2(\frac{1}{3})^k - 3(-\frac{1}{5})^{k+1}] = 2\sum_{k=0}^{\infty} (\frac{1}{3})^k -$

$3\sum_{k=0}^{\infty} (-\frac{1}{5})^{k+1} = 2(\frac{1}{1-\frac{1}{3}}) - 3(\frac{-1/5}{1-(-\frac{1}{5})}) = 3 - 3(-\frac{1}{6}) = \frac{7}{2}.$

13. $\sum_{k=1}^{\infty} [\frac{2^k + 3^k}{6^k} - \frac{1}{7^{k+1}}] = \sum_{k=1}^{\infty} (\frac{1}{3})^k + \sum_{k=1}^{\infty} (\frac{1}{2})^k -$

$\sum_{k=1}^{\infty} \frac{1}{7^{k+1}} = \frac{1/3}{1-\frac{1}{3}} + \frac{1/2}{1-\frac{1}{2}} - \frac{1/49}{1-\frac{1}{7}} = \frac{1}{2} + 1 - \frac{1}{42} = \frac{31}{21}.$

14. $\sum_{k=1}^{\infty} [\sin\frac{1}{k} + 2^{-k} - \sin\frac{1}{k+1}] =$

$\sum_{k=1}^{\infty} (\sin\frac{1}{k} - \sin\frac{1}{k+1}) + \sum_{k=1}^{\infty} 2^{-k} =$

$\lim_{n\to+\infty} (\sin 1 - \sin\frac{1}{n+1}) + \frac{\frac{1}{2}}{1-\frac{1}{2}} = \sin 1 + 1.$

15. No. But $\sum_{k=1}^{\infty} \frac{1}{k}$ <u>may</u> converge. Later (Section 11.4), we see that it actually diverges.

16. By Theorem 1, $\lim_{n\to+\infty} \frac{c^n}{n!} = 0.$

17. $-2 + 1 - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6} - \frac{2}{7} + \ldots = -2\sum_{k=1}^{\infty} (-1)^{k+1}(\frac{1}{k}) =$ $-2 \ln 2.$

18. The linearity property (Theorem 3, Section 11.3) has been used here when there is no guarantee that it is applicable, since we do not know whether or not $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} b_{k+1}$ converge.

19. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$ (see Problem 11). $-\sum_{k=1}^{\infty} \ln\frac{k}{k+1} =$

$-\sum_{k=1}^{\infty} [\ln k - \ln(k+1)] = -\lim_{n\to+\infty} [0 - \ln(n+1)] = +\infty.$

Since $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges and $\sum_{k=1}^{\infty} \ln\frac{k}{k+1}$ diverges, the given series diverges.

20. $\sum_{k=1}^{\infty} ar^{k-1} = \sum_{j=0}^{\infty} ar^j.$

21. $\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \sum_{j=1}^{\infty} \frac{1}{j(j+1)}.$

22. $\sum_{k=M}^{\infty} a_k = \sum_{j=1}^{\infty} a_{j+M-1}.$

23. $\sum_{k=1}^{\infty} a_k = \sum_{j=M}^{\infty} a_{j-M+1}.$

24. $\sum_{k=M+1}^{\infty} (b_k - b_{k+1}) = \lim_{n\to+\infty} \sum_{k=M+1}^{n} (b_k - b_{k+1}) =$

$\lim_{n\to+\infty} (b_{M+1} - b_{n+1}) = b_{M+1} - \lim_{n\to+\infty} b_{n+1} = b_{M+1} -$ $\lim_{n\to+\infty} b_n.$

25. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{M} \frac{1}{k(k+1)} + \sum_{k=M+1}^{\infty} \frac{1}{k(k+1)}.$

Thus, $1 = 1 - (\frac{1}{M+1}) + \sum_{k=M+1}^{\infty} \frac{1}{k(k+1)}.$ Hence, $\sum_{k=M+1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{M+1}.$

26. (a) Let the series be $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, and suppose $\sum_{k=M+1}^{\infty} a_k = \sum_{k=M+1}^{\infty} b_k$. Now by Theorem 5, $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=M+1}^{\infty} a_k$ converges if and only if $\sum_{k=M+1}^{\infty}$ bk converges (since they are equal) if and only if $\sum_{k=1}^{\infty} b_k$ converges.

(b) Suppose $\sum_{k=1}^{\infty} a_k$ is modified by changing, deleting or adding a single term in its r'th position to obtain a series $\sum_{k=1}^{\infty} b_k$. (Note that if a term is deleted, it can be thought of as being changed to a 0; if a term is added, then the modified series is affected in the rth position, and the

original series in position r can be thought of as having term 0.) $\sum_{k=r+1}^{\infty} b_k = \sum_{k=r+1}^{\infty} a_k$ and so by part (a), the convergence or divergence of the series is not affected.

27. $e = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$.

Thus, $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = e - 1$.

28. By Theorem 3 of Section 11.2, $S_M \le \lim_{n\to+\infty} s_n$ for all positive integers M. Now $S_M = \sum_{k=1}^{M} a_k$ and $\lim_{n\to+\infty} s_n = \sum_{k=1}^{\infty} a_k$. Therefore, $\sum_{k=1}^{M} a_k \le \sum_{k=1}^{\infty} a_k$ holds for all positive integers M.

29. $s_n = \sum_{k=1}^{n} \frac{k}{(k+1)\cdot 3^k} \le \sum_{k=1}^{n} \frac{1}{3^k} \le \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$.

Hence, $\sum_{k=1}^{\infty} \frac{k}{(k+1)3^k}$ converges.

30. $s_n = \sum_{k=1}^{n} \frac{(k-1)\ln 3}{4^{k-1}} = \sum_{k=1}^{n} \frac{\ln 3^{k-1}}{4^{k-1}} < \sum_{k=1}^{n} \frac{3^{k-1}}{4^{k-1}} \le$

$\sum_{k=1}^{\infty} \frac{3^{k-1}}{4^{k-1}} = \sum_{k=1}^{\infty} (\frac{3}{4})^{k-1} = \frac{1}{1 - \frac{3}{4}} = 4$. Thus,

$\sum_{k=1}^{\infty} \frac{(k-1)\ln 3}{4^{k-1}}$ converges.

31. $s_n = \sum_{k=0}^{n} \frac{4^{-k} k}{k^2 + 1} \le \sum_{k=0}^{n} 4^{-k} \le \sum_{k=0}^{\infty} 4^{-k} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$.

Hence, $\sum_{k=0}^{\infty} \frac{4^{-k} k}{k^2 + 1}$ converges.

32. $s_n = \sum_{k=0}^{n} \frac{k}{5^k} < \sum_{k=0}^{n} \frac{4^k}{5^k} = \sum_{k=0}^{n} (\frac{4}{5})^k \le \sum_{k=0}^{\infty} (\frac{4}{5})^k =$

$\frac{1}{1 - \frac{4}{5}} = 5$. Thus, $\sum_{k=0}^{\infty} \frac{k}{5^k}$ converges.

33. $\sum_{k=1}^{n} \frac{1}{k^2} \le 1 + \sum_{k=2}^{n} \frac{1}{(k-1)k} \le 1 + \sum_{k=2}^{\infty} (\frac{1}{k-1} - \frac{1}{k}) =$

$1 - \sum_{k=2}^{\infty} (\frac{1}{k} - \frac{1}{k-1}) = 1 - (\frac{1}{2} - \lim_{n\to+\infty} \frac{1}{n-1}) = 1 - \frac{1}{2} =$

$\frac{1}{2}$. Thus, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

34. $s_n = \sum_{k=1}^{n} \frac{1}{k!} \le \sum_{k=1}^{n} \frac{1}{2^{k-1}} \le \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{1 - \frac{1}{2}} = 2$, so that $\{s_n\}$ is bounded above by $M = 2$ and the series $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges.

35. Let $s_n = \sum_{k+1}^{n} a_k$ and let $t_n = \sum_{k=1}^{n} b_k$. Then $s_n - t_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k - b_k)$ is the nth partial sum of the series $\sum_{k=1}^{n} (a_k - b_k)$. Now $\lim_{n\to+\infty} (s_n - t_n) = \lim_{n\to+\infty} s_n - \lim_{n\to+\infty} t_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$ since $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent. Thus, since the nth partial sum of $\sum_{k=1}^{\infty} (a_k - b_k)$ has a limit, the series $\sum_{k=1}^{\infty} (a_k - b_k)$ converges and $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$.

36. By Problem 28, the sum of the series is an upper bound for the sequence of partial sums, and 0 is a lower bound.

37. Let c be a constant and assume that $\sum_{k=1}^{\infty} a_k$ is a convergent series with nth partial sum $s_n = \sum_{k=1}^{n} a_k$. Then $cs_n = \sum_{k=1}^{n} ca_k$ is the nth partial sum of the series $\sum_{k=1}^{\infty} ca_k$; hence, $\sum_{k=1}^{\infty} ca_k$ converges and $\sum_{k=1}^{\infty} ca_k = \lim_{n\to+\infty} cs_n = c \lim_{n\to+\infty} s_n = c \sum_{k=1}^{\infty} a_k$. Now, suppose that c is a non-zero constant and that $\sum_{k=1}^{\infty} a_k$ is divergent. Then $\sum_{k=1}^{\infty} ca_k$ must also be divergent, for otherwise, by what has just been proved, $\sum_{k=1}^{\infty} (\frac{1}{c})ca_k = \sum_{k=1}^{\infty} a_k$ would be convergent.

Problem Set 11.4, page 667

1. Define $f(x) = \frac{1}{x\sqrt[3]{x}}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now $\int_{1}^{+\infty} \frac{1}{x\sqrt[3]{x}}\,dx =$

$\lim_{b\to+\infty} \int_{1}^{b} x^{-4/3}dx = \lim_{b\to+\infty} (-3x^{-1/3})\Big|_{1}^{b} =$

$\lim_{b\to+\infty} \left(\frac{-3}{3\sqrt{b}} + 3\right) = 3$. Thus, $\int_1^\infty \frac{1}{x\sqrt[3]{x}}\,dx$ converges, and so $\sum_{k=1}^{\infty} \frac{1}{k\sqrt[3]{k}}$ converges.

2. Let $f(x) = \frac{1}{x^2+4}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now, $\int_1^\infty \frac{1}{x^2+4}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^2+4}\,dx = \lim_{b\to+\infty} \left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)\Big|_1^b = \lim_{b\to+\infty} \left[\frac{1}{2}\tan^{-1}\frac{b}{2} - \frac{1}{2}\tan^{-1}\frac{1}{2}\right] = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\tan^{-1}\frac{1}{2}$. Since $\int_1^\infty \frac{1}{x^2+4}\,dx$ converges, then $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$ converges.

3. Define $f(x) = \frac{3x^2}{x^3+16}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now, $\int_1^\infty \frac{3x^2}{x^3+16}\,dx = \lim_{b\to+\infty} \int_1^b \frac{3x^2}{x^3+16}\,dx = \lim_{b\to+\infty} \ln(x^3+16)\Big|_1^b = \lim_{b\to+\infty} [\ln(b^3+16) - \ln 17] = +\infty$. Thus, $\int_1^\infty \frac{3x^2}{x^3+16}\,dx$ diverges, and so $\sum_{k=1}^{\infty} \frac{3k^2}{k^3+16}$ diverges.

4. Define $f(x) = \frac{3x}{x^2+8}$. f(x) is continuous, decreasing, and nonnegative on $[2,\infty)$. Now $\int_2^\infty \frac{3x\,dx}{x^2+8} = \lim_{b\to+\infty} \int_2^b \frac{3x\,dx}{x^2+8} = \lim_{b\to+\infty} \frac{3}{2}\ln(x^2+8)\Big|_2^b = \lim_{b\to+\infty} \frac{3}{2}\ln(b^2+8) - \frac{3}{2}\ln 12 = +\infty$. Thus, $\int_2^\infty \frac{3x\,dx}{x^2+8}$ diverges, so $\sum_{k=1}^{\infty} \frac{3k}{k^2+8}$ diverges.

5. Define $f(x) = \frac{2x}{(5+3x^2)^{3/2}}$. f(x) is continuous, decreasing, and nonnegative on $[1,\infty)$. Now $\int_1^\infty \frac{2x\,dx}{(5+3x^2)^{3/2}} = \lim_{b\to+\infty} \int_1^b \frac{2x\,dx}{(5+3x^2)^{3/2}} = \lim_{b\to+\infty} -\frac{2}{3}(5+3x^2)^{-1/2}\Big|_1^b = \lim_{b\to+\infty} \frac{1}{3\sqrt{2}} - \frac{2}{3}(5+3b^2)^{-1/2} = \frac{1}{3\sqrt{2}}$. Thus, $\int_1^\infty \frac{2x\,dx}{(5+3x^2)^{3/2}}$ converges, so $\sum_{k=1}^{\infty} \frac{2k}{(5+3k^2)^{3/2}}$ converges.

6. Define $f(x) = \frac{1}{x\sqrt{x^2-1}}$. f(x) is continuous, decreasing, and nonnegative on $[2,\infty)$. Now $\int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to+\infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}$. Now by the trig substitution $x = \sec\theta$, $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta} = \int d\theta = \theta = \sec^{-1}x$, so $\int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to+\infty} \sec^{-1}x\Big|_2^b = \lim_{b\to+\infty} \sec^{-1}b - \sec^{-1}2 = \pi/2 - \pi/3 = \pi/6$. Thus, $\int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$ converges, so $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}}$ converges.

7. Let $f(x) = \left(\frac{1000}{x}\right)^2$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now, $\int_1^\infty \left(\frac{1000}{x}\right)^2 dx = \lim_{b\to+\infty} \int_1^b \frac{(1000)^2}{x^2}\,dx = \lim_{b\to+\infty} (1000)^2\left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to+\infty} \left[(1000)^2 - \frac{(1000)^2}{b}\right] = (1000)^2$. Thus, $\int_1^\infty \left(\frac{1000}{x}\right)^2 dx$ converges, and so $\sum_{n=1}^{\infty} \left(\frac{1000}{n}\right)^2$ converges.

8. Define $f(x) = e^{-x}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Here, $\int_1^\infty e^{-x}dx = \lim_{b\to+\infty} \int_1^b e^{-x}dx = \lim_{b\to+\infty} -e^{-x}\Big|_1^b = \lim_{b\to+\infty} (e^{-1} - e^{-b}) = \frac{1}{e}$. Thus, $\int_1^\infty e^{-x}dx$ converges and so does $\sum_{m=1}^{\infty} e^{-m}$.

9. Define $f(x) = \frac{\ln x}{x}$. f is continuous, decreasing, and nonnegative $[2,\infty)$. Now $\int_2^\infty \frac{\ln x}{x}\,dx = \lim_{b\to+\infty} \int_2^b \frac{\ln x}{x}\,dx = \lim_{b\to+\infty} \frac{(\ln x)^2}{2}\Big|_2^b = \lim_{b\to+\infty} \left[\frac{(\ln b)^2}{2} - \frac{(\ln 2)^2}{2}\right] = +\infty$. Thus, $\int_2^\infty \frac{\ln x}{x}\,dx$ diverges, so $\sum_{k=2}^{\infty} \frac{\ln k}{k}$ diverges.

10. Define $f(x) = \frac{1}{x\ln x}$. f is continuous, decreasing, and nonnegative on $[2,\infty)$. Thus, $\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to+\infty} \int_2^b \frac{1}{x\ln x}\,dx = \lim_{b\to+\infty} \ln(\ln x)\Big|_2^b = \lim_{b\to+\infty} [\ln(\ln b) - \ln(\ln 2)] = +\infty$. Thus,

$\int_2^{\infty} \frac{1}{x \ln x} dx$ diverges. Hence, $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges.

11. Define $f(x) = xe^{-x}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now, $\int_1^{\infty} xe^{-x}dx =$
$\lim_{b\to+\infty} [-xe^{-x}\Big|_1^b + \int_1^b e^{-x}dx] =$
$\lim_{b\to+\infty} (-be^{-b} + e^{-1} - e^{-b} + e^{-1}) = [\lim_{b\to+\infty} \frac{-1}{e^b}] +$
$2e^{-1} - 0 = \frac{2}{e}$. Thus, $\int_1^{\infty} xe^{-x}dx$ converges, and so does $\sum_{j=1}^{\infty} je^{-j}$.

12. Define $f(x) = xe^{-x^2}$. Then f(x) is continuous, decreasing, and nonnegative on $[1,\infty)$. Now
$\int_1^{\infty} xe^{-x^2}dx = \lim_{b\to\infty} \int_1^b xe^{-x^2}dx = \lim_{b\to\infty} -\frac{1}{2} e^{-x^2}\Big|_1^b =$
$\lim_{b\to\infty} -\frac{1}{2}[e^{-b^2} - e^{-1}] = -\frac{1}{2}[0 - e^{-1}] = \frac{1}{2e}$. Thus,
$\int_1^{\infty} xe^{-x^2}dx$ converges, so $\sum_{k=1}^{\infty} ke^{-k^2}$ converges also.

13. Define $f(x) = \frac{\tan^{-1}x}{1 + x^2}$. f is decreasing, continuous, and nonnegative on $[1,\infty)$. Thus, $\int_1^{\infty} \frac{\tan^{-1}x}{1 + x^2} dx =$
$\lim_{b\to+\infty} \int_1^b \frac{\tan^{-1}x}{1 + x^2} dx = \lim_{b\to+\infty} [\frac{(\tan^{-1}x)^2}{2}]\Big|_1^b =$
$\lim_{b\to+\infty} [\frac{(\tan^{-1}b)^2}{2} - \frac{\pi^2}{32}] = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}$. Therefore,
$\sum_{m=1}^{\infty} \frac{\tan^{-1}m}{1 + m^2}$ converges.

14. Define $f(x) = \frac{x}{2^x}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now $\int_1^{\infty} \frac{x}{2^x} dx = \lim_{b\to+\infty} \int_1^b \frac{x}{2^x}dx =$
$\lim_{b\to+\infty} [-\frac{x}{2 \ln 2}\Big|_1^b + \int_1^b \frac{1}{\ln 2} (\tfrac{1}{2})^x dx] =$
$\lim_{b\to+\infty} [-\frac{b}{2 \ln 2} + \frac{1}{2 \ln 2} - \frac{1}{(\ln 2)^2} (\tfrac{1}{2})^b + \frac{1}{\ln 2} (\tfrac{1}{2})] =$
$-\infty$. Thus, $\sum_{r=1}^{\infty} \frac{r}{2^r}$ diverges.

15. Define $f(x) = \frac{1}{(2x + 1)(3x + 1)}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now
$\int_1^{\infty} \frac{1}{(2x + 1)(3x + 1)} = \lim_{b\to+\infty} \int_1^b \frac{1}{(2x + 1)(3x + 1)} dx =$
$\lim_{b\to+\infty} [\int_1^b \frac{-2}{2x + 1} dx + \int_1^b \frac{3}{3x + 1} dx] =$
$\lim_{b\to+\infty} [-\ln(2x + 1)\Big|_1^b + \ln(3x + 1)\Big|_1^b] =$
$\lim_{b\to+\infty} \ln \frac{(3x + 1)}{2x + 1}\Big|_1^b = \ln[\lim_{b\to+\infty} \frac{3b + 1}{2b + 1}] - \lim_{b\to+\infty} [\ln \frac{4}{3}] =$
$\ln \frac{3}{2} - \ln \frac{4}{3}$. Hence, $\sum_{k=1}^{\infty} \frac{1}{(2k + 1)(3k + 1)}$ converges.

16. Define $f(x) = \frac{1}{x(x + 1)(x + 2)}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now
$\int_1^{\infty} \frac{1}{x(x + 1)(x + 2)} dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x(x + 1)(x + 2)} dx =$
$\lim_{b\to+\infty} [\int_1^b \frac{\frac{1}{2}}{x} dx + \int_1^b \frac{-1}{x + 1} dx + \int_1^b \frac{\frac{1}{2}}{x + 2} dx] =$
$\lim_{b\to+\infty} [\tfrac{1}{2} \ln x\Big|_1^b - \ln (x + 1)\Big|_1^b + \tfrac{1}{2} \ln(x + 2)\Big|_1^b] =$
$\lim_{b\to+\infty} [\tfrac{1}{2} \ln b - \ln(b + 1) + \ln 2 + \tfrac{1}{2} \ln(b + 2) -$
$\tfrac{1}{2} \ln 3] = \lim_{b\to+\infty} [\tfrac{1}{2} \ln \frac{(b)(b + 2)}{(b + 1)^2} + \ln 2 - \tfrac{1}{2} \ln 3] =$
$\tfrac{1}{2} \ln[\lim_{b\to+\infty} \frac{b^2 + 2b}{b^2 + 2b + 1}] + \ln 2 - \tfrac{1}{2} \ln 3 =$
$[\tfrac{1}{2} \ln 1 + \ln 2 - \frac{\ln 3}{2}] = \ln 2 - \frac{\ln 3}{2}$. Thus,
$\int_1^{\infty} \frac{1}{x(x + 1)(x + 2)} dx$ converges, and so
$\sum_{n=1}^{\infty} \frac{1}{n(n + 1)(n + 2)}$ converges.

17. Let $f(x) = \coth x$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now $\int_1^{\infty} \coth x\, dx =$
$\lim_{b\to+\infty} \int_1^b \coth x\, dx = \lim_{b\to+\infty} \ln (\cosh x)\Big|_1^b =$
$\lim_{b\to+\infty} [\ln(\cosh b) - \ln(\cosh 1)] = +\infty$. Thus,
$\int_1^{\infty} \coth x\, dx$ diverges, and so does $\sum_{n=1}^{\infty} \coth n$.

18. Define $f(x) = \frac{1}{1 + \sqrt{x}}$. f is continuous, decreasing, and nonnegative on $[1,\infty)$. Now $\int_1^{\infty} \frac{1}{1 + \sqrt{x}} dx =$
$\lim_{b\to+\infty} \int_1^b \frac{1}{1 + \sqrt{x}} dx = \lim_{b\to+\infty} [2\sqrt{x} - 2 \ln(1 + \sqrt{x})]\Big|_1^b =$
$\lim_{b\to+\infty} [2\sqrt{b} - 2 \ln(1 + \sqrt{b}) - 2 - 2 \ln 2] =$
$2 \lim_{b\to+\infty} [\ln e^{\sqrt{b}} - \ln(1 + \sqrt{b})] - 2 - 2 \ln 2 =$

$2 \lim_{b\to+\infty} [\ln \frac{e^{\sqrt{b}}}{1+\sqrt{b}}] - 2 - 2 \ln 2$. But $\lim_{b\to+\infty} \frac{e^{\sqrt{b}}}{1+\sqrt{b}} =$

$\lim_{b\to+\infty} \frac{\frac{1}{2\sqrt{b}} e^{\sqrt{b}}}{\frac{1}{2\sqrt{b}}} = \lim_{b\to+\infty} e^{\sqrt{b}} = +\infty$. Hence, the integral diverges and so does the given series.

19. Define $f(x) = \frac{1}{x\sqrt{\ln x}}$. f is continuous, decreasing, and nonnegative on $[2,\infty)$. Now $\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx =$ $\lim_{b\to+\infty} \int_2^b \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to+\infty} 2\sqrt{\ln x}\Big|_2^b =$ $\lim_{b\to+\infty} [2\sqrt{\ln b} - 2\sqrt{\ln 2}] = +\infty$. So $\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx$ diverges, and so $\sum_{k=2}^\infty \frac{1}{k\sqrt{\ln k}}$ diverges.

20. Define $f(x) = \frac{1}{x \ln x \ln(\ln x)}$. f is continuous, decreasing, and nonnegative on $[3,\infty)$. Thus, $\int_3^\infty \frac{1}{x \ln x \ln(\ln x)}\,dx = \lim_{b\to+\infty} \int_3^b \frac{1}{x \ln x \ln(\ln x)}\,dx$ [let $u = \ln(\ln x)$] $= \lim_{b\to+\infty} \ln[\ln(\ln x)]\Big|_3^b =$ $\lim_{b\to+\infty} [\ln(\ln[\ln b]) - \ln(\ln[\ln 3])] = +\infty$. Hence, $\sum_{k=3}^\infty \frac{1}{k \ln k \ln(\ln k)}$ diverges.

21. We are going to compare the given series with the convergent p series $\sum_{k=1}^\infty \frac{1}{k^2}$. We want to show that $\frac{k^2}{k^4 + 3k + 1} \le \frac{1}{k^2}$, but this is equivalent to $k^4 \le k^4 + 3k + 1$. Hence, $\sum_{k=1}^\infty \frac{1}{k^2}$ dominates $\sum_{k=1}^\infty \frac{k^2}{k^4 + 3k + 1}$, and so $\sum_{k=1}^\infty \frac{k^2}{k^4 + 3k + 1}$ converges.

22. Here, $\frac{k}{k^3 + 2k + 7} \le \frac{1}{k^2}$ is equivalent to $k^3 \le k^3 + 2k + 7$. Hence, $\sum_{k=1}^\infty \frac{1}{k^2}$ dominates $\sum_{k=1}^\infty \frac{k}{k^3 + 2k + 7}$. Hence, $\sum_{k=1}^\infty \frac{k}{k^3 + 2k + 7}$ is convergent.

23. We compare the given series with the convergent geometric series $\sum_{k=1}^\infty \frac{1}{5^k}$. Now $\frac{1}{k5^k} \le \frac{1}{5^k}$ since $5^k \le k5^k$. Hence, $\sum_{k=1}^\infty \frac{1}{5^k}$ dominates $\sum_{k=1}^\infty \frac{1}{k5^k}$, and so $\sum_{k=1}^\infty \frac{1}{k5^k}$ converges.

24. We compare $\sum_{n=1}^\infty \frac{5}{(n+1)3^n}$ with the convergent geometric series $\sum_{n=1}^\infty \frac{5}{3^n}$. Now $\frac{5}{(n+1)3^n} < \frac{5}{3^n}$ since $(n+1)3^n > 3^n$. Thus, $\sum_{n=1}^\infty \frac{5}{(n+1)3^n}$ converges.

25. We compare the given series with the convergent geometric series $\sum_{j=1}^\infty \frac{1}{7^j}$. Now $\frac{j+1}{j+2} \le 1$, and so $(\frac{j+1}{j+2})(\frac{1}{7^j}) \le \frac{1}{7^j}$. Thus, $\sum_{j=1}^\infty \frac{1}{7^j}$ dominates $\sum_{j=1}^\infty \frac{j+1}{(j+2)\cdot 7^j}$. Hence, $\sum_{j=1}^\infty \frac{j+1}{(j+2)\cdot 7^j}$ is convergent.

26. Since the p series $\sum_{r=1}^\infty \frac{1}{r^{4/3}}$ converges, then $\sum_{r=1}^\infty \frac{5}{r^{4/3}}$ converges by Theorem 3 of Section 11.3. Since $\frac{5r}{\sqrt[3]{r^7+3}} \le \frac{4}{r^{4/3}}$ is equivalent to $r^{7/3} \le \sqrt[3]{r^7} + 3$ is equivalent to $r^7 \le r^7 + 3$, then $\sum_{r=1}^\infty \frac{5}{r^{4/3}}$ dominates $\sum_{r=1}^\infty \frac{5r}{\sqrt[3]{r^7+3}}$ and so $\sum_{r=1}^\infty \frac{5r}{\sqrt[3]{r^7+3}}$ converges.

27. We compare $\sum_{k=1}^\infty \frac{8}{\sqrt[3]{k+1}}$ with the divergent p series $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{k}}$. Thus, $\frac{8}{\sqrt[3]{k+1}} \ge \frac{1}{\sqrt[3]{k}}$ since $8\sqrt[3]{k} \ge \sqrt[3]{k+1}$, which is equivalent to $512k \ge k + 1$. Thus the given series dominates $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{k}}$ and so $\sum_{k=1}^\infty \frac{8}{\sqrt[3]{k+1}}$ diverges.

28. Since the harmonic series $\sum_{k=1}^\infty \frac{1}{k}$ diverges, then so does $\frac{1}{5}\sum_{k=1}^\infty \frac{1}{k} = \sum_{k=1}^\infty \frac{1}{5k}$ by Theorem 3 of Section 11.3. Now $\frac{1}{4k+6} \ge \frac{1}{5k}$ is equivalent to $5k \ge 4k + 6$, which is equivalent to $k \ge 6$. Thus, $\sum_{k=1}^\infty \frac{1}{4k+6}$ eventually dominates $\sum_{k=1}^\infty \frac{1}{5k}$. Hence, $\sum_{k=1}^\infty \frac{1}{4k+6}$ diverges.

29. We compare the given series with the divergent

series $\sum_{j=1}^{\infty} \frac{1}{5j}$, the harmonic series $\frac{1}{5}$ (see Problem 28). Now $\frac{j^2}{j^3 + 4j + 3} \geq \frac{1}{5j}$ is equivalent to $5j^3 \geq j^3 + 4j + 3$, which is equivalent to $4j^3 \geq 4j + 3$, or $j^3 \geq j + \frac{3}{4}$ for $j \geq 2$. Thus, $\sum_{j=1}^{\infty} \frac{j^2}{j^3 + 4j + 3}$ eventually dominates $\sum_{j=1}^{\infty} \frac{1}{5j}$. Hence, the given series diverges.

30. Clearly $\frac{\ln k}{k} \geq \frac{1}{k}$ for $k \geq 2$. Thus, $\sum_{k=2}^{\infty} \frac{\ln k}{k}$ dominates $\sum_{k=2}^{\infty} \frac{1}{k}$ and so $\sum_{k=2}^{\infty} \frac{\ln k}{k}$ diverges.

31. The p series $\sum_{q=1}^{\infty} \frac{1}{\sqrt{q}}$ diverges and by Theorem 3 of Section 11.3, so does $\sum_{q=1}^{\infty} \frac{1}{3\sqrt{q}}$. Thus, $\frac{\sqrt{q}}{q + 2} \geq \frac{1}{3\sqrt{q}}$ is equivalent to $3q \geq q + 2$, which is equivalent to $q \geq 1$. Therefore, $\sum_{q=1}^{\infty} \frac{\sqrt{q}}{q + 2}$ dominates $\sum_{q=1}^{\infty} \frac{1}{3\sqrt{q}}$ and so the given series diverges.

32. We compare the given series with the convergent geometric series $\sum_{j=1}^{\infty} \frac{2}{e^j}$ $(a = 2, r = \frac{1}{e})$. Now $\frac{1 + e^{-j}}{e^j} \leq \frac{2}{e^j}$ is equivalent to $e^j + 1 \leq 2e^j$, which is equivalent to $1 \leq e^j$. Thus, the given series $\sum_{j=1}^{\infty} \frac{1 + e^{-j}}{e^j}$ is dominated by $\sum_{j=1}^{\infty} \frac{2}{e^j}$ and so it converges.

33. We use the divergent p series $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}}$ for the limit comparison test. Since $\lim_{n\to+\infty} \frac{\frac{1}{\sqrt[3]{n^2 + 5}}}{\frac{1}{\sqrt[3]{n^2}}} =$

$$\lim_{n\to+\infty} \sqrt[3]{\frac{n^2}{n^2 + 5}} = \lim_{n\to+\infty} \sqrt[3]{\frac{1}{1 + \frac{5}{n^2}}} = 1, \text{ then}$$

$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2 + 5}}$ diverges.

34. We use the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ for the limit comparison test. Since $\lim_{n\to+\infty} \frac{\frac{1}{3\cdot 2^n + 2}}{\frac{1}{2^n}} =$

$$\lim_{n\to+\infty} \frac{2^n}{3\cdot 2^n + 2} = \lim_{n\to+\infty} \frac{1}{3 + \frac{2}{2^n}} = \frac{1}{3}, \text{ then } \sum_{k=1}^{\infty} \frac{1}{3\cdot 2^k + 2}$$

converges.

35. We use the convergent p series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ for the limit comparison test. Since

$$\lim_{n\to+\infty} \frac{\frac{5n^2}{(n + 1)(n + 2)(n + 3)(n + 4)}}{\frac{1}{n^2}} =$$

$$\lim_{n\to+\infty} \frac{5n^4}{(n + 1)(n + 2)(n + 3)(n + 4)} =$$

$$\lim_{n\to+\infty} \frac{5n^4}{(n^2 + 3n + 2)(n^2 + 7n + 12)} =$$

$$\lim_{n\to+\infty} \frac{5}{(1 + \frac{3}{n} + \frac{2}{n^2})(1 + \frac{7}{n} + \frac{12}{n^2})} = 5, \text{ then}$$

$\sum_{k=1}^{\infty} \frac{5k^2}{(k + 1)(k + 2)(k + 3)(k + 4)}$ converges.

36. We use the convergent geometric series $\sum_{k=1}^{\infty} (\frac{e}{5})^j$ for the limit comparison test. Since, $\lim_{n\to+\infty} \frac{\frac{1 + e^n}{n + 5^n}}{(\frac{e}{5})^n} =$

$$\lim_{n\to+\infty} \frac{5^n + 5^n e^n}{ne^n + 5^n e^n} = \lim_{n\to+\infty} \frac{\frac{1}{e^n} + 1}{\frac{n}{5^n} + 1} = 1, \text{ then } \sum_{j=1}^{\infty} \frac{1 + e^j}{j + 5^j}$$

converges. (Note that $\lim_{n\to+\infty} \frac{n}{5^n} = \lim_{x\to+\infty} \frac{x}{5^x} = \lim_{x\to+\infty} \frac{1}{(\ln 5)5^x} = 0$.)

37. We use the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ for the limit comparison test. Since $\lim_{n\to+\infty} \frac{\frac{n^2}{1 + n^3}}{\frac{1}{n}} =$

$$\lim_{n\to+\infty} \frac{n^3}{1 + n^3} = \lim_{n\to+\infty} \frac{1}{\frac{1}{n^3} + 1} = 1, \text{ then the given}$$

series diverges.

38. We use the convergent p series $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ for the

limit comparison test. Since $\lim_{n\to+\infty} \dfrac{\dfrac{1}{n\sqrt{2n^3+5}}}{\dfrac{1}{n^{5/2}}} =$

$\lim_{n\to+\infty} \dfrac{n^{5/2}}{\sqrt{2n^5+5n^2}} = \lim_{n\to+\infty} \sqrt{\dfrac{n^5}{2n^5+5n^2}} = \lim_{n\to+\infty} \sqrt{\dfrac{1}{2+\dfrac{5}{n^3}}} =$

$\sqrt{\tfrac{1}{2}}$, then the given series converges.

39. Comparison Test: Clearly, $\dfrac{k-1}{k\cdot 2^k} < \dfrac{1}{2^k}$, and $\sum_{k=2}^{\infty} \dfrac{1}{2^k}$ is a convergent geometric series with $a = \dfrac{1}{4}$ and $r = \dfrac{1}{2}$. Therefore, $\sum_{k=2}^{\infty} \dfrac{k-1}{k\cdot 2^k}$ converges also.

40. Define $f(x) = \dfrac{1}{x(\ln x)^2}$. $f(x)$ is continuous, decreasing, and nonnegative on $[2,\infty)$. Now

$\int_2^{\infty} \dfrac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \dfrac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \dfrac{-1}{\ln x}\Big|_2^b =$

$\lim_{b\to\infty} -\left[\dfrac{1}{\ln b} - \dfrac{1}{\ln 2}\right] = \dfrac{1}{\ln 2}$. Thus, $\int_2^{\infty} \dfrac{dx}{x(\ln x)^2}$ converges, so $\sum_{k=2}^{\infty} \dfrac{1}{k(\ln k)^2}$ converges also.

41. Comparison Test: $\dfrac{1}{(3k-1)3^k} < \dfrac{1}{3^k}$ for $k \geq 1$, and $\sum_{k=1}^{\infty} \dfrac{1}{3^k}$ is a convergent geometric series. Therefore, $\sum_{k=1}^{\infty} \dfrac{1}{(3k-1)3^k}$ converges also.

42. We compare the given series with the convergent geometric series $\sum_{k=1}^{\infty} 2(3/5)^k$. Now $\dfrac{1+3^k}{1+5^k} < \dfrac{2\cdot 3^k}{1+5^k} < \dfrac{2\cdot 3^k}{5^k}$. Thus, $\sum_{k=1}^{\infty} 2(3/5)^k$ dominates $\sum_{k=1}^{\infty} \dfrac{1+3^k}{1+5^k}$, so the given series converges.

43. Comparison Test: $\dfrac{1}{1+7^k} < \dfrac{1}{7^k}$, and $\sum_{k=1}^{\infty} \dfrac{1}{7^k}$ is a convergent geometric series. Therefore, $\sum_{k=1}^{\infty} \dfrac{1}{1+7^k}$ converges also.

44. We compare the given series with the divergent p series $\sum_{k=1}^{\infty} \dfrac{1}{3\sqrt{k}}$. Now $\dfrac{5\sqrt{k}}{2k+5} > \dfrac{\sqrt{k}}{2k+5} > \dfrac{\sqrt{k}}{3k} = \dfrac{1}{3\sqrt{k}}$ for $k > 5$, so $\sum_{k=1}^{\infty} \dfrac{5\sqrt{k}}{2k+5}$ eventually dominates $\sum_{k=1}^{\infty} \dfrac{1}{3\sqrt{k}}$. Thus, the given series diverges.

45. Consider $\lim_{k\to\infty} a_k = \lim_{k\to\infty} e^{-1/k} = y$. Since $f(x) = e^{-1/x}$ is continuous for $x \geq 1$, $\ln y = \lim_{x\to\infty} \ln(e^{-1/x}) = \lim_{x\to\infty} -1/x = 0$, so $y = e^0 = 1$. Thus, $\lim_{k\to\infty} a_k = 1 \neq 0$, so the series diverges.

46. Define $f(x) = x^3e^{-x^4}$. Then $f(x)$ is continuous, decreasing, and nonnegative on $[1,\infty)$. Now

$\int_1^{\infty} x^3e^{-x^4}\,dx = \lim_{b\to\infty} \int_1^b x^3e^{-x^4}\,dx = \lim_{b\to\infty} -\dfrac{1}{4}e^{-x^4}\Big|_1^b =$

$\lim_{b\to\infty} -\dfrac{1}{4}\left[e^{-b^4} - e^{-1}\right] = \dfrac{1}{4e}$. Thus, $\sum_{k=1}^{\infty} k^3e^{-k^4}$ converges by the integral test.

47. Limit Comparison Test: Let $b_n = \dfrac{1}{k}$ and consider the limit $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\dfrac{n}{n^2+n-2}}{\dfrac{1}{n}} = \lim_{n\to\infty} \dfrac{n^2}{n^2+n-2} =$ $1 > 0$. Since $\sum_{k=2}^{\infty} \dfrac{1}{k}$ diverges, so does $\sum_{k=2}^{\infty} \dfrac{k}{(k-1)(k+2)}$.

48. We use the limit comparison test with the divergent harmonic series $\sum_{k=5}^{\infty} \dfrac{1}{k}$. Now $\lim_{n\to\infty} \dfrac{\dfrac{2\sqrt{n}+3}{\sqrt{n^3-5n^2+1}}}{\dfrac{1}{n}} =$

$\lim_{n\to\infty} \dfrac{2n\sqrt{n}+3n}{\sqrt{n^3-5n^2+1}} = \lim_{n\to\infty} \dfrac{2+3/\sqrt{n}}{\sqrt{1-5/n+1/n^3}} = 2 > 0$, so the given series diverges.

49. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2+1}{n^2+4} = 1 \neq 0$, $\sum_{k=1}^{\infty} \dfrac{n^2+1}{n^2+4}$ diverges.

50. We use the limit comparison test with the convergent p series $\sum_{k=1}^{\infty} \dfrac{1}{16k^4}$. Now $\lim_{n\to\infty} \dfrac{16n^4}{(8n^3+7n+1)^{4/3}} =$

$\lim_{n\to\infty} \left(\dfrac{8n^3}{8n^3+7n+1}\right)^{4/3} = 1^{4/3} = 1 > 0$, so the given series diverges.

51. We use the convergent geometric series $\sum_{j=1}^{\infty} \dfrac{1}{7^j}$ for

the limit comparison test. Since $\lim_{n\to+\infty} \dfrac{\dfrac{1}{7^n - \cos n}}{\dfrac{1}{7^n}} =$

$\lim_{n\to+\infty} \dfrac{7^n}{7^n - \cos n} = \lim_{n\to+\infty} \dfrac{1}{1 - \dfrac{\cos n}{7^n}} = 1$ (since

$-1 \le \cos n \le 1$ and 7^n becomes large without bound),

then the given series converges.

52. We use the convergent p series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ for the modified limit comparison test. Since $\lim_{n\to+\infty} \dfrac{\dfrac{\ln n}{n^2+4}}{\dfrac{1}{n^{3/2}}} =$

$\lim_{n\to+\infty} \dfrac{(\ln n)n^{3/2}}{n^2 + 4} = \lim_{n\to+\infty} \dfrac{\ln n}{n^{\frac{1}{2}} + 4n^{-3/2}} =$

$\lim_{x\to+\infty} \dfrac{\ln x}{\sqrt{x} + 4x^{-3/2}} = \lim_{x\to+\infty} \dfrac{\dfrac{1}{x}}{\dfrac{1}{2\sqrt{x}} - 6x^{-5/2}} =$

$\lim_{x\to+\infty} \dfrac{1}{\dfrac{\sqrt{x}}{2} - \dfrac{6}{x^{3/2}}} = 0$, then the given series converges.

53. Limit Comparison Test: Let $b_n = \frac{1}{n^2}$ and consider

$\lim_{n\to\infty} \left(\dfrac{\dfrac{1}{n\sqrt{n^2 - 1}}}{\dfrac{1}{n^2}} \right) = \lim_{n\to\infty} \dfrac{n}{\sqrt{n^4 - n^2}} = \lim_{n\to\infty} \dfrac{1}{\sqrt{1 - 1/n^2}} =$

$1 > 0$. Since $\sum_{k=2}^{\infty} \frac{1}{k^2}$ is a convergent p series,

$\sum_{k=2}^{\infty} \dfrac{1}{k\sqrt{k^2 - 1}}$ converges also.

54. Since $\ln(1 + 1/k) = \ln(\frac{k+1}{k}) = \ln(k + 1) - \ln k$, the series can be rewritten as a telescoping series $\sum_{k=1}^{\infty} [\ln(k + 1) - \ln k]$ whose n^{th} partial sum is

$s_n = \ln(n + 1) - \ln(1) = \ln(n + 1)$. Since

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n + 1) = +\infty$, the series diverges.

55. Limit Comparison Test: Let $b_n = \frac{1}{n^8}$. $\lim_{n\to\infty} \frac{a^n}{b^n} =$

$\lim_{n\to\infty} \dfrac{\dfrac{2n+1}{(n^3+1)^3}}{\dfrac{1}{n^8}} = \lim_{n\to\infty} \dfrac{2n^9 + n^8}{n^9 + 3n^6 + 3n^3 + 1} = 2 > 0.$

Since $\sum_{k=1}^{\infty} \frac{1}{k^8}$ is a convergent p series,

$\sum_{k=1}^{\infty} \dfrac{2k + 1}{(k^3 + 1)^3}$ converges also.

56. We compare the series $\sum_{k=1}^{\infty} \frac{k+3}{k!}$ with the convergent geometric series $\sum_{k=1}^{\infty} \frac{10}{2^k}$. Now $\frac{k+3}{k!} \le \frac{10}{2^k}$ is equivalent to $2^k(k + 3) \le 10k!$ which holds for all $k \ge 1$, since by inspection $\underbrace{2\cdot 2 \ldots 2}_{k \text{ factors}}(k + 3) \le 10\cdot k\cdot(k - 1)\ldots 3\cdot 2$ (or we could prove it by induction). Hence, the series $\sum_{k=1}^{\infty} \frac{k+3}{k!}$ is dominated by $\sum_{k=1}^{\infty} \frac{10}{2^k}$ and so converges.

57. Limit Comparison Test: Let $b_n = \frac{1}{\sqrt{n}}$. $\lim_{n\to\infty} \frac{a_n}{b_n} =$

$\lim_{n\to\infty} \dfrac{\dfrac{\sqrt{n}}{n+1}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \dfrac{n}{n+1} = 1 > 0.$ Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p series, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ diverges also.

58. We use the comparison test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{3^k}$. Here $0 \le \sin^2 k \le 1$, $\frac{\sin^2 k}{3^k} < \frac{1}{3^k}$; so the given series is dominated by $\sum_{k=1}^{\infty} \frac{1}{3^k}$, and hence converges.

59. Limit Comparison Test: Let $b_n = \frac{1}{n^{4/3}}$. $\lim_{n\to\infty} \frac{a_n}{b_n} =$

$\lim_{n\to\infty} \dfrac{\dfrac{1}{\sqrt[3]{n(n^2+1)(2n-1)}}}{\dfrac{1}{n^{4/3}}} =$

$\lim_{n\to\infty} \dfrac{\sqrt[3]{n^4}}{\sqrt[3]{2n^4 - n^3 + 2n^2 - n}} \Big/ \left(\dfrac{\sqrt[3]{n^4}}{\sqrt[3]{n^4}}\right) = \dfrac{1}{\sqrt[3]{2}} > 0.$

Since $\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$ is a convergent p series,

$\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{n(n^2 + 1)(2n - 1)}}$ converges also.

60. We use the limit comparison test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$. Now $\lim_{n\to\infty} \dfrac{\sin^{-1}(1/2^n)}{1/2^n} =$

$\lim_{x\to\infty} \dfrac{\sin^{-1}(2^{-x})}{2^{-x}} = \lim_{x\to\infty} \dfrac{\dfrac{(-\ln 2)2^{-x}}{\sqrt{1 - 2^{-2x}}}}{(-\ln 2)\cdot 2^{-x}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1 - 2^{-2x}}} =$

$1 > 0$; so the original series converges also.

61. Limit Comparison Test: Let $b_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$. Since the harmonic series $\sum_{k=1}^{\infty} 1/k$ diverges, so does $\sum_{k=1}^{\infty} \sin(1/k)$.

62. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 \sin^2(1/n) = \lim_{n\to\infty} \frac{\sin^2(1/n)}{(1/n)^2} = \lim_{t\to 0} \frac{\sin^2 t}{t^2}$, where $t = 1/n$, and $\lim_{t\to 0} \left(\frac{\sin t}{t}\right)^2 = 1 \neq 0$, so the series diverges.

63. Limit Comparison Test: Let $a_n = \sin(1/n)$ and $b_n = \tan(1/n)$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{\tan(1/n)} = \lim_{n\to\infty} \cos(1/n) = 1 > 0$. Now $\sum_{k=1}^{\infty} \sin(1/k)$ diverges by Problem 61, so $\sum_{k=1}^{\infty} \tan(1/k)$ diverges also.

64. We use the convergent p series $\sum_{j=1}^{\infty} \frac{1}{j^2}$ for the modified limit comparison test. Since $\lim_{n\to+\infty} \frac{\frac{n!}{(2n)!}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{\frac{n!}{(2n)(2n-1)\ldots(n+1)(n!)}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2}{(2n)(2n-1)(2n-2)\ldots(n+1)} = \lim_{n\to+\infty} \frac{1}{2(2-\frac{1}{n})(2n-2)\ldots(n+1)} = 0$, then the given series converges.

65. (a) By the mean value theorem for integrals, there exists a c, with $k - 1 \leq c \leq k$, such that $f(c)[k-(k-1)] = \int_{k-1}^{k} f(x)dx$, or $f(c) = \int_{k=1}^{k} f(x)dx$.

(b) $k \geq c \geq k - 1$ and f is decreasing.

(c) By parts (a) and (b), $f(k) \leq \int_{k-1}^{k} f(x)dx \leq f(k-1)$.

66. (a) By part (c) of Problem 65, we know that $f(2) \leq \int_1^2 f(x)dx$, $f(3) \leq \int_2^3 f(x)dx, \ldots, f(n) \leq \int_{n-1}^{n} f(x)dx$. Thus, $f(2) + f(3) + \ldots + f(n) \leq \int_1^2 f(x)dx + \int_2^3 f(x)dx + \ldots + \int_{n-1}^{n} f(x)dx$ or $\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)dx$.

(b) By part (c) of Problem 65, we know that $\int_1^2 f(x)dx \leq f(1)$, $\int_2^3 f(x)dx \leq f(2), \ldots, \int_n^{n+1} f(x)dx \leq f(n)$. Thus, $\int_1^2 f(x)dx + \int_2^3 f(x)dx + \ldots + \int_n^{n+1} f(x)dx \leq f(1) + f(2) + \ldots + f(n)$ or $\int_1^{n+1} f(x)dx \leq \sum_{k=1}^{n} f(k) = \sum_{k=2}^{n+1} f(k-1)$.

(c) By part (b), $\int_1^{n+1} f(x)dx \leq \sum_{k=1}^{n} f(k) = f(1) + \sum_{k=2}^{n} f(k) \leq f(1) + \int_1^n f(x)dx$ by part (a).

67. Since $\int_1^{n+1} f(x)dx \leq \sum_{k=1}^{n} f(k) \leq f(1) + \int_1^n f(x)dx$, then $\lim_{n\to+\infty} \int_1^{n+1} f(x)dx \leq \lim_{n\to+\infty} \sum_{k=1}^{n} f(k) \leq f(1) + \lim_{n\to+\infty} \int_1^n f(x)dx$ or $\int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} f(k) \leq f(1) + \int_1^{\infty} f(x)dx$.

68. By Problem 67, $\int_1^{\infty} \frac{1}{x^2+1} dx \leq \sum_{k=1}^{\infty} \frac{1}{k^2+1} \leq f(1) + \int_1^{\infty} \frac{1}{x^2+1} dx$. Thus, $\lim_{b\to+\infty} \int_1^b \frac{1}{x^2+1} dx \leq \sum_{k=1}^{\infty} \frac{1}{k^2+1} \leq \frac{1}{2} + \lim_{b\to+\infty} \int_1^b \frac{1}{x^2+1} dx$. Now $\lim_{b\to+\infty} \int_1^b \frac{1}{x^2+1} dx = \lim_{b\to+\infty} \tan^{-1}x\Big|_1^b = \lim_{b\to+\infty} (\tan^{-1}b - \tan^{-1}1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. Thus, $\frac{\pi}{4} \leq \sum_{k=1}^{\infty} \frac{1}{k^2+1} \leq \frac{1}{2} + \frac{\pi}{4}$.

69. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent since it is the p series with $p > 1$, yet $\sum_{k=1}^{\infty} \frac{1}{k}$ - the harmonic series - is divergent.

70. Suppose $\sum_{k=1}^{\infty} b_k$ eventually dominates $\sum_{k=1}^{\infty} a_k$; that is $\sum_{k=M}^{\infty} b_k$ dominates $\sum_{k=M}^{\infty} a_k$. Now $\sum_{k=1}^{\infty} b_k$ converges if and only if $\sum_{k=M}^{\infty} b_k$ converges. Thus, $\sum_{k=M}^{\infty} a_k$ converges, and it converges if and only if $\sum_{k=1}^{\infty} a_k$ converges. Now if $\sum_{k=1}^{\infty} b_k$ eventually dominates $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges, otherwise we have the previous situation and a contradiction.

71. Suppose $s_n = \sum_{k=1}^{n} a_k$ and $\{s_n\}$ is bounded. Then, by Theorem 6 of Section 11.3, the series $\sum_{k=1}^{\infty} a_k$ is convergent. This contradicts the fact that $\sum_{k=1}^{\infty} a_k$ is divergent. Thus, $\{s_n\}$ is not bounded and s_n becomes large without bound as $n \to +\infty$.

72. Suppose $\lim_{n\to+\infty} \frac{a_n}{b_n} = 0$ and $\sum_{k=1}^{\infty} b_k$ converges. Given any $\varepsilon > 0$, say 1, there exists N such that $\frac{a_n}{b_n} < 1$ for $n > N$. Thus, $a_n < b_n$ for $n > N$, so that $\sum_{k=1}^{\infty} b_k$ eventually dominates $\sum_{k=1}^{\infty} a_k$ and so $\sum_{k=1}^{\infty} a_k$ converges. Now suppose $\lim_{n\to+\infty} \frac{a_n}{b_n} = +\infty$ and $\sum_{k=1}^{\infty} b_k$ diverges. For n large enough, $1 < \frac{a_n}{b_n}$ since $\lim_{n\to+\infty} \frac{a_n}{b_n} = +\infty$. Thus, $b_n < a_n$, and so $\sum_{k=1}^{\infty} b_k$ is dominated by $\sum_{k=1}^{\infty} a_k$. Hence, $\sum_{k=1}^{\infty} a_k$ diverges.

73. For $m \le k - 1 < k \le M$, we have $f(k) \le \int_{k-1}^{k} f(x)dx \le f(k - 1)$. Thus, $f(m + 1) \le \int_{m}^{m+1} f(x)dx \le f(m)$, $f(m + 2) \le \int_{m+1}^{m+2} f(x)dx \le f(m + 1)$, $f(m + 3) \le \int_{m+2}^{m+3} f(x)dx \le f(m + 2),\ldots,f(M) \le \int_{M-1}^{M} f(x)dx \le f(M - 1)$. Adding these inequalities, we obtain $\sum_{k=m+1}^{M} f(k) \le \int_{m}^{M} f(x)dx \le \sum_{k=m}^{M-1} f(k)$. Adding $f(m)$ to the first inequality, we find that $\sum_{k=m}^{M} f(k) \le f(m) + \int_{m}^{M} f(x)dx$, while adding $f(M)$ to the second inequality, we find that $f(M) + \int_{m}^{M} f(x)dx \le \sum_{k=m}^{M} f(k)$. Therefore, $f(M) + \int_{m}^{M} f(x)dx \le \sum_{k=m}^{M} f(k) \le f(m) + \int_{m}^{M} f(x)dx$.

Problem Set 11.5, page 678

1. $\left\{\frac{1}{n^2}\right\}$ is a decreasing sequence of positive terms and $\lim_{n\to+\infty} \frac{1}{n^2} = 0$. Hence, the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ converges by Leibniz's theorem.

2. $\left\{\frac{1}{(2n)!}\right\}$ is a decreasing sequence of positive terms and $\lim_{n\to+\infty} \frac{1}{(2n)!} = 0$. Hence, the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!}$ converges by Leibniz's theorem.

3. Let f be defined by $f(x) = \frac{x}{x^3 + 2}$. Now $f'(x) = \frac{-2x^3 + 2}{(x^3 + 2)^2}$, so f is decreasing for $x \ge 1$. Thus, the sequence $\{a_n\} = \left\{\frac{n}{n^3 + 2}\right\}$ is decreasing for $n \ge 1$; also, each a_n is positive. Now $\lim_{n\to+\infty} \frac{n}{n^3 + 2} = \lim_{n\to+\infty} \frac{1}{n^2 + \frac{2}{n}} = 0$. Hence, the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{k^3 + 2}$ converges by Leibniz's theorem.

4. $\left\{\frac{1}{n^3}\right\}$ is a decreasing sequence of positive terms and $\lim_{n\to\infty} \frac{1}{n^3} = 0$. Hence, the alternating series $\sum_{k=1}^{\infty} \frac{-\cos k\pi}{k^3}$ converges by Leibniz's theorem.

5. Consider $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{\sqrt{k^5+7}}$ and define $f(x) = \frac{x}{\sqrt{x^5+7}}$. Now $f'(x) = \frac{-3x^5+14}{(x^5+7)^{3/2}} < 0$ for $x \geq 2$. Thus $\{a_n\} = \left\{\frac{n}{\sqrt{n^5+7}}\right\}$ is decreasing for $n \geq 2$. But $a_1 = \frac{1}{\sqrt{8}}$ and $a_2 = \frac{2}{\sqrt{39}}$, and $a_1 < a_2$. Hence, $\{a_n\}$ is decreasing for all $n \geq 1$. Each a_n is positive and $\lim_{n\to+\infty} a_n = 0$. Hence, the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{\sqrt{k^5+7}}$ converges by Leibniz's theorem, and so $-\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{\sqrt{k^5+7}} = \sum_{k=1}^{\infty} \frac{(-1)^k k}{\sqrt{k^5+7}}$ converges.

6. Consider $\sum_{k=5}^{\infty} \frac{(-1)^{k+1}}{k^2-10k+26}$. Now $k^2 - 10k + 26 = (k-5)^2 + 1 \geq 0$ for all k, and so $\frac{1}{k^2-10k+26}$ is positive for all k. Now for $f(x) = \frac{1}{x^2-10x+26}$, $f'(x) = \frac{-(2x-10)}{(x^2-10x+26)^2} \leq 0$ for $x \geq 5$. Hence, $\left\{\frac{1}{n^2-10n+26}\right\}$ is decreasing for $n \geq 5$. Also, $\lim_{n\to+\infty} \frac{1}{n^2-10n+26} = 0$. Thus, by Leibniz's theorem, the alternating series $\sum_{k=5}^{\infty} \frac{(-1)^{k+1}}{k^2-10k+26}$ converges, and so does $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2-10k+26}$ since a finite number of terms does not affect the convergence of a series.

7. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k+7}$ diverges since $\lim_{n\to+\infty} \frac{n+1}{n+7} = 1 \neq 0$.

8. $\sum_{k=1}^{\infty} (-1)^k \frac{3k^2}{4k^2+1}$ diverges since $\lim_{n\to+\infty} \frac{3n^2}{4n^2+1} = \frac{3}{4} \neq 0$.

9. Since the terms corresponding to k = 0 will not affect the convergence of the series, it is enough to consider $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(k+2)}$. $\left\{\frac{1}{\ln(n+2)}\right\}$ is a decreasing sequence of positive terms, and $\lim_{n\to+\infty} \frac{1}{\ln(n+2)} = 0$. Hence, by Leibniz's theorem, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(k+2)}$ converges, and so does $\sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}$.

10. Consider $f(x) = \frac{\ln(x+1)}{x\sqrt{x}}$. Now $f'(x) = \frac{x - \frac{3}{2}(x+1)\ln(x+1)}{x^{5/2}}$, and $f'(x) \leq 0$ for all x. Thus, $\left\{\frac{\ln(n+1)}{n\sqrt{n}}\right\}$ is a decreasing sequence of positive terms. Also, $\lim_{n\to+\infty} \frac{\ln(n+1)}{n\sqrt{n}} = \lim_{x\to+\infty} \frac{\ln(x+1)}{x^{3/2}} = \lim_{x\to+\infty} \frac{\frac{1}{x+1}}{\frac{3}{2}\sqrt{x}} = 0$. Hence, by Leibniz's theorem, the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln(k+1)}{k\sqrt{k}}$ converges.

11. Consider $\sum_{k=2}^{\infty} (-1)^{k+1} \sin \frac{\pi}{k}$. For $n \geq 2$, $\{\sin \frac{\pi}{n}\}$ is a decreasing sequence of positive terms, and $\lim_{n\to+\infty} \sin \frac{\pi}{n} = \lim_{n\to+\infty} \frac{\frac{\pi}{n} \sin \frac{\pi}{n}}{\frac{\pi}{n}} = (\lim_{x\to+\infty} \frac{\pi}{x}) \lim_{x\to+\infty} \frac{\sin \frac{\pi}{x}}{\frac{\pi}{x}} = (0)(1) = 0$. Hence, by Leibniz's theorem, $\sum_{k=2}^{\infty} (-1)^{k+1} \sin \frac{\pi}{k}$ converges, and so does $\sum_{k=1}^{\infty} (-1)^{k+1} \sin \frac{\pi}{k}$ since one term does not affect the convergence of a series.

12. $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{k}{\ln k}$ diverges since $\lim_{n\to+\infty} \frac{n}{\ln n} = \lim_{x\to+\infty} \frac{x}{\ln x} = \lim_{x\to+\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to+\infty} x = +\infty \neq 0$.

13. Consider $\sum_{k=3}^{\infty} \frac{(-1)^k\sqrt{k}}{k+3}$. Clearly $\frac{\sqrt{k}}{k+3}$ is positive for $k \geq 1$. Define $f(x) = \frac{\sqrt{x}}{x+3}$. Then $f'(x) = \frac{3-x}{2\sqrt{x}(x+3)^2} \leq 0$ for $x \geq 3$. Thus, $\{\frac{\sqrt{n}}{n+3}\}$ is a decreasing sequence for $n \geq 3$, and $\lim_{n\to+\infty} \frac{\sqrt{n}}{n+3} = \lim_{n\to+\infty} \frac{1}{\sqrt{n} + \frac{3}{\sqrt{n}}} = 0$. Hence, by Leibniz's theorem, the

alternating series $\sum_{k=3}^{\infty} \frac{(-1)^k\sqrt{k}}{k+3}$ converges, and so does $\sum_{k=1}^{\infty} \frac{(-1)^k\sqrt{k}}{k+3}$ since a finite number of terms does not affect the convergence of a series.

14. $\sum_{k=1}^{\infty} (\ln k)\cos k\pi$ diverges, since $\lim_{n\to+\infty} (\ln k)\cos n\pi \neq 0$.

15. $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(n+1)^2(3/7)^{n+1}}{n^2(3/7)^n}\right| =$
$\lim_{n\to+\infty} \frac{3}{7} \cdot \frac{(n+1)^2}{n^2} = \frac{3}{7} < 1$, so the series $\sum_{k=1}^{\infty} k^2(3/7)^k$ converges absolutely.

16. $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(-1)^{n+2}3^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^{n+1}3^n}\right| =$
$\lim_{n\to+\infty} \frac{3}{n+1} = 0 < 1$. Hence, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}3^k}{k!}$ converges absolutely.

17. $\lim_{n\to+\infty} \left|\frac{\left[\frac{(-1)^{n+2}5^{n+1}}{(n+1)4^{n+1}}\right]}{\left[\frac{(-1)^{n+1}5^n}{n\cdot 4^n}\right]}\right| = \lim_{n\to+\infty} \frac{5^{n+1}}{(n+1)4^{n+1}} \cdot \frac{4^n(n)}{5^n} =$
$\lim_{n\to+\infty} \frac{5n}{4(n+1)} = \frac{5}{4} > 1$. Hence, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}5^k}{k\cdot 4^k}$ diverges by the ratio test.

18. $\lim_{n\to+\infty} \left|\frac{\frac{(-1)^{n+2}[(n+1)^3+1]}{(n+1)!}}{\frac{(-1)^{n+1}(n^3+1)}{n!}}\right| =$
$\lim_{n\to+\infty} \frac{n^3+3n^2+3n+2}{(n^3+1)(n+1)} = \lim_{n\to+\infty} \frac{1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}}{(1+\frac{1}{n^3})(n+1)} =$
$0 < 1$, so $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k^3+1)}{k!}$ converges absolutely by the ratio test.

19. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n}\right| = \lim_{n\to\infty} \frac{1}{e}\frac{n+1}{n} =$
$\frac{1}{e} < 1$, so $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{e^k}$ converges absolutely by the ratio test.

20. $\lim_{n\to\infty} \left|\frac{\frac{1}{(5n+7)3^{n+1}}}{\frac{1}{(5n+2)3^n}}\right| = \lim_{n\to\infty} \left|\frac{5n+2}{3(5n+7)}\right| = \frac{5}{15} < 1$, so
$\sum_{k=1}^{\infty} \frac{1}{(5k+2)3^k}$ converges absolutely by the ratio test.

21. $\lim_{n\to\infty} \left|\frac{(n+1)!}{(2n+2)!} \cdot \frac{2n!}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = 0 < 1$,
so $\sum_{k=1}^{\infty} \frac{k!}{(2k)!}$ converges absolutely by the ratio test.

22. $\lim_{n\to\infty} \left|\frac{(n+2)!}{7^{n+1}} \cdot \frac{7^n}{(n+1)!}\right| = \lim_{n\to\infty} \frac{n+2}{7} = +\infty$, so
$\sum_{k=1}^{\infty} \frac{(k+1)!}{7^k}$ diverges by the ratio test.

23. $\lim_{n\to+\infty} \left|\frac{\frac{(-1)^{n+2}7^{n+1}}{(3n+3)!}}{\frac{(-1)^{n+1}7^n}{(3n)!}}\right| = \lim_{n\to+\infty} \frac{7}{(3n+3)(3n+2)(3n+1)} =$
$0 < 1$, and the given series converges absolutely by the ratio test.

24. $\lim_{n\to+\infty} \left|\frac{\frac{(-1)^{n+1}(2n+1)!}{e^{n+1}}}{\frac{(-1)^n(2n-1)!}{e^n}}\right| = \lim_{n\to+\infty} \frac{(2n+1)(2n)}{e} = +\infty$,
so the series $\sum_{k=1}^{\infty} \frac{(-1)^k(2k-1)!}{e^k}$ diverges by the ratio test.

25. $\lim_{n\to+\infty} \left|\frac{\frac{(-1)^{n+2}(n+1)^4}{(1.02)^{n+1}}}{\frac{(-1)^{n+1}(n)^4}{(1.02)^n}}\right| = \lim_{n\to+\infty} \left(\frac{n+1}{n}\right)^4\left(\frac{1}{1.02}\right) =$
$\frac{1}{1.02} < 1$, so the given series converges absolutely by the ratio test.

26. $\lim_{n\to+\infty} \left|\frac{\left[\frac{(-1)^{n+1}(1+e^{n+1})}{2^{n+1}}\right]}{\left[\frac{(-1)^n(1+e^n)}{2^n}\right]}\right| = \lim_{n\to+\infty} \frac{1}{2}\left(\frac{1+e^{n+1}}{1+e^n}\right) =$
$\lim_{n\to+\infty} \frac{1}{2}\left(\frac{e^{-n}+e}{e^{-n}+1}\right) = \frac{e}{2} > 1$; hence, the series diverges by the ratio test.

27. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{8n/n^n} = \lim_{n\to\infty} 8/n = 0 < 1$, so $\sum_{k=1}^{\infty} \frac{8^k}{k^k}$ converges absolutely by the root test.

28. $\lim_{n\to\infty} \sqrt[n]{|\frac{1}{2} + \frac{1}{n}|^n} = \lim_{n\to\infty} (\frac{1}{2} + \frac{1}{n}) = \frac{1}{2} < 1$, so $\sum_{k=1}^{\infty} (\frac{1}{2} + \frac{1}{k})^k$ converges absolutely by the root test.

29. $\lim_{n\to\infty} \sqrt[n]{(\frac{7n}{5n+1})^n} = \lim_{n\to\infty} \frac{7n}{5n+1} = 7/5 > 1$, so $\sum_{k=1}^{\infty} (\frac{7k}{5k+1})^k$ diverges by the root test.

30. $\lim_{n\to\infty} \sqrt[n]{|n^n(2/3)^n|} = \lim_{n\to\infty} 2/3 \cdot n = +\infty$, so $\sum_{k=1}^{\infty} k^k(2/3)^k$ diverges by the root test.

31. $\lim_{n\to+\infty} \sqrt[n]{|(-1)^{n+1}(\frac{n}{3n+1})^n|} = \lim_{n\to+\infty} \frac{n}{3n+1} = \lim_{n\to+\infty} \frac{1}{3 + \frac{1}{n}} = \frac{1}{3} < 1$. Hence, the series converges absolutely by the root test.

32. $\lim_{n\to+\infty} \sqrt[n]{\left|\frac{(-1)^n n^n}{(\ln n)^n}\right|} = \lim_{n\to+\infty} \frac{n}{\ln n} = \lim_{x\to+\infty} \frac{x}{\ln x} = \lim_{x\to+\infty} \frac{1}{\frac{1}{x}} = +\infty$. Hence, the given series diverges by the root test.

33. $\lim_{n\to+\infty} \sqrt[n]{(\sqrt[n]{n} - 1)^n} = \lim_{n\to+\infty} (\sqrt[n]{n} - 1) = [\lim_{n\to+\infty} e^{\frac{1}{n}\ln n}] - 1$. But $\lim_{x\to+\infty} \frac{1}{x}\ln x = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$, and so $\lim_{n\to+\infty} e^{\frac{1}{n}\ln n} = e^0 = 1$. Hence, $\lim_{n\to+\infty} (\sqrt[n]{n} - 1) = 1 - 1 = 0 < 1$, and so the given series converges absolutely by the root test.

34. $\lim_{n\to+\infty} \sqrt[n]{\frac{n^n}{(2n + \frac{1}{n})^n}} = \lim_{n\to+\infty} \frac{n}{2n + \frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{2 + \frac{1}{n^2}} = \frac{1}{2} < 1$. Hence, the given series converges absolutely by the root test.

35. $\lim_{n\to+\infty} \left|\frac{(-1)^{n+1} \frac{3^{n+1}}{(n+1)!}}{(-1)^n \frac{3^n}{n!}}\right| = \lim_{n\to+\infty} \frac{3}{n+1} = 0 < 1$, so the given series converges absolutely by the ratio test.

36. $\lim_{n\to+\infty} \frac{(n+1)(\frac{3}{5})^{n+1}}{n(\frac{3}{5})^n} = \lim_{n\to+\infty} (\frac{n+1}{n})(\frac{3}{5}) = \frac{3}{5} < 1$, so given series converges absolutely by the ratio test.

37. $\lim_{n\to+\infty} \left|\frac{\frac{(-1)^{n+2}}{\ln(n+2)}}{\frac{(-1)^{n+1}}{\ln(n+1)}}\right| = \lim_{n\to+\infty} \frac{\ln(n+1)}{\ln(n+2)} = \lim_{x\to+\infty} \frac{\ln(x+1)}{\ln(x+2)} = \lim_{x\to+\infty} \frac{\frac{1}{x+1}}{\frac{1}{x+2}} = \lim_{x\to+\infty} \frac{x+2}{x+1} = 1$. The ratio test is inconclusive. Now, $\sum_{k=1}^{\infty} \frac{1}{\ln(k+1)}$ diverges by comparison with the harmonic series. By Leibniz's theorem, the given series converges, since $\{\frac{1}{\ln(n+1)}\}$ is a decreasing sequence of positive terms and $\lim_{n\to+\infty} \frac{1}{\ln(n+1)} = 0$. Hence, the given series is conditionally convergent.

38. We compare $\sum_{k=1}^{\infty} \frac{k^2}{k^3 + 10}$ with $\sum_{k=1}^{\infty} \frac{1}{100\,k}$ which diverges: $\frac{k^2}{k^3+10} > \frac{1}{100\,k}$ since $100\,k^3 > k^3 + 10$ and $99\,k^3 > 10$ for all k. Hence, $\sum_{k=1}^{\infty} \frac{k^2}{k^3+10}$ diverges. But $\sum_{k=3}^{\infty} \frac{(-1)^{k+1}k^2}{k^3+10}$ converges by Leibniz's theorem, and consequently $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^2}{k^3+10}$ converges, since two terms will not affect convergence. Hence, the given series is conditionally convergent.

39. Since $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 2$, then $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges, and hence $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges, too. But $\sum_{n=3}^{\infty} \frac{(-1)^{n+1}\ln n}{n}$ converges by Leibniz's theorem, and hence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln n}{n}$ converges. Thus, the given series converges conditionally.

40. $\lim_{n\to+\infty} \left|\frac{(-1)^{n+1} \frac{(n+1)!}{(2n+3)!}}{(-1)^n \frac{n!}{(2n+1)!}}\right| = \lim_{n\to+\infty} \frac{n+1}{(2n+3)(2n+2)} =$

$\lim_{n\to+\infty} \dfrac{1 + \frac{1}{n}}{4n + 8 + \frac{6}{n^2}} = 0 < 1$, and so the series is absolutely convergent by the ratio test.

41. We compare $\sum_{j=1}^{\infty} \dfrac{1}{j^2 + 1}$ with the convergent p series $\sum_{j=1}^{\infty} \dfrac{1}{j^2}$. Now $\dfrac{1}{j^2 + 1} < \dfrac{1}{j^2}$ for all j, since $j^2 + 1 > j^2$. Hence, the series $\sum_{j=1}^{\infty} \dfrac{1}{j^2 + 1}$ converges. Hence, $\sum_{j=1}^{\infty} \dfrac{(-1)^j}{j^2 + 1}$ is absolutely convergent.

42. $\lim_{n\to+\infty} \dfrac{\left[\frac{2\cdot4\cdot6 \cdots (2n)(2n+2)}{1\cdot4\cdot7\cdots(3n-2)(3n+1)}\right]}{\left[\frac{2\cdot4\cdot6 \cdots 2n}{1\cdot4\cdot7\cdots(3n-2)}\right]} = \lim_{n\to+\infty} \dfrac{2n + 2}{3n + 1} =$ $\dfrac{2}{3} < 0$. Hence, the given series converges absolutely.

43. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{(n+1)!}{1\cdot3\cdot5 \cdots (2n+1)} \cdot \dfrac{1\cdot3\cdot5\cdots(2n-1)}{n!} =$ $\lim_{n\to\infty} \dfrac{n+1}{2n+1} = 1/2 < 1$, so the series $\sum_{k=1}^{\infty} \dfrac{(-1)^k k!}{1\cdot3\cdot5\cdots(2k-1)}$ is absolutely convergent.

44. $\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{3} + \frac{1}{2n}\right|^n} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n}\right) = 1/3 < 1$, so $\sum_{k=1}^{\infty} \left(\frac{1}{3} + \frac{1}{2k}\right)^k$ converges absolutely by the root test.

45. $\lim_{n\to\infty} \dfrac{(n+1)!}{(n+1)^2(n+2)^2} \cdot \dfrac{n^2(n+1)^2}{n!} = \lim_{n\to\infty} \dfrac{(n+1)n^2}{(n+2)^2} =$ $+\infty$, so the series $\sum_{k=1}^{\infty} \dfrac{k!}{k^2(k+1)^2}$ diverges.

46. $\lim_{n\to\infty} \left|\dfrac{(2n+2)!}{4^{n+1}3^n} \cdot \dfrac{4^n 3^{n-1}}{(2n)!}\right| = \lim_{n\to\infty} \dfrac{(2n+2)(2n+1)}{4\cdot 3} =$ $+\infty$, so the series $\sum_{k=1}^{\infty} \dfrac{(2k)!}{4^k 3^{k-1}}$ diverges by the root test.

47. $\lim_{n\to\infty} \left|\dfrac{(n+3)!}{3^{n+1}(n+1)!} \cdot \dfrac{3^n n!}{(n+2)!}\right| = \lim_{n\to\infty} \dfrac{n+3}{3(n+1)} = \dfrac{1}{3} < 1$, so $\sum_{k=1}^{\infty} \dfrac{(k+2)!}{3^k k!}$ converges absolutely by the ratio test.

48. $\lim_{n\to\infty} \left|\dfrac{(n+1)^2(n+1)!}{(2n+2)!} \cdot \dfrac{(2n)!}{n^2 n!}\right| =$ $\lim_{n\to\infty} \dfrac{(n+1)^3}{n^2(2n+2)(2n+1)} = 0 < 1$, so the series $\sum_{k=1}^{\infty} \dfrac{k^2 k!}{(2k)!}$ converges absolutely by the ratio test.

49. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+2}{3n+1}\right)^n} = \lim_{n\to\infty} \dfrac{n+2}{3n+1} = \dfrac{1}{3} < 1$, so $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+2}{3k+1}\right)^k$ converges absolutely by the root test.

50. $\lim_{n\to\infty} \left|\dfrac{(-1)^{n+1}(4n+4)!}{[(n+1)!]^2} \cdot \dfrac{(n!)^2}{(-1)^n(4n)!}\right| =$ $\lim_{n\to\infty} \dfrac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^2} = +\infty$, so $\sum_{k=0}^{\infty} (-1)^k \dfrac{(4k)!}{(k!)^2}$ diverges by the ratio test.

51. $\lim_{n\to+\infty} \dfrac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to+\infty} \dfrac{(n+1)^{n+1}}{(n+1)(n^n)} =$ $\lim_{n\to+\infty} \dfrac{(n+1)^n}{n^n} = \lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$. The given series diverges.

52. $\lim_{n\to+\infty} \dfrac{\frac{[(n+1)!]^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to+\infty} \left[\frac{(n+1)!}{n!}\right]^2 \cdot \dfrac{(2n)!}{(2n+2)!} =$ $\lim_{n\to+\infty} \dfrac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to+\infty} \dfrac{n^2 + 2n + 1}{4n^2 + 6n + 2} =$ $\lim_{n\to+\infty} \dfrac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} = \dfrac{1}{4} < 0$. The given series converges absolutely.

53. $\lim_{n\to\infty} \left|\dfrac{(-1)^{n+1}e^{n+1}}{n+1} \cdot \dfrac{n}{(-1)^n e^n}\right| = \lim_{n\to\infty} e\left(\frac{n+1}{n}\right) = e > 1$, so the series $\sum_{k=1}^{\infty} (-1)^k \dfrac{e^k}{k}$ diverges by the ratio test.

54. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^n}{n!}\right)^n} = \lim_{n\to\infty} \dfrac{n^n}{n!} = +\infty$ by Problem 42, Section 11.1. Therefore, the series $\sum_{k=1}^{\infty} \left(\frac{k^k}{k!}\right)^k$ diverges by the root test.

55. $s_5 = \frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} = \frac{1249}{3080} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k-1}$. The absolute value of the error will not exceed $\frac{1}{17}$. The approximation $\frac{1249}{3080}$ overestimates $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k-1}$.

56. $s_{100} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots + \frac{1}{2^{99}} - \frac{1}{2^{100}} =$ $\frac{1}{2}\left[\frac{1-(-\frac{1}{2})^{100}}{1-(-\frac{1}{2})}\right] = \frac{2^{100}-1}{3(2^{100})} \approx \frac{1}{3} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$. The absolute value of the error will not exceed $\frac{1}{2^{101}}$. The approximation $\frac{2^{100}-1}{3(2^{100})}$ underestimates $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$.

57. $s_4 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} = \frac{115}{144} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. The absolute value of the error will not exceed $\frac{1}{25}$. The approximation $\frac{115}{144}$ underestimates the true value.

58. We consider $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3+1}$. For this series, $s_4 = \frac{1}{2} - \frac{1}{9} + \frac{1}{28} - \frac{1}{65} = \frac{6703}{16,380} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3+1}$. The absolute value of the error does not exceed $\frac{1}{126}$. Here, $\frac{6703}{16,380}$ underestimates $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3+1}$, so that $-\frac{6703}{16,380}$ overestimates $-\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3+1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3+1}$ with an error whose absolute value does not exceed $\frac{1}{126}$.

59. We consider $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 5^k}$. Here, $s_3 = \frac{1}{5} - \frac{1}{2(5^2)} + \frac{1}{3(5^3)} = \frac{137}{750} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 5^k}$ with an error which in absolute value does not exceed $\frac{1}{4(5^4)} = \frac{1}{2500}$. The approximation $\frac{137}{750}$ overestimates the sum of the series under consideration, and so $-\frac{137}{750}$ underestimates $-\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k \cdot 5^k}$ with an error whose absolute value does not exceed $\frac{1}{2500}$.

60. $\sum_{k=1}^{\infty} \frac{\sin(k+\frac{1}{2})\pi}{2k!} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k!}$. Now consider $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k!}$. Here, $s_3 = \frac{1}{2} - \frac{1}{4} + \frac{1}{12} = \frac{1}{3} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k!}$ with an error which in absolute value does not exceed $\frac{1}{48}$. The approximation $\frac{1}{3}$ overestimates $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k!}$, and so $-\frac{1}{3}$ underestimates $-\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k!} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k!}$ with an error whose absolute value does not exceed $\frac{1}{48}$.

61. $\frac{1}{n \cdot 2^n} \le \frac{5}{10^4}$ for $n = 8$. Hence, s_7 will approximate the sum of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 2^k}$ with the absolute value of the error not exceeding 5×10^{-4}. Now $s_7 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384} + \frac{1}{896} \approx 0.406$.

62. $\frac{n}{(2n)!} \le \frac{5}{10^4}$ for $n = 4$. Thus, s_3 will approximate the sum of the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{(2k)!}$ with an error in absolute value not exceeding 5×10^{-4}. $s_3 = -\frac{1}{2} + \frac{2}{24} - \frac{3}{720} = -\frac{1}{2} + \frac{1}{12} - \frac{1}{240} \approx -0.421$.

63. Since $\lim_{m\to+\infty} s_{2m} = S$, then $|S - s_n| < \varepsilon$ when n is even and bigger than some number, say N_0. Since $\lim_{m\to+\infty} s_{2m-1} = S$, then $|S - s_n| < \varepsilon$ for n odd and bigger than some number, say N_1. Now choose N bigger than both N_0 and N_1. If $n > N$, then $|S - s_n| < \varepsilon$ for n even and odd. Thus, $\lim_{n\to+\infty} s_n = S$.

64. (a) Suppose $j = 1$. Then, since $N + 1 > N$, then $|a_{N+1}| < |a_N|r$. Now suppose k is any positive integer and that $|a_{N+k}| < |a_N|r^k$. Then, since $N + k > N$, $|a_{(N+k)+1}| < |a_{(N+k)}| \cdot r < |a_N|r^k \cdot r = |a_N|r^{k+1}$. Hence, $|a_{N+j}| < |a_N|r^j$ holds for all positive integers j.

(b) The argument is exactly the same as that in (a) except that the inequality is reversed throughout.

65. Yes. We will show that $\frac{a_k^2}{1 + a_k^2} \le |a_k|$ for $k \ge 1$,

and by the comparison test $\sum_{k=1}^{\infty} \frac{a_k^2}{1 + a_k^2}$ does converge. Now, $\frac{a_k^2}{1 + a_k^2} \le |a_k|$ is equivalent to $a_k^2 \le |a_k| \cdot a_k^2 + |a_k|$. This inequality is true if $a_k = 0$. Now let $|a_k| = x$. We want to show that $x^2 < x^3 + x$ for $x > 0$. But $x < x^2 + 1$ means $x^2 - x + 1 > 0$ and $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4} > 0$ for all x. Hence, $\frac{a_k^2}{1 + a_k^2} \le |a_k|$ for $k \ge 1$.

66. Suppose $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L < 1$. Choose r with $L < r < 1$, and let $\varepsilon = r - L$. Then, since $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L$, there exists a positive integer N such that $|\sqrt[n]{|a_n|} - L| < \varepsilon$ for $n \ge N$; that is, $L - \varepsilon < \sqrt[n]{|a_n|} < L + \varepsilon = r$. Thus, $|a_n| < r^n$, and so the geometric series $\sum_{k=1}^{\infty} r^k$ dominates $\sum_{k=1}^{\infty} |a_k|$ and so $\sum_{k=1}^{\infty} |a_k|$ converges. Therefore, $\sum_{k=1}^{\infty} a_k$ converges absolutely. Now suppose $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L > 1$. Choose r so that $1 < r < L$, and put $\varepsilon = L - r$. Then there is an N such that $L - \varepsilon < \sqrt[n]{|a_n|} < L + \varepsilon$ for $n \ge N$. Hence, $r = L - \varepsilon < \sqrt[n]{|a_n|}$ and $r^n < |a_n|$. But $\lim_{n\to+\infty} r^n = +\infty$. Hence, $\lim_{n\to+\infty} |a_n| = +\infty$. Thus, the series $\sum_{k=1}^{\infty} a_k$ diverges. To see part (iii), we choose the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$. Now $\lim_{n\to+\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{x\to+\infty} \frac{1}{x^{2/x}} = \lim_{x\to+\infty} \frac{1}{e^{2/x \ln x}}$. But $\lim_{x\to+\infty} \frac{2 \ln x}{x} = \lim_{x\to+\infty} \frac{2}{x} = 0$, and so $\lim_{x\to+\infty} \frac{1}{e^{(2/x)\ln x}} = \frac{1}{e^0} = 1$. Now $\lim_{n\to+\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{n^{1/n}} = \lim_{x\to+\infty} \frac{1}{x^{1/x}} = \lim_{x\to+\infty} \frac{1}{e^{(1/x)\ln x}} = 1$ since $\lim_{x\to+\infty} \frac{\ln x}{x} = 0$. Thus the $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1$ does not imply convergence or divergence conclusively.

67. By the generalized triangle inequality (Problem 30, Section 5.1), $|\sum_{k=1}^{n} a_k| \le \sum_{k=1}^{n} |a_k|$, and $\sum_{k=1}^{n} |a_k| \le \sum_{k=1}^{\infty} |a_k|$ by the absolute convergence of $\sum_{k=1}^{\infty} a_k$ and Problem 28, Section 11.3. Thus, $|\sum_{k=1}^{n} a_k| \le \sum_{k=1}^{\infty} |a_k|$. Taking the limit, we have $\lim_{n\to+\infty} |\sum_{k=1}^{n} a_k| \le \lim_{n\to+\infty} \sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} |a_k|$. But $\lim_{n\to+\infty} |\sum_{k=1}^{n} a_k| = |\lim_{n\to+\infty} \sum_{k=1}^{n} a_k| = |\sum_{k=1}^{\infty} a_k|$. Therefore, $|\sum_{k=1}^{\infty} a_k| \le \sum_{k=1}^{\infty} |a_k|$.

Problem Set 11.6, page 685

1. The center is a = 0. $\lim_{n\to+\infty} \left|\frac{7^{n+1}}{7^n}\right| = \lim_{n\to+\infty} 7 = 7$, so by Theorem 1, $R = \frac{1}{7}$. When $x = \frac{1}{7}$, the series becomes $\sum_{k=0}^{\infty} 7^k(\frac{1}{7^k}) = \sum_{k=0}^{\infty} 1$, which diverges. Now when $x = -\frac{1}{7}$, the series becomes $\sum_{k=0}^{\infty} (-1)^k$, which diverges. Hence, $I = (-\frac{1}{7}, \frac{1}{7})$.

2. $\sum_{k=0}^{\infty} \frac{x^{k+1}}{\sqrt{k+1}} = x \sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k+1}}$ for a fixed value of x. A constant times a series does not affect its convergence, so we will find I for $\sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k+1}}$. The center is a = 0. $\lim_{n\to+\infty} \left|\frac{\frac{1}{\sqrt{n+2}}}{\frac{1}{\sqrt{n+1}}}\right| = \lim_{n\to+\infty} \sqrt{\frac{n+1}{n+2}} = 1$, so R = 1, by Theorem 1. Now for x = 1, $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}}$ diverges by comparison with the divergent series $\sum_{k=1}^{\infty} \frac{1}{2\sqrt{k}}$. When x = -1, the series is $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}}$, which converges by Leibniz's theorem. Hence, the given series is conditionally convergent for

$x = -1$. $I = [-1,1)$.

3. $a = 0$. $\lim_{n\to+\infty}\left|\dfrac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to+\infty}\dfrac{1}{n+1} = 0$, so $R = +\infty$ by Theorem 1. $I = (-\infty, +\infty)$.

4. $a = 0$. $\lim_{n\to+\infty}\left|\dfrac{3^n\sqrt{n}}{3^{n+1}\sqrt{n+1}}\right| = \lim_{n\to+\infty}\dfrac{1}{3}\sqrt{\dfrac{n}{n+1}} = \dfrac{1}{3}$, so $R = 3$ by Theorem 1. The endpoints $a - R = -3$ and $a + R = 3$ must be tested. When $x = 3$, the series becomes $\sum_{n=1}^{\infty}\dfrac{3^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$, which is a divergent p series. When $x = -3$, the series is $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$, which converges conditionally by Leibniz's test. Thus, $I = [-3,3)$.

5. $a = 0$. $\lim_{n\to+\infty}\left|\dfrac{(n+1)!}{n!}\right| = \lim_{n\to+\infty} n + 1 = +\infty$, so $R = 0$ by Theorem 1 and I consists of the single number 0.

6. $a = 0$. $\lim_{n\to\infty}\left|\dfrac{(n+1)^3}{5^{n+1}}\cdot\dfrac{5^n}{n^3}\right| = \lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^3\cdot\dfrac{1}{5} = \dfrac{1}{5}$, so by Theorem 1, $R = 5$. We must test the endpoints $x = 5$ and $x = -5$. When $x = 5$, we have $\sum_{n=0}^{\infty} n^3$, which diverges. When $n = -5$, we have $\sum_{n=0}^{\infty} - n^3$, which also diverges, so $I = (-5,5)$.

7. $a = 0$. We use the original ratio test: $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{3^{n+1}x^{2n+2}}{n+3}\cdot\dfrac{n+2}{3^n x^{2n}}\right| = \lim_{n\to\infty}\dfrac{(n+2)}{(n+3)}|3x^2| = 3x^2 < 1$ when $|x| < 1/\sqrt{3}$, so $R = 1/\sqrt{3}$. The endpoints $a + R = 1/\sqrt{3}$ and $a - R = -1/\sqrt{3}$ must be tested. When $x = \pm 1/\sqrt{3}$, the series becomes $\sum_{n=0}^{\infty}\dfrac{1}{n+2}$, which diverges since it is the harmonic series minus the 1st term. Thus, $I = \left(\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.

8. $a = 0$. $\lim_{n\to\infty}\left|\dfrac{(n+2)3^n}{(n+3)3^{n+1}}\right| = \lim_{n\to\infty}\dfrac{n+2}{3n+9} = 1/3$, so $R = 3$ by Theorem 1. We test the endpoints: When $x = 3$, we have $\sum_{n=0}^{\infty}\dfrac{1}{n+2}$, which is a divergent harmonic series. When $x = -3$, we have $\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{n+2}$, which converges conditionally by Leibniz's test. Thus, $I = [-3,3)$.

9. $a = 0$. $\lim_{n\to\infty}\left|\dfrac{3+n^2}{3+(n+1)^2}\right| = 1$, so $R = 1$. The endpoints are 1 and -1. When $x = 1$, $\sum_{n=1}^{\infty}\dfrac{1}{3+n^2}$ converges absolutely by comparison with the p series $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$. When $x = -1$, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{3+n^2}$ converges absolutely. Thus, $I = [-1,1]$.

10. $a = 0$. We use the original ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0 \text{ for all } x, \text{ so } R = \infty.$$

Thus, $I = (-\infty,\infty)$.

11. $a = 0$. We use the original ratio test:

$$\lim_{n\to+\infty}\left|\frac{\frac{(-1)^{n+2}x^{2n+1}}{(2n+1)!}}{\frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}}\right| = \lim_{n\to+\infty}\frac{x^2}{(2n+1)(2n)} = 0 < 1$$

for all x. $R = +\infty$ and $I = (-\infty,\infty)$.

12. $a = 0$. We use the original ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{5n+5}}{(n+1)^{5/2}}\cdot\frac{n^{5/2}}{(-1)^n x^{5n}}\right| = \lim_{n\to\infty}\frac{n^{5/2}}{(n+1)^{5/2}}|x^5| =$$

$|x^5| < 1$ when $|x| < 1$, so $R = 1$. When $x = 1$, we have $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{k^{5/2}}$, which converges absolutely (p series). When $x = -1$, we have $\sum_{n=1}^{\infty}\dfrac{(-1)^n(-1)^{5n}}{n^{5/2}} = \sum_{n=1}^{\infty}\dfrac{1}{n^{5/2}}$, which is a convergent p series. Thus, $I = [-1,1]$.

13. $a = 2$. $\lim_{n\to\infty}\left|\dfrac{3^n}{3^{n+1}}\right| = 1/3$, so $R = 3$. The endpoints are $a + R = 5$ and $a - R = -1$. When $x = 5$, the series becomes $\sum_{n=0}^{\infty} 3^n/3^n = \sum_{n=0}^{\infty} 1$, which diverges. When $x = -1$, the series is $\sum_{n=0}^{\infty}\dfrac{(-3)^n}{3^n}$, which looks

like 1 + (-1) + 1 + (-1) + ... and hence diverges. So I = (-1,5).

14. a = 1. $\lim_{n\to+\infty}\left|\dfrac{\dfrac{(-1)^{n+1}}{n+2}}{\dfrac{(-1)}{n+1}}\right| = \lim_{n\to+\infty}\dfrac{n+1}{n+2} = 1$, so R = 1 by Theorem 1. Now a - R = 0 and a + R = 2 are the endpoints to be tested. When x = 0, the series becomes $\sum_{k=0}^{\infty}\frac{1}{k+1}$, which diverges by comparison with $\sum_{k=1}^{\infty}\frac{1}{2}(\frac{1}{k})$. When x = 2, the series becomes $\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$, which converges by Leibniz's theorem. So the series converges conditionally for x = 2. I = (0,2].

15. a = -3. $\lim_{n\to+\infty}\left|\dfrac{\dfrac{(-1)^{n+2}(n+1)}{7^n}}{\dfrac{(-1)^{n+1}n}{7^{n-1}}}\right| = \lim_{n\to+\infty}(\frac{n+1}{7n}) =$ $\lim_{n\to+\infty}\dfrac{1+\frac{1}{n}}{7} = \frac{1}{7}$, so R = 7 by Theorem 1. The endpoints a - R = -10 and a + R = 4 must be tested. When x = 4, the series becomes $\sum_{k=1}^{\infty}(-1)^{k+1}k$, which diverges since the nth term does not approach 0 as $n\to+\infty$. When x = -10, the series becomes $\sum_{k=1}^{\infty}(-1)^{2k}k = \sum_{k=1}^{\infty}k$, which diverges. I = (-10,4).

16. a = -2. $\lim_{n\to+\infty}\left|\dfrac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}}\right| = \lim_{n\to+\infty}(\frac{n}{n+1})^2 = 1$, so R = 1. The endpoints to be tested are a - R = -3 and a + R = -1. Thus, when x = -1, the series becomes $\sum_{k=1}^{\infty}\frac{1}{k^2}$, which converges absolutely; and so the series converges absolutely at x = -3. Hence, I = [-3,-1].

17. a = -1. $\lim_{n\to\infty}\left|\dfrac{2^{n+1}}{\ln(n+2)}\cdot\dfrac{\ln(n+1)}{2^n}\right| =$ $\lim_{n\to\infty} 2\cdot\dfrac{\ln(n+1)}{\ln(n+2)} = 2\lim_{x\to\infty}\dfrac{\ln(x+1)}{\ln(x+2)} = 2\lim_{x\to\infty}\dfrac{\frac{1}{x+1}}{\frac{1}{x+2}} =$ $2\lim_{x\to\infty}\dfrac{x+2}{x+1} = 2\cdot 1 = 2$, so R = 1/2. The endpoints are -3/2 and -1/2. When x = -3/2, the series becomes $\sum_{k=1}^{\infty}\frac{(-1)^k}{\ln(k+1)}$, which converges conditionally by Leibniz's test. When x = -1/2, the series is $\sum_{k=1}^{\infty}\frac{1}{\ln(k+1)}$, which diverges by comparison with the series $\sum_{k=1}^{\infty}\frac{1}{k+1}$. Thus, I = [-3/2, 1/2).

18. a = 3/2. $\lim_{n\to\infty}\left|\dfrac{(2x-3)^{n+1}}{4^{2n+2}}\cdot\dfrac{4^{2n}}{(2x-3)^n}\right| =$ $\lim_{n\to\infty}\dfrac{4^{2n}}{4^{2n+2}}|2x-3| = \dfrac{|2x-3|}{16} < 1$ when $|2x-3| < 16$, or $|x-\frac{3}{2}| < 8$, so R = 8. The endpoints are a - R = $-\frac{13}{2}$ and a + R = $\frac{19}{2}$. When x = $\frac{19}{2}$, the series is $\sum_{k=1}^{\infty}\frac{16^k}{4^{2k}} = \sum_{k=1}^{\infty}1$, which diverges. When x = $-\frac{13}{2}$, we have $\sum_{k=1}^{\infty}\frac{(-16)^k}{4^{2k}} = \sum_{k=1}^{\infty}(-1)^k$, which diverges. Thus, I = $(-\frac{13}{2}, \frac{19}{2})$.

19. a = -1. $\lim_{n\to\infty}\left|\dfrac{\sqrt{n}}{\sqrt{n+1}}\right| = 1$, so R = 1. The endpoints are -2 and 0. When x = -2, the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ converges by Leibniz's test. When x = 0, the series $\sum_{k=1}^{\infty}1/\sqrt{k}$ is a divergent p series. Thus, I = [-2,0).

20. a = -1. $\lim_{n\to+\infty}\left|\dfrac{\dfrac{1}{(n+1)\sqrt{n+2}}}{\dfrac{1}{n\sqrt{n+1}}}\right| = \lim_{n\to+\infty}\dfrac{n}{n+1}\cdot\sqrt{\dfrac{n+1}{n+2}} = 1$, so R = 1. The endpoints to be tested are a - R = -2 and a + R = 0. For x = 0, the series becomes $\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k+1}}$ which converges absolutely by comparison with the p series $\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$. Thus, I = [-2,0], since the series will be absolutely convergent at x = -2 also.

21. $a = -5$. $\lim_{n\to+\infty} \left| \dfrac{\left[\frac{1}{(2n+1)(2n+2)}\right]}{\left[\frac{1}{(2n-1)(2n)}\right]} \right| =$

$\lim_{n\to+\infty} \dfrac{4n^2 - 2n}{4n^2 + 6n + 2} = 1$, so $R = 1$. We test the endpoints $a - R = -6$ and $a + R = -4$. When $x = -6$, the series becomes $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{(2k-1)(2k)}$ which converges absolutely by comparison with the p series $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$. So does the series for $x = -4$. Thus, $I = [-6,-4]$.

22. $a = 1$. We use the original ratio test. For $n \geq 2$,

$$\lim_{n\to+\infty} \left| \dfrac{\frac{(x-1)^{2n}}{(2n-2)!}}{\frac{(x-1)^{2n-2}}{(2n-4)!}} \right| = \lim_{n\to+\infty} \dfrac{|x-1|^2}{(2n-2)(2n-3)} = 0 \text{ for}$$

all x. Hence, $R = +\infty$ and $I = (-\infty,\infty)$.

23. $a = -2$. $\lim_{n\to\infty} \left| \dfrac{2^{n+1}}{(n+2)^3} \cdot \dfrac{(n+1)^3}{2^n} \right| = \lim_{n\to\infty} \dfrac{(n+1)^3}{(n+2)^3} \cdot 2 =$ 2, so $R = 1/2$. The endpoints $x = -5/2$ and $x = -3/2$ must be tested. When $x = -5/2$, the series is $\sum_{j=0}^{\infty} \dfrac{(-1)^j(-1)^j}{(j+1)^3} = \sum_{j=0}^{\infty} \dfrac{1}{(j+1)^3}$, which converges by comparison with $\sum_{j=1}^{\infty} \dfrac{1}{j^3}$. When $x = -3/2$, we have $\sum_{j=0}^{\infty} \dfrac{(-1)^j}{(j+1)^3}$, which converges by Leibniz's test. Thus, $I = [-5/2, -3/2]$.

24. $a = -1$. $\lim_{n\to\infty} \left| \dfrac{\sqrt{n+1}\,(x+1)^{n+1}}{1\cdot3\cdot5\cdots(2n+3)} \cdot \dfrac{1\cdot3\cdot5\cdots(2n+1)}{\sqrt{n}\,(x+1)^n} \right| =$

$\lim_{n\to\infty} \sqrt{\dfrac{n+1}{n}}\, \dfrac{|x+1|}{2n+3} = 0$ for all values of x, so $R = \infty$ and $I = (-\infty,\infty)$.

25. $a = 1$. $\lim_{n\to+\infty} \left| \dfrac{\frac{1}{(n+3)!}}{\frac{1}{(n+2)!}} \right| = \lim_{n\to+\infty} \dfrac{1}{n+3} = 0$, so $R = +\infty$. Hence, $I = (-\infty,\infty)$.

26. $a = 0$. $\lim_{n\to+\infty} \left| \dfrac{\frac{(-1)^{n+1}}{2n+3} \cdot (\frac{x}{2})^{2n+2}}{\frac{(-1)^n}{2n+1} (\frac{x}{2})^{2n}} \right| = \lim_{n\to+\infty} \dfrac{2n+1}{2n+3} \cdot \dfrac{|x|^2}{4} =$

$\dfrac{|x|^2}{4} < 1$, so $|x| < 2$. So $R = 2$. The endpoints of I are $a - R = -2$ and $a + R = 2$. When $x = 2$, the series becomes $\sum_{k=0}^{\infty} \dfrac{(-1)^k}{2k+1}$, which converges by Leibniz's theorem. When $x = -2$, we get the same series. Since $\sum_{k=0}^{\infty} \dfrac{1}{2k+1}$ diverges by the integral test, we have conditional convergence at the endpoints. $I = [-2,2]$.

27. $a = -1$. $\lim_{n\to+\infty} \left| \dfrac{\frac{(x+1)^{5n+5}}{(n+2)5^{n+1}}}{\frac{(x+1)^{5n}}{(n+1)5^n}} \right| = \lim_{n\to+\infty} \dfrac{|x+1|^5}{5} \cdot$

$(\frac{n+1}{n+2}) = \dfrac{|x+1|^5}{5} < 1$ provided $|x+1|^5 < 5$ and $|x+1| < \sqrt[5]{5}$, and so $R = \sqrt[5]{5}$. So the endpoints of I are $-1 - \sqrt[5]{5}$ and $-1 + \sqrt[5]{5}$. When $x = -1 - \sqrt[5]{5}$, the series becomes $\sum_{k=0}^{\infty} \dfrac{(-5)^k}{(k+1)5^k}$, which converges absolutely by comparison with $\sum_{k=0}^{\infty} \dfrac{1}{5^k}$. Thus, $I = [-1 - \sqrt[5]{5}, -1 + \sqrt[5]{5}]$.

28. $a = 1$. $\sum_{k=0}^{\infty} \dfrac{(1-x)^k}{(k+1)3^k} = \sum_{k=0}^{\infty} \dfrac{(-1)^k(x-1)^k}{(k+1)3^k} =$

$$\lim_{n\to+\infty} \left| \dfrac{\frac{(-1)^{n+1}}{(n+2)3^{n+1}}}{\frac{(-1)^n}{(n+1)3^n}} \right| = \lim_{n\to+\infty} \dfrac{(n+1)}{(n+2)\cdot 3} = \dfrac{1}{3}, \text{ so } R = 3.$$

The endpoints of I are $a - R = -2$ and $a + R = 4$. When $x = -2$, the series $\sum_{k=0}^{\infty} \dfrac{1}{k+1}$ diverges by the integral test. When $x = 4$, the series $\sum_{k=0}^{\infty} \dfrac{(-1)^k}{k+1}$ is convergent, so we have conditional convergence at $x = 4$. Thus, $I = (-2,4]$.

29. $\sum_{k=1}^{\infty} \dfrac{1}{k}(\dfrac{x}{4} - 1)^k = \sum_{k=1}^{\infty} \dfrac{(x-4)^k}{k \cdot 4^k}$. So $a = 4$.

$$\lim_{n\to+\infty} \left| \dfrac{\frac{1}{(n+1)4^{n+1}}}{\frac{1}{n \cdot 4^n}} \right| = \lim_{n\to+\infty} \dfrac{n}{(n+1)\cdot 4} = \dfrac{1}{4}, \text{ so } R = 4.$$

The endpoints of I are 0 and 8. When $x = 0$, the

series becomes $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which converges by Leibniz's theorem. For x = 8, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. So we have conditional convergence for x = 0. Thus, I = [0,8).

30. $\sum_{n=0}^{\infty} \frac{1}{3n-1}\left(\frac{x}{3}+\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{3n-1}\frac{(x+2)^n}{3^n}$. a = -2.

$$\lim_{n\to+\infty}\left|\frac{\frac{1}{3n+2}\cdot\frac{1}{3^{n+1}}}{\frac{1}{3n-1}\cdot\frac{1}{3^n}}\right| = \lim_{n\to+\infty}\frac{(3n-1)}{(3n+2)3} = \frac{1}{3}, \text{ so } R = 3.$$

The endpoints of I are -5 and 1. For x = -5, the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{3n-1}$ is convergent by Leibniz's theorem. When x = 1, the series $\sum_{n=0}^{\infty}\frac{1}{3n-1}$ diverges by the integral test. Thus, we have conditional convergence at x = -5. Hence, I = [-5,1).

31. Here a = 5. Using the original ratio test, we have

$$\lim_{n\to+\infty}\left|\frac{\frac{(-1)^{n+1}2^{n+1}(x-5)^{2n+2}}{(n+1)^3}}{\frac{(-1)^n 2^n (x-5)^{2n}}{n^3}}\right| = \lim_{n\to+\infty}\frac{2|x-5|^2}{(n+1)^3}\cdot n^3 =$$

$2\cdot|x-5|^2 < 1$ when $|x-5|^2 < \frac{1}{2}$ or $|x-5| < \frac{1}{\sqrt{2}}$, so $R = \frac{1}{\sqrt{2}}$. The endpoints of I are $5 - \frac{1}{\sqrt{2}}$ and $5 + \frac{1}{\sqrt{2}}$. For $x = 5 - \frac{1}{\sqrt{2}}$, the series becomes $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}$ which converges absolutely. We get the same series when $x = 5 + \frac{1}{\sqrt{2}}$. Thus, $I = [5 - \frac{1}{\sqrt{2}}, 5 + \frac{1}{\sqrt{2}}]$.

32. $\sum_{j=1}^{\infty}\frac{(3-x)^{j-1}}{\sqrt{j}} = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}(x-3)^{j-1}}{\sqrt{j}}$. a = 3.

$$\lim_{n\to+\infty}\left|\frac{\frac{(-1)^n}{\sqrt{n+1}}}{\frac{(-1)^{n-1}}{\sqrt{n}}}\right| = \lim_{n\to+\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 1, \text{ so } R = 1.$$ The

endpoints of I are 2 and 4. When x = 2, the series becomes $\sum_{j=1}^{\infty}\frac{1}{\sqrt{j}}$, which diverges (p series with p < 1). When x = 4, the series $\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{\sqrt{j}}$ converges by Leibniz's theorem. Hence, the series is conditionally convergent at x = 4. I = (2,4].

33. a = 1. $\lim_{n\to+\infty}\left|\frac{\tan^{-1}(n+1)}{\tan^{-1}n}\right| = \frac{\frac{\pi}{2}}{\frac{\pi}{2}} = 1$, so R = 1. The endpoints of I are 0 and 2. When x = 2, the series becomes $\sum_{k=1}^{\infty}\tan^{-1}k$, which diverges since the nth term does not go to zero. When x = 0, the series $\sum_{k=1}^{\infty}(-1)^k\tan^{-1}k$ diverges for the same reason. Thus, I = (0,2).

34. a = 3. We use the original ratio test:

$$\lim_{n\to+\infty}\left|\frac{\frac{(x-3)^{4n+4}}{\sqrt[n+1]{n+1}}}{\frac{(x-3)^{4n}}{\sqrt[n]{n}}}\right| = \lim_{n\to+\infty}|x-3|^4\cdot\frac{\sqrt[n]{n}}{\sqrt[n+1]{n}} =$$

$|x-3|^4$ since $\lim_{n\to+\infty}\sqrt[n]{n} = 1$ and $\lim_{n\to+\infty}\sqrt[n+1]{n} = 1$ by l'Hôpital's rule. Now $|x-3|^4 < 1$, so that $|x-3| < 1$ and so R = 1. The endpoints of I are 2 and 4. When x = 4, the series becomes $\sum_{k=1}^{\infty}\frac{1}{\sqrt[k]{k}}$, which diverges since $\lim_{n\to+\infty}\frac{1}{\sqrt[n]{n}} = 1 \neq 0$. For x = 2, the series is identical. Hence, I = (2,4).

35. a = 2. $\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{n^n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot(n+1) = e\cdot\lim_{n\to\infty}(n+1) = +\infty$, so R = 0 and I consists of the single number 2.

36. Here a = -1. Now we use the original ratio test:

$$\lim_{n\to+\infty}\left|\left(\frac{5^{n+1}+5^{-n-1}}{5^n+5^{-n}}\right)\cdot\frac{(x+1)^{3n+1}}{(x+1)^{3n-2}}\right| =$$

$$\lim_{n\to+\infty}\left(\frac{1+\frac{1}{5^{2n+2}}}{\frac{1}{5}+\frac{1}{5^{2n+1}}}\right)|x+1|^3 = 5|x+1|^3 < 1 \text{ for}$$

$|x+1|^3 < \frac{1}{5}$ or $|x+1| < \frac{1}{\sqrt[3]{5}}$, so $R = \frac{1}{\sqrt[3]{5}}$ and the endpoints of I are $-1 - \frac{1}{\sqrt[3]{5}}$ and $-1 + \frac{1}{\sqrt[3]{5}}$. When $x = -1 + \frac{1}{\sqrt[3]{5}}$, the series becomes

$\sum_{k=1}^{\infty} (5^{2/3} + 5^{(2/3)-2k})$, which diverges since the nth term does not approach 0 as n approaches $+\infty$. When $x = -1 - \frac{1}{\sqrt[3]{5}}$, the resulting series, $\sum_{k=1}^{\infty} -(5^{2/3} + 5^{(2/3)-2k})$, diverges for the same reason. Hence, $I = (-1 - \frac{1}{\sqrt[3]{5}}, -1 + \frac{1}{\sqrt[3]{5}})$.

37. $a = 3$. $\lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-3)^{n+2}}{9^{n+1}} \cdot \frac{9^n}{(-1)^n(x-3)^{2n}}\right| = \lim_{n\to\infty} \frac{(x-3)^2}{9} < 1$ when $(x-3)^2 < 9$, or $|x - 3| < 3$, so $R = 3$. The endpoints of I are 0 and 6. For $x = 0$, the series becomes $\sum_{k=1}^{\infty} \frac{(-1)^k(-3)^{2k}}{9^k} = \sum_{k=1}^{\infty} (-1)^k$, which diverges. For $x = 6$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^k 3^{2k}}{9^k} = \sum_{k=1}^{\infty} (-1)^k$, which diverges. Thus, $I = (0,6)$.

38. $a = 0$. $\lim_{n\to+\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to+\infty} \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to+\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to+\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}$, so $R = e$. When $x = -e$, the series becomes $\sum_{k=1}^{\infty} \frac{k!}{k^k}(-e)^k$. Now, we consider $\left|\frac{n!}{n^n}(-e)^n\right| = \frac{n!}{n^n} \cdot e^n$. By Stirling's formula, Problem 58, Section 11.1, $\sqrt{2n\pi}\left(\frac{n}{e}\right)^n < n!$, so that $\frac{n!}{n^n} > e^n \frac{\sqrt{2n\pi}\,\frac{n^n}{e^n} \cdot e^n}{n^n} = \sqrt{2n\pi}$. Thus, since $\lim_{n\to+\infty} \sqrt{2n\pi} = +\infty$, $\lim_{n\to+\infty} \frac{n!}{n^n} e^n = +\infty$, and so $\sum_{k=1}^{\infty} \frac{k!}{k^k}(-e)^k$ diverges. Similarly, $\sum_{k=1}^{\infty} \frac{k!}{k^k} e^k$ diverges. Hence, $I = (-e,e)$.

39. $a = 0$. $\lim_{n\to\infty} \left|\frac{(-1)^{n+1} 1\cdot3\cdot5\cdots(2n+1)x^{2n+3}}{2\cdot4\cdot6\cdots(2n+2)} \cdot \frac{}{(-1)^n 1\cdot3\cdot5\cdots(2n-1)x^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{2n+1}{2n+2}x^2\right| = x^2 < 1$ when $|x| < 1$, so $R = 1$. The endpoints are $x = 1$ and $x = -1$. When $x = -1$, the series is $\sum_{k=1}^{\infty} (-1)^k \frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}(-1)^{2k+1} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}$. Now $a_k = \frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}$ is clearly decreasing, since $\frac{2k-1}{2k} < 1$. Also, $\lim_{n\to\infty} a_n = 0$, which we will prove by showing that $\lim_{n\to\infty} \ln a_n = -\infty$: $\ln a_k = \ln\left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{(2k-1)}{2k}\right) = \ln(1/2) + \ln(3/4) + \ln(5/6) + \ldots + \ln\left(\frac{2k-1}{2k}\right) = \sum_{i=1}^{k} \ln\left(\frac{2i-1}{2i}\right) = \sum_{i=1}^{k} \ln(1 - 1/2i)$, so $\lim_{k\to\infty} \ln a_k = \sum_{i=1}^{\infty} \ln(1 - 1/2i)$. But we claim that $\ln(1 - 1/2i) < -1/2i$: consider the function $f(x) = \ln x - x + 1$. The reader can easily verify that $f(x)$ has a maximum of 0 at $x = 1$, so $f(x) < 0$ for $0 < x < 1$. Now let $x = 1 - 1/2i$, so that $1/2 \leq x \leq 1$ for $i = 1,2,\ldots,\infty$. So $\ln(1 - 1/2i) - (1 - 1/2i) + 1 < 0$, or $\ln(1 - 1/2i) < -1/2i$. Therefore $\sum_{i=1}^{\infty} \ln(1 - 1/2i) < \sum_{i=1}^{\infty} - 1/2i = -\infty$, so $\lim_{k\to\infty} \ln a_k = -\infty$ and $\lim_{k\to\infty} a_k = 0$. So by Leibniz's test, $\sum_{k=1}^{\infty} (-1)^k \frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots2k}$ converges. When $x = 1$, $a_k = \frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots2k} = \left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\cdots\left(\frac{2k-1}{2k-2}\right) \cdot \frac{1}{2k} \geq \frac{1}{2k}$, and $\sum_{k=1}^{\infty} \frac{1}{2k}$ diverges, so $\sum_{k=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots2k}$ diverges by comparison. Thus, $I = [-1,1)$.

40. $a = 8$. The series is $\sum_{k=0}^{\infty} k!(x-8)^{k+1}$. Now $\lim_{n\to+\infty} \left|\frac{(n+1)!}{n!}\right| = \lim_{n\to+\infty} (n+1) = +\infty$, so $R = 0$ by Theorem 1. Hence, the interval of convergence is $\{8\}$.

41. For a fixed value of x, $(x - a)^p$ is a constant. Hence, $\sum_{k=0}^{\infty} c_k(x-a)^{p+k} = (x-a)^p \sum_{k=0}^{\infty} c_k(x-a)^k$ has the same radius of convergence as that of the series $\sum_{k=0}^{\infty} c_k(x-a)^k$, since a constant multiple of

a series has no effect on its convergence. Thus, the radius of convergence of $\sum_{k=0}^{\infty} c_k(x-a)^{p+k}$ is R.

42. The radius of convergence of $\sum_{k=0}^{\infty} c_k(x-a)^k$ is R, which is the radius of convergence of $\sum_{k=0}^{\infty} c_k x^k$. Thus, $\sum_{k=0}^{\infty} c_k t^k$ converges if $|t| < R$ and diverges if $|t| > R$. Replace t by $(x-a)^p$, so that $\sum_{k=0}^{\infty} c_k(x-a)^{pk}$ converges if $|x-a|^p < R$ and diverges if $|x-a|^p > R$; that is, if $|x-a| < \sqrt[p]{R}$, the series converges, and if $|x-a| > \sqrt[p]{R}$, the series diverges. Hence, the radius of convergence of $\sum_{k=0}^{\infty} c_k(x-a)^{pk}$ is $\sqrt[p]{R}$.

43. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c(c-1)(c-2)\cdots(c-n)(c-n-1)}{(n+1)!}\cdot\frac{n!}{c(c-1)\cdots(c-n)}\right| = \lim_{n\to\infty}\left|\frac{c-n-1}{n+1}\right| = 1$, so R = 1. Since a = 0, by Theorem 1, the series converges on (-1,1), or for $|x| < 1$.

(b) By Theorem 1 in Section 11.3, since the series converges for $|x| < 1$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{c(c-1)(c-2)\cdots(c-n)}{n!}x^n = 0$ for $|x| < 1$.

44. (a) By the root test, for $x \neq a$, $\lim_{n\to+\infty}\sqrt[n]{|c_n(x-a)^n|} = \lim_{n\to+\infty}\sqrt[n]{|c_n|}\sqrt[n]{|x-a|^n} = \lim_{n\to+\infty}\sqrt[n]{|c_n|}\cdot|x-a| = +\infty$. Hence, the series diverges for all $x \neq a$. The series converges only for x = a. Thus, the radius of convergence is zero.

(b) By the root test, $\lim_{n\to+\infty}\sqrt[n]{|c_n||x-a|^n} = \lim_{n\to+\infty}\sqrt[n]{|c_n|}|x-a| = 0 < 1$ for all x. Hence, the series converges for all x. Thus, the radius of convergence is infinite.

(c) By the root test, $\lim_{n\to+\infty}\sqrt[n]{|c_n||x-a|^n} = \lim_{n\to+\infty}\sqrt[n]{|c_n|}\cdot|x-a| = L\cdot|x-a| < 1$ provided $|x-a| < \frac{1}{L}$, so that the series converges for all x, where $|x-a| < \frac{1}{L}$ and diverges for $|x-a| > \frac{1}{L}$. Thus, the radius of convergence is $R = \frac{1}{L}$.

45. $\lim_{n\to+\infty}\left|\frac{\frac{1}{1+b^{n+1}}}{\frac{1}{1+b^n}}\right| = \lim_{n\to+\infty}\frac{1+b^n}{1+b^{n+1}} = \lim_{n\to+\infty}\frac{\frac{1}{b^n}+1}{\frac{1}{b^n}+b} = \frac{1}{b}$, so that by Theorem 1, R = b. The endpoints of I are b and -b. When x = b or when x = -b, the resulting series $\sum_{k=0}^{\infty}\frac{b^k}{1+b^k}$ or $\sum_{k=0}^{\infty}\frac{(-b)^k}{1+b^k}$, respectively, diverges, since the nth term does not approach 0 as $n \to +\infty$. Thus, I = (-b,b).

46. Part (ii). By the original ratio test, since $\lim_{n\to+\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to+\infty}\left|\frac{c_{n+1}}{c_n}\right||x-a| = \sqrt[p]{r}$. $|x-a| = 0$ for all x, then the series converges for all x. Thus, the radius of convergence is infinite.

Part (iii). By the original ratio test, since $\lim_{n\to+\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to+\infty}\left|\frac{c_{n+1}}{c_n}\right|\cdot|x-a| = +\infty$ for all $x \neq a$, we know the series diverges for all x except a. Thus, the radius of convergence is 0.

47. $\lim_{n\to+\infty}\left|\frac{\frac{1}{a^{n+1}+b^{n+1}}}{\frac{1}{a^n+b^n}}\right| = \lim_{n\to+\infty}\frac{a^n+b^n}{a^{n+1}+b^{n+1}} = \lim_{n\to+\infty}\frac{1+(\frac{b}{a})^n}{a+(\frac{b}{a})^n\cdot a} = \frac{1}{a}$. Thus, by Theorem 1, the radius of convergence is a.

Problem Set 11.7, page 691

1. $D_x(x + x^2/2 + x^3/3 + x^4/4 + \ldots) = 1 + x + x^2 + x^3 + \ldots$, which converges for $|x| < 1$ since the original series $\sum_{k=1}^{\infty} x^k/k$ converges for $|x| < 1$; so R = 1.

2. $D_x(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$,

which converges for all x since the original series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}$ converges for all x; $R = +\infty$.

3. $\int (1 + 2x + 3x^2 + 4x^3 + \ldots)dx = x + x^2 + x^3 + x^4 + \ldots$, which converges for $|x| < 1$ since the original series $\sum_{k=1}^{\infty} kx^{k-1}$ does; so $R = 1$.

4. $\int (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots)dx = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \ldots$, which converges for all x since the original series does; so $R = +\infty$.

5. Replacing x by x^4, we obtain $\frac{1}{1 - x^4} = \sum_{k=1}^{\infty} x^{4k} = 1 + x^4 + x^3 + \ldots$ for $|x^4| < 1$ or $|x| < 1$; so $R = 1$.

6. $\frac{x}{1 - x^4} = x \cdot \frac{1}{1 - x^4} = x \cdot \sum_{k=0}^{\infty} x^{4k} = \sum_{k=0}^{\infty} x^{4k+1} = x + x^5 + x^9 + \ldots$ for $|x| < 1$; so $R = 1$.

7. Replacing x by $4x$, $\frac{1}{1 - 4x} = \sum_{k=0}^{\infty} (4x)^k = 1 + 4x + 16x^2 + \ldots$ for $|4x| < 1$ or $|x| < \frac{1}{4}$; so $R = \frac{1}{4}$.

8. $\frac{x3}{(1 - x^4)^2} = \frac{1}{4} D_x (\frac{1}{1 - x^4}) = \frac{1}{4} D_x \sum_{k=0}^{\infty} x^{4k} =$

$\frac{1}{4} \sum_{k=1}^{\infty} (4k)x^{4k-1} = \sum_{k=1}^{\infty} kx^{4k-1} = x^3 + 2x^7 + 3x^{11} + 4x^{15} + \ldots$ for $|x| < 1$ by Property I; $R = 1$.

9. $\frac{1}{1 - x^2} = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k}$ for $|x^2| < 1$, or $|x| < 1$. Now $\frac{x}{1 - x^2} = x \cdot \sum_{k=0}^{\infty} x^{2k} = \sum_{k=0}^{\infty} x^{2k+1} = x + x^3 + x^5 + \ldots$ for $|x| < 1$; so $R = 1$.

10. $\int_0^x \frac{t}{1 - t^2} dt = \int_0^x [\sum_{k=0}^{\infty} t^{2k+1}]dt = \sum_{k=0}^{\infty} \int_0^x t^{2k+1}dt = \sum_{k=0}^{\infty} \frac{x^{2k+2}}{2k + 2} = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots$ for $|x| < 1$ by Property III. Here, $R = 1$.

11. $\frac{1}{2 + x} = \frac{1}{2}(\frac{1}{1 + \frac{x}{2}}) = \frac{1}{2}[\frac{1}{1 - (-\frac{x}{2})}] = \frac{1}{2} \sum_{k=0}^{\infty} (-\frac{x}{2})^k = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2^{k+1}} = \frac{1}{2} - \frac{x}{2^2} + \frac{x^2}{2^3} - \frac{x^3}{2^4} + \ldots$ for $|-\frac{x}{2}| < 1$ or $|x| < 2$; so $R = 2$.

12. $\frac{1 + x^2}{(1 - x^2)^2} = D_x[\frac{x}{1 - x^2}] = D_x \sum_{k=0}^{\infty} x^{2k+1} = \sum_{k=1}^{\infty} (2k + 1)x^{2k} = 3x^2 + 5x^4 + 7x^6 + \ldots$ for $|x| < 1$ by Property I; $R = 1$.

13. $\ln(1 - x) = -\int_0^x \frac{1}{1 - t} dt = -\int_0^x [\sum_{k=0}^{\infty} t^k]dt = -\sum_{k=0}^{\infty} \int_0^x t^k dt = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k + 1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$ for $|x| < 1$; $R = 1$.

14. $\ln(\frac{1 + x}{1 - x}) = \ln(1 + x) - \ln(1 - x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} - \sum_{k=0}^{\infty} \frac{-x^{k+1}}{k + 1} = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j} + \sum_{j=1}^{\infty} \frac{x^j}{j} = \sum_{j=1}^{\infty} [(-1)^{j+1} + 1] \frac{x^j}{j} = \sum_{j=2n-1}^{\infty} \frac{2 \cdot x^{2n-1}}{2n - 1} = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots$ for $|x| < 1$; $R = 1$.

15. $\int_0^x \ln(1 - t)dt = \int_0^x [-\sum_{k=0}^{\infty} \frac{t^{k+1}}{k + 1}]dt = -\sum_{k=0}^{\infty} \frac{x^{k+2}}{(k + 1)(k + 2)} = -\frac{x^2}{2} - \frac{x^3}{2 \cdot 3} - \frac{x^4}{3 \cdot 4} - \frac{x^5}{4 \cdot 5} - \ldots$ for $|x| < 1$; $R = 1$.

16. $\tanh^{-1} x = \int_0^x \frac{dt}{1 - t^2} = \int_0^x [\sum_{k=0}^{\infty} t^{2k}]dt = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k + 1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$ for $|x| < 1$; $R = 1$.

17. $\tan^{-1} t = \int_0^t \frac{du}{1 + u^2} = \int_0^t (1 - u^2 + u^4 - u^6 + \ldots)du = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots$, so $\int_0^x \tan^{-1} t \, dt = \int_0^x (t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots)dt = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k}}{(2k)(2k - 1)}$. The radius of convergence for this series is the same as the radius of convergence for the original series $\frac{1}{1 + u^2}$, which converges for $|u| < 1$; so $R = 1$.

18. $\frac{1}{6 - x - x^2} = \frac{1}{(3 + x)(2 - x)} = \frac{1}{5(3 + x)} + \frac{1}{5(2 - x)}$. Now $\frac{1}{3 + x} = \frac{1}{3[1 - (-\frac{x}{3})]} = \frac{1}{3} \sum_{k=0}^{\infty} (-\frac{x}{3})^k = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^{k+1}}$ for $|x| < 3$. Also, $\frac{1}{2 - x} = \frac{1}{2(1 - \frac{x}{2})} = \frac{1}{2} \sum_{k=0}^{\infty} (\frac{x}{2})^k =$

$\sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}$ for $|x| < 2$. Hence, $\frac{1}{6 - x - x^2} =$

$\frac{1}{5}[\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^{k+1}} + \sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}]$ for $|x| < 2$, or

$\frac{1}{5}\sum_{k=0}^{\infty} \frac{[(-1)^k 2^{k+1} + 3^{k+1}]x^k}{6^{k+1}} = \frac{1}{5}(\frac{5}{6} + \frac{5x}{6^2} + \frac{35x^2}{6^3} +$

$\frac{65x^3}{6^4} + \frac{275x^4}{6^5} + \ldots$ for $|x| < 2$, or $\frac{1}{6} + \frac{x}{6^2} + \frac{7x^2}{6^3} + \frac{13x^3}{6^4} +$

$\frac{55x^4}{6^5} + \ldots$ for $|x| < 2$, since $(-1)^k 2^{k+1} + 3^{k+1}$ is divisible by 5 for all k, as can be verified by mathematical induction. Here, $R = 2$.

19. $\int_0^x \frac{dt}{6 - t - t^2} = \int_0^x \sum_{k=0}^{\infty} (\frac{[(-1)^k 2^{k+1} + 3^{k+1}]t^k}{(5)6^{k+1}})dt =$

$\sum_{k=0}^{\infty} \frac{[(-1)^k 2^{k+1} + 3^{k+1}]x^{k+1}}{(5)(k + 1)6^{k+1}} =$

$\frac{1}{5}(\frac{x}{6} + \frac{x^2}{2\cdot 6^2} + \frac{7x^3}{3\cdot 6^3} + \frac{13x^4}{4\cdot 6^4} + \ldots)$ for $|x| < 2$; $R = 2$.

20. $\int_0^x \tanh^{-1} t\, dt = \int_0^x [\sum_{k=0}^{\infty} \frac{t^{2k+1}}{2k + 1}]dt =$

$\sum_{k=0}^{\infty} \frac{x^{2k+2}}{(2k + 1)(2k + 2)} = \frac{x^2}{2} + \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} + \frac{x^8}{7\cdot 8} + \ldots$

for $|x| < 1$; $R = 1$.

21. $\sum_{k=0}^{\infty} (-1)^{k+1}x^k = -1 + x - x^2 + x^3 - x^4 + \ldots = \frac{-1}{1 + x}$,

so $R = 1$.

22. $\sum_{k=0}^{\infty} (-1)^k(x - 1)^k = \frac{1}{1 + (x - 1)} = \frac{1}{x}$ for $|x - 1| < 1$

or $0 < x < 2$.

23. $\sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1 - x^2}$, which

converges for $x^2 < 1$, or $|x| < 1$; $R = 1$.

24. $\sum_{k=0}^{\infty} x^{k+1} = x \sum_{k=0}^{\infty} x^k = \frac{x}{1 - x}$, which converges for

$|x| < 1$; $R = 1$.

25. $\sum_{k=1}^{\infty} kx^{2k-1} = x + 2x^3 + 3x^5 + 4x^7 + \ldots =$

$\frac{1}{2}\frac{d}{dx}(x^2 + x^4 + x^6 + x^8 + \ldots) =$

$\frac{1}{2}\frac{d}{dx}[x^2(1 + x^2 + x^4 + x^6 + \ldots)] = \frac{1}{2}\frac{d}{dx}(\frac{x^2}{1 - x^2}) =$

$\frac{x}{(1 - x^2)^2}$. The series $1 + x^2 + x^4 + x^6 + \ldots$

converges for $|x| < 1$; therefore, so does the derived series and so does $\sum_{k=1}^{\infty} kx^{2k-1}$. Thus, $R = 1$.

26. $\sum_{k=0}^{\infty} \frac{x^{k+1}}{k + 1} = \sum_{k=0}^{\infty} (\int x^k\, dx) = \int(\sum_{k=0}^{\infty} x^k)dx = \int\frac{dx}{1 - x} =$

$-\ln|1 - x|$, valid for $|x| < 1$; $R = 1$.

27. By Example 2, $\frac{1}{(1 - x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$ for $|x| < 1$. So

for $x = \frac{1}{2}$, $\frac{1}{(1 - \frac{1}{2})^2} = \sum_{k=1}^{\infty} \frac{k}{2^{k-1}}$, so $4 = \sum_{k=1}^{\infty} \frac{k}{2^{k-1}}$.

Thus, $\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{k}{2^{k-1}} = \frac{1}{2}(4) = 2$.

28. Since $\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$, $|x| < 1$, then $\sum_{k=0}^{\infty} (1 - x)^k =$

$\frac{1}{x}$ for $|1 - x| < 1$ or $0 < x < 2$. Thus for $0 < x < 2$,

$D_x \sum_{k=0}^{\infty} (1 - x)^k = \sum_{k=1}^{\infty} -k(1 - x)^{k-1} = -\frac{1}{x^2}$, and so

$\sum_{k=1}^{\infty} kx(1 - x)^{k-1} = \frac{1}{x}$ for $0 < x < 2$. Thus, for

$0 < p < 1$, $\sum_{k=1}^{\infty} kp(1 - p)^{k-1} = \frac{1}{p}$. When $p = 0$, the

sum is 0. When $p = 1$, the sum is 0.

29. $f'(x) = \sum_{k=1}^{\infty} k^2\cdot kx^{k-1} = \sum_{k=1}^{\infty} k^3x^{k-1}$. By Theorem 1

in Section 11.6, $\lim_{n\to+\infty} \frac{(n + 1)^3}{n^3} =$

$\lim_{n\to+\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^3} = \lim_{n\to+\infty} 1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3} = 1$,

so that $R = 1$.

30. $f'(x) = \sum_{k=1}^{\infty} (-1)^{k+1}k^2\cdot k(x - 2)^{k-1} =$

$\sum_{k=1}^{\infty} (-1)^{k+1}k^3(x - 2)^{k-1}$. Now $\lim_{n\to+\infty}\left|\frac{(-1)^{n+2}(n + 1)^3}{(-1)^{n+1}(n^3)}\right| =$

$\lim_{n\to+\infty} [\frac{(n + 1)^3}{n^3}] = 1$, so that $R = 1$.

31. $f'(x) = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k!} = \sum_{k-1}^{\infty} \frac{x^{k-1}}{(k - 1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Now

$\lim_{n\to+\infty} \frac{\frac{1}{(n + 1)!}}{\frac{1}{n!}} = \lim_{n\to+\infty} \frac{1}{n + 1} = 0$, so $R = +\infty$.

32. $f'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k + 1)x^{2k}}{(2k + 1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{2k}}{(2k)!}$.

By the ratio test, $\lim_{n\to+\infty}\left|\dfrac{\frac{(-1)^{n+2}x^{2n+2}}{(2n+2)!}}{\frac{(-1)^{n+1}x^{2n}}{(2n)!}}\right| =$

$\lim_{n\to+\infty}\dfrac{|x^2|}{(2n+2)(2n+1)} = 0 < 1$ for all x. Hence,

$R = +\infty$.

33. $f'(x) = \sum_{k=1}^{\infty} 2^{\frac{k}{2}}(2k)(x+1)^{2k-1} =$

$\sum_{k=1}^{\infty} 2^{k+2/2}(k)(x+1)^{2k-1}$. By the ratio test,

$\lim_{n\to+\infty}\left|\dfrac{2^{\frac{n+3}{2}}(n+1)(x+1)^{2n+1}}{2^{\frac{n+2}{2}}(n)(x+1)^{2n-1}}\right| =$

$\lim_{n\to+\infty}\left|\dfrac{2^{\frac{1}{2}}(n+1)(x+1)^2}{n}\right| = [\lim_{n\to+\infty}(\frac{n+1}{n})]\cdot|(x+1)^2|\sqrt{2}$

$= \sqrt{2}\,|x+1|^2 < 1$ if and only if

$|x+1|^2 < \dfrac{1}{\sqrt{2}}$ or $|x+1| < \dfrac{1}{\sqrt[4]{2}}$. Hence, $R = \dfrac{1}{\sqrt[4]{2}}$.

34. $f'(x) = \sum_{k=1}^{\infty}\dfrac{k^3(x-1)^{k^3-1}}{k^3} = \sum_{k=1}^{\infty}(x-1)^{k^3-1}$. By

the ratio test, $\lim_{n\to+\infty}\left|\dfrac{(x-1)^{(n+1)^3-1}}{(x-1)^{n^3-1}}\right| =$

$\lim_{n\to+\infty}|x-1|^{3n^2+3n+1} = 0$ provided $|x-1| < 1$. When

$|x-1| > 1$, then the limit is $+\infty$. Hence, $R = 1$.

35. $\int_0^x f(t)dt = \int_0^x [\sum_{k=0}^{\infty}\dfrac{(-1)^k t^{2k}}{(2k)!}]dt = \sum_{k=0}^{\infty}\int_0^x \dfrac{(-1)^k t^{2k}}{(2k)!}dt =$

$\sum_{k=0}^{\infty}\dfrac{(-1)^k x^{2k+1}}{(2k+1)!}$. By the ratio test,

$\lim_{n\to+\infty}\left|\dfrac{\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to+\infty}\dfrac{|x^2|}{(2n+3)(2n+2)} = 0$ for

all x. Hence, $R = +\infty$.

36. $\int_0^x f(t)dt = \int_0^x (\sum_{k=0}^{\infty}\dfrac{t^k}{2^{k+1}})dt = \sum_{k=0}^{\infty}\int_0^x \dfrac{t^k}{2^{k+1}}dt =$

$\sum_{k=0}^{\infty}\dfrac{x^{k+1}}{(k+1)2^{k+1}}$. By Theorem 1 in Section 11.6,

$\lim_{n\to+\infty}\left|\dfrac{\frac{1}{(n+2)2^{n+2}}}{\frac{1}{(n+1)2^{n+1}}}\right| = \lim_{n\to+\infty}\dfrac{n+1}{(n+2)\cdot 2} = \lim_{n\to+\infty}\dfrac{1+1/n}{2+4/n} =$

$\frac{1}{2}$, so that $R = 2$.

37. $\int_0^x f(t)dt = \int_0^x (\sum_{k=0}^{\infty}\dfrac{t^{2k+1}}{(2k+1)!})dt =$

$\sum_{k=0}^{\infty}\int_0^x \dfrac{t^{2k+1}}{(2k+1)!}dt = \sum_{k=0}^{\infty}\dfrac{x^{2k+2}}{(2k+2)!}$. By the ratio

test, $\lim_{n\to+\infty}\left|\dfrac{\frac{x^{2n+4}}{(2n+4)!}}{\frac{x^{2n+2}}{(2n+2)!}}\right| = \lim_{n\to+\infty}\dfrac{|x^2|}{(2n+4)(2n+3)} =$

$0 < 1$ for all x. Hence, $R = +\infty$.

38. $\int_0^x f(t)dt = \int_0^x (\sum_{k=1}^{\infty}\dfrac{t^k}{k^3})dt = \sum_{k=1}^{\infty}\int_0^x \dfrac{t^k}{k^3}dt =$

$\sum_{k=1}^{\infty}\dfrac{x^{k+1}}{(k+1)k^3}$. By Theorem 3 in Section 11.6,

$\lim_{n\to+\infty}\left|\dfrac{\frac{1}{(n+2)(n+1)^3}}{\frac{1}{(n+1)n^3}}\right| = \lim_{n\to+\infty}\dfrac{(n+1)(n^3)}{(n+2)(n+1)^3} =$

$\lim_{n\to+\infty}\dfrac{n^3}{(n+2)(n+1)^2} = \lim_{n\to+\infty}\dfrac{n^3}{n^3+4n^2+5n+2} =$

$\lim_{n\to+\infty}\dfrac{1}{1+\frac{4}{n}+\frac{5}{n^2}+\frac{2}{n^3}} = 1$. Thus, $R = 1$.

39. (a) $f(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$, so that $f(0) = 1$.

(b) $f'(x) = -x + \dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \ldots$, so that $f'(0) = 0$.

(c) $f''(x) = -1 + \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \ldots$, so that $f''(0) = -1$.

(d) $f'''(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$, so that $f'''(0) = 0$.

40. We want the absolute value of the error to be less than $\dfrac{5}{10^5}$, and since the power series expansion for $\tan^{-1}x$ is alternating, the error in absolute value will be less than the absolute value of the first omitted term. Thus, since $\dfrac{(\frac{1}{7})^n}{n} < \dfrac{5}{10^5}$ for $n = 5$, then $\tan^{-1}\frac{1}{7} \approx \frac{1}{7} - \dfrac{(\frac{1}{7})^3}{3} + \dfrac{(\frac{1}{7})^5}{5} - \dfrac{(\frac{1}{7})^7}{7} \approx 0.141897$.

Also, since $\dfrac{(\frac{1}{3})^n}{n} < \dfrac{5}{10^5}$ for $n = 8$, $\tan^{-1}\frac{1}{3} \approx \frac{1}{3} - \dfrac{(\frac{1}{3})^3}{3} + \dfrac{(\frac{1}{3})^5}{5} - \dfrac{(\frac{1}{3})^7}{7} + \dfrac{(\frac{1}{3})^9}{9} - \dfrac{(\frac{1}{3})^{11}}{11} + \dfrac{(\frac{1}{3})^{13}}{13} \approx$

0.321751. Hence, $\frac{\pi}{4} = \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} \approx$

$0.141897 + 2(0.321751) = 0.785399$, and so $\pi \approx 3.141596$. Rounding off to four decimal places, we have $\pi \approx 3.1416$. (The correct value of π rounded off to six decimal places is 3.141593.)

41. $\sum_{k=0}^{\infty} x^k$ is a geometric series with initial term 1 and ratio x. Therefore its n^{th} partial sum can be expressed $S_n(x) = \dfrac{a_1(1 - r^{n+1})}{1 - r} = \dfrac{1(1 - x^{n+1})}{1 - x}$, for $x \neq 1$. Then $R_n(x) = \dfrac{1}{1 - x} - S_n(x) = \dfrac{1}{1 - x} - \dfrac{1 - x^{n+1}}{1 - x} = \dfrac{x^{n+1}}{1 - x}$.

42. $\dfrac{1}{1 - x} = S_n(x) + R_n(x) = \sum_{k=0}^{n} x^k + \dfrac{x^{n+1}}{1 - x}$, so $\int_{-1}^{0} \dfrac{dx}{1 - x} = \int_{-1}^{0} (\sum_{k=0}^{n} x^k)dx + \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx$, or $-\ln|1 - x|\Big|_{-1}^{0} = \sum_{k=0}^{n} \int_{-1}^{0} x^k dx + \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx$. Thus $-\ln|1| + \ln|2| = \sum_{k=0}^{n} \dfrac{x^{k+1}}{k + 1}\Big|_{-1}^{0} + \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx$, or $\ln 2 = \sum_{k=0}^{n} \dfrac{(-1)^{k+1}}{k + 1} + \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx$, so $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots + \dfrac{(-1)^n}{n + 1} + \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx$.

43. Although it is true that the series on the right converges, we cannot conclude that its sum is given by the expression on the left, i.e., ln 2. Since the equality was obtained by integrating the series representation for $\dfrac{1}{1 + x}$, which is valid only for $|x| < 1$, we can only assert that the new series converges to $\ln(1 + x)$ on the same interval.

44. $\left|\int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx\right| \leq [0 - (-1)]M_n$, where M_n is the maximum value of $\left|\dfrac{x^{n+1}}{1 - x}\right|$ on the interval $[-1,0]$. Now if $-1 \leq x \leq 0$, then $1 \leq 1 - x \leq 2$, so $1/2 \leq \dfrac{1}{1 - x} \leq 1$. Therefore $\left|\dfrac{x^{n+1}}{1 - x}\right| = |x^{n+1}|\left|\dfrac{1}{1 - x}\right| \leq |x^{n+1}|$, so $M_n \leq |x^{n+1}|$, and $0 \leq \left|\int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx\right| \leq 1 \cdot M_n \leq |x^{n+1}|$. But $\lim_{n\to\infty} |x^{n+1}| = 0$, so by the squeeze theorem, $\lim_{n\to\infty} \left|\int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx\right| = 0$ also. Now by Problem 42, $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots + \dfrac{(-1)^n}{n + 1} + \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx$ for all n, so $\ln 2 = \lim_{n\to\infty} (1 - 1/2 + 1/3 - 1/4 + \ldots + \dfrac{(-1)^n}{n + 1} + \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx) = \lim_{n\to\infty} (\sum_{k=1}^{n} \dfrac{(-1)^k}{k + 1}) + \lim_{n\to\infty} \int_{-1}^{0} \dfrac{x^{n+1}}{1 - x}\,dx = \sum_{k=1}^{\infty} \dfrac{(-1)^k}{k + 1}$.

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1. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^4(x) = \sin x$, and so forth. Thus, $f(\frac{\pi}{6}) = \frac{1}{2}$, $f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, $f''(\frac{\pi}{6}) = -\frac{1}{2}$, $f'''(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$, $f^4(\frac{\pi}{6}) = \frac{1}{2}$, and so forth. The Taylor series for f at $a = \frac{\pi}{6}$ is $f(\frac{\pi}{6}) + f'(\frac{\pi}{6})(x - \frac{\pi}{6}) + \dfrac{f''(\frac{\pi}{6})}{2!}(x - \frac{\pi}{6})^2 + \dfrac{f'''(\frac{\pi}{6})(x - \frac{\pi}{6})^3}{3!} + \dfrac{f^4(\frac{\pi}{6})}{4!}(x - \frac{\pi}{6})^4 + \ldots = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{\frac{1}{2}}{2!}(x - \frac{\pi}{6})^2 - \frac{\sqrt{3}/2}{3!}(x - \frac{\pi}{6})^3 + \frac{\frac{1}{2}}{4!}(x - \frac{\pi}{6})^4 + \ldots$.

2. $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\dfrac{1}{2^2}x^{-3/2}$, $f'''(x) = \dfrac{3}{2^3}x^{-5/2}$, $f^4(x) = -\dfrac{3\cdot5}{2^4}x^{-7/2}$, $f^5(x) = \dfrac{3\cdot5\cdot7}{2^5}x^{-9/2}$, and so forth. Thus, $f(9) = 3$, $f'(9) = \dfrac{1}{2\cdot3}$, $f''(9) = -\dfrac{1}{2^2(3^3)}$, $f'''(9) = \dfrac{3}{2^3}\cdot\dfrac{1}{3^5}$, $f^4(9) = \dfrac{-3\cdot5}{2^4\cdot3^7}$, $f^5(9) = \dfrac{3\cdot5\cdot7}{2^5\cdot3^9}$, and so forth. The Taylor series for f at a = 9 is $3 + \dfrac{1}{2\cdot3}(x - 9) - \dfrac{1}{2!}\dfrac{1}{2^2\cdot3^3}(x - 9)^2 + \dfrac{1}{3!}\dfrac{1}{2^3\cdot3^5}(x - 9)^3 - \dfrac{3\cdot5}{4!2^4\cdot3^7}(x - 9)^4 + \dfrac{3\cdot5\cdot7}{5!2^5\cdot3^9}(x - 9)^5 + \ldots = 3 + \dfrac{x - 9}{2\cdot3} + \sum_{k=2}^{\infty} (-1)^{k-1}\dfrac{1\cdot3\cdot5\cdots(2k - 3)(x - 9)^k}{k!2^k\cdot3^{2k-1}}$.

3. $f(x) = \frac{1}{x}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -3\cdot 2x^{-4}$, $f^4(x) = 4\cdot 3\cdot 2x^{-5}$, $f^5(x) = -5\cdot 4\cdot 3\cdot 2x^{-6}$, and so forth. Thus, $f(2) = \frac{1}{2}$, $f'(2) = -2^{-2}$, $f''(2) = 2(2^{-3})$, $f'''(2) = -3\cdot 2\cdot(2^{-4})$, $f^4(2) = 4\cdot 3\cdot 2\cdot(2^{-5})$, $f^5(2) = -5!(2)^{-6}$, and so forth. The Taylor series for f at a = 2 is $\frac{1}{2} - \frac{(x-2)}{2^2} + \frac{2(x-2)^2}{2!2^3} - \frac{3!(x-2)^3}{3!2^4} + \frac{4!}{4!}\frac{(x-2)^4}{2^5} - \frac{5!(x-2)^5}{5!2^6} + \ldots = \frac{1}{2} - \frac{x-2}{2^2} + \frac{(x-2)^2}{2^3} - \frac{(x-2)^3}{2^4} + \frac{(x-2)^4}{2^5} - \frac{(x-2)^5}{2^6} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{(x-2)^k}{2^{k+1}}$.

4. $f(x) = x^{3/2}$, $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$, $f''(x) = \frac{3}{2^2}x^{-\frac{1}{2}}$, $f'''(x) = -\frac{3}{2^3}x^{-3/2}$, $f^4(x) = \frac{3\cdot 3}{2^4}x^{-5/2}$, $f^5(x) = \frac{-3\cdot 3\cdot 5}{2^5}x^{-7/2}$, and so forth. Thus, $f(1) = 1$, $f'(1) = \frac{3}{2}$, $f''(1) = \frac{3}{2^2}$, $f'''(1) = -\frac{3}{2^3}$, $f^4(1) = \frac{3^2}{2^4}$, $f^5(1) = \frac{-3^2\cdot 5}{2^5}$, $f^6(1) = \frac{3^2\cdot 5\cdot 7}{2^6}$, and so forth. The Taylor series for f at a = 1 is $1 + \frac{3}{2}(x-1) + \frac{3}{2!2^2}(x-1)^2 - \frac{3\cdot 1}{3!2^3}(x-1)^3 + \frac{3\cdot 3}{4!2^4}(x-1)^4 - \frac{3\cdot 3\cdot 5}{5!2^5}(x-1)^5 + \frac{3\cdot 3\cdot 5\cdot 7}{6!2^6}(x-1)^6 - \ldots = 1 + \frac{3}{2}(x-1) + \frac{3}{2!2^2}(x-1)^2 + 3\sum_{k=3}^{\infty}(-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-5)(x-1)^k}{k!2^k}$.

5. $f(x) = e^x$, $f'(x) = e^x$, and so forth. $f(4) = e^4$, $f'(4) = e^4$, and so forth. The Taylor series for f at a = 4 is $e^4 + e^4(x-4) + \frac{e^4}{2!}(x-4)^2 + \frac{e^4(x-4)^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{e^4(x-4)^k}{k!}$.

6. We obtain the Taylor series for $f(x) = \cos x$ at $a = \frac{\pi}{6}$ by differentiating the corresponding series for sin x in Problem 1. Thus we get $\frac{\sqrt{3}}{2} - \frac{1}{2!}(x - \frac{\pi}{6}) - \frac{\frac{\sqrt{3}}{2}}{2!}(x - \frac{\pi}{6})^2 + \frac{\frac{1}{2}}{3!}(x - \frac{\pi}{6})^3 + \frac{\frac{\sqrt{3}}{2}}{4!}(x - \frac{\pi}{6})^4 - \ldots$.

7. $f(x) = (x-1)^{\frac{1}{2}}$, $f'(x) = (x-1)^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{2^2}(x-1)^{-3/2}$, $f'''(x) = \frac{3}{2^3}(x-1)^{-5/2}$, $f^4(x) = -\frac{3\cdot 5}{2^4}(x-1)^{-7/2}$, and so forth. $f(2) = 1$, $f'(2) = \frac{1}{2}$, $f''(2) = -\frac{1}{2^2}$, $f'''(2) = \frac{3}{2^3}$, $f^4(2) = \frac{-3\cdot 5}{2^4}$, and so forth. The Taylor series for f at a = 2 is $1 + \frac{1}{2}(x-2) - \frac{1}{2!2^2}(x-2)^2 + \frac{3}{3!2^3}(x-2)^3 - \frac{3\cdot 5}{4!2^4}(x-2)^4 + \ldots = 1 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \sum_{k=3}^{\infty} \frac{(-1)^{k+1}1\cdot 3\cdots(2k-3)}{k!2^k}(x-2)^k$.

8. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^4(x) = \cos x$, and so forth. $f(\frac{\pi}{3}) = \frac{1}{2}$, $f'(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$, $f''(\frac{\pi}{3}) = -\frac{1}{2}$, $f'''(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $f^4(\frac{\pi}{3}) = \frac{1}{2}$, and so forth. The Taylor series for f at $a = \frac{\pi}{3}$ is $\frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{2}\cdot\frac{1}{2!}(x - \frac{\pi}{3})^2 + \frac{\sqrt{3}}{2}\cdot\frac{1}{3!}(x - \frac{\pi}{3})^3 + \frac{1}{2}\cdot\frac{1}{4!}(x - \frac{\pi}{3})^4$.

9. $f(x) = (1+x)^{-2}$, $f'(x) = -2(1+x)^{-3}$, $f''(x) = 6(1+x)^{-4}$, $f'''(x) = -24(1+x)^{-5}$, $f^4(x) = 120(1+x)^{-6}$, etc. Thus, $f(-2) = 1$, $f'(-2) = 2$, $f''(-2) = 6$, $f'''(-2) = 24$, $f^4(-2) = 120$, etc. The Taylor series for f at a = -2 is $f(-2) + f'(-2)(x+2) + f''(-2)/2!(x+2)^2 + f'''(-2)/3!(x+2)^3 + f^4(-2)/4!(x+2)^4 + \ldots = 1 + 2(x+2) + 3(x+2)^2 + 4(x+2)^3 + 5(x+2)^4 + \ldots \sum_{k=0}^{\infty}(k+1)(x+2)^k$.

10. $f(x) = \ln(1+x)$, $f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$, $f^4(x) = \frac{-6}{(1+x)^4}$, etc. $f(1) = \ln 2$, $f'(1) = 1/2$, $f''(1) = -1/4$, $f'''(1) = 1/4$, $f^{(4)}(1) = -3/8$, etc.; so the Taylor series for f at a = 1 is $\ln 2 + 1/2(x-1) - \frac{1}{4\cdot 2!}(x-1)^2 + \frac{1}{4\cdot 3!}(x-1)^3 - \frac{3}{8\cdot 4!}(x-1)^4 + \ldots = \ln 2 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\cdot 2^k}(x-1)^k$.

11. $f(x) = \csc x$, $f'(x) = -\csc x \cot x$, $f''(x) = \csc^3 x + \csc x \cot^2 x$, $f'''(x) = -5 \csc^3 x \cot x - \csc x \cot^3 x$, $f^4(x) = 18 \csc^3 x \cot^2 x + 5 \csc^5 x + \csc x \cot^4 x$, etc. Thus, $f(\pi/6) = 2$, $f'(\pi/6) = -2\sqrt{3}$, $f''(\pi/6) = 14$, $f'''(\pi/6) = -43\sqrt{3}$, $f^4(\pi/6) = 610$, etc. The Taylor series for f at $a = \pi/6$ is $2 - 2\sqrt{3}(x - \pi/6) + 14/2!(x - \pi/6)^2 - \frac{46\sqrt{3}}{3!}(x - \pi/6)^3 + \frac{610}{4!}(x - \pi/6)^4 + \ldots = 2 - 2\sqrt{3}(x - \pi/6) + 7(x - \pi/6)^2 - \frac{23\sqrt{3}}{3}(x - \pi/6)^3 + \frac{305}{12}(x - \pi/6)^4 + \ldots$.

12. $f(x) = \cot x$, $f'(x) = -\csc^2 x$, $f''(x) = 2 \csc^2 x \cot x$, $f'''(x) = -4 \csc^2 x \cot^2 x - 2 \csc^4 x$, $f^4(x) = 8 \csc^2 x \cot^3 x + 16 \csc^4 x \cot x$, etc. $f(\pi/4) = 1$, $f'(\pi/4) = -2$, $f''(\pi/4) = 4$, $f'''(\pi/4) = -16$, $f^4(\pi/4) = 80$; so the Taylor series for f at $a = \pi/4$ is $1 - 2(x - \pi/4) + \frac{4}{2!}(x - \pi/4)^2 - \frac{16}{3!}(x - \pi/4)^3 + \frac{80}{4!}(x - \pi/4)^4 + \ldots$.

13. $f(x) = \tan x$, $f'(x) = \sec^2 x$, $f''(x) = 2 \sec^2 x \tan x$, $f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$, $f^4(x) = 16 \sec^4 x \tan x + 8 \sec^2 x \tan^3 x$, etc. Thus, $f(\pi/4) = 1$, $f'(\pi/4) = 2$, $f''(\pi/4) = 4$, $f'''(\pi/4) = 16$, $f^4(\pi/4) = 80$, etc.; so the Taylor series for f at $a = \pi/4$ is $1 + 2(x - \pi/4) + 4/2!(x - \pi/4)^2 + 16/3!(x - \pi/4)^3 + 80/4!(x - \pi/4)^4 + \ldots = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2 + 8/3(x - \pi/4)^3 + 10/3(x - \pi/4)^4 + \ldots$.

14. $f(x) = \sec x$, $f'(x) = \sec x \tan x$, $f''(x) = \sec x \tan^2 x + \sec^3 x$, $f'''(x) = \sec x \tan^3 x + 5 \sec^3 x \tan x$, $f^4(x) = 5 \sec^5 x + 18 \sec^3 x \tan^2 x + \sec x \tan^4 x$, etc. So $f(\pi/3) = 2$, $f'(\pi/3) = 2\sqrt{3}$, $f''(\pi/3) = 14$, $f'''(\pi/3) = 46\sqrt{3}$, $f^4(\pi/3) = 610$, etc. Thus, the Taylor series for f at $a = \pi/3$ is $2 + 2\sqrt{3}(x - \pi/3) + 14/2!(x - \pi/3)^2 + \frac{46\sqrt{3}}{3!}(x - \pi/3)^3 + \frac{610}{4!}(x - \pi/3)^4 + \ldots = 2 + 2\sqrt{3}(x - \pi/3) + 7(x - \pi/3)^2 + \frac{23\sqrt{3}}{3}(x - \pi/3)^3 + \frac{305}{12}(x - \pi/3)^4 + \ldots$.

15. $f(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$. The Maclaurin series for f is $\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) - \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x^k}{k!}[1 + (-1)^{k+1}] = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2x^{2n-1}}{(2n - 1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n - 1)!} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k + 1)!}$.

16. Since $\cosh x$ is the derivative of $\sinh x$, then the Taylor series of f at $a = 0$ is $\sum_{n=1}^{\infty} \frac{(2n - 1)x^{2n-2}}{(2n - 1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n)!}$, which is obtained by differentiating the series in Problem 15.

17. Since $e^x = 1 + x + \frac{x^2}{x!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ for all x, then $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$ for all x.

18. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $|x| < 1$.
$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$ for $|x| < 1$.
Thus, $f(x) = \ln \frac{1 + x}{1 - x} = \ln(1 + x) - \ln(1 - x) = [x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots] + [x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots] = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \ldots = 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k - 1}$ for $|x| < 1$.

19. Since $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ for all x, then $\cos^2 x = 1 - x^4/2! + x^8/4! - x^{12}/6! + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!}$ for all x.

20. Since $e^x = 1 + x + x^2/2! + x^3/3! + \ldots = \sum_{k=0}^{\infty} x^k/k!$ for all x, then $e^{-3x} = 1 - 3x + 9x^2/2! - 27x^3/3! + \ldots$ and $xe^{-3x} = x - 3x^2 + 9x^3/2! - 27x^4/3! + \ldots = \sum_{k=0}^{\infty} \frac{(-3)^k x^{k+1}}{k!}$ for all x.

21. Since $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!}$ for all x, $\sin 2x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k+1}}{(2k + 1)!}$ and $x^2 \sin 2x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{2k+3}}{(2k + 1)!}$ for all x.

22. Since $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$ for $|x| < 1$, then $\ln(1 + x^2) = x^2 - x^4/2 + x^6/3 -$

$x^8/4 + \ldots, = \sum \frac{(-1)^k x^{k+3}}{(k+1)}$ for $|x^2| < 1$, or $|x| < 1$.

23. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Now, since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ for all x, then, $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots = 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \ldots$, and $1 - \cos 2x = \frac{2^2x^2}{2!} - \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} - \frac{2^8x^8}{8!} + \ldots$. Thus, $\frac{1}{2}(1 - \cos 2x) = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \frac{2^7x^8}{8!} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}2^{2k-1}x^{2k}}{(2k)!}$.

24. Since $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$ for all x, $e^{-x^3} = 1 - x^3 + x^6/2! - x^9/3! + \ldots$ and $x^2e^{-x^3} = x^2 - x^5 + x^8/2! - x^{11}/3! + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+2}}{k!}$ for all x.

25. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ for all x. Thus, for $x \neq 0$, $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$. But the series $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$ is 1 when $x = 0$. So $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}$ for all x.

26. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ for $|x| < 1$. Thus, $\frac{\tan^{-1}x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots$ for $|x| < 1$ and $x \neq 0$. But when $x = 0$, the series $1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots$ is 1. Hence, $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k+1}$ for $|x| < 1$.

27. $\sin t^2 = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \ldots$. Thus, $\int_0^x \sin t^2\, dt = \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!}$.

28. $\ln(1 + x)^{\frac{1}{x}} = \frac{1}{x}\ln(1 + x)$. Now $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $|x| < 1$. For $x \neq 0$, $\frac{\ln(1 + x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$ $|x| < 1$. But when $x = 0$, the series $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$ is 1. Thus $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k+1}$ for $|x| < 1$.

29. $e^t = 1 + t + t^2/2! + t^3/3! + \ldots$ for all x, so $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \ldots$ for all x, and $f(x) = \int_0^x (1 - t^2 + t^4/2! - t^6/3! + \ldots)dt = x - x^3/3 + x^5/5\cdot 2! - x^7/7\cdot 3! + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$ for all x.

30. $\sin t = t - t^3/3! + t^5/5! - t^7/7! + \ldots$, so $1 - t^2/3! + t^4/5! - t^6/7! + \ldots = \frac{\sin t}{t}$ for $t \neq 0$. When $t = 0$, $1 - t^2/3! + t^4/5! - t^6/7! + \ldots = 1$, so $g(t) = 1 - t^2/3! + t^4/5! - t^6/7! + \ldots$ for all t. Thus, $\int_0^x g(t)dt = \int_0^x (1 - t^2/3! + t^4/5! - t^6/7! + \ldots)dt = x - x^3/3\cdot 3! + x^5/5\cdot 5! - x^7/7\cdot 7! + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k\cdot k!}$.

31. According to Leibniz's theorem, the error of the estimate is no greater than the absolute value of the first neglected term. Since $|(-0.02)^3/3!| = 1.3\times 10^{-6} < 5\times 10^{-5}$, we may use $e^{-0.02} \approx 1 + (-0.02) + (-0.002)^2/2! = 1 - 0.02 + 0.0002 = 0.9802$.

32. By Leibniz's theorem, since $(0.1)^5/5! \approx 8.3\times 10^{-8} < 5\times 10^{-5}$, we may use $\sin(0.1) \approx 0.1 - (0.1)^3/3! = 0.1 - 1.6667\times 10^{-4} = 0.0998$.

33. We use the series expansion for $\ln(1 + x)$ with $x = -0.1$. Since this series is not alternating, we use the Lagrange form of the remainder to bound the error. Now $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - 0)^{n+1} = \frac{(-1)^n n!}{(n+1)!(c+1)^{n+1}} x^{n+1}$, so $|R_n(x)| = \frac{1}{n+1}\left|\frac{x}{c+1}\right|^{n+1}$; and $|R_n(-0.1)| = \frac{1}{n+1}(0.1)^{n+1}\frac{1}{(c+1)^{n+1}}$ where $-0.1 < c < 0$, so $|c + 1| > 0.9$. Thus, $|R_n(-0.1)| < \frac{1}{n+1}\left(\frac{0.1}{0.9}\right)^{n+1} = \frac{1}{n+1}\left(\frac{1}{9}\right)^{n+1}$. The first value of n for which $|R_n(-0.1)| < 5\times 10^{-5}$ is $n = 4$, since $|R_4(-0.1)| = \frac{1}{5}\left(\frac{1}{9}\right)^5 = 3.387\times 10^{-6}$. Thus, $\ln(0.9) \approx -0.1 -$

$(0.1)^2/2 - (0.1)^3/3 + (0.1)^4/4 =$

$-[0.1 + 0.005 + (0.000333\ldots) + 0.000025] = -0.1054.$

34. $\cos 59° = \cos 1.0297443$ radians. We use the series expansion for $\cos x$ around $a = \pi/3$, given in Problem 8: $\cos x = 1/2 - \sqrt{3}/2(x - \pi/3) - \frac{1}{2\cdot 2!}(x - \pi/3)^2 + \frac{\sqrt{3}}{2\cdot 3!}(x - \pi/3)^3 + \frac{1}{2\cdot 4!}(x - \pi/3)^4 - \ldots$. This may be regarded as an alternating series by taking the terms in pairs. Since $\frac{1}{2\cdot 4!}(0.0174533)^4 + \frac{\sqrt{3}}{2\cdot 5!}(0.0174533)^5 \approx 1.9332\times 10^{-9} < 5\times 10^{-5}$, we may use $\cos(1.0297433) \approx 1/2 + \sqrt{3}/2(0.0174533) - \frac{1}{2\cdot 2!}(0.0174533)^2 - \frac{\sqrt{3}}{2\cdot 3!}(0.0174533)^3 = 0.5 + 0.015115 - 7.6154\times 10^{-5} - 7.6738\times 10^{-7} = 0.5150.$

35. $\int_0^t \cos x^2 dx = \int_0^t (1 - x^4/2! + x^8/4! - x^{12}/6! + \ldots)dx = t - t^5/5\cdot 2! + t^9/9\cdot 4! - t^{13}/13\cdot 6! + \ldots$. Since $(0.5)^9/9\cdot 4! \approx 9\times 10^{-6} < 5\times 10^{-5}$, we may use $\int_0^{0.5} \cos x^2\, dx \approx 0.5 - (0.5)^5/5\cdot 2! = 0.5 - 0.003125 = 0.4969.$

36. $x \cos\sqrt{x} = x - x^2/2! + x^3/4! - x^4/6! + \ldots$, so $\int_0^{0.25} x\cos\sqrt{x}\, dx = [x^2/2 - x^3/3\cdot 2! + x^4/4\cdot 4! - x^5/5\cdot 6! + \ldots]_0^{0.25} = (0.25)^2/2 - (0.25)^3/3\cdot 2! + (0.25)^4/4\cdot 4! - (0.25)^5/5\cdot 6! + \ldots$. Since $(0.25)^4/4\cdot 4! = 4.069\times 10^{-5} < 5\times 10^{-5}$, by Leibniz's theorem we may use $\int_0^{0.25} x\cos\sqrt{x}\, dx \approx (0.25)^2/2 - (0.25)^3/3\cdot 2! = 0.03125 - 0.0026042 = 0.0286.$

37. $\int_0^1 e^{-x^2} dx = \int_0^1 (1 - x^2 + x^4/2! - x^6/3! + \ldots)dx = 1 - 1/3 + 1/5\cdot 2! - 1/7\cdot 3! + \ldots$. Since $1/15\cdot 7! \approx 1.32\times 10^{-5} < 5\times 10^{-5}$, we may use $\int_0^1 e^{-x^2} dx \approx 1 - 1/3 + 1/5\cdot 2! - 1/7\cdot 3! + 1/9\cdot 4! - 1/11\cdot 5! + 1/13\cdot 6! = 1 - (0.333\ldots) + 0.1 - 0.0238095 + 0.0046296 - 0.00075758 + 0.00010684 = 0.7468.$

38. $\frac{\sin x}{x} = 1 - x^2/3! + x^4/5! - x^6/7! + \ldots$ so $\int_{0.1}^1 \frac{\sin x}{x} dx = [x - x^3/3\cdot 3! + x^5/5\cdot 5! - x^7/7\cdot 7! + \ldots]_{0.1}^1 = (1 - 1/3\cdot 3! + 1/5\cdot 5! - 1/7\cdot 7! + \ldots) - (0.1 - (0.1)^3.3\cdot 3! + (0.1)^5/5\cdot 5! - (0.1)^7/7\cdot 7! + \ldots).$ By the triangle inequality, the error of the estimate is less than the sum of the errors of each series. Now $1/7\cdot 7! + (0.1)^7/7\cdot 7! = 2\ 8345\times 10^{-5} + 2.8345\times 10^{-12} < 5\times 10^{-5}$, so we may use $\int_{0.1}^1 \frac{\sin x}{x} dx \approx (1 - 1/3\cdot 3! + 1/5\cdot 5!) - (0.1 - (0.1)^3/3\cdot 3! + (0.1)^5/5\cdot 5!) = 1 - 0.0555556 + 1.67\times 10^{-3} - 0.1 + 5.5556\times 10^{-5} + 1.667\times 10^{-8} = 0.8462.$

39. $\int_{0.1}^1 \frac{1 - e^{-x}}{x} dx = \int_{0.1}^1 (1 - x/2! + x^2/3! - x^3/4! + \ldots)dx = [x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \frac{x^4}{4\cdot 4!} + \ldots]_{0.1}^1 = (1 - 1/4 + 1/18 - 1/96 + \ldots) - (0.1 - (0.1)^2/4 + (0.1)^3/8 - (0.1)^4/96 + \ldots)$. To insure accuracy within 5×10^{-5}, the sum of the errors of the two series must be no greater than 5×10^{-5}. Now since $\frac{1}{7\cdot 7!} + \frac{(0.1)^7}{7\cdot 7!} \approx 2.8345\times 10^{-5}$, we may use $\int_{0.1}^1 \frac{1 - e^{-x}}{x} dx \approx (1 - 1/4 + 1/18 - 1/96 + 1/600 - 1/4320) - (0.1 - (0.1)^2/4 + (0.1)^3/18 - (0.1)^4/96 + (0.1)^5/600 - (0.1^6/4320) = 0.6990.$

40. $\int_{0.2}^{0.5} \frac{\ln(1 + x)}{x} dx = \int_{0.2}^{0.5} (1 - x/2 + x^2/3 - x^3/4 + \ldots)dx = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots]_{0.2}^{0.5} = (0.5 - (0.5)^2/4 + (0.5)^3/9 - (0.5)^4/16 + \ldots) - (0.2 - (0.2)^2/4 + (0.2)^3/9 - (0.2)^4/16 + \ldots)$. Since $(0.5^9/81 + (0.2)^9/81 = 2.4119\times 10^{-5} < 5\times 10^5$, we may use $\int_{0.2}^{0.5} \frac{\ln(1+x)}{x} dx \approx (0.5 - (0.5)^2/4 + (0.5)^3/9 - (0.5)^4/16 + (0.5)^5/25 - (0.5)^6/36 + (0.5)^7/49 - (0.5)^8/64) - (0.2 - (0.2)^2/4 + (0.2)^3/9 - (0.2)4/16 + (0.2)^5/25 - (0.2)^6/36 + (0.2)^7/49 - (0.2)^8/64) = 0.2576.$

41. $\int_{0.3}^{1.1} \frac{1 - \cos x}{x} dx = \int_{0.3}^{1.1} (x/2! - x^3/4! + x^5/6! -$

$x^7/8! + \ldots)dx = \left[\frac{x^2}{2\cdot 2!} - \frac{x^4}{4\cdot 4!} + \frac{x^6}{6\cdot 6!} - \ldots\right]_{0.3}^{1.1} =$ $\left(\frac{(1.1)^2}{2\cdot 2!} - \frac{(1.1)^4}{4\cdot 4!} + \frac{(1.1)^6}{6\cdot 6!} - \ldots\right) - \left(\frac{(0.3)^2}{2\cdot 2!} - \frac{(0.3)^4}{4\cdot 4!} + \frac{(0.3)^6}{4\cdot 4!} - \ldots\right)$. Since $\frac{(1.1)^8}{8\cdot 8!} + \frac{(0.3)^8}{8\cdot 8!} = 6.6\times 10^{-6} + 2.034\times 10^{-10} < 5\times 10^{-5}$, we may use $\int_{.3}^{1.1} \frac{1 - \cos x}{x}\,dx \approx \left(\frac{(1.1)^2}{2\cdot 2!} - \frac{(1.1)^4}{4\cdot 4!} + \frac{(1.1)^6}{6\cdot 6!}\right) - \left(\frac{(0.3)^2}{2\cdot 2!} - \frac{(0.3)^4}{4\cdot 4!} + \frac{(0.3)^6}{6\cdot 6!}\right) = 0.3025 - 0.015251 - 0.00041 - 0.0225 + 8.4375\times 10^{-5} - 8.6806\times 10^{-7} = 0.02652.$

42. $\ln(1 + \sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + \ldots$ for $|\sin x| < 1$, so $\int_0^{0.2} \ln(1 + \sin x)dx = \int_0^{0.2} \left(\sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} + \frac{\sin^4 x}{4} + \ldots\right)dx -$ $\cos x + (-x/4 + \frac{1}{8}\sin 2x) + (\frac{1}{9}\cos^3 x - \frac{1}{3}\cos x) + (\frac{-3x}{32} + \frac{1}{16}\sin 2x - \frac{1}{128}\sin 4x) + \ldots]_0^{0.2} =$ $\Big(-\cos 0.2 + [\frac{-0.2}{4} + (1/8)\sin 0.4] + [(1/9)\cos^3 0.2 - (1/3)\cos 0.2] + [-3(0.2)/32 + (1/16)\sin 0.4 - (1/128)\sin 0.8] + \ldots\Big) - \Big(-\cos(0) + [0 + (1/8)\sin(0)] + [(1/9)\cos^3(0) - (1/3)\cos(0)] + [0 + (1/16)\sin(0) - (1/128)\sin(0)] + \ldots\Big)$. Neither Leibniz's theorem nor the Lagrange form of the remainder can be used to bound the error for this series. However, after evaluating the first five terms of the expansion, the reader will observe the sums converging to 0.0187.

Alternative solution: Let $f(x) = \int_0^x \ln(1 + \sin t)dt$ for $-\pi/2 < x < \pi/2$. Then $f'(x) = \ln(1 + \sin x)$, $f''(x) = (1 + \sin x)^{-1}\cos x$, $f'''(x) = -(1 + \sin x)^{-1}$, $f^{(4)}(x) = (1 + \sin x)^{-2}\cos x$, and $f^{(5)}(x) = -(1 + \sin x)^{-3}(2\cos^2 x + \sin x + \sin^2 x)$. Now, for $0 < c < 0.2$, $|f^{(5)}(c)| < 2\cos^2 c + \sin c + \sin^2 c < 2 + c + c^2$, since $\sin c < c$ and $\cos c < 1$. Thus, $|f^{(5)}(c)| < 2.24$. By Taylor's theorem, p. 622, $\int_0^{0.2} \ln(1 + \sin x)dx = f(0.2) = f(0) + \frac{f'(0)}{1!}(0.2) + \frac{f''(0)}{2!}(0.2)^2 + \frac{f'''(0)}{3!}(0.2)^3 + \frac{f^{(4)}(0)}{4!}(0.2)^4 + \frac{f^{(5)}(c)}{5!}(0.2)^5 = 0 + 0 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{3!} + \frac{(0.2)^4}{4!} + R_4$ with error $|R_4| = \left|\frac{f^{(5)}(c)}{5!}(0.2)^5\right| < \frac{2.24}{120}\left(\frac{32}{10^5}\right) < 5\times 10^{-5}$. Thus, $\int_0^{0.2} \ln(1 + \sin x)dx \approx \frac{(0.2)^2}{2} - \frac{(0.2)^3}{3!} + \frac{(0.2)^4}{4!} = 0.0187$, rounded off to four decimal places.

43. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$. When n is odd, $c_n = \frac{(-1)^{\frac{n+3}{2}}}{n}$, $n = 1,3,5,7,\ldots$; and when n is even, $c_n = 0$. Thus, by Definition 2, since $c_n = \frac{f^{(n)}(0)}{n!}$, then $f^{(n)}(0) = n!c_n = 0$ if n is even; and $f^n(0) = n!c_n = \frac{n!(-1)^{\frac{n+3}{2}}}{n} = (-1)^{\frac{n+3}{2}}(n - 1)!$ if n is odd.

44. (a) Define g by $g(x) = f(x + a)$. Since $g(x)$ is a polynomial of degree n, we can write $g(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$. Therefore, $f(x) = f(x + a - a) = g(x - a) = c_0 + c_1(x - a) + c_2(x - a)^2 + \ldots + c_n(x - a)^n$.

(b) Since f is represented by a power series, where $c_i = 0$ for each integer $i > n$, then by Theorem 1 and Definition 2, $c_k = \frac{f^k(a)}{k!}$ for $k = 0, 1, 2, \ldots, n$.

(c) $c_0 = f(2) = 153$, $c_1 = f'(2) = 141$, $c_2 = f''(2)/2 = 75$, $c_3 = f'''(2)/6 = 29$, $c_4 = f^4(2)/24 = 5$ So $f(x) = 153 + 141(x - 2) + 75(x - 2)^2 + 29(x - 2)^3 + 5(x - 2)^4$.

45. $f(x) = x\sin x = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}x^{2j}}{(2j - 1)!}$ for all x. $c_k = \frac{f^k(0)}{k!}$, so $f^k(0) = c_k k!$. Now when k is odd, $c_k = 0$, and so $f^{15}(0) = 0$.

46. $f(x) = \cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \ldots$ for all x, $= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \ldots$ for all x.

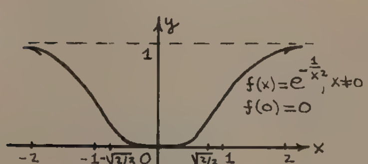
Now $c_k = \frac{f^k(0)}{k!}$, so that $f^{16}(0) = c_{16} \cdot 16! = \frac{16!}{8!}$.

47. $\int_0^x e^{-t^2}dt = x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} - \ldots =$ $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$ for all x by Problem 29. $f^{17}(0) =$ $17!c_{17} = \frac{17!(-1)^8}{17(8!)} = \frac{16!}{8!}$.

48. Using Example 3, we have $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots =$ $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$ for all x. Thus $xe^{-x} =$ $\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k!}$. Therefore, $f^{19}(0) = 19!c_{19} =$ $\frac{19!(-1)^{18}}{18!} = 19$.

49. By Problem 22, $\ln(1 + x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{k}$ for $|x| < 1$. Thus, $f^{20}(0) = 20!c_{20} = \frac{20!(-1)^{11}}{10} =$ $(-2)19!$.

50. (a) To aid in sketching the graph of f, we find, for $x \neq 0$, $f'(x)$ and $f''(x)$. Now $f'(x) = \frac{2e^{-1/x^2}}{x^3}$ and so f is increasing for $x > 0$ and decreasing for $x < 0$. $f''(x) = e^{-\frac{1}{x^2}}\left[\frac{4 - 6x^2}{x^6}\right]$ and $4 - 6x^2 > 0$ for $|x| < \sqrt{\frac{2}{3}}$ and $4 - 6x^2 < 0$ for $|x| > \sqrt{\frac{2}{3}}$. Thus, the graph of f is concave upward for $-\sqrt{\frac{2}{3}} < x < \sqrt{\frac{2}{3}}$ and concave downward for $x < -\sqrt{\frac{2}{3}}$ and for $x > -\sqrt{\frac{2}{3}}$. $\left(\sqrt{\frac{2}{3}} \approx 0.82\right)$. Since $\lim_{x\to+\infty} e^{-1/x^2} = e^0 = 1$, then $y = 1$ is a horizontal asymptote. The graph is symmetric about the y axis.



(b) From part (a) we have, $f'(x) = \frac{2e^{-\frac{1}{x^2}}}{x^3}$, $x \neq 0$ and $f''(x) = e^{-\frac{1}{x^2}}\left(\frac{4 - 6x^2}{x^6}\right)$, $x \neq 0$. Now, $f'''(x) =$ $e^{-\frac{1}{x^2}}\left[\frac{24x^4 - 36x^2 + 8}{x^9}\right]$.

(c) For $n = 1$, $f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}} = 2 \cdot \left(\frac{1}{x}\right)^3 \cdot e^{-\frac{1}{x^2}} =$ $P_1\left(\frac{1}{x}\right) \cdot f(x)$. Now assume $f^k(x) = P\left(\frac{1}{x}\right) \cdot f(x)$. Then $f^{k+1}(x) = D_x\left[P\left(\frac{1}{x}\right) \cdot f(x)\right] = P'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \cdot f(x) +$ $P\left(\frac{1}{x}\right) \cdot f'(x) = Q\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \cdot f(x) + P\left(\frac{1}{x}\right) \cdot 2 \cdot \left(\frac{1}{x}\right)^3 \cdot f(x) =$ $q\left(\frac{1}{x}\right) \cdot f(x) + r\left(\frac{1}{x}\right) \cdot f(x) = \left[q\left(\frac{1}{x}\right) + r\left(\frac{1}{x}\right)\right] \cdot f(x) =$ $p\left(\frac{1}{x}\right) \cdot f(x)$. Hence, $f^n(x) = P\left(\frac{1}{x}\right)f(x)$ holds for $x \neq 0$.

(d) $\frac{f^n(x)}{x} = \frac{1}{x}P\left(\frac{1}{x}\right) \cdot f(x) = p\left(\frac{1}{x}\right) \cdot e^{-\frac{1}{x^2}}$. Let $t = \frac{1}{x}$. As $x \to 0$, then $t \to +\infty$. Thus, we want to show that $\lim_{t\to+\infty} p(t) \cdot e^{-t^2} = 0$; that is, $\lim_{t\to+\infty} (c_0 + c_1t + c_2t^2 + \ldots + c_kt^k) \cdot e^{-t^2}$. So we need only show that $\lim_{t\to+\infty} \frac{t^n}{e^{t^2}} = 0$ for each $n =$ $0, 1, 2, \ldots, k$. But $0 \leq \frac{t^n}{e^{t^2}} \leq \frac{t^n}{e^t}$, so we will show that $\lim_{t\to+\infty} \frac{t^n}{e^t} = 0$. By repeated application of l'Hôpital's rule, $\lim_{t\to+\infty} \frac{t^n}{e^t} = \lim_{x\to+\infty} \frac{x^n}{e^x} = \frac{n!}{e^x} = 0$.

(e) By part (c) we know all derivatives of f exist for $x \neq 0$. Now we show $f^n(0) = 0$ by induction on n. For $n = 1$, $f'(0) = \lim_{x\to0} \frac{f(x) - f(0)}{x} = \lim_{x\to0} \frac{f(x)}{x} =$ $\lim_{x\to0} \frac{e^{-\frac{1}{x^2}}}{e} = \lim_{t\to+\infty} \frac{e^{-t^2}}{\frac{1}{t}} = \lim_{t\to+\infty} \frac{t}{e^{t^2}} = \lim_{x\to+\infty} \frac{x}{e^{x^2}} =$ $\lim_{x\to+\infty} \frac{1}{2xe^{x^2}} = 0$. Now assume $f^k(0) = 0$ and show that $f^{k+1}(0) = 0$. Now $f^{k+1} = \lim_{x\to0} \frac{f^k(x) - f^k(0)}{0} =$ $\lim_{x\to0} \frac{f^k(x)}{x} = 0$ by part (d). Therefore, $f^n(0) = 0$

for all positive integers n. Hence, f is infinitely differentiable on $(-\infty,\infty)$.

(f) The Maclaurin series for f at 0 is $0 + 0 + 0 + \ldots + 0 + \ldots$ since $f^n(0) = 0$ for all n. But $f(x) = 0$ only for $x = 0$, and so $f(x) \neq 0 + 0 + \ldots + 0 + \ldots$ for $x \neq 0$. Thus, f cannot be expanded into a power series about 0.

51. $f(x) = \sum_{k=0}^{\infty} c_k(x - a)^k$, so that $f'(x) = \sum_{k=1}^{\infty} kc_k(x - a)^{k-1} = \sum_{k=n}^{\infty} k(k - 1)\ldots(k - n + 1)c_k(x - a)^{k-n}$ when $n = 1$. Now assume that $f^n(x) = \sum_{k=n}^{\infty} k(k - 1)\ldots(k - n + 1)c_k(x - a)^{k-n}$. Hence, $f^{n+1}(x) = \sum_{k=n+1}^{\infty} k(k - 1)\ldots(k - n + 1)(k - n)c_k(x - a)^{k-n-1} = \sum_{k=n+1}^{\infty} (k(k - 1)\ldots[k - (n + 1) + 1]\cdot c_k(x - a)^{k-(n+1)})$. Therefore, the result holds for all positive integers n.

52. Define $f(x) = \sum_{k=0}^{\infty} b_k(x - a)^k = \sum_{k=0}^{\infty} c_k(x - a)^k$. Now in Theorem 1, take $r = \varepsilon$, so that the hypotheses hold and we can conclude that $b_k = \frac{f^k(a)}{k!}$ and $c_k = \frac{f^k(a)}{k!}$. Hence, $b_k = c_k$ for all nonnegative integers k.

53. $x = a_0 + a_1 10^{-1} + a_2 10^{-2} + \ldots + a_k 10^{-k} + \ldots$, where a_0 is the whole number part of x and a_k is the digit in the 10^{-k} position in the decimal expansion of x. Rounding off to the nearest 10^{-n} is accomplished as follows: if $a_{n+1} < 5$, $r = x - \sum_{k=n+1}^{\infty} a_k 10^{-k}$; if $a_{n+1} \geq 5$, $r = x + 10^{-n} - \sum_{k=n+1}^{\infty} a_k 10^{-k}$. In the first case, $a_{k+1} \leq 4$, so $|x - r| = \sum_{k=n+1}^{\infty} a_k(1/10)^k \leq 4(1/10)^{n+1} + 10^{-(n+2)} \sum_{k=0}^{\infty} 9(1/10)^k = 4\cdot 10^{-(n+1)} + 10^{-(n+2)} \frac{9}{1 - (1/10)} = 4\cdot 10^{-(n+1)} + 10^{-(n+1)} = 5\times 10^{-(n+1)}$. In the second case, $a_{n+1} \geq 5$, so $|x - r| = |x - x - 10^{-n} + \sum_{k=n+1}^{\infty} a_k 10^{-k}| = |-10^{-n} + \sum_{k=n+1}^{\infty} a_k 10^{-k}| = 10^{-n} - \sum_{k=n+1}^{\infty} a_k 10^{-k} = 10^{-n} - (a_{n+1}10^{-(n+1)} + \sum_{k=n+2}^{\infty} a_k(1/10)^k) \leq 10^{-n} - 5\cdot 10^{-(n+1)} - 10^{-(n+2)} \sum_{k=0}^{\infty} (1/10)^k = 10^{-n} - 5\cdot 10^{-(n+1)} - 10^{-(n+2)} \frac{1}{1 - 1/10} = 10^{-n} - (5\cdot 10^{-(n+1)} + 1/9\cdot 10^{-(n+1)}) \leq 10^{-n} - 5.1\times 10^{-(n+1)} \leq 5\times 10^{-(n+1)}$. Thus, in either case, the error $|x - r|$ does not exceed $5\times 10^{-(n+1)}$.

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1. Take $p = 1/4$ in Theorem 1. Then $c_0 = 1$, $c_2 = -3/32$, $c_3 = 7/128, \ldots$. In general, $c_n = \frac{\frac{1}{4}(\frac{1}{4} - 1)(\frac{1}{4} - 2)\ldots(\frac{1}{4} - n + 1)}{n!} = \frac{1(1 - 4)(1 - 8)\ldots[1 - 4(n - 1)]}{4^n n!} = \frac{(-1)^{n+1}(3\cdot 7\cdot 11\cdots(4n - 5))}{4^n n!}$ for $n \geq 2$, so $\sqrt[4]{1 + x} = 1 + (1/4)x + \sum_{k=2}^{\infty} \frac{(-1)^{k+1} 3\cdot 7\cdot 11\cdots(4k - 5)}{4^k k!} x^k$ for $|x| < 1$.

2. Since $\sqrt{1 + x} = (1 + x)^{1/2}$, we take $p = \frac{1}{2}$ in Theorem 1, $c_n = \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)(\frac{1}{2} - 3)\ldots(\frac{1}{2} - n + 1)}{n!} = \frac{(-1)(-3)(-5)\ldots(3 - 2n)}{2^n\cdot n!} = \frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n - 3)]}{2^n\cdot n!}$ for $n \geq 2$. $\sqrt{1 + x} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!} x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-3/2)}{3!} x^3 + \ldots = 1 + \frac{1}{2}x + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{1\cdot 3\cdot 5\cdots(2k - 3)x^k}{2^k k!}$. Thus, $\sqrt{1 + x^2} =$

$1 + \frac{1}{2}x^2 + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k-3)x^{2k}}{2^k k!}$ for $|x| < 1$.

3. Since $\frac{1}{\sqrt[3]{1+x}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k[1\cdot4\cdot7\cdots(3k-2)]x^k}{3^k\cdot k!}$, then $\frac{1}{\sqrt[3]{1-x^2}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k[1\cdot4\cdot7\cdots(3k-2)](-x^2)^k}{3^k\cdot k!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{2k}[1\cdot4\cdot7\cdots(3k-2)]x^{2k}}{3^k\cdot k!} = 1 + \sum_{k=1}^{\infty} \frac{1\cdot4\cdot7\cdots(3k-2)x^{2k}}{3^k\cdot k!}$ for $|x| < 1$.

4. Since $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \sum_{k=2}^{\infty} \frac{(-1)^k 3\cdot5\cdot7\cdots(2k-1)}{2^k k!}x^k$, then $\frac{1}{\sqrt{1-x}} = 1 + (\frac{1}{2})x + \sum_{k=2}^{\infty} \frac{(-1)^k 3\cdot5\cdot7\cdots(2k-1)}{2^k k!}\cdot(-x)^k = 1 + (\frac{1}{2})x + \sum_{k=2}^{\infty} \frac{3\cdot5\cdot7\cdots(2k-1)}{2^k k!}x^k$, and $\frac{2x}{\sqrt{1-x}} = 2x + x^2 + \sum_{k=2}^{\infty} \frac{3\cdot5\cdot7\cdots(2k-1)}{2^{k-1}k!}x^{k+1}$ for $|x| < 1$.

5. $\frac{1}{\sqrt[3]{1+x}} = (1+x)^{-1/3}$. For $p = -\frac{1}{3}$ in Theorem 1, $c_n = \frac{-\frac{1}{3}(-\frac{4}{3})(-\frac{7}{3})\ldots(-\frac{1}{3} - n + 1)}{n!} = \frac{(-1)(-4)(-7)\ldots(-3n+2)}{3^n\cdot n!} = \frac{(-1)^n[1\cdot4\cdot7\cdots(3n-2)]}{3^n\cdot n!}$ for $n \geq 1$. $\frac{1}{\sqrt[3]{1+x}} = 1 - \frac{1}{3}x + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}x^2 + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!}x^3 + \ldots = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k[1\cdot4\cdot7\cdots(3k-2)]x^k}{3^k\cdot k!}$ for $|x| < 1$.

6. $\frac{1}{\sqrt[3]{1-x^2}} = x \cdot \frac{1}{\sqrt[3]{1-x^2}} = x \cdot [1 + \sum_{k=1}^{\infty} \frac{1\cdot4\cdot7\cdots(3k-2)x^{2k}}{3^k\cdot k!}]$ for $|x| < 1$, $= x + \sum_{k=1}^{\infty} \frac{1\cdot4\cdot7\cdots(3k-2)x^{2k+1}}{3^k\cdot k!}$ for $|x| < 1$.

7. We first find $(1+x)^{-1}$, then $(1+2x)^{-1}$, and since $D_x(1+2x)^{-1} = \frac{-2}{(1+2x)^2}$, then the given series is just $\frac{-x}{2}\cdot D_x(1+2x)^{-1}$. Now $(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!} + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k k! x^k}{k!} = \sum_{k=0}^{\infty} (-1)^k x^k$ for $|x| < 1$. Thus, $(1+2x)^{-1} = \sum_{k=0}^{\infty} (-1)^k 2^k x^k$ for $|2x| < 1$, or $|x| < \frac{1}{2}$. Now $D_x(1+2x)^{-1} = \sum_{k=1}^{\infty} (-1)^k 2^k k x^{k-1}$. Therefore, $\frac{-x}{2}\cdot D_x(1+2x)^{-1} = \sum_{k=1}^{\infty} (-1)^{k+1}2^{k-1}kx^k$ for $|x| < \frac{1}{2}$.

8. $(1+x)^{3/2} = 1 + \frac{3}{2}x + \frac{\frac{3}{2}(\frac{1}{2})}{2!}x^2 + \frac{\frac{3}{2}(\frac{1}{2})(-\frac{1}{2})}{3!}x^3 + \frac{(\frac{3}{2})\cdot(\frac{1}{2})\cdot(-\frac{1}{2})\cdot(-\frac{3}{2})x^4}{4!} + \ldots = 1 + \frac{3}{2}x + \frac{3}{2^2\cdot2!}x^2 + \sum_{k=3}^{\infty} \frac{3(-1)^k 1\cdot3\cdots(2k-5)x^k}{2^k\cdot k!}$ for $|x| < 1$. Now $(9+x)^{3/2} = 9^{3/2}(1 + \frac{x}{9})^{3/2} = 27(1 + \frac{x}{9})^{3/2}$. Hence, $(9+x)^{3/2} = 27[1 + \frac{3}{2}(\frac{x}{9}) + \frac{3}{2^2\cdot2!}(\frac{x}{9})^2 + \sum_{k=3}^{\infty} \frac{3\cdot(-1)^k 1\cdot3\cdots(2k-5)}{2^k\cdot k!}(\frac{x}{9})^k$ for $|\frac{x}{9}| < 1$. $81[\frac{1}{3} + \frac{1}{2}(\frac{x}{9}) + \frac{x^2}{2^2\cdot2!9^2} + \sum_{k=3}^{\infty} \frac{(-1)^k 1\cdot3\cdots(2k-5)x^k}{2^k\cdot k!9^k}$ for $|x| < 9$.

9. $\sqrt[3]{27+x} = \sqrt[3]{27(1 + \frac{x}{27})} = 3\sqrt[3]{1 + \frac{x}{27}} = 3[1 + \frac{1}{3}(\frac{x}{27}) + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot5\cdot8\cdot11\cdots(3k-4)}{3^k\cdot k!}(\frac{x}{27})^k] = 3 + \frac{x}{27} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot5\cdot8\cdot11\cdots(3k-4)x^k}{3^{4k-1}\cdot k!}$ for $|\frac{x}{27}| < 1$ or $|x| < 27$.

10. $\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1\cdot3\cdot5\cdot7\cdots(2k-1)}{2^k\cdot k!}x^k$ for $|x| < 1$. So $\frac{1}{\sqrt{1+x^3}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1\cdot3\cdot5\cdots(2k-1)x^{3k}}{2^k\cdot k!}$ for $|x^3| < 1$, or $|x| < 1$.

11. $\sqrt[5]{1+x} = 1 + (1/5)x + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}4\cdot9\cdot14\cdots(5k-6)}{5^k k!}x^k$ for $|x| < 1$, so $\sqrt[5]{1+x^3} = 1 + (1/5)x^3 +$

$\sum_{k=2}^{\infty} \frac{(-1)^{k+1}4\cdot9\cdot14\cdots(5k-6)}{5^k k!}x^{3k}$ for $|x^3| < 1$, or $|x| < 1$.

12. $\frac{1}{\sqrt[3]{1+x}} = 1 - (1/3)x + \sum_{k=2}^{\infty} \frac{(-1)^k 4\cdot7\cdot10\cdots(3k-2)}{3^k k!}x^k$ for $|x| < 1$, so $\frac{1}{\sqrt[3]{1+x^2}} = 1 - (1/3)x^2 + \sum_{k=2}^{\infty} \frac{(-1)^k 4\cdot7\cdot10\cdots(3k-2)}{3^k k!}x^{2k}$ for $|x^2| < 1$, or $|x| < 1$, and $\frac{x}{\sqrt[3]{1+x^2}} = x - (1/3)x^3 + \sum_{k=2}^{\infty} \frac{(-1)^k 4\cdot7\cdot10\cdots(3k-2)}{3^k k!}x^{2k+1}$ for $|x| < 1$.

13. Note $\sqrt{101} = \sqrt{100+1} = \sqrt{100(1+1/100)} = 10\sqrt{1+1/100}$. Now $\sqrt{1+x} = 1 + (1/2)x - (1/8)x^2 + \ldots$, so $\sqrt{1+1/100} \approx 1 + 1/2(0.01) - 1/8(0.01)^2 = 1.0049875$, with an error no greater than $1/16(0.01)^3$, by Leibniz's theorem. Thus $\sqrt{101} = 10\sqrt{1+1/100} \approx 10.049875$, with an error not exceeding $10\cdot(1/16)(0.01)^3 = 6.25\times10^{-7}$.

14. $\sqrt{99} = 100 - 1 = 10\sqrt{1-1/100}$. Substituting $x = -1/100$ into the binomial expansion for $\sqrt{1+x}$, we obtain $\sqrt{1-1/100} \approx 1 - 1/2(0.01) - 1/8(0.01)^2 = 0.9949875$, so $\sqrt{99} \approx 10\cdot(0.9949875) = 9.949875$. We use the Lagrange form of the remainder to conclude that the error is no greater than $10\cdot|R_2(x)| = \left|\frac{f'''(c)}{3!}\right| |0.01|^3$, where $-0.01 < c < 0$. Now $\left|\frac{f'''(c)}{3!}\right| = \frac{1}{16}\left|\frac{1}{(1+c)^{5/2}}\right|$ and $0.99 < 1 + c < 1$; so $(0.99)^{5/2} < (1+c)^{5/2} < 1$ and $\frac{1}{(1+c)^{5/2}} < \frac{1}{(0.99)^{5/2}}$, so the error does not exceed $10\cdot\frac{1}{16(0.99)^{5/2}}(0.01)^3 = 6.409\text{x}10^{-7}$.

15. $\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k-3)x^k}{2^k\cdot k!}$, $|x| < 1$. Thus, for $x = 0.03$, $\sqrt{1+0.03} \approx 1 + \frac{0.03}{2} - \frac{(0.03)^2}{2^2\cdot 2!} = 1.0148875$, with an error no larger than the omitted term (since the series alternates after the first term). Thus, the error does not exceed $\frac{1\cdot3\cdot(0.03)^3}{2^3\cdot3!} = \frac{(0.03)^3}{2^4} \approx 0.0000017 = 1.7\times10^{-6}$. (The true value of $\sqrt{1.03}$ rounded off to six places is 1.014889.)

16. For $|x| < 1$, $\sqrt[5]{1+x} \approx 1 + \frac{1}{5}x + \frac{\frac{1}{5}(\frac{1}{5}-1)}{2!}x^2 = 1 + \frac{1}{5}x - \frac{4}{5^2\cdot2!}x^2$. Thus, $\sqrt[5]{32+1} = (32)^{1/5}\sqrt[5]{1+\frac{1}{32}} \approx 2[1 + \frac{\frac{1}{32}}{5} - \frac{4}{50}(\frac{1}{32})^2] \approx 2(1.006172) = 2.012344$. The error does not exceed $2[\frac{\frac{1}{5}(-\frac{4}{5})(-\frac{9}{5})(\frac{1}{32})^3}{3!}] = \frac{12}{5^3\cdot(32)^3} \approx 0.000003 = 3\times10^{-6}$. (The true value of $\sqrt[5]{33}$ rounded off to seven decimals is 2.0123466.)

17. $\sqrt[4]{1+x} \approx 1 + \quad x + \frac{(\quad - 1)x^2}{2!} = 1 + \frac{x}{4} - \frac{3}{4^2\cdot2!}x^2$ for $|x| < 1$. Thus, $\sqrt[4]{17} = \sqrt[4]{16+1} = \sqrt[4]{16}\cdot\sqrt[4]{1+\frac{1}{16}} \approx 2[1 + \frac{\frac{1}{16}}{4} - \frac{3(\frac{1}{16})^2}{4^2\cdot2!}] \approx 2.030518$. The error does not exceed $2[\frac{\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})}{3!}(\frac{1}{16})^3] = \frac{7}{4^3(16)^3} \approx 0.000027 = 2.7\times10^{-5}$. (The true value of $\sqrt[4]{17}$ rounded off to seven decimals is 2.0305432.)

18. $\frac{1}{\sqrt[3]{1+x}} \approx 1 - \frac{x}{3} + \frac{1\cdot4}{3^2\cdot2}x^2$, for $|x| < 1$. Thus, $\frac{1}{\sqrt[3]{100}} = \frac{1}{\sqrt[3]{125-25}} = \frac{1}{\sqrt[3]{125}\sqrt[3]{1-25/125}} = \frac{1}{5\sqrt[3]{1+(-\frac{1}{5})}} \approx \frac{1}{5}[1 + \frac{1/5}{3} + \frac{2}{9}(\frac{1}{5})^2] = 0.215111$. The error does not exceed $\frac{1}{5}[\frac{|f^3(c)||-1/5|^3}{3!}] = \frac{f^3(c)}{5^3\cdot30}$, where $-\frac{1}{5} \le c \le 0$. Now $f'''(c) = \frac{28}{27}(1+c)^{-10/3}$ and $\frac{4}{5} \le 1 + c \le 1$, so that $\frac{|f^3(c)|}{5^3\cdot30} = \frac{28}{5^3\cdot27\cdot30(1+c)^{10/3}} \le \frac{28(5)^{10/3}}{5^3\cdot27\cdot30\cdot4^{10/3}} = \frac{7\cdot5^{1/3}}{27\cdot30\cdot4^{7/3}} < \frac{7\cdot2}{27\cdot30\cdot4^2} = 0.001080$. Hence, the error does not exceed $\frac{1}{5}(0.001080) \approx 0.000216 = 2.16\times10^{-4}$. (The true value of $\frac{1}{\sqrt[3]{100}}$ rounded off to seven decimal places is 0.2154435.)

19. $\sqrt{1 + x} = 1 + \frac{x}{2} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k - 3)x^k}{2^k k!}$ for $|x| < 1$; hence, $\sqrt{1 + x^3} = 1 + \frac{x^3}{2} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k - 3)}{2^k k!}x^{3k}$ for $|x| < 1$. Thus, $\int_0^{2/3} \sqrt{1 + x^3}\, dx = \frac{2}{3} + \frac{(\frac{2}{3})^4}{8} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k - 3)}{2^k\ k!(3k + 1)}(\frac{2}{3})^{3k+1}$. Since the series alternates (after the first term) and the terms decrease in absolute value, the error involved in using only the first n terms will not exceed $\frac{1\cdot3\cdot5\cdots(2n - 3)}{2^n\ n!(3n + 1)}(\frac{2}{3})^{3n+1}$ in absolute value. For $n = 3$, we have $\frac{1\cdot3\cdot5\cdots(2n - 3)}{2^n\ n!(3n + 1)}(\frac{2}{3})^{3n+1} = \frac{1}{160}(\frac{2}{3})^{10} < \frac{5}{10^4}$. Hence, $\int_0^{2/3} \sqrt{1 + x^3}\, dx \approx \frac{2}{3} + \frac{(\frac{2}{3})^4}{8} - \frac{(\frac{2}{3})^7}{56} \approx$ 0.690.

20. $\frac{1}{\sqrt{1 + x^3}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1\cdot3\cdot5\cdots(2k - 1)x^{3k}}{2^k\cdot k!}$, $|x| < 1$.
$\int_0^{1/2} \frac{1}{\sqrt{1 + x^3}}\, dx = (\frac{1}{2}) + \sum_{k=1}^{\infty} \frac{(-1)^k 1\cdot3\cdot5\cdots(2k-1)(\frac{1}{2})^{3k+1}}{(3k + 1)\ 2^k\cdot k!}$.
The series alternates, so that the error in absolute value does not exceed the value of the first omitted term. Thus, we want $\frac{1\cdot3\cdot5\cdots(2n - 1)}{(3n + 1)2^n\cdot n!2^{3n+1}} \le \frac{5}{10^4}$; and it holds for n = 2. Thus, $\int_0^{1/2} \frac{1}{\sqrt{1 + x^3}}\, dx \approx \frac{1}{2} - \frac{1}{128} \approx 0.492$.

21. $\int_0^{0.5} \sqrt{1 - x^4}\, dx = \int_0^{0.5} (1 - \frac{1}{2}x^4 - \frac{1}{8}x^8 - \frac{1}{16}x^{12} - \dots)dx = x - x^5/10 - x^9/72 - x^{13}/208 - \dots]_0^{0.5} = 0.5 - \frac{(0.5)^5}{10} - \frac{(0.5)^9}{72} - \frac{(0.5)^{13}}{208} - \dots$. Since this series is not alternating, we cannot estimate the error; however, the reader will observe that the series rapidly converges to 0.497 (rounded to three decimal places).

22. $\frac{1}{\sqrt{1 + x^4}} = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \dots$, so $\int_0^{0.5} \frac{dx}{\sqrt{1 + x^4}} = \int_0^{0.5} (1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \dots)dx = [x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \dots]_0^{0.5} = 0.5 - \frac{1}{6}(0.5)^3 + \frac{3}{40}(0.5)^5 - \frac{5}{112}(0.5)^7 + \dots$. Since $\frac{5}{112}(0.5)^7 = 3.4877\times10^{-4} < 5\times10^{-4}$, we may use $\int_0^{0.5} \frac{dx}{\sqrt{1 + x^4}} \approx 0.5 - \frac{1}{6}(0.5)^3 + \frac{3}{40}(0.5)^5 \approx 0.482$.

23. $\int_0^{0.4} \sqrt[3]{1 + x^4}\, dx = \int_0^{0.4} (1 + \frac{1}{3}x^4 - \frac{1}{9}x^8 + \frac{5}{81}x^{12} - \dots)dx = x + x^5/15 - x^9/81 + 5x^{13}/1053 - \dots]_0^{0.4} = 0.4 + (0.4)^5/15 - (0.4)^9/81 + 5(0.4)^{13}/1053 - \dots$. Since $(0.4)^9/81 = 3.2\times10^{-6} < 5\times10^{-4}$, we may use $\int_0^{0.4} \sqrt[3]{1 + x^4}\, dx \approx 0.4 + (0.4)^5/15 \approx 0.401$.

24. $\int_0^{0.2} \frac{dx}{\sqrt[4]{1 + x^2}} = \int_0^{0.2} (1 - \frac{1}{4}x^2 + \frac{5}{32}x^4 - \frac{15}{128}x^6 + \dots)dx = [x - \frac{1}{12}x^3 + \frac{1}{32}x^5 - \frac{15}{896}x^7 + \dots]_0^{0.2} = 0.2 - \frac{1}{12}(0.2)^3 + \frac{1}{32}(0.2)^5 - \frac{15}{896}(0.2)^7 + \dots$. Since $\frac{1}{32}(0.2)^5 = 10^{-5} < 5\times10^{-4}$, we may use $\int_0^{0.2} \frac{dx}{\sqrt[4]{1 + x^2}} \approx 0.2 - 1/12(0.2)^3 \approx 0.199$.

25. $\int_0^1 \sqrt[3]{27 + x^3}\, dx = 3\int_0^1 \sqrt[3]{1 + (x/3)^3}\, dx = 3\int_0^1 (1 + \frac{1}{3}(x/3)^3 - \frac{1}{9}(x/3)^6 + \frac{5}{81}(\frac{x}{3})^9 - \dots)dx = 3[x + \frac{x^4}{324} - \frac{x^7}{45,927} + \frac{5x^{10}}{15,943,230} - \dots]_0^1 = 3[1 + \frac{1}{324} - \frac{1}{45,927} + \frac{5}{15,943,230} - \dots]$. Since $\frac{1}{45,927} = 2.2\times10^{-5} < 5\times10^{-4}$, we may use $\int_0^1 \sqrt[3]{27 + x^3}\, dx \approx 3[1 + \frac{1}{324}] \approx 3.009$.

26. $\frac{1}{\sqrt[4]{16 - x^2}} = \frac{1}{2\sqrt[4]{1 - (x/4)^2}} = \frac{1}{2}(1 + \frac{1}{4}(\frac{x}{4})^2 + \frac{5}{32}(\frac{x}{4})^4 + \frac{15}{128}(\frac{x}{4})^6 + \dots] = \frac{1}{2}(1 + \frac{x^2}{64} + \frac{5x^2}{8192} + \frac{15x^6}{524,288} + \dots)$, so $\int_0^1 \frac{dx}{\sqrt[4]{16 - x^2}} = \frac{1}{2}\int_0^1 (1 + \frac{x^2}{64} + \frac{5x^4}{8192} + \frac{15x^6}{524,288} + \dots)dx =$

$\frac{1}{2}[x + \frac{x^3}{192} + \frac{x^5}{8192} + \frac{15x^6}{3,670,016} + \cdots]_0^1 =$

$\frac{1}{2}[1 + \frac{1}{192} + \frac{1}{8192} + \frac{15}{3,670,016} + \cdots]$. Since this series is not alternating, we cannot estimate the error; however, by evaluating several terms the reader will observe that the series converges to 1.005.

27. Suppose $n = 1$. Then $c_1 = \frac{p - 0}{0 + 1} c_0 = p \cdot c_0 = p \cdot 1 = p$. But $c_1 = \frac{1}{1!} p \cdot (p - 1) \ldots (p - n + 1) = p$ for $n = 1$. Now suppose $c_k = \frac{1}{k!} p(p - 1) \ldots (p - k + 1)$ for $k > 1$. We know that $c_{k+1} = \frac{p - k}{k + 1} c_k$, and so now $c_{k+1} = \frac{(p - k)}{k + 1} \cdot \frac{1}{k!} p \cdot (p - 1) \ldots (p - k + 1) = \frac{1}{(k + 1)!} p \cdot (p - 1)(p - 2) \ldots [p - (k + 1) + 1]$. Hence, the result holds for all n.

28. Replacing n by $n + 1$, we have

(a) $c_{n+1} = \frac{1}{(n+1)!} p \cdot (p - 1)(p - 2) \ldots [p - (n+1) + 1] = \frac{1}{n!(n + 1)} p \cdot (p - 1)(p - 2) \ldots (p - n + 1)(p - n) = \frac{p - n}{n + 1} \cdot c_n$ for $n \geq 0$.

(b) By part (a), $(n + 1)c_{n+1} = p \cdot c_n - nc_n$, so that $(n + 1)c_{n+1} + nc_n = pc_n$. Now $(1 + x)D_x \sum_{k=0}^{\infty} c_k x^k = (1 + x) \sum_{k=1}^{\infty} kc_k x^{k-1} = \sum_{k=1}^{\infty} kc_k x^{k-1} + \sum_{k=1}^{\infty} kc_k x^k = \sum_{n=0}^{\infty} (n + 1)c_{n+1} x^n + \sum_{n=0}^{\infty} (n)c_n x^n = \sum_{n=0}^{\infty} [(n + 1)c_{n+1} + nc_n]x^n = \sum_{n=0}^{\infty} pc_n x^n$ (from above) $= p \sum_{k=0}^{\infty} c_k x^k$.

29. In Problem 7, we found the binomial series for $(1 + x)^{-1} = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - \ldots$, $|x| < 1$. The geometric series expansion is $(1 + x)^{-1} = \frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \ldots$, $|x| < 1$, from Section 11.7. They are identical.

30. $(a + x)^p = [a(1 + \frac{x}{a})]^p = a^p(1 + \frac{x}{a})^p$. Since $|x| < a$, then $|\frac{x}{a}| < 1$. Thus, expanding into a binomial series, we have $(1 + \frac{x}{a})^p = 1 + p(\frac{x}{a}) + \frac{p(p - 1)}{2!}(\frac{x}{a})^2 + \frac{p(p - 1)(p - 2)}{3!}(\frac{x}{a})^3 + \ldots$. Therefore, $a^p(1 + \frac{x}{a})^p = a^p + a^{p-1} \cdot p \cdot x + \frac{a^{p-2}p(p - 1)}{2!}x^2 + \frac{a^{p-3}p(p - 1)(p - 2)}{3!}x^3 + \ldots = a^p + \sum_{k=1}^{\infty} \frac{p \cdot (p - 1)(p - 2) \ldots (p - k + 1)}{k!} a^{p-k} x^k$ for $|x| < a$.

31. Since $n > p$, then $p - n < 0$. Thus, $\frac{p - n}{n + 1} < 0$, and so whichever sign c_{n+1} has, the successive term $\frac{p - n}{n + 1} c_{n+1}$ will have the opposite sign. Hence, $\sum_{k=n+1}^{\infty} c_k x^k$ is an alternating series.

32. We need only show that $|\frac{p - n}{n + 1}| < 1$. Now $n > p$, so that $|p - n| = n - p$. Thus, $\frac{n - p}{n + 1} < 1$ is equivalent to $n - p < n + 1$, or $-p < 1$, or $p > -1$. Thus, since $p > -1$, then $|\frac{p - n}{n + 1}| < 1$.

33. Suppose $p = n$. Then $c_n = 0$ for $n \geq p + 1$. The expansion is correct for all x, since $(1 + x)^p$ in this case is a polynomial, and the binomial theorem is applicable.

34. $\frac{1}{\sqrt{1 + x^2}} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k - 1)}{2^k\, k!} x^{2k}$, so $\int_0^x \frac{dx}{\sqrt{1 + t^2}} = \int_0^x (1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k + 1)}{2^k\, k!} t^{2k})dt =$ $x + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k - 1)}{2^k\, k!(2k + 1)} x^{2k+1} = \sinh^{-1}x$, or $\sinh^{-1}x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$.

35. $\frac{1}{\sqrt{1 + x}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k - 1)}{2^k \cdot k!} x^k$ for $|x| < 1$. Thus, $\frac{1}{\sqrt{1 - x^2}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1 \cdot 3 \cdot 5 \cdots (2k - 1)}{2^k \cdot k!} (-x^2)^k$ for $|-x^2| \leq 1$, and so $\frac{1}{\sqrt{1 - x^2}} = 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)x^{2k}}{2^k \cdot k!}$ for $|x| < 1$. $\sin^{-1}x = \int_0^x \frac{dt}{\sqrt{1 - t^2}} = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)x^{2k+1}}{(2k + 1)2^k \cdot k!}$ for $|x| < 1$.

Review Problem Set, Chapter 11, page 705

1. $\lim_{n\to+\infty} \frac{n(n+1)}{3n^2+7n} = \lim_{n\to+\infty} \frac{1+1/n}{3+\frac{7}{n}} = \frac{1}{3}$. The sequence converges with limit $\frac{1}{3}$.

2. $\lim_{n\to+\infty} \frac{\sin n}{n} = 0$ since $|\sin n| \le 1$ for all n and $n \to +\infty$. The sequence converges with limit 0.

3. $\lim_{n\to+\infty} \frac{\sqrt{n+1}}{\sqrt{3n+1}} = \lim_{n\to+\infty} \sqrt{\frac{n+1}{3n+1}} = \lim_{n\to+\infty} \sqrt{\frac{1+\frac{1}{n}}{3+\frac{1}{n}}} = \sqrt{\frac{1}{3}}$. The sequence converges with limit $\sqrt{\frac{1}{3}}$.

4. $\lim_{n\to+\infty} \frac{7n^3+3n^2-n^3(\frac{1}{2})^n}{3n^2+n^2(\frac{3}{4})^n} = \lim_{n\to+\infty} \frac{7n^3+3n^2-n^3(\frac{2}{3})^n}{3n^2+n^2} =$ $\lim_{n\to+\infty} \frac{7+\frac{3}{n}-(\frac{2}{3})^n}{\frac{3}{n}+\frac{1}{n}} = +\infty$ since $(\frac{2}{3})^n \to 0$ as $n \to +\infty$. The sequence diverges.

5. Each term is either 0 or $\frac{2}{n}$. But $\lim_{n\to+\infty} \frac{2}{n} = 0$. Hence, the sequence converges with limit 0.

6. $\lim_{n\to+\infty} (50+\frac{1}{n})^2 \cdot (1+\frac{n-1}{n^2})^{50} = \lim_{n\to+\infty} (50+\frac{1}{n})^2 \cdot$ $\lim_{n\to+\infty} (1+\frac{1}{n}-\frac{1}{n^2})^{50} = 50^2 \cdot 1 = 50^2$. The sequence converges with limit 2500.

7. Since $\cos\frac{n\pi}{2}$ is either 0, 1, or -1 and since $\sqrt{n}$ approaches $+\infty$ as $n \to +\infty$, it follows that $\lim_{n\to+\infty} \frac{\cos\frac{n\pi}{2}}{\sqrt{n}} = 0$. Thus, the sequence converges with limit 0.

8. Each term is either 0 or 2n. But 2n gets larger as $n \to +\infty$. Thus, the sequence diverges.

9. Each term is either n^2+2n or n^2-2n. $\lim_{n\to+\infty} (n^2+2n) = +\infty$ and $\lim_{n\to+\infty} n(n-2) = +\infty$. Hence, the sequence diverges.

10. The sequence diverges, since for n odd, $\lim_{n\to+\infty} \frac{1}{(n+1)-(1-n)} = \lim_{n\to+\infty} \frac{1}{2n} = 0$; for n even, $\lim_{n\to+\infty} \frac{1}{(n+1)+(1-n)} = \lim_{n\to+\infty} \frac{1}{2} = \frac{1}{2}$.

11. We have already observed that $\lim_{n\to+\infty} \frac{k^n}{n!} = 0$. Hence, $\lim_{n\to+\infty} (1-\frac{3^n}{n!}) = 1$. The sequence converges with limit 1.

12. $\lim_{n\to+\infty} \frac{2^n \cdot n!}{(2n+1)!} =$

$\lim_{n\to+\infty} \frac{2^n \cdot n!}{(2n+1)(2n)(2n-1)(2n-2)\ldots(n+1)n!} =$

$\lim_{n\to+\infty} \frac{2^n}{(2n+1)(2n)(2n-1)(2n-2)\ldots(n+1)} = 0$, since there are n + 1 factors in the denominator, each much larger than 2 for n large, and only n 2's in the numerator. The sequence converges with limit 0.

13. $\{2^n\}$ is increasing.

14. $\{\frac{1}{2^n}\}$ is decreasing.

15. $\{\frac{(-1)^n}{n}\}$ is nonmonotone.

16. $\{(-1)^n\}$ is nonmonotone.

17. No, since the sequence increases and then decreases: $a_1 = -1$, $a_2 = 0$, $a_3 = \frac{1}{3}$.

18. (a) $a_{n+1} - a_n = (\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}) - (\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n} = \frac{2n+2+2n+1}{(2n+1)(2n+2)} - \frac{1}{n} =$

$\frac{4n^2+3n-4n^2-6n-2}{n(2n+1)(2n+2)} = \frac{-3n-2}{n(2n+1)(2n+2)} < 0.$

Thus $a_{n+1} < a_n$, so $\{a_n\}$ is decreasing.

(b) $\{a_n\}$ is bounded below by 0. Hence, $\{a_n\}$ converges, since a decreasing sequence bounded below always converges.

19. Here $\{s_n\}$ is the sequence of partial sums of the geometric series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^{n-1}}$. The sum of the series is $\frac{1}{1-(-\frac{1}{4})} = \frac{4}{5}$. Thus, the sequence is bounded; it is bounded above by 1 and below by $\frac{3}{4}$. The sequence is nonmonotone and it is convergent with limit $\frac{4}{5}$.

20. We want to show that $\lim_{n\to+\infty} a_n \le \lim_{n\to+\infty} b_n$—that is, $\lim_{n\to+\infty} (b_n - a_n) \ge 0$ where $\lim_{n\to+\infty} (b_n - a_n)$ exists—

since $\{a_n\}$ and $\{b_n\}$ converge. Since $a_n \le b_n$, then $b_n - a_n \ge 0$, and we need only show that a sequence of nonnegative terms has a nonnegative limit. Call $\{b_n - a_n\} = \{c_n\}$, and suppose that $\lim\limits_{n\to+\infty} c_n = L < 0$. Then for $\varepsilon = -L$, there exists a number N such that for all $n > N$, $|c_n - L| < -L$; that is, $c_n - L < -L$ or $c_n < 0$, which contradicts the fact that $c_n = b_n - a_n \ge 0$. Thus, $\lim\limits_{n\to+\infty} c_n = \lim\limits_{n\to+\infty} (b_n - a_n)$ is nonnegative, and so $\lim\limits_{n\to+\infty} a_n \le \lim\limits_{n\to+\infty} b_n$.

21. A sequence is a succession of numbers listed in a definite order, whereas a series is an indicated sum of terms of a particular sequence.

22. $s_n = \sum\limits_{k=1}^{n} \frac{k}{(k+1)(k+2)(k+3)} = \sum\limits_{k=1}^{n} [-\frac{1}{2(k+1)} + \frac{2}{k+2} - \frac{3}{2(k+3)}] = \sum\limits_{k=1}^{n} [(-\frac{1}{2(k+1)} + \frac{3}{2(k+2)}) + (\frac{1}{2(k+2)} - \frac{3}{2(k+3)})] = \sum\limits_{k=1}^{n} [\frac{2k+1}{2(k+1)(k+2)} - \frac{(2k+3)}{2(k+2)(k+3)}] = \sum\limits_{k=1}^{n} (\frac{2k+1}{2(k+1)(k+2)} - \frac{[2(k+1)+1]}{2[(k+1)+1][(k+1)+2]}) = \frac{3}{12} - \frac{2n+3}{2(n+2)(n+3)}$. Now $\lim\limits_{n\to+\infty} s_n = \lim\limits_{n\to+\infty} [\frac{1}{4} - \frac{2n+3}{2n^2+10n+12}] = \frac{1}{4}$, since $\lim\limits_{n\to+\infty} \frac{2n+3}{2n^2+10n+2} = \lim\limits_{n\to+\infty} \frac{2+\frac{3}{n}}{2n+10+\frac{2}{n}} = 0$. Thus, $\sum\limits_{k=1}^{\infty} \frac{k}{(k+1)(k+2)(k+3)} = \frac{1}{4}$.

23. $s_n = \sum\limits_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = \sum\limits_{k=1}^{n} [\frac{\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}} - \frac{\sqrt{k}}{\sqrt{k}\sqrt{k+1}}] = \sum\limits_{k=1}^{n} (\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}) = 1 - \frac{1}{\sqrt{n+1}}$. Now $\lim\limits_{n\to+\infty} s_n = \lim\limits_{n\to+\infty} (1 - \frac{1}{\sqrt{n+1}}) = 1$. Therefore, $\sum\limits_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = 1$.

24. $s_n = \sum\limits_{k=1}^{n} \frac{4}{(2k-1)(2k+3)} = \sum\limits_{k=1}^{n} [\frac{1}{(2k-1)} + \frac{-1}{(2k+3)}] = \sum\limits_{k=1}^{n} [(\frac{1}{(2k-1)} - \frac{1}{2k+1}) + (\frac{1}{2k+1} - \frac{1}{2k+3})] = \sum\limits_{k=1}^{n} (\frac{1}{2k-1} - \frac{1}{2k+1}) + \sum\limits_{k=1}^{n} (\frac{1}{2k+1} - \frac{1}{2k+3}) = 1 - \frac{1}{2n+1} + \frac{1}{3} - \frac{1}{2n+3} = \frac{4}{3} - \frac{1}{2n+1} - \frac{1}{2n+3}$. Now $\lim\limits_{n\to+\infty} s_n = \lim\limits_{n\to+\infty} (\frac{4}{3} - \frac{1}{2n+1} - \frac{1}{2n+3}) = \frac{4}{3}$. Hence, $\sum\limits_{k=1}^{\infty} \frac{4}{(2k-1)(2k+3)} = \frac{4}{3}$.

25. We have a telescoping series:

$s_n = \sum\limits_{k=1}^{\infty} [\sin\frac{1}{k} - \sin\frac{1}{k+1}] = \sin 1 - \sin\frac{1}{n+1}$. Now $\lim\limits_{n\to+\infty} [\sin 1 - \sin\frac{1}{n+1}] = \sin 1 - 0 = \sin 1$. Hence, $\sum\limits_{k=1}^{\infty} [\sin\frac{1}{k} - \sin\frac{1}{k+1}] = \sin 1$.

26. $s_n = \sum\limits_{k=1}^{n} (b_k - b_{k+p}) = \sum\limits_{k=1}^{n} [(b_k - b_{k+1}) + (b_{k+1} - b_{k+2}) + \dots + (b_{k+p-1} - b_{k+p})] = \sum\limits_{k=1}^{n} (b_k - b_{k+1}) + \sum\limits_{k=1}^{n} (b_{k+1} - b_{k+2}) + \dots + \sum\limits_{k=1}^{n} (b_{k+p-1} - b_{k+p}) = (b_1 - b_{n+1}) + (b_2 - b_{n+2}) + (b_3 - b_{n+3}) + \dots + (b_p - b_{n+p})$. Now, $\sum\limits_{k=1}^{n} (b_k - b_{k+p}) = \lim\limits_{n\to+\infty} s_n = \lim\limits_{n\to+\infty} [(b_1 - b_{n+1}) + (b_2 - b_{n+2}) + \dots + (b_p - b_{n+p})] = b_1 + b_2 + b_3 + \dots + b_p - \lim\limits_{n\to+\infty} b_{n+1} - \lim\limits_{n\to+\infty} b_{n+2} - \dots - \lim\limits_{n\to+\infty} b_{n+p} = b_1 + b_2 + \dots + b_p - pL$, since $\lim\limits_{n\to+\infty} b_n = L$.

27. $a_n = s_n - s_{n-1} = \frac{3n}{2n+5} - \frac{3(n-1)}{2(n-1)+5} = \frac{3n}{2n+5} - \frac{3n-3}{2n+3} = \frac{15}{(2n+3)(2n+5)}$. The desired series is $\sum\limits_{k=1}^{\infty} \frac{15}{(2k+3)(2k+5)}$. Since $\lim\limits_{n\to+\infty} \frac{3n}{2n+5} = \frac{3}{2}$, the series converges and its sum is $\frac{3}{2}$.

28. $\sum\limits_{k=2}^{\infty} [5(\frac{1}{2})^k + 3(\frac{1}{3})^k] = \sum\limits_{k=2}^{\infty} 5(\frac{1}{2})^k + \sum\limits_{k=2}^{\infty} 3(\frac{1}{3})^k = \frac{\frac{5}{4}}{1 - \frac{1}{2}} + \frac{\frac{3}{9}}{1 - \frac{1}{3}} = \frac{5}{2} + \frac{1}{2} = 3$.

29. $\sum\limits_{k=1}^{\infty} \frac{3}{10^k} = \frac{3/10}{1 - \frac{1}{10}} = \frac{1}{3}$.

30. $\sum_{k=1}^{\infty} 2(-\frac{1}{3})^{k+7} = \frac{2(-\frac{1}{3})^8}{1-(-\frac{1}{3})^9} = \frac{2(3)}{3^9+1} = \frac{6}{19,684} = \frac{3}{9,842}$.

31. $\sum_{k=0}^{\infty}[2(\frac{1}{4})^k + 7(\frac{1}{7})^{k+1}] = \sum_{k=0}^{\infty} 2(\frac{1}{4})^k + \sum_{k=0}^{\infty} 7(\frac{1}{7})^{k+1} =$

$\frac{2}{1-\frac{1}{4}} + \frac{1}{1-\frac{1}{7}} = \frac{8}{3} + \frac{7}{6} = \frac{23}{6}$.

32. Assume $|x| < \frac{1}{B}$, so that $B|x| < 1$. Hence, the geometric series $\sum_{k=0}^{\infty} AB^k x^k$ converges, since the absolute value of the ratio, $|Bx|$, is less than 1. Now, $|a_k x^k| \le AB^k \cdot x^k$, and so $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent by the direct comparison test.

33. The argument is not valid since $\lim_{n\to+\infty} \frac{2n+1}{2n-1} = 0$ is not a sufficient condition for convergence. In fact, $\sum_{k=1}^{\infty} \ln \frac{2k+1}{2k-1}$ is divergent, since it is telescoping, and $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} [-(\ln 1 - \ln(2n+1))] = +\infty$.

34. Since $\{a_n\}$ converges, say to L, we can find N large enough so that for all $n \ge N$, $a_n \approx L$. Thus, the series $\sum_{k=N}^{\infty} a_k b_k$ looks very much like $\sum_{k=N}^{\infty} L\cdot b_k$, which converges since a constant times a convergent series is still convergent. Thus, $\sum_{k=1}^{\infty} a_k b_k$ must converge, since a finite number of terms does not affect convergence. Thus, we have convinced ourselves informally that the series in question converges. More facts about the real numbers are needed for a rigorous proof, which can be found in a mathematical analysis course.

35. $\int_2^{\infty} \frac{1}{x(\ln x)^6}\,dx = \lim_{b\to+\infty} \int_2^b \frac{dx}{x(\ln x)^6} =$

$\lim_{b\to+\infty} [\frac{-(\ln x)^{-5}}{5}]\Big|_2^b = \lim_{b\to+\infty} [\frac{1}{5(\ln 2)^5} - \frac{1}{5(\ln b)^5}] =$

$\frac{1}{5(\ln 2)^5}$. Hence, the integral converges, and so $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^6}$ converges.

36. $\int_1^{\infty} \frac{x}{10+x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{x}{10+x^2}\,dx =$

$\lim_{b\to+\infty} \frac{1}{2}\ln(10+x^2)\Big|_1^b = \lim_{b\to+\infty} [\frac{1}{2}\ln(10+b^2) - \frac{1}{2}\ln 11] =$

$+\infty$. Hence, the integral converges, and so $\sum_{k=1}^{\infty} \frac{k}{10+k^2}$ diverges.

37. $\int_1^{\infty} \frac{x^2}{e^x}\,dx = \lim_{b\to+\infty} \int_1^b x^2e^{-x}dx =$

$\lim_{b\to+\infty} (-x^2e^{-x} - 2xe^{-x} - 2e^{-x})\Big|_1^b =$

$\lim_{b\to+\infty} [-\frac{b^2}{e^b} - \frac{2b}{e^b} - \frac{2}{e^b} + \frac{5}{e}] = 0 - 0 - 0 + \frac{5}{e}$. (The limit of $\frac{-b^2}{e^b}$ and $\frac{-2b}{e^b}$ is obtained by l'Hôpital's rule, and the integration was by parts.) The integral converges, and so the series converges.

38. Consider $f(x) = \frac{\ln x}{x^2}$. f(x) is continuous, decreasing, and nonnegative on $[2,\infty)$. Now $\int_2^{\infty} \frac{\ln x}{x^2}\,dx =$

$\lim_{b\to\infty} \int_2^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} [\frac{-\ln x}{x} - \frac{1}{x}]_2^b = \lim_{b\to\infty} \frac{1+\ln 2}{2} -$

$\frac{1+\ln b}{b} = \lim_{b\to\infty} \frac{1+\ln 2}{2} - \frac{1/b}{1} = \frac{1+\ln 2}{2}$. Thus, $\int_2^{\infty} \frac{\ln x}{x^2}\,dx$ converges; so $\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$ converges also.

39. We compare $\sum_{k=1}^{\infty} \frac{k^2}{k^2+2}(\frac{1}{3})^k$ with the convergent geometric series $\sum_{k=1}^{\infty} (\frac{1}{3})^k$. Now, $(\frac{n^2}{n^2+2})(\frac{1}{3})^n <$ $(\frac{1}{3})^n$ since $\frac{n^2}{n^2+2} < 1$. Hence, the given series converges.

40. We compare the given series with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Now $\frac{1}{3+n!} < \frac{1}{n^2}$ since $n^2 \le n! + 3$ for all $n \ge 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p series, then $\sum_{k=1}^{\infty} \frac{1}{3+k!}$ converges.

41. Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, then so does $\sum_{k=1}^{\infty} \frac{1}{6k}$. Now $\frac{1}{5n+1} \ge \frac{1}{6n}$, $6n \ge 5n+1$ for all $n \ge 1$; therefore, $\sum_{k=1}^{\infty} \frac{1}{5k+1}$ diverges.

42. We compare the series with $\sum_{k=1}^{\infty} \frac{1}{10\sqrt{k}}$; $\frac{1}{\sqrt{10k}} > \frac{1}{10\sqrt{k}}$

since $10\sqrt{k} > \sqrt{10k}$ for all k. Since $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a p series with p < 1, then it diverges and so does any multiple of it. Thus, the given series diverges.

43. Consider $f(x) = \frac{\sqrt{x}}{x + 10}$ and $f'(x) = \frac{10 - x}{2\sqrt{x}(x + 10)^2}$. Thus, f is decreasing for $x \geq 10$. Hence, $\frac{\sqrt{n}}{n + 10}$ decreases for $n \geq 10$. Also, $\lim_{n\to+\infty} \frac{\sqrt{n}}{n + 10} = \lim_{n\to+\infty} \frac{1}{\sqrt{n} + \frac{10}{\sqrt{n}}} = 0$. Hence, the alternating series $\sum_{k=10}^{\infty} \frac{(-1)^k\sqrt{k}}{k + 10}$ converges by Leibniz's theorem, so $\sum_{k=1}^{\infty} \frac{(-1)^k\sqrt{k}}{k + 10}$ converges, too. Now $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k + 10}$ diverges by comparison with $\sum_{k=1}^{\infty} \frac{1}{10\sqrt{k}}$. Hence, the given series is conditionally convergent.

44. Consider $\sum_{k=2}^{\infty} \frac{1}{k^2 + (-1)^k}$. If n is even, then $\frac{1}{n^2 + 1} < \frac{2}{n^2}$ since $n^2 < 2n^2 + 2$ for all n; and if n is odd, then $\frac{1}{n^2 - 1} < \frac{2}{n^2}$ since $n^2 < 2n^2 - 1$ for all n. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, then so is $\sum_{k=2}^{\infty} \frac{2}{k^2}$. Thus, by the direct comparison test, $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + (-1)^k}$ converges absolutely.

45. Since $\frac{n}{n + 1} < 1$ for all n, then $(\frac{n}{n + 1})(\frac{1}{9})^n < (\frac{1}{9})^n$ for all n. Now $\sum_{k=1}^{\infty} (\frac{1}{9})^k$ converges since it is a geometric series with ratio less than 1. Hence, $\sum_{k=1}^{\infty} (\frac{k}{k + 1})(\frac{1}{9})^k$ converges absolutely by the direct comparison test.

46. $\lim_{n\to+\infty} \frac{1}{\ln(1 + \frac{1}{n})} = \frac{1}{\ln[\lim_{n\to\infty} (1 + \frac{1}{n})]} = +\infty$. Hence, the series diverges.

47. $\sum_{k=1}^{\infty} \frac{1 + (-1)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$. Now $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges since it is the harmonic series. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by Leibniz's theorem. Since one series diverges and the other converges, the sum of the two series diverges by Theorem 4, Section 11.3. Hence $\sum_{k=1}^{\infty} \frac{1 + (-1)^k}{k}$ diverges.

48. $\lim_{n\to+\infty} \frac{1}{\ln(e^n + e^{-n})} = \frac{1}{\ln[\lim_{n\to+\infty} (e^n + e^{-n})]} = 0$. Now we want to show that $\frac{1}{\ln(e^{n+1} + e^{-n-1})} \leq \frac{1}{\ln(e^n + e^{-n})}$. But this means we must show $\ln(e^{n+1} + e^{-n-1}) \geq \ln(e^n + e^{-n})$; that is, $e^{n+1} + e^{-n-1} \geq e^n + e^{-n}$. But dividing by e^n, $e + \frac{1}{e^{2n+1}} \geq 1 + \frac{1}{e^{2k}}$ since $e \geq 1 + \frac{1}{e^{2n}} - \frac{1}{e^{2n+1}}$ for all n, since the number on the right is always less than 2. Hence, the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(e^k + e^{-k})}$ converges by Leibniz's theorem. Now, the series $\sum_{k=1}^{\infty} \frac{1}{\ln(e^k + e^{-k})}$ diverges by comparison with $\sum_{k=1}^{\infty} \frac{1}{\ln e^{3k}} = \sum_{k=1}^{\infty} \frac{1}{3k}$. Hence, the given series is conditionally convergent.

49. By the ratio test, $\lim_{n\to+\infty} \frac{\left[\frac{1\cdot3\cdot5\cdots(2n + 1)}{3^{n+1}(n + 1)!}\right]}{\left[\frac{1\cdot3\cdot5\cdots(2n - 1)}{3^n\cdot(n!)}\right]} = \lim_{n\to+\infty} \frac{2n + 1}{3(n + 1)} = \lim_{n\to+\infty} \frac{2 + \frac{1}{n}}{3 + \frac{3}{n}} = \frac{2}{3} < 1$, so that the series converges absolutely.

50. Put $I_k = \int_0^{\pi/2} \sin^k x\, dx$. By a standard reduction formula, $I_k = \frac{k - 1}{k} I_{k-2}$. Also, since $0 \leq \sin x \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$, then $\sin^{2k+1}x \leq \sin^{2k}x \leq \sin^{2k-1}x$ for $0 \leq x \leq \frac{\pi}{2}$, and therefore $I_{2k+1} \leq I_{2k} \leq I_{2k-1}$. From $I_k = \frac{k - 1}{k} I_{k-2}$, we can prove by induction that $\frac{2\cdot4\cdot6\cdots(2k)}{1\cdot3\cdot5\cdots(2k - 1)} = \frac{\pi}{2I_{2k}}$ and that $I_{2k+1} = \frac{1}{2k + 1}\cdot\frac{\pi}{2I_{2k}}$, so that $I_{2k-1} = \frac{2k + 1}{2k} I_{2k+1} =$

$\frac{1}{2k}\,\frac{\pi}{2I_{2k}}$. Hence, $\frac{1}{2k+1}\cdot\frac{\pi}{2I_{2k}} \le I_{2k} \le \frac{1}{2k}\,\frac{\pi}{2I_{2k}}$, and so

$\frac{1}{2k+1}\cdot\frac{\pi}{2} \le I_{2k}{}^2 \le \frac{1}{2k}\cdot\frac{\pi}{2}$ so that $\pi k \le$

$\left[\frac{2\cdot4\cdot6\cdots(2k)}{1\cdot3\cdot5\cdots(2k-1)}\right]^2 \le \pi\cdot\frac{2k+1}{2}$. It follows that

$\lim\limits_{k\to+\infty}\left[\frac{2\cdot4\cdot6\cdots(2k)}{1\cdot3\cdot5\cdots(2k-1)}\right]^2 = +\infty$, so the given series diverges.

51. Consider $\sum\limits_{k=1}^{\infty} e^{-k^2}$. Now $\frac{1}{e^{n^2}} < \frac{1}{n^2}$, since $n^2 < e^{n^2}$. Hence, since $\sum\limits_{k=1}^{\infty}\frac{1}{k^2}$ converges, then by the direct comparison test, $\sum\limits_{k=1}^{\infty} e^{-k^2}$ converges. Thus, the given series is absolutely convergent.

52. $\sin(\pi k + \frac{1}{\ln k}) = \sin(\pi k)\cos\frac{1}{\ln k} + \cos(\pi k)\sin\frac{1}{\ln k} = \frac{\cos \pi k}{\ln k} = \frac{(-1)^{k+1}}{\ln k}$ for $k \ge 2$. Thus, $\sum\limits_{k=2}^{\infty}\sin(\pi k + \frac{1}{\ln k}) = \sum\limits_{k=2}^{\infty}\frac{(-1)^{k+1}}{\ln k}$. Now, $\lim\limits_{n\to+\infty}\frac{1}{\ln n} = 0$, and $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$ for all $n \ge 2$. Thus, the alternating series $\sum\limits_{k=2}^{\infty}\frac{(-1)^{k+1}}{\ln k}$ converges by Leibniz's theorem. Since $\sum\limits_{k=2}^{\infty}\frac{1}{\ln k}$ diverges by comparison with $\sum\limits_{k=2}^{\infty}\frac{1}{k}$, then the given series is conditionally convergent.

53. (a) Since the series is alternating, the error in absolute value will not exceed the absolute value of the first omitted term. Now $\frac{1}{n2^n} < \frac{5}{10^4}$ for $n = 8$.

Thus, we estimate the series by the first seven terms. Hence, $\sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\cdot2^k} \approx \frac{1}{2} - \frac{1}{2^3} + \frac{1}{3\cdot2^3} - \frac{1}{4\cdot2^4} + \frac{1}{5\cdot2^5} - \frac{1}{6\cdot2^6} + \frac{1}{7\cdot2^7} \approx 0.4058$.

(b) We want $\frac{1}{(3n)^3} < \frac{5}{10^4}$. This holds for $n = 5$. Hence, $\sum\limits_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{1}{(3k)^3} = \frac{1}{3^3} - \frac{1}{6^3} + \frac{1}{9^3} - \frac{1}{12^3} = 0.0332$.

54. A series which satisfies the given condition is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{9} + \frac{1}{8} + \frac{1}{27} + \ldots$ where

$$a_k = \begin{cases} (\frac{1}{3})^{k/2} & \text{if } k \text{ is even} \\ (\frac{1}{2})^{\frac{k+1}{2}} & \text{if } k \text{ is odd.}\end{cases}$$

The series converges since it is the sum of two geometric series, each of which converges. Now consider $\lim\limits_{n\to+\infty}\frac{a_{n+1}}{a_n}$ for n even. Then $\frac{a_{n+1}}{a_n} = \frac{(\frac{1}{2})^{\frac{n+2}{2}}}{(\frac{1}{3})^{n/2}} = \frac{3^{n/2}}{2^{n/2}\cdot2} = (\frac{3}{2})^{n/2}\cdot\frac{1}{2}$. Thus, $\lim\limits_{n\to+\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to+\infty}(\frac{3}{2})^n\cdot\frac{1}{2} = +\infty$. If n is odd, then $\frac{a_{n+1}}{a_n} = \frac{(\frac{1}{3})^{\frac{n+1}{2}}}{(\frac{1}{2})^{\frac{n+1}{2}}} = \frac{2^{\frac{n+1}{2}}}{3^{\frac{n+1}{2}}} = (\frac{2}{3})^{\frac{n+1}{2}}$. Then $\lim\limits_{n\to+\infty}(\frac{2}{3})^{\frac{n+1}{2}} = 0$.

Thus, $\lim\limits_{n\to+\infty}\frac{a_{n+1}}{a_n}$ does not exist.

55. $a = 1$. We use the ratio test. $\lim\limits_{n\to+\infty}\left|\dfrac{\frac{(x-1)^{2n+2}}{(n+1)5^{n+1}}}{\frac{(x-1)^{2n}}{n\cdot5^n}}\right| = \lim\limits_{n\to+\infty}\left[\frac{n\cdot5^n}{(n+1)5^{n+1}}\right]|x-1|^2 = \frac{|x-1|^2}{5} < 1$ for $|x-1|^2 < 5$ or $|x-1| < \sqrt{5}$. Thus, the series converges for all x such that $|x-1| < \sqrt{5}$ and diverges for $|x-1| > \sqrt{5}$. Hence, $R = \sqrt{5}$. When $x = 1 - \sqrt{5}$, then the series becomes $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ which diverges because it is a harmonic series. When $x = 1 + \sqrt{5}$, then the series becomes $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ again, and so diverges at this endpoint. Thus, $I = (1 - \sqrt{5}, 1 + \sqrt{5})$.

56. $a = 0$. $\sum\limits_{k=0}^{\infty}(\sin\frac{\pi k}{2})x^k = \sum\limits_{k=0}^{\infty}(-1)^k x^{2k+1}$. Now, we use the ratio test: $\lim\limits_{n\to+\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(-1)^n x^{2n+1}}\right| = \lim\limits_{n\to+\infty}|x^2| = |x^2| < 1$ for $|x| < 1$ or for $-1 < x < 1$. $R = 1$. When $x = 1$ or when $x = -1$, we have divergence since the general term does not approach 0 as n approaches $+\infty$. $I = (-1,1)$.

57. $\sum\limits_{k=0}^{\infty}(\cos \pi k)(x+2)^k = \sum\limits_{k=0}^{\infty}(-1)^k(x+2)^k$. Now

$a = -2$. $\lim_{n\to+\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\right| = \lim_{n\to+\infty} 1 = 1$. So $R = 1$.

The endpoints are $a - R = -2 - 1 = -3$ and $a + R = -1$. When $x = -3$, the series $\sum_{k=0}^{\infty}(-1)^k(-1)^k = \sum_{k=0}^{\infty} 1$ diverges. When $x = -1$, the series $\sum_{k=0}^{\infty}(-1)^k(1)^k = \sum_{k=0}^{\infty}(-1)^k$ diverges. (In both cases, the nth term does not approach 0 as $n \to +\infty$.) Thus, $I = (-3,-1)$.

58. $a = 0$. Now $\lim_{n\to+\infty}\left|\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)(2n+1)}{1\cdot3\cdot5\cdot7\cdots(2n-1)}\right| = \lim_{n\to+\infty}(2n+1) = +\infty$. Hence, $R = 0$. Thus, I consists of the single number 0.

59. $a = 10$. $\lim_{n\to+\infty}\frac{\left[\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)(2n+1)}{2^{3n+4}}\right]}{\left[\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2^{3n+1}}\right]} = \lim_{n\to+\infty}\frac{(2n+1)}{2^3} = +\infty$, and so $R = 0$ by Theorem 1, part (iii), Section 11.6. Thus, I consists of the single number 10.

60. $a = -4$. $\lim_{n\to+\infty}\frac{2^{n+1}}{2^n} = \lim_{n\to+\infty} 2 = 2$. Hence, $R = \frac{1}{2}$ by Theorem 3, Section 11.6. We test the endpoints, $a - R = -\frac{9}{2}$ and $a + R = -\frac{7}{2}$. When $x = -\frac{9}{2}$, then the series becomes $\sum_{k=0}^{\infty}2^k(-\frac{1}{2})^k = \sum_{k=0}^{\infty}(-1)^k$, which diverges. When $x = -\frac{7}{2}$, then the series becomes $\sum_{k=0}^{\infty} 1$, which diverges. Hence, $I = (-\frac{9}{2},-\frac{7}{2})$.

61. $a = -\pi$. $\lim_{n\to+\infty}\left|\frac{\left[\frac{(-1)^{n+1}10^{n+1}}{(n+1)!}\right]}{\left[\frac{(-1)^n10^n}{n!}\right]}\right| = \lim_{n\to+\infty}\frac{10}{n+1} = 0$, so that $R = +\infty$. Thus, $I = (-\infty,\infty)$.

62. $a = -6$. $\lim_{n\to+\infty}\left|\frac{\left[\frac{1\cdot5\cdot9\cdot13\cdots(4n-3)(4n+1)}{2\cdot4\cdot6\cdot8\cdots(2n)(2n+2)}\right]}{\left[\frac{1\cdot5\cdot9\cdot13\cdots(4n-3)}{2\cdot4\cdot6\cdot8\cdots(2n)}\right]}\right| = \lim_{n\to+\infty}\frac{4n+1}{2n+2} = \lim_{n\to+\infty}\frac{4+\frac{1}{n}}{2+\frac{2}{n}} = 2$. Thus, $R = \frac{1}{2}$ by Theorem 1, part (i), Section 11.6. We test the endpoints $a - R = -\frac{13}{2}$ and $a + R = -\frac{11}{2}$. When $x = -\frac{13}{2}$, then the series becomes $\sum_{k=1}^{\infty}\frac{1\cdot5\cdot9\cdot13\cdots(4k-3)(-1)^k}{2\cdot4\cdot6\cdots(2k)2^k}$. Call $c_k = \frac{1\cdot5\cdot9\cdots(4k-3)}{2\cdot4\cdot6\cdots(2k)}$. Since $\frac{c_{k+1}}{2^{k+1}} = \frac{4k+1}{4k+4}\cdot\frac{c_k}{2^k}$, then the terms of the alternating series $\sum_{k=1}^{\infty}c_k(-\frac{1}{2})^k$ are decreasing in absolute value. Also, $\frac{c_k}{2^k} = \frac{1\cdot5\cdot9\cdot13\cdots(4k-3)}{2\cdot4\cdot6\cdot8\cdots(2k)2^k} \le \frac{2\cdot6\cdot10\cdot14\cdots(4k-2)}{2\cdot4\cdot6\cdot8\cdots(2k)\cdot2^k} = \frac{1\cdot3\cdot5\cdot7\cdots(2k-1)}{2\cdot4\cdot6\cdot8\cdots(2k)} \le \sqrt{\frac{1}{\pi k}}$ by the solution to Problem 50. Hence, $\lim_{k\to+\infty}\frac{c_k}{2^k} = 0$, so that $\sum_{k=1}^{\infty}c_k(-\frac{1}{2})^k$ converges by Leibniz's theorem. When $x = -\frac{11}{2}$, the series becomes $\sum_{k=1}^{\infty}c_k\cdot(\frac{1}{2})^k$. Now $c_k\cdot(\frac{1}{2})^k = \frac{1\cdot5\cdot9\cdot13\cdots(4k-3)}{2\cdot4\cdot6\cdot8\cdots(2k)\cdot2^k} \ge \frac{4\cdot8\cdot12\cdots(4k-4)}{1\cdot2\cdot3\cdots k\cdot4^k} = \frac{1\cdot2\cdot3\cdots(k-1)}{1\cdot2\cdot3\cdots k\cdot4} = \frac{1}{4k}$, so the series $\sum_{k=1}^{\infty}c_k(\frac{1}{2})^k$ diverges by comparison with $\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k}$. $I = [-\frac{13}{2},-\frac{11}{2})$.

63. $a = 3$. We use the ratio test.

$\lim_{n\to+\infty}\left|\frac{\left[\frac{(-1)^{n+1}2^{2n+3}(x-3)^{2n+2}}{2n+3}\right]}{\left[\frac{(-1)^n2^{2n+1}(x-3)^{2n}}{2n+1}\right]}\right| = \lim_{n\to+\infty}\frac{2^2|x-3|^2(2n+1)}{(2n+3)} = 4\cdot|x-3|^2 < 1$ provided $|x-3|^2 < \frac{1}{4}$ or $|x-3| < \frac{1}{2}$. Hence, $R = \frac{1}{2}$. The endpoints of the interval of convergence are $\frac{5}{2}$ and $\frac{7}{2}$. When $x = \frac{7}{2}$, the series becomes $\sum_{k=0}^{\infty}\frac{(-1)^k2^{2k+1}}{(2k+1)2^{2k}} = \sum_{k=0}^{\infty}\frac{(-1)^k\cdot2}{2k+1}$, which converges by Leibniz's theorem. When $x = \frac{5}{2}$, we get the same series. Hence, $I = [\frac{5}{2},\frac{7}{2}]$.

64. $a = 0$. We use the fact that $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$. The series can be written $\sum_{k=1}^{\infty}\frac{k(k+1)}{2}x^{2k-1}$. By the ratio test,

$\lim_{n\to+\infty} \left| \frac{\frac{(n+1)(n+2)}{2}}{\frac{n(n+1)}{2}} \cdot \frac{x^{2n+1}}{x^{2n-1}} \right| =$

$\lim_{n\to+\infty} \left(\frac{n^2+3n+2}{n^2+n}\right)|x|^2 = |x|^2 < 1$ or $|x| < 1$, so that $R = 1$. When $x = 1$ or when $x = -1$, the series diverges since the nth term does not approach 0 as $n \to +\infty$. Thus, $I = (-1,1)$.

65. Since $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $|x| < 1$, then $\ln x = \ln[1 + (x - 1)] = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$ for $|x - 1| < 1$ or $0 < x < 2$.

66. By Problem 2 of Problem Set 11.9, we know that $\sqrt{1+x} = 1 + \frac{1}{2}x + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{1\cdot3\cdot5\cdots(2k-3)x^k}{2^k\cdot k!}$ for $|x| < 1$. $\sqrt{4+x} = 2\sqrt{1+\frac{x}{4}} =$ $2[1 + \frac{1}{2}(\frac{x}{4}) + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k-3)x^k}{2^k\cdot k!\,4^k}]$ for $|\frac{x}{4}| < 1$ or $|x| < 4$. Now $\sqrt{x} = \sqrt{4 + (x-4)} =$ $2[1 + \frac{1}{2^3}(x-4) + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k-3)(x-4)^k}{2^{3k}\cdot k!}] =$ $2 + \frac{x-4}{2^2} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot3\cdot5\cdots(2k-3)(x-4)^k}{2^{3k-1}\cdot k!}$ for $|x - 4| < 4$ or $0 < x < 8$.

67. $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, $f'''(x) = e^x$. $f(-1) = \frac{1}{e}$, $f'(-1) = \frac{1}{e}$, $f''(-1) = \frac{1}{e}$, and $f'''(-1) = \frac{1}{e}$. The first four terms of the Taylor series for f at $a = -1$ are $\frac{1}{e} + \frac{1}{e}(x+1) + \frac{1}{e}\frac{(x+1)^2}{2!} + \frac{1}{e}\frac{(x+1)^3}{3!}$.

68. $f(x) = \tan x$, $f'(x) = \sec^2 x$, $f''(x) = 2\sec^2 x \tan x$, $f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$. Therefore, $f(\frac{\pi}{4}) = 1$, $f'(\frac{\pi}{4}) = 2$, $f''(\frac{\pi}{4}) = 2(2)(1) = 4$, $f'''(\frac{\pi}{3}) = 8$. The first four terms of the Taylor series for f at $a = \frac{\pi}{4}$ are $1 + 2(x - \frac{\pi}{4}) + \frac{4(x-\frac{\pi}{4})^2}{2!} + \frac{8(x-\pi/4)^3}{3!}$.

69. $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{2^2}x^{-3/2}$, $f'''(x) = \frac{3}{2^3}x^{-5/2}$. $f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{2^2}$, $f'''(1) = \frac{3}{2^3}$. The first four terms of the Taylor series for f at $a = 1$ are $1 + \frac{1}{2}(x - 1) - \frac{(x-1)^2}{2!\cdot2^2} + \frac{3}{2^3}\cdot\frac{(x-1)^3}{3!}$.

70. $f(x) = \ln\frac{1}{x} = -\ln x$, $f'(x) = -\frac{1}{x}$, $f''(x) = \frac{1}{x^2}$, $f'''(x) = -\frac{2}{x^3}$. $f(2) = -\ln 2$, $f'(2) = -\frac{1}{2}$, $f''(2) = \frac{1}{4}$, $f'''(2) = -2/2^3 = -\frac{1}{4}$. The first four terms of the Taylor series for f at $a = 2$ are $-\ln 2 - \frac{1}{2}(x-2) + \frac{1}{4}\frac{(x-2)^2}{2!} - \frac{1}{4}\frac{(x-2)^3}{3!}$.

71. $g(x) = \sin 2x$, $g'(x) = 2\cos 2x$, $g''(x) = -4\sin 2x$, $g'''(x) = -8\cos 2x$. $g(\frac{\pi}{4}) = 1$, $g'(\frac{\pi}{4}) = 0$, $g''(\frac{\pi}{4}) = -4$, $g'''(\frac{\pi}{4}) = 0$. The first four terms of the Taylor series for g at $a = \frac{\pi}{4}$ are $1 + 0\cdot(x - \frac{\pi}{4}) - \frac{4(x-\frac{\pi}{4})^2}{2!} + \frac{0\cdot(x-\frac{\pi}{4})^3}{3!} = 1 + 0 - \frac{4}{2!}(x-\frac{\pi}{4})^2 + 0$.

72. $h(x) = \sec x$, $h'(x) = \sec x \tan x$, $h''(x) = \sec x \tan^2 x + \sec^3 x$, $h'''(x) = \sec x \tan^3 x + 2\tan x \sec^3 x + 3\sec^3 x \tan x$. $h(\frac{\pi}{6}) = \frac{2}{\sqrt{3}}$, $h'(\frac{\pi}{6}) = \frac{2}{3}$, $h''(\frac{\pi}{6}) = \frac{10}{3\sqrt{3}}$, $h'''(\frac{\pi}{6}) = \frac{42}{9}$. The first four terms of the Taylor series for h at $a = \frac{\pi}{6}$ are $\frac{2}{\sqrt{3}} + \frac{2}{3}(x - \frac{\pi}{6}) + \frac{10}{3\sqrt{3}}\frac{(x-\frac{\pi}{6})^2}{2!} + \frac{42}{9}\cdot\frac{(x-\frac{\pi}{6})^3}{3!}$.

73. Let r be any positive number. For x in the interval $(-1 - r, -1 + r)$, we have $|f^n(x)| = |e^x| = e^x \le e^{r-1}$ since $x < r - 1$. We take $M = e^{r-1}$ in Theorem 1 of Section 11.8, so $f(x) = e^x = \sum_{k=0}^{\infty} \frac{f^k(-1)(x+1)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{e^k}\frac{(x+1)^k}{k!}$ holds for all values of x between $-1 - r$ and $-1 + r$. Since we can choose r as large as we please, then $e^x = \sum_{k=0}^{\infty} \frac{(x+1)^k}{e^k\cdot k!}$ holds for all x.

74. (a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all x.

$f(x) = \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots =$ $\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$ for all x. From Section 11.7, page 687, a power series is continuous on the interior of its interval of convergence. Here $I = (-\infty,\infty)$. Hence f is continuous for all x.

(b) $f'(x) = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{(k+1)!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k+1)(k-1)!} =$ $\frac{1}{2} + \frac{x}{3} + \frac{x^2}{4(2!)} + \frac{x^3}{5(3!)} + \frac{x^4}{6(4!)} + \ldots$ for all x.

(c) For $x = 1$, $\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k+1)(k-1)!} = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} =$ $f'(1)$. Now $f(x) = \frac{e^x - 1}{x}$, so that $f'(x) =$ $\frac{xe^x - e^x + 1}{x^2}$ and $f'(1) = 1$. Thus, $\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1$.

75. $\tan^{-1}t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots$ $|t| < 1$, so that $\int_0^x \tan^{-1}t\,dt = \frac{x^2}{2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \ldots =$ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{(2k-1)2k}$, $|x| < 1$. $x\tan^{-1}x = x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \ldots$ for $|x| < 1$. $\ln(1 + x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$ for $|x| < 1$. Now $x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) =$ $[x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \ldots] -$ $[\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \ldots] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{2k-1} -$ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{2k} = \sum_{k=1}^{\infty} \frac{(2k - 2k + 1)(-1)^{k+1}x^{2k}}{(2k-1)2k} =$ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{(2k-1)2k} = \int_0^x \tan^{-1}t\,dt$ for $|x| < 1$.

76. The binomial series for $\sqrt{1+x}$ is given in Problem 66. Thus, $\sqrt{1 + x^2} = 1 + \frac{x^2}{2} +$ $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot 3\cdot 5\cdots(2k-3)x^{2k}}{2^k\cdot k!}$ for $|x| < 1$.

$\int_0^x \sqrt{1 + t^2}\,dt = x + \frac{x^3}{2\cdot 3} +$ $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}1\cdot 3\cdot 5\cdots(2k-3)x^{2k+1}}{(2k+1)2^k\cdot k!}$ for $|x| < 1$.

77. In Section 11.9, Problem 4, we found that $\frac{1}{\sqrt{1+x}} =$ $1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1\cdot 3\cdot 5\cdot 7\cdots(2k-1)}{2^k\cdot k!}x^k$ for $|x| < 1$. Thus, $\frac{1}{\sqrt{1+x^2}} = 1 + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-1)x^{2k}}{2^k\cdot k!}$ for $|x| < 1$.

78. In Example 1, Section 11.9, we found that $\sqrt[3]{1+x} =$ $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \ldots = 1 + \frac{1}{3}x +$ $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot 5\cdot 8\cdot 11\cdots(3k-4)}{3^k\cdot k!}x^k$, $|x| < 1$. So $\sqrt[3]{1 + x^2} = 1 + \frac{x^2}{3} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot 5\cdot 8\cdot 11\cdots(3k-4)x^{2k}}{3^k\cdot k!}$, $|x| < 1$. Thus, $D_x \sqrt[3]{1 + x^2} = \frac{2x}{3} +$ $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}(2k)2\cdot 5\cdot 8\cdot 11\cdots(3k-4)x^{2k-1}}{3^k\cdot k!}$, $|x| < 1$.

79. $(1 + x)^{2/3} = 1 + \frac{2}{3}x + \frac{\frac{2}{3}(\frac{2}{3} - 1)}{2!}x^2 + \frac{\frac{2}{3}(\frac{2}{3} - 1)(\frac{2}{3} - 2)}{3!}x^3 +$ $\ldots$ for $|x| < 1, = 1 + \frac{2}{3}x + \frac{\frac{2}{3}(-\frac{1}{3})}{2!}x^2 + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{3!}x^3 +$ $\frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{4!}x^4 + \ldots$ for $|x| < 1, = 1 + \frac{2}{3}x +$ $\sum_{k=1}^{\infty} \frac{(-1)^k 2\cdot 1\cdot 4\cdot 7\cdots(3k-2)}{3^{k+1}(k+1)!}x^{k+1}$. $(1 - 2x)^{2/3} =$ $1 + \frac{2}{3}(-2x) + \sum_{k=1}^{\infty} \frac{(-1)^k\cdot 2\cdot 1\cdot 4\cdot 7\cdots(3k-2)(-2x)^{k+1}}{3^{k+1}(k+1)!} =$ $1 - \frac{4}{3}x - \sum_{k=1}^{\infty} \frac{2^{k+2}1\cdot 4\cdot 7\cdots(3k-2)x^{k+1}}{3^{k+1}(k+1)!}$ for $|-2x| < 1$, or $|x| < \frac{1}{2}$.

80. $(16 + x^4)^{1/4} = 16^{1/4}(1 + \frac{x^4}{16})^{1/4} = 2(1 + \frac{x^4}{16})^{1/4}$. $(1 + x)^{1/4} =$ $1 + \frac{1}{4}x + \frac{\frac{1}{4}(-\frac{3}{4})}{2!}x^2 + \frac{1}{4}\frac{(-\frac{3}{4})(-\frac{7}{4})}{3!}x^3 + \ldots$ for $|x| < 1, =$ $1 + \frac{1}{4}x - \frac{1\cdot 3}{2!4^2}x^2 + \frac{1\cdot 3\cdot 7}{3!4^3}x^3 + \ldots$ for $|x| < 1$, $=$ $1 + \frac{1}{4}x + \sum_{k=1}^{\infty} \frac{(-1)^k 3\cdot 7\cdot 11\cdots(4k-1)x^{k+1}}{(k+1)!4^{k+1}}$ for $|x| < 1$.

$(16 + x^4) = 2[1 + \frac{x^4}{16}]^{1/4} = 2(1 + \frac{1}{4}\cdot\frac{x^4}{16} +$ $\sum_{k=1}^{\infty} \frac{(-1)^k 3\cdot 7\cdot 11\cdots(4k-1)x^{4k+4}}{(k+1)!4^{k+1}(16)^{k+1}}$ for $|\frac{x^4}{16}| < 1$, $=$ $2 + \frac{x^4}{32} + \sum_{k=1}^{\infty} \frac{(-1)^k 3\cdot 7\cdot 11\cdots(4k-1)x^{4k-4}}{(k+1)!2^{6k+5}}$ for $|x| < 2$.

81. In Example 1, Section 11.9, we found that $\sqrt[3]{1+x} = 1 + \frac{1}{3}x + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot5\cdot8\cdot11\cdots(3k-4)x^k}{3^k\cdot k!}$, $|x| < 1$. Thus, $\sqrt[3]{1+t^3} = 1 + \frac{1}{3}t^3 + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot5\cdot8\cdot11\cdots(3k-4)t^{3k}}{3^k\cdot k!}$, $|t^3| < 1$, or $|t| < 1$. Therefore, $\int_0^x \sqrt[3]{1+t^3}\,dt = x + \frac{x^4}{3\cdot4} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot5\cdot8\cdots(3k-4)x^{3k+1}}{(3k+1)3^k\cdot k!}$ for $|x| < 1$.

82. We assume that $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots = \sum_{k=0}^{\infty} c_kx^k$. $f(0) = 0$, so that $c_0 = 0$. $f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots$, so that $f'(0) = c_1 = \sqrt{a}$. Now $f''(x) = 2c_2 + 3\cdot2c_3x + 4\cdot3c_4x^2 + \ldots$. Thus, $f''(x) + af(x) = 0$ means $[2c_2 + 3\cdot2c_3x + 4\cdot3c_4x^2 + \ldots] + [ac_0 + ac_1x + ac_2x^2 + ac_3x^3 + \ldots] = 0$, or $[2c_2 + 3\cdot2c_3x + 4\cdot3c_4x^2 + \ldots] + [a\sqrt{a}x + ac_2x^2 + ac_3x^3 + \ldots] = 0$. Thus, $2c_2 = 0$, $3\cdot2c_3 + a\sqrt{a} = 0$, $4\cdot3c_4 + ac_2 = 0$, $5\cdot4c_5 + ac_3 = 0$, so that $c_2 = 0$ and so $c_{2n} = 0$ for all $n \geq 0$.
$c_1 = \sqrt{a}$, $c_3 = \frac{-a\sqrt{a}}{3\cdot2}$, $c_5 = \frac{-ac_3}{5\cdot4} = \frac{a^2\sqrt{a}}{5!}$, $c_7 = \frac{-a^3\sqrt{a}}{7!}$, and so forth. Hence, $c_{2n+1} = \frac{(-1)^n\cdot\sqrt{a}(a)^n}{(2n+1)!}$. Thus, we have $f(x) = \sqrt{a}x - \frac{a\sqrt{a}}{3!}x^3 + \frac{a^2\cdot\sqrt{a}}{5!}x^5 - \frac{a^3\sqrt{a}}{7!}x^7 + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^ka^k\cdot\sqrt{a}\,x^{2k+1}}{(2k+1)!}$.

83. By Problem 16 in Problem Set 11.9, $\sqrt[5]{1+x} \approx 1 + \frac{1}{5}x - \frac{4}{5^2\cdot2!}x^2$ for $|x| < 1$. Thus, $\sqrt[5]{30} = \sqrt[5]{32+(-2)} = 32^{1/5}\sqrt[5]{1+(-\frac{1}{16})} \approx 2[1 + \frac{1}{5}(-\frac{1}{16}) - \frac{4}{50}(-\frac{1}{16})^2] \approx 2[0.987188] = 1.974375$. The error does not exceed $2[\frac{|f^{(3)}(c)||-\frac{1}{16}|^3}{3!}] = \frac{f^{(3)}(c)}{16^3\cdot3}$, where $-\frac{1}{16} \leq c \leq 0$.
Now $f'''(c) = \frac{36}{5^3}(1+c)^{-14/5}$ and $\frac{15}{16} \leq 1 + c \leq 1$, so that $\frac{f^3(c)}{16^3\cdot3} = \frac{36}{5^3\cdot16^3\cdot3(1+c)^{14/5}} \leq \frac{36(16)^{14/5}}{(15)^{14/5}\cdot5^3\cdot16^3\cdot3} = \frac{12}{(15)^{14/5}\cdot125\cdot16^{1/5}} < \frac{12}{(15)^{14/5}\cdot125\cdot15^{1/5}} = \frac{12}{15^3\cdot125} \approx 2.85\times10^{-5}$.

84. In Problem 78, we found $\sqrt[3]{1+x^2} = 1 + \frac{x^2}{3} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2\cdot5\cdot8\cdot11\cdots(3k-4)x^{2k}}{3^k\cdot k!}$ for $|x| < 1$. Hence, $\int_0^{1/2} \sqrt[3]{1+x^2}\,dx \approx 1 + \frac{(\frac{1}{2})^2}{3} - \frac{2(\frac{1}{2})^4}{3^2\cdot2} \approx 1.07639$.
The error in absolute value does not exceed $\left|\frac{(-1)^42\cdot5}{3^3\cdot3!2^6}\right| = \frac{10}{(27)(6)64} = 0.00096 \approx 0.001$.

85. (a) $\sin x + \cos x = \sum_{k=0}^{\infty} \frac{(-1)^kx^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^kx^{2k}}{(2k)!} = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \ldots$ for all x.

(b) $\cos^2x - \sin^2x = \cos 2x = \sum_{k=0}^{\infty} \frac{(-1)^k(2x)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k4^kx^{2k}}{(2k)!} = 1 - 2x^2 + \frac{4^2x^4}{4!} - \frac{4^3}{6!}x^6 + \frac{4^4}{8!}x^8 - \ldots = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8 - \ldots$ for all x.

(c) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, so that $\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^kx^{2k}$, $|x| < 1$. $\tan^{-1}x = \int_0^x \sum_{k=0}^{\infty} (-1)^kx^{2k}dx = \sum_{k=0}^{\infty} \frac{(-1)^kx^{2k+1}}{2k+1}$. Thus, $\tan^{-1}x^3 = \sum_{k=0}^{\infty} \frac{(-1)^kx^{6k+3}}{2k+1} = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \frac{x^{21}}{7} + \frac{x^{27}}{9} - \frac{x^{33}}{11} + \ldots$, $|x| < 1$.

(d) $10^x = e^{\ln 10x} = e^{(\ln 10)x}$. Now $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ for all x. Therefore, $e^{(\ln 10)x} = 10^x = 1 + (\ln 10)x + \frac{(\ln 10)^2x^2}{2!} + \frac{(\ln 10)^3x^3}{3!} + \ldots$ for all x.

86. $f(x) = f(-x) = \sum_{k=0}^{\infty} c_k(-x)^k = \sum_{k=0}^{\infty} (-1)^kc_kx^k = \sum_{k=0}^{\infty} c_kx^k$ for $|x| < R$. By Problem 52 of Problem Set 11.8, the coefficients must be equal, so that $(-1)^kc_k = c_k$ and so $c_k = 0$.

87. $\sin x$.

88. $\cos x$.

89. $f(x) = \frac{1}{x}(x^2 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots) =$

$\frac{1}{x}(x^2 - x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots) =$

$\frac{1}{x}(x^2 - x + \sin x) = x - 1 + \frac{\sin x}{x}.$

90. $e^{\sin x}$.

91. $e^{x \ln 2} = 2^x$.

92. $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k 3^k x^k}{2^k k!} + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k!} x^k =$

$\sum_{k=1}^{\infty} \frac{(-\frac{3x}{2})^k}{k!} + \sum_{k=1}^{\infty} \frac{(-\frac{x}{2})^k}{k!} = e^{-3x/2} - 1 + e^{-x/2} - 1 =$

$e^{-3x/2} + e^{-x/2} - 2$ for all x.